A REPRESENTATION THEOREM FOR MEASURES ON INFINITE DIMENSIONAL SPACES

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If \( X \) is a locally compact, regular topological space, then the well known Riesz representation theorem sets up an isomorphism between the family of all bounded Radón outer measures on \( X \) and the set of continuous positive linear functionals on the family of continuous functions with compact support in \( X \). In this isomorphism corresponding elements, \( l \) a linear functional and \( \mu \) a measure, satisfy the relationship
\[
l(f) = \int f \, d\mu
\]
for all continuous functions \( f \) with compact support in \( X \).

Since an infinite product of locally compact, regular spaces is in general no longer locally compact with respect to the product topology, the Riesz representation theorem fails to hold for such spaces. In this paper, an analogue of the Riesz representation theorem is obtained for this case.

The main idea is to replace the various families mentioned above by the following:

(i) A family \( ^\mathcal{C} \) of cylinders whose elements act like compact sets for a "pseudo-topology" \( \mathcal{G} \), where \( \mathcal{G} \) is closed under finite intersections and countable unions and is a subset of the product topology.

(ii) A family \( M \) of bounded outer measures, related to \( ^\mathcal{C} \) and \( \mathcal{G} \) in much the same way as bounded Radón outer measures are related to compact and open sets.

(iii) A family \( F \) of functions depending only on a finite number of coordinates, with respect to which they are continuous and have compact support.

(iv) A family \( L \) of positive linear functionals on the linear span of \( F \).

Under the added hypothesis of \( \sigma \)-compactness of the coordinate spaces, we show that \( L \) and \( M \) are isomorphic in such a way that corresponding elements, \( l \) in \( L \) and \( \mu \) in \( M \), satisfy the relationship
\[
l(f) = \int f \, d\mu
\]
for all \( f \) in \( F \).

Moreover we show that the elements of \( M \) can be viewed as the projective limit measures of projective systems of bounded regular Borel measures.

From the integrability of the members of \( F \), it follows that all bounded Borel functions which depend only on a finite number of coordinates are also integrable. Thus the simple functions used by Šilov [7] and the tame functions used by Segal [6] and Gross [2] in the development of an
integration theory on Hilbert space are included among the integrable functions of the measures considered here. (For a good guide to the literature in this area see the bibliography in Gross [3].) Our results therefore not only characterize an important class of linear functionals in terms of projective limits of regular Borel measures, but also enable one to extend these functionals to a much wider class of functions through a standard integral with respect to a measure, thereby obviating the need to develop a special theory of integration in infinite dimensional spaces for this purpose.

1. General notation.
(1) $\emptyset$ is the empty set.
(2) $\omega$ is the set of natural numbers.
(3) $\mathbb{R}$ is the set of real numbers.
(4) $\mathscr{C}$ is a compact family if and only if for every subfamily $\mathcal{A}$ of $\mathscr{C}$, if the intersection of any finite number of members of $\mathcal{A}$ is nonvoid, then the intersection of all members of $\mathcal{A}$ is nonvoid.
(5) For $f$ a function on $X$ to $\mathbb{R}$ and $A \subset X$, $f|A$ is the restriction of $f$ to $A$,
\[1_A\] is the characteristic function of $A$,
\[\|f\|_\infty = \sup \{|f(x)| : x \in X\},\]
\[f^+(x) = \max \{0, f(x)\}\] for $x \in X$,
support $f =$ closure $\{x: f(x) > 0\}$ if $X$ is a topological space.
(6) If for $n \in \omega$, $\alpha_n$ is a set, $a_n \in \mathbb{R}$, $f_n$ is a function on $X$ to $\mathbb{R}$, then
$\alpha_n \uparrow \alpha$ if and only if $\alpha_n \subset \alpha_{n+1}$ and $\bigcup_{n \in \omega} \alpha_n = \alpha$,
$a_n \uparrow a$ if and only if $a_n \leq a_{n+1}$ and $\lim_{n \in \omega} a_n = a$,
\[f_n \uparrow f\] if and only if for all $x \in X$, $f_n(x) \leq f_{n+1}(x)$ and $\lim_{n \in \omega} f_n(x) = f(x)$.
(7) For $I$ an index set and $X_i$ a set for each $i \in I$,
$\Pi_{i \in I} X_i = \{x : x_i \in X_i \text{ for each } i \in I\}$.
(8) $\mu$ is a Carathéodory measure on $X$ if and only if $\mu$ is a function on the family of all subsets of $X$ such that $\mu(\emptyset) = 0$ and $0 \leq \mu(A) \leq \sum_{n \in \omega} \mu(B_n) \leq \infty$ whenever $A \subset \bigcup_{n \in \omega} B_n \subset X$.
(9) For $\mu$ a Carathéodory measure on $X$, $A$ is $\mu$-measurable if and only if $A \subset X$ and for every $B \subset X$, $\mu(B) = \mu(B \cap A) + \mu(B - A)$.
$\mathcal{M}_\mu = \{A : A \text{ is } \mu\text{-measurable}\}$.
(10) $\mu$ is a $\mathcal{G}$-outer measure on $X$ if and only if $\mu$ is a Carathéodory measure on $X$, $\mathcal{G} \subset \mathcal{M}_\mu$, and for every $A \subset X$, $\mu(A) = \inf \{\mu(B) : B \in \mathcal{G} \text{ and } A \subset B\}$.
(11) $\mu$ is the Carathéodory measure on $X$ generated by $\tau$ and $\mathcal{G}$ if and only if $\mathcal{G}$ is a family of subsets of $X$, $\tau(A) \geq 0$ for every $A \in \mathcal{G}$, and for $B \subset X$, $\mu(B) = \inf \{\sum_{A \in \mathcal{G}} \tau(A) : \mathcal{H} \subset \mathcal{G}, \mathcal{H} \text{ is countable and } B \subset \bigcup_{A \in \mathcal{H}} A\}$.


