Pacific Journal of Mathematics

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RICHARD EARL HODEL

Vol. 30, No. 1 September 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 1, 1969

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R. E. HODEL

This paper is a study of Sum Theorems for various classes of topological spaces. Specifically, suppose that X is a topological space and $\{F_\alpha\}$ is a cover of X such that each F_α belongs to some class Q of topological spaces. When can we assert that X is in Q? We shall concentrate our attention on those cases where the elements of $\{F_\alpha\}$ are either all open or all closed and the collection $\{F_\alpha\}$ is a σ-locally finite cover of X.

Throughout this paper Q will denote a class of topological spaces; e.g., normal spaces, paracompact spaces, etc. Perhaps the best known Sum Theorem is the so called Locally Finite Sum Theorem, hereinafter denoted (Σ).

(Σ): *Let X be a topological space and let $\{F_\alpha\}$ be a locally finite closed cover of X such that each F_α is in Q. Then X is in Q.*

It is known that (Σ) holds when Q is the class of regular spaces [14], normal spaces [13], collectionwise normal spaces [13], paracompact spaces [11], stratifiable spaces [3], or metrizable spaces [14]. In §5 we show that (Σ) also holds for pointwise paracompact spaces.

The main results of the paper are in §3. In that section we prove three Sum Theorems, each of which holds for any class of topological spaces which satisfies (Σ) and is hereditary with respect to closed subsets. These results illustrate the importance of (Σ) in our study of Sum Theorems.

In §4 we give an application of one of the Sum Theorems, namely a Subset Theorem for totally normal spaces. This theorem closely parallels the result in [9].

The reader is referred to the following papers for definitions: collectionwise normal [1]; paracompact [11]; point finite collection [12]; stratifiable [2]. A topological space X is *pointwise paracompact* if every open cover of X has a point finite open refinement. According to Dowker [5] a normal space X is *totally normal* if every open subset U of X can be written as a locally finite (in U) collection of open F_σ subsets of X.

2. Examples. In this section we discuss two examples which nullify several conjectures and in addition will serve as a guide in selecting appropriate hypotheses for the Sum Theorems appearing in §3.

EXAMPLE 1. This example is of particular interest for those classes of topological spaces between normal and metrizable. Let S denote the screenable, nonnormal Moore space given by Heath in [8]. The space S can be exhibited as (1) the union of two open metrizable spaces or (2) the union of a countable closure preserving collection of closed sets, each of which is discrete (and hence metrizable.) From (1) we see that to obtain interesting Sum Theorems involving open covers we must make additional assumptions about the sets; from (2) we see that in general locally finite cannot be replaced by closure preserving in (Σ).

EXAMPLE 2. One might expect that the trouble caused by Example 1 is due to the nonnormality of X. And in fact, every *normal* space which is a *locally finite* union of open metrizable spaces is metrizable [15]. On the other hand, Example H given by Bing in [1] is a perfectly normal space which is not collectionwise normal, not pointwise paracompact [12], but can be exhibited as the union of a countable closure preserving collection of closed sets, each of which is discrete.

3. **The Sum Theorems.** In this section we state and prove three Sum Theorems. In the statement of each theorem Q denotes a class of topological spaces which satisfies (Σ) and is hereditary with respect to closed subsets (i.e., if X is in Q and F is a closed subset of X then F is in Q). Consider the following six classes of topological spaces: normal, collectionwise normal, paracompact, stratifiable, metrizable, pointwise paracompact. Each of these classes is hereditary with respect to closed sets, and it is known that (Σ) holds for the first five classes listed. In §5 we show that (Σ) also holds for pointwise paracompact spaces. Thus the three Sum Theorems hold for each of the six classes of spaces.

SUM THEOREM I. *Let X be a topological space and let $\mathscr{V}$ be a σ-locally finite open cover of X such that the closure of each element of $\mathscr{V}$ is in Q. Then X is in Q.*

Proof. Let $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i$, where $\mathscr{V}_i$ is a locally finite collection. For each positive integer i let $V_i = \cup \{V: V \in \mathscr{V}_i\}$. Then $\{\bar{V}: V \in \mathscr{V}_i\}$ is a locally finite closed cover of $\bar{V}_i$, each element of which is in Q and so by (Σ) $\bar{V}_i$ is in Q. Now let $F_1 = \bar{V}_1$ and for $i = 2, 3, \cdots$ let $F_i = \bar{V}_i - \bigcup_{j<i} V_j$. Then $\{F_i: i = 1, 2, \cdots\}$ is a locally finite closed cover of X, each element of which belongs to Q and so again by (Σ) X is in Q.

A subset V of a topological space is called *elementary* in case V is open and $V = \bigcup_{i=1}^{\infty} V_i$, where each V_i is open and $\bar{V}_i \subseteq V$ for all

i. As for examples every cozero set [6, p. 15] is an elementary set and every open F_σ subset of a normal space is an elementary set. The following result follows without difficulty from Sum Theorem I.