Theorem for Measures on Infinite Dimensional Spaces

(12) For $X$ a topological space, $\mu$ is a Radon outer measure on $X$ if and only if $\mu$ is a Carathéodory measure on $X$ such that

1. open sets are $\mu$-measurable,
2. if $C$ is compact then $\mu(C) < \infty$,
3. if $A$ is open then $\mu(A) = \sup \{\mu(C) : C \text{ is compact}, C \subset A\}$,
4. if $B \subset X$ then $\mu(B) = \inf \{\mu(A) : A \text{ is open}, B \subset A\}$.

(13) For $X$ a topological space, $\mu$ is the topological measure cranked by $\tau$ if and only if $\tau$ is a function on the family of closed compact subsets of $X$, $\tau_*(A) = \sup \{\tau(C) : C \text{ is closed compact and } C \subset A\}$ for $A$ an open subset of $X$, and $\mu$ is the Carathéodory measure on $X$ generated by $\tau_*$ and the family of open subsets of $X$.

(14) Remarks. We mention here two well known facts about Carathéodory measures:

1. The Carathéodory measure on $X$ generated by $\tau$ and $\mathcal{C}$ is in fact a Carathéodory measure on $X$.
2. If $X$ is locally compact and regular and $\tau$ is a function on the family of closed compact subsets of $X$ such that for $A, B$ closed and compact we have $0 \leq \tau(A) \leq \tau(A \cup B) \leq \tau(A) + \tau(B) < \infty$ and $\tau(A \cup B) = \tau(A) + \tau(B)$ if $A \cap B = \emptyset$, then the topological measure cranked by $\tau$ is a Radon outer measure on $X$. (See for example Sion [8].)

2. The family $\mathcal{C}$ of cylinders. Throughout this paper we suppose that $T$ is any index set and that for each $t \in T$, $Y_t$ is a locally compact, $\sigma$-compact and regular topological space.

2.1. Definitions.

1. $X = \prod_{t \in T} Y_t$.
2. $I$ is the set of nonvoid finite subsets of $T$, ordered by inclusion.

For $i, j \in I$ with $i \subseteq j$

3. $X_i = \prod_{t \in i} Y_t$ is equipped with the product topology (which is locally compact, $\sigma$-compact and regular),
4. $\mathcal{K}_i$ is the family of closed compact subsets of $X_i$,
5. $\pi_i$ (respectively $\pi_{ij}$) is the canonical projection of $X$ (respectively $X_i$) onto $X_i$,
6. For $A \subset X_i$, $\text{cyl} \ A = \pi_i^{-1}[A]$.

If no confusion is possible we will for $t \in T$ identify $t$ and $\{t\}$, $Y_t$ and $X_{\{t\}}$. Thus $Y_t = X_{\{t\}} = X_t$ and $\mathcal{K}_{\{t\}} = \mathcal{K}_t$.

2.2. Definitions.

1. $\mathcal{C} = \{\alpha : \text{there exists } i \in I \text{ and } \beta \in \mathcal{K}_i \text{ with } \alpha = \text{cyl} \beta\}$. Thus $\mathcal{C}$ is the family of cylinder sets which for some $i \in I$ have a compact base in $X_i$. 


(2) \( \mathcal{C}_o \) is the closure under finite intersections of the family of complements of sets in \( \mathcal{C} \).

(3) \( \mathcal{C} \) is the closure of \( \mathcal{C}_o \) under countable unions.

The essential properties of \( \mathcal{C} \) are the following:

**Theorem 2.3.** \( \mathcal{C} \) is a compact family.

**Corollary 2.4.** The closure of \( \mathcal{C} \) under finite unions is a compact family.

**Corollary 2.5.** If \( \alpha \in \mathcal{C} \) and, for each \( n \in \omega \), \( B_n \in \mathcal{C}_o \) with \( \alpha \subset \bigcup_{n \in \omega} B_n \) then there exists \( N \in \omega \) such that \( \alpha \subset \bigcup_{n=0}^N B_n \).

**Proof of 2.3.** Let \( \mathcal{A} \) be any subfamily of \( \mathcal{C} \) such that for any nonvoid finite \( B \subset \mathcal{A} \) we have \( \bigcap_{\alpha \in B} \alpha \neq \emptyset \). For each \( \alpha \in \mathcal{A} \) let \( i_\alpha \in I \) be such that \( \alpha = \text{cycl} \beta \) for some \( \beta \in \mathcal{H}_{i_\alpha} \). Let \( S = \bigcup_{\alpha \in \mathcal{A}} i_\alpha \) and for each \( t \in S \) choose \( \alpha_t \in \mathcal{A} \) with \( t \in i_{\alpha_t} \) and let \( C_t = \pi_t[\alpha_t] \). Then \( C_t \) is compact in \( X_t \) and \( C_t \neq \emptyset \). Let \( z \) be a fixed point in \( X \) with \( z_t \in C_t \) for each \( t \in S \).

Then \( C = \{ x \in X : x_t \in C_t \text{ for } t \in S \text{ and } x_t = z_t \text{ for } t \in T-S \} \) is a compact subset of \( X \) with respect to the product topology. Now let \( \mathcal{B} \) be the family of nonvoid finite subsets of \( \mathcal{A} \). Then \( \mathcal{B} \) is directed by inclusion. If for each \( B \in \mathcal{B} \) we let \( T_B = \bigcup_{\alpha \in B} i_\alpha \), then \( T_B \) is finite. For each \( B \in \mathcal{B} \) choose \( y_B^B \in \bigcap_{\beta \in B} \beta \cap \bigcap_{t \in T_B} \alpha_t \). Then \( y_B^B \in C_t \) for each \( t \in T_B \) and if \( x_B^B \) is defined by \( x_t^B = y_t^B \) for \( t \in T_B \) and \( x_t^B = z_t \) for \( t \in T - T_B \), then \( x_B^B \in \bigcap_{\beta \in B} \beta \) and \( x_B^B \in C \). Hence \( \{ x_B^B ; B \in \mathcal{B} \} \) is a net in \( C \), and since \( C \) is compact, this net has a cluster point \( x \). If \( \alpha \in \mathcal{A} \) then \( x_B^B \in \alpha \) for any \( B \in \mathcal{B} \) with \( \{ \alpha \} \subset B \). Therefore the net \( \{ x_B^B ; B \in \mathcal{B} \} \) is eventually in \( \alpha \) for each \( \alpha \in \mathcal{A} \). Hence \( x \in \alpha \) for each \( \alpha \in \mathcal{A} \) and so \( \bigcap_{\alpha \in \mathcal{A}} \alpha \neq \emptyset \).


**Proof of 2.5.** Immediate from the definition of \( \mathcal{C}_o \) and 2.4.

The following well known elementary lemma will be needed later:

**Lemma 2.6.** If \( i \in I \), \( A, B \) are open in \( X_i \), \( \gamma \in \mathcal{H}_i \) and \( \gamma \subset A \cup B \), then there exist \( \alpha, \beta \in \mathcal{H}_i \) with \( \alpha \subset A, \beta \subset B \) and \( \alpha \cup \beta = \gamma \).

3. The family \( \mathcal{M} \) of measures.

3.1. **Definition.** \( \mathcal{M} = \{ \mu : \mu \) is a bounded outer measure on \( X \) such that
(1) \( C \subset \mathcal{M}_i \),
(2) \( \mu(A) = \sup \{ \mu(\alpha); \alpha \in C \text{ and } \alpha \subset A \} \text{ for } A \in \mathcal{C} \),
(3) \( \mu(B) = \inf \{ \mu(A); A \in \mathcal{C} \text{ and } B \subset A \} \text{ for } B \subset X \).

3.2. DEFINITIONS. For any set function \( \tau \) on \( \mathcal{C} \)
(1) \( \tau \) satisfies condition (a) if and only if \( \tau \) is bounded, \( \tau(\emptyset) = 0 \) and for every \( i \in I \) and \( \alpha, \beta \in \mathcal{K}_i \) \( 0 \leq \tau(\text{cyl } \alpha) \leq \tau(\text{cyl } \alpha \cap \text{cyl } \beta) \leq \tau(\text{cyl } \alpha) + \tau(\text{cyl } \beta) \) and \( \tau(\text{cyl } \alpha \cup \text{cyl } \beta) = \tau(\text{cyl } \alpha) + \tau(\text{cyl } \beta) \) if \( \alpha \cap \beta = \emptyset \).
(2) \( \tau \) satisfies condition (b) if and only if for every \( i \in I, \alpha \in \mathcal{K}_i, t \in T-i \) and sequence \( C \in \mathcal{K}_t \) with \( C_n \subset \text{interior } C_{n+1} \) for \( n \in \omega \) and \( C_n \uparrow X_t \), if \( j = i \cup \{t\} \) and \( \beta_n = \{x \in X_j; x \in \alpha \text{ and } x \in C_n\} \) then \( \tau(\text{cyl } \beta_n) \uparrow \tau(\text{cyl } \alpha) \). (Note that we certainly have \( \text{cyl } \beta_n \supset \text{cyl } \alpha \).)
(3) \( \tau_*(A) = \sup \{ \tau(\alpha); \alpha \in \mathcal{C} \text{ and } \alpha \subset A \} \text{ for } A \in \mathcal{C} \).

The key results of this section are summed up in the following

**Theorem 3.3.** Let \( \tau \) satisfy conditions (a) and (b) and \( \mu \) be the Carathéodory measure on \( X \) generated by \( \tau_* \) and \( \mathcal{G} \). Then

(1) \( \mu \in \mathcal{M} \) and \( \mu \) agrees with \( \tau_* \) on \( \mathcal{C} \),
(2) if \( i \in I, \mu_i(A) = \mu(\text{cyl } A) \) for \( A \subset X_i, \tau_i(\alpha) = \tau(\text{cyl } \alpha) \) for \( \alpha \in \mathcal{K}_i \) and \( \nu_i \) is the topological outer measure on \( X_i \) cranked by \( \tau_i \), then \( \nu_i \) is a bounded Radon outer measure and \( \mu_i \) agrees with \( \nu_i \) on \( \mathcal{M}_{\nu_i} \).

For the proof of this theorem two preliminary lemmas are needed.

**Lemma A.** Let \( \tau \) satisfy condition (b). Then for \( i, j \in I \) with \( i \subset j \) and \( \alpha \in \mathcal{K}_i \) we have \( \tau(\text{cyl } \alpha) = \sup \{ \tau(\text{cyl } \beta); \beta \in \mathcal{K}_j \text{ and } \text{cyl } \beta \subset \text{cyl } \alpha \} \).

**Proof.** Follows easily from condition (b) and induction.

**Lemma B.** If \( \tau \) satisfies conditions (a) and (b) then \( \tau_* \) is countably subadditive on \( \mathcal{C} \).

**Proof.** Let \( A_n \in \mathcal{C} \) for \( n \in \omega, \varepsilon > 0 \) and \( \alpha \in \mathcal{C} \) with \( \alpha \subset \bigcup_{n \in \omega} A_n \).
For each \( n \in \omega, A_n = \bigcup_{m \in \omega} B_{nm} \) where \( B_{nm} \in \mathcal{G}_n \).
So \( \alpha \subset \bigcup_{n \in \omega} \bigcup_{m \in \omega} B_{nm} \) and hence by Corollary 2.5 there exist \( N, M \in \omega \) such that \( \alpha \subset \bigcup_{n=0}^N \bigcup_{m=0}^M B_{nm} \). Let, for \( 0 \leq n \leq N, E_n = \bigcup_{m=0}^M B_{nm} \) and \( i_n \) be such that \( E_n = \text{cyl } A_{i_n} \) for some \( A_n \subset X_i \). Let \( i_n \in I \) be such that \( \alpha = \text{cyl } \gamma \) for some \( \gamma \in \mathcal{K}_i \). Let \( i = i_n, i^N = 0, i^N, \) then \( i \) is finite and \( i_n \subset i \). By Lemma A of this section choose \( \beta \in \mathcal{K}_i \) with \( \text{cyl } \beta \subset \alpha \) and \( \tau(\alpha) \leq \tau(\text{cyl } \beta) + \varepsilon \). Now for \( 0 \leq n \leq N, \pi_i[E_n] \) is open in \( X_i \) and \( \beta \subset \pi_i[\alpha] \subset \bigcup_{n=0}^N \pi_i[E_n] \).
By Lemma 2.6 and induction we can, for \(0 \leq n \leq N\), find \(\beta_n \in \mathcal{N}_n\) with \(\beta_n \subset \pi_{[E_n]}\) and \(\beta = \bigcup_{n=0}^{N} \beta_n\). Then \(\text{cyl} \beta = \bigcup_{n=0}^{N} \text{cyl} \beta_n \subset \bigcup_{n=0}^{N} E_n\).

By condition (a) we have \(\tau(\text{cyl} \beta) \leq \sum_{n=0}^{N} \tau(\text{cyl} \beta_n)\). Hence \(\tau(\alpha) \leq \tau(\text{cyl} \beta) + \varepsilon \leq \sum_{n=0}^{N} \tau(\text{cyl} \beta_n) + \varepsilon \leq \sum_{n=0}^{\infty} \tau_*(E_n) + \varepsilon\). It follows that \(\tau_*(\bigcup_{n \in \omega} \alpha_n) = \sup \{\tau(\alpha): \alpha \in \mathcal{E} \text{ and } \alpha \subset \bigcup_{n \in \omega} \alpha_n\} \leq \sum_{n=0}^{\infty} \tau_*(A_n)\).

We now proceed to prove Theorem 3.3.

Proof of 3.3(1) Since, by Lemma B, \(\tau_*\) is countably subadditive on \(\mathcal{E}\) and since \(\text{gr}_1\) is closed under countable unions we have for \(A \in \mathcal{E}\), \(\mu(A) = \tau_*A\) and therefore for \(B \subseteq X\), \(\mu(B) = \inf \{\mu(A): A \in \mathcal{E} \text{ and } B \subseteq A\}\). Furthermore, since clearly \(\mu(\alpha) \leq \tau(\alpha)\) for each \(\alpha \in \mathcal{E}\), it follows that for \(A \in \mathcal{E}\), \(\mu(A) = \sup \{\mu(\alpha): \alpha \in \mathcal{E} \text{ and } \alpha \subset A\}\). Now, since \(\mu\) certainly is a bounded Carathéodory measure on \(X\), all that remains is to show \(\mathcal{E} \subseteq \mathcal{M}_\mu\). So let \(A \in \mathcal{E}, B \subseteq X\) and \(\varepsilon > 0\). Choose \(B' \in \mathcal{E}\) with \(B \subseteq B'\) and \(\mu(B') \leq \mu(B) + \varepsilon\). Let \(\alpha \in \mathcal{E}\) with \(\alpha \subseteq B' \cap A\) and \(\mu(B' \cap A) \leq \mu(\alpha) + \varepsilon\). Let \(\beta \in \mathcal{E}\) with \(\beta \subseteq B' - \alpha\) and \(\mu(B' - \alpha) \leq \mu(\beta) + \varepsilon\). By Lemma A we can suppose that \(\alpha \cup \beta \in \mathcal{E}\) also. Then \(\mu(B \cap A) + \mu(B - A) \leq \mu(B' \cap A) + \mu(B' - \alpha) \leq \mu(\alpha) + \mu(\beta) + 2\varepsilon = \mu(\alpha \cup \beta) + 2\varepsilon \leq \mu(B') + 2\varepsilon \leq \mu(B) + 3\varepsilon\).

Hence \(\mu(B \cap A) + \mu(B - A) = \mu(B)\) for all \(B \subseteq X\). It follows that \(\mathcal{E} \subseteq \mathcal{M}_\mu\).

Proof of 3.3(2) Let \(\nu_i\) be the topological outer measure on \(X_i\), cranked by \(\tau_i\). By condition (a) and Remark 1.14.2 we have that \(\nu_i\) is a bounded Radón outer measure on \(X_i\). Let \(A \subseteq X_i\) be open in \(X_i\). Since \(X_i - A\) is closed and \(X_i\) is \(\sigma\)-compact, we can for \(n \in \omega\) choose \(C_n \in \mathcal{N}_i\) such that \(C_n \uparrow (X_i - A)\). Then \(A = \bigcap_{n \in \omega} (X_i - C_n)\). Since \(\text{cyl} (X_i - C_n) \in \mathcal{E}\) we have

\[
\mu_i(X_i - C_n) = \mu(\text{cyl} (X_i - C_n)) = \tau_*(\text{cyl} (X_i - C_n)) = \sup \{\tau(\beta): \beta \in \mathcal{E} \text{ and } \beta \subseteq \text{cyl} (X_i - C_n)\} = \sup \{\tau(\text{cyl} \alpha): \alpha \in \mathcal{N}_i \text{ and } \alpha \subseteq (X_i - C_n)\} = \nu_i(X_i - C_n).
\]

Furthermore since the \(X_i - C_n\) are \(\mu_i\)-measurable as well as \(\nu_i\)-measurable we have \(\mu_i(A) = \lim_{n \to \omega} \mu_i(X_i - C_n) = \lim_{n \in \omega} \nu_i(X_i - C_n) = \nu_i(A)\). Hence \(\mu_i\) and \(\nu_i\) agree on open sets. If \(D \subseteq X_i\) then \(\nu_i(D) = \inf \{\nu_i(A): A \text{ is open in } X_i \text{ and } D \subseteq A\} = \inf \{\mu_i(A): A \text{ is open in } X_i \text{ and } D \subseteq A\} \geq \mu_i(D)\). Hence \(\mu_i \leq \nu_i\) always.

Now let \(B \in \mathcal{M}_\mu\). Given \(\varepsilon > 0\) choose \(A \subseteq X_i\) with \(B \subseteq A\) and \(\nu_i(A) < \nu_i(B) + \varepsilon\). Since \(\nu_i(A) = \nu_i(B) + \nu_i(A - B)\) we have \(\nu_i(A - B) < \varepsilon\) and consequently \(\mu_i(A - B) < \varepsilon\). But \(\mu_i(A) \leq \mu_i(A - B) + \varepsilon\).
\[ \mu_i(B) < \mu_i(B) + \varepsilon. \] Hence
\[
\mu_i(B) = \inf \{ \mu_i(A) : A \text{ is open in } X_i \text{ and } B \subset A \}
= \inf \{ \nu_i(A) : A \text{ is open in } X_i \text{ and } B \subset A \}
= \nu_i(B). \]

3.4. \textit{M as related to projective limit measures.} Suppose that for each \( i \in I, \mathcal{B}_i \) is the \( \sigma \)-ring generated by \( \mathcal{X}_i \) and \( \nu_i \) is a measure on \( \mathcal{B}_i \). We call \( \{ \nu_i : i \in I \} \) a projective system of measures if whenever \( i, j \in I \) with \( i < j \) we have for \( A \in \mathcal{B}_i \)
\[
\nu_i(A) = \nu_j(\tau_i^{-1}(A)).
\]
We say that the projective system \( \{ \nu_i : i \in I \} \) admits a projective limit measure \( \nu \) if \( \nu \) is a measure on the \( \sigma \)-ring \( \mathcal{B} \) of subsets of \( X \) generated by \( \{ \text{cyl } B : B \in \mathcal{B}_i \text{ for some } i \in I \} \) such that for each \( i \in I \) and \( A \in \mathcal{B}_i \),
\[
\nu(\text{cyl } A) = \nu_i(A).
\]
Such a measure \( \nu \), if it exists, is unique and can thus be called the projective limit measure of the system \( \{ \nu_i : i \in I \} \).

For more general definitions of projective or inverse systems of measures see Choksi [1], Mallory [4] or Meyer [5].

Now, if for \( i \in I \) we call \( \nu_i \) a bounded regular Borel measure whenever \( \nu_i \) is a bounded measure on \( \mathcal{B}_i \) such that for every \( A \in \mathcal{B}_i \)
\[
\nu_i(A) = \inf \{ \nu_i(B) : B \text{ is open and } A \subset B \}
= \sup \{ \nu_i(C) : C \in \mathcal{X}_i \text{ and } C \subset A \}
\]
we then have

**Theorem 3.4.1.** \( \mu \in M \) if and only if \( \mu \) is a \( \mathcal{C} \)-outer measure on \( X \) and \( \mu \mid \mathcal{B} \) is the projective limit measure of a projective system \( \{ \nu_i : i \in I \} \) of bounded regular Borel measures \( \mu_i \) on \( \mathcal{B}_i \).

**Proof.** Suppose \( \mu \in M \). Then \( \mu \) is a \( \mathcal{C} \)-outer measure on \( X \). If \( \mu_i(A) = \mu(\text{cyl } A) \) for \( A \in \mathcal{B}_i \) then clearly \( \{ \mu_i : i \in I \} \) forms a projective system of measures and \( \mu \mid \mathcal{B} \) is clearly the projective limit measure of this system. Using 3.3(2) one can easily check that each \( \mu_i \) is in fact a bounded regular Borel measure on \( \mathcal{B}_i \).

Conversely let \( \mu \) be a \( \mathcal{C} \)-outer measure on \( X \) and \( \mu \mid \mathcal{B} \) be the projective limit measure of a projective system \( \{ \mu_i : i \in I \} \) of bounded regular Borel measures \( \mu_i \) on \( \mathcal{B}_i \). Let for each \( i \in I \) and \( \alpha \in \mathcal{X}_i \),
\[
\tau(\text{cyl } \alpha) = \mu_i(\alpha).
\]
Then \( \tau \) is a set function on \( \mathcal{C} \) satisfying conditions (a) and (b). Let \( \nu \) be the Carathéodory measure on \( X \) generated by \( \tau \) and \( \mathcal{G} \). Then by 3.3(1), \( \nu \in M \). Clearly \( \nu \mid \mathcal{B} \) is the projective limit measure of the system \( \{ \nu_i : i \in I \} \) where \( \nu_i(A) = \nu(\text{cyl } A) \) for \( A \in \mathcal{B}_i \).
From 3.3(2), we see that \( \mu_i = \nu_i \) for each \( i \in I \). Hence \( \mu \mid \mathcal{B} = \nu \mid \mathcal{B} \).
Since \( \mathcal{G} \subset \mathcal{B}, \mu \mid \mathcal{G} = \nu \mid \mathcal{G} \) and therefore, since both \( \mu \) and \( \nu \) are \( \mathcal{G} \)-outer measures on \( X \), we have \( \mu = \nu \). Hence \( \mu \in M \).

4. The representation theorem.

4.1. **Definitions.** (1) For \( i \in I, C_0(X_i) \) is the set of continuous real valued functions on \( X_i \) with compact support.

(2) For \( i \in I \) and \( h \in C_0(X_i) \), \( \text{cyl} \, h \) is the function on \( X \) given by \( (\text{cyl} \, h)(x) = h(x \mid i) \) for every \( x \in X \).

(3) \( F = \{ f: \) there exists \( i \in I \) and \( h \in C_0(X_i) \) with \( f = \text{cyl} \, h \} \).

4.2. **Definitions.** (1) \( L = \{ l: l \) is a positive linear functional on the linear span of \( F \) such that

(1) there exists \( K > 0 \) with \( |l(f)| \leq K \| f \|_{\infty} \) for all \( f \in F \),

(2) if \( i, j \in I \) with \( i \subset j, f \in C_0(X_i) \) and, for \( n \in \omega, f_n \in C_0(X_j) \) with \( \text{cyl} \, f_n \uparrow \text{cyl} \, f \) then \( l(\text{cyl} \, f_n) \uparrow l(\text{cyl} \, f) \).

(Note that in the definition of \( L \) above, condition (1) does not necessarily imply condition (2).)

(2) For \( l \in L, \tau^l \) is the set function on \( \mathcal{G} \) given by \( \tau^l(\alpha) = \inf \{ l(f): 1_{\alpha} \leq f \in F \} \) for \( \alpha \in \mathcal{G} \).

Our basic theorem now is

**Theorem 4.3.** For each \( l \in L \) there exists a unique \( \mu^l \in M \) such that the relationship \( l(f) = \int f \, d\mu^l \) holds for all \( f \in F \). Moreover the mapping \( l \mapsto \mu^l \) is an isomorphism between \( L \) and \( M \).

For the proof of this theorem we will need three preliminary lemmas.

**Lemma C.** For \( l \in L, i \in I \) and \( \alpha \in \mathcal{A}_i \),

\[ \tau^l(\text{cyl} \, \alpha) = \inf \{ l(\text{cyl} \, f): 1_{\alpha} \leq f \in C_0(X_i) \} \] .

**Proof.** Suppose \( h \in F \) and \( 1_{\text{cyl} \, \alpha} \leq h \). We want to find \( f \in C_0(X_i) \) with \( 1_{\alpha} \leq f \) and \( \text{cyl} \, f \leq h \). By definition there exists \( j \in I \) and \( g \in C_0(X_j) \) such that \( h = \text{cyl} \, g \). Let \( k = i \cup j \). For \( z \in X_k \) let \( h_k(z) = h(y) \) for some \( y \in X \) with \( y \mid k = z \). (Note that \( h_k(z) \) is independent of \( y \) provided \( y \mid k = z \), and that \( \text{cyl} \, h_k = h \).) Since \( g \in C_0(X_j) \) and \( h_k(z) = g(z \mid j) \) we have that \( h_k \) is uniformly continuous on \( X_k \). Hence if for \( x \in X_i \)

\[ f^*(x) = \inf \{ h_k(z): z \in X_k \text{ and } z \mid i = x \} \]
\[ = \inf \{ h(y): y \in X \text{ and } y \mid i = x \} \]
then \( f^* \) is continuous on \( X_i \). Moreover it is clear that \( \text{cyl} f^* \leq h \) and \( 1_{\alpha} \leq f^* \). Since \( X_i \) is locally compact and regular there exists \( f \in C_0(X_i) \) with \( 1_{\alpha} \leq f \leq f^* \). Hence \( 1_{\alpha} \leq f \) and \( \text{cyl} f \leq h \). It follows that

\[
\tau'(\text{cyl} \alpha) = \inf \{l(h) : 1_{\text{cyl} \alpha} \leq h \in F\} \\
= \inf \{l(\text{cyl} f) : 1_{\alpha} \leq f \in C_0(X_i)\}.
\]

**Lemma D.** Let \( l \in L, i \in I, \alpha \in \mathcal{K}_i \). Then for every \( \varepsilon > 0 \) there exists \( A \) open in \( X_i \) with \( \alpha \subset A \) such that for any \( j \in I \) with \( i \subset j \) and \( f \in C_0(X_j) \) with \( \|f\|_\infty \leq 1 \) and \( \{x : f(x) > 0\} \subset \pi_{ij}[A - \alpha] \) we have \( l(\text{cyl} f) \leq \varepsilon \).

**Proof.** By Lemma C choose \( h \in C_0(X_i) \) with \( 1_{\alpha} \leq h \) and \( l(\text{cyl} h) \leq \tau'(\text{cyl} \alpha) + \varepsilon/2 \). Let

\[
A = \{x : (1 + \varepsilon/1 + 2l(\text{cyl} h))h(x) > 1\}.
\]

Then \( A \) is open and \( \alpha \subset A \). Now let \( j \in I \) with \( i \subset j \). Suppose first that \( g \in C_0(X_j) \) with \( 0 \leq g \leq 1 \) and support \( g \subset \pi_{ij}[A - \alpha] \). Let \( \beta = \pi_{ij}[\text{support } g] \). Then \( \alpha, \beta \) are disjoint compact subsets of \( A \) and so let \( V, W \) be disjoint neighborhoods of \( \alpha \) and \( \beta \) respectively with \( V \cup W \subset A \). Let \( v, w \in C_0(X_i) \) with \( 1_{\alpha} \leq v \leq 1_{\beta} \) and \( 1_{\beta} \leq w \leq 1_{\alpha} \). Then \( v + w \leq (1 + \varepsilon/1 + 2l(\text{cyl} h))h \) and therefore

\[
l(\text{cyl} v) + l(\text{cyl} w) \leq l(\text{cyl} h) + \varepsilon/2 \\
\leq \tau'(\text{cyl} \alpha) + \varepsilon \leq l(\text{cyl} v) + \varepsilon.
\]

Hence \( l(\text{cyl} w) \leq \varepsilon \) and since \( \text{cyl} g \leq \text{cyl} w \) we have by condition (2) of 4.2(1), that \( l(\text{cyl} g) \leq l(\text{cyl} w) \). Thus \( l(\text{cyl} g) \leq \varepsilon \).

Now let \( f \in C_0(X_j) \) with \( \|f\|_\infty \leq 1 \) and \( \{x : f(x) > 0\} \subset \pi_{ij}[A - \alpha] \). For \( n \in \omega \) let \( \beta_n = \{x : f(x) \geq 1/n\} \). Then \( \beta_n \in \mathcal{K}_j \) and \( \beta_n \subset \text{interior } \beta_n+1 \). Let \( g_n \in C_0(X_j) \) with \( 1_{\beta_n} \leq g_n \leq 1_{\beta_n+1} \) and let \( f_n = f \cdot g_n \). Then support \( f_n \subset \beta_n+1 \subset \pi_{ij}[A - \alpha] \), \( 0 \leq f_n \leq 1 \), and hence by the above argument, \( l(\text{cyl} f_n) \leq \varepsilon \). Since \( f_n \uparrow f^+ \) we have by condition (2) of 4.2(1), that \( l(\text{cyl} f_n) \uparrow l(\text{cyl} f^+) \). It follows that \( l(\text{cyl} f^+) \leq \varepsilon \) and therefore \( l(\text{cyl} f) \leq \varepsilon \).

**Lemma E.** For \( l \in L, \tau^i \) satisfies conditions (a) and (b).

**Proof.** Condition (a) follows easily from Lemma C using well known standard arguments. To prove condition (b), let \( i \in I, \alpha \in \mathcal{K}_i, \) \( t \in T - i \) and \( C \) be a sequence in \( \mathcal{K}_i \) with \( C_n \subset \text{interior } C_{n+1} \) for \( n \in \omega \), and \( C_n \uparrow X_i \). Let \( j = i \cup \{t\} \) and \( \beta_n = \{x \in X_j : x | i \in \alpha \) and \( x_i \in C_n\). Then \( \text{cyl} \beta_n \uparrow \text{cyl} \alpha \). Given \( \varepsilon > 0 \), by Lemma D, there exists \( A \) open in
Xi with \( \alpha \subset A \) such that for any \( g \in C_0(X_i) \) with \( \|g\|_\infty \leq 1 \) and 
\( \{x: g(x) > 0\} \subset \pi_{ij}[A - \alpha] \) we have \( l(\text{cyl } g) \leq \varepsilon \). Choose \( f \in C_0(X_i) \) with 
\( 1_n \leq f \leq 1 \) and let \( k_n \in C_0(X_i) \) with \( 1_c \leq k_n \leq 1_c \).

For \( x \in X_i \) let \( f_n(x) = f(x) \cdot k_n(x) \). Then \( f_n \in C_0(X_i), 1_{\beta_n} \leq f_n \leq 1 \) and 
\( \text{cyl } f_n \uparrow \text{cyl } f \). By Lemma C choose \( h_n \in C_0(X_i) \) with 
\( 1_{\beta_n} \leq h_n \leq f_n \) and 
\( l(\text{cyl } h_n) \leq \tau_l(\text{cyl } \beta_n) + \varepsilon \). We note that \( f_n - h_n+1 \in C_0(X_i), \|f_n - h_n+1\|_\infty \leq 1 \) and 
\( \{x: (f_n - h_n+1)(x) > 0\} \subset \pi_{ij}[A - \alpha] \). Hence by Lemma D,

\[
\tau_l(\text{cyl } (f_n - h_n+1)) < \varepsilon .
\]

Since \( f_n = f_n - h_n+1 + h_n+1 \) we have 
\[
l(\text{cyl } f_n) = l(\text{cyl } (f_n - h_n+1)) + l(\text{cyl } h_n+1) \]
\[
\leq l(\text{cyl } h_n+1) + \varepsilon \leq \tau_l(\text{cyl } \beta_n+1) + 2\varepsilon .
\]

Hence 
\[
\tau_l(\text{cyl } \alpha) \leq \lim_{n \in \omega} l(\text{cyl } f_n)
\]
\[
\leq \lim_{n \in \omega} l(\text{cyl } h_n+1) + \varepsilon \leq \lim_{n \in \omega} \tau_l(\text{cyl } \beta_n+1) + 2\varepsilon .
\]

Thus \( \tau_l(\text{cyl } \alpha) \leq \lim_{n \in \omega} \tau_l(\text{cyl } \beta_n) \) and since certainly the reverse inequality holds, we have 
\( \tau_l(\text{cyl } \alpha) = \lim_{n \in \omega} \tau_l(\text{cyl } \beta_n) \).

**Proof of 4.3.** Let \( l \in L \). By Lemma E, \( \tau_l \) satisfies conditions (a) and (b) and hence by 3.3.1 the Carathéodory outer measure \( \mu_l \) on \( X \) generated by \( \tau_l \) and \( \mathcal{E} \) is in \( M \).

Now suppose \( f \in F \). By definition there exists \( i \in I \) and \( h \in C_0(X_i) \) such that \( f = \text{cyl } h \). If for every \( A \subset X_i \) we let 
\( \mu_i(A) = \mu_l(\text{cyl } A) \) then 
\( \int \text{cyl } h d\mu_i = \int h d\mu_i \). If for \( \alpha \in \mathcal{K}_i \) we let \( \tau_i(\alpha) = \tau_l(\text{cyl } \alpha) \) and let 
\( \nu_i \) be the topological outer measure on \( X_i \) cranked by \( \tau_i \), then by 
3.3(2), \( \nu_i \) is a Radon outer measure on \( X_i \) and \( \mu_i \) agrees with \( \nu_i \) on all \( \nu_i \)-measurable sets. Hence since \( h \in C_0(X_i) \) we have 
\[\int h d\nu_i = \int h d\mu_i .\]

Furthermore if \( l_i(g) = l(\text{cyl } g) \) for \( g \in C_0(X_i) \) then \( l_i \) is a positive continuous linear functional on \( C_0(X_i) \) and by Lemma C

\[\tau_i(\alpha) = \inf \{l_i(g): 1_n \leq g \in C_0(X_i)\} \text{ for } \alpha \in \mathcal{K}_i .\]

Hence by the Riesz Representation Theorem \( l_i \) and \( \nu_i \) satisfy the relationship

\[l_i(g) = \int g d\nu_i \text{ for all } g \in C_0(X_i) .\]

Hence 
\( l(f) = l(\text{cyl } h) = l_i(h) = \int h d\nu_i = \int h d\mu_i \) 
\[
= \int \text{cyl } h d\mu_i = \int f d\mu_i .
\]
To show uniqueness, suppose $\mu \in M$ and $l(f) = \int fd\mu$ for all $f \in F$. For each $i \in I$ let $\mu_i(A) = \mu(\text{cyl } A)$ for $A \subset X_i$, $\tau_i(\alpha) = \mu(\text{cyl } \alpha)$ for $\alpha \in X_i$ and $\nu_i$ be the topological outer measure on $X_i$ cranked by $\tau_i$. By 3.3(2), $\nu_i$ is a Radon outer measure on $X_i$ and $\mu_i$ agrees with $\nu_i$ on $\mathcal{H}_i$, hence also on $\mathcal{B}_i$. Furthermore for all $f \in \mathcal{C}_0(X_i)$

$$\int fd\nu_i = \int fd\mu_i = l(\text{cyl } f) = \int fd\nu_i,$$

and therefore by the Riesz representation theorem $\nu_i = \nu_i$. It follows that $\mu_i$ and $\mu_i^1$ agree on $\mathcal{B}_i$. Hence the projective systems $\{\mu_i | \mathcal{B}_i : i \in I\}$ and $\{\mu_i^1 | \mathcal{B}_i : i \in I\}$ are equal and so their respective projective limit measures, which by 3.4.1 are $\mu | \mathcal{B}$ and $\mu_i^1 | \mathcal{B}_i$ are also equal. Since $\mathcal{C}_i \subset \mathcal{B}$ we have that $\mu$ and $\mu_i^1$ agree on $\mathcal{C}$ and so $\mu = \mu_i^1$. The mapping $l \rightarrow \mu_i^1$ is now clearly an isomorphism between $L$ and $M$.

5. Example to show that $\sigma$-compactness of the coordinate spaces is needed. Let $R$ have the discrete topology (which is not $\sigma$-compact) and consider $R^2$ with the product topology. For $h \in \mathcal{C}_0(R)$ and $x \in R^2$, let $(\text{cyl}_1 h)(x) = h(x_1)$ and $(\text{cyl}_2 h)(x) = h(x_2)$.

Let $F_0 = \mathcal{C}_0(R^2)$

$$F_1 = \{f : f = \text{cyl}_1 h \text{ for some } h \in \mathcal{C}_0(R)\}$$

$$F_2 = \{f : f = \text{cyl}_2 h \text{ for some } h \in \mathcal{C}_0(R)\}.$$  

Using the notations of this paper, we let $T = \{1, 2\}$, $Y_1 = Y_2 = R$ with the discrete topology and define $X, \mathcal{C}, \mathcal{D}, M, F$ and $L$ as before.

First we note that $F = F_0 \cup F_1 \cup F_2$ and that since pairwise intersections of $F_0, F_1$ and $F_2$ consist of the zero element only, every $f$ in the linear span of $F$ has a unique representation as $f = f_0 + f_1 + f_2$ where $f_n \in F_n$ for $n = 0, 1, 2$. For fixed $z \in R^2$ (which equals $X$) define $l$ by $l(f) = f_0(z) + 2f_1(z) + 2f_2(z)$ for $f$ in the linear span of $F$. Then $l \in L$ but we shall show that there is no $\mu \in M$ such that $l(f) = \int fd\mu$ for all $f \in F$. Suppose we did find such a $\mu \in M$. Then if $A = \{x \in R^2 : x_1 = z_1\}$ we have $1_A \in F_1 \subset F$ and hence

$$\mu(A) = \int 1_A d\mu = l(1_A) = 2 \cdot 1_A(z) = 2.$$  

We next note that $1_{\{z\}} \in F_0 \subset F$ and so

$$\mu(\{z\}) = \int 1_{\{z\}} d\mu = l(1_{\{z\}}) = 1_{\{z\}}(z) = 1.$$  

Furthermore since $A, \{z\}$ and $A - \{z\}$ are all $\mu$-measurable we have
\[ \mu(A - \{z\}) = \mu(A) - \mu(\{z\}) = 2 - 1 = 1 \]

On the other hand \(A - \{z\} \subset \mathbb{R}^2 - \{z\}\) which is in \(\mathcal{C}\). Hence

\[ \mu(A - \{z\}) \leq \mu(\mathbb{R}^2 - \{z\}) \]
\[ = \sup \{\mu(\alpha): \alpha \in \mathcal{C} \text{ and } \alpha \subset \mathbb{R}^2 - \{z\}\} \]
\[ = \sup \{l(1_{\alpha}): \alpha \in \mathcal{C} \text{ and } \alpha \subset \mathbb{R}^2 = \{z\}\} = 0 \]

since \(l(1_{\alpha}) = 0\) for any \(\alpha \in \mathcal{C}\) with \(z \notin \alpha\). Hence \(A - \{z\}\) would have to have measure zero and one simultaneously, which is impossible.

**BIBLIOGRAPHY**


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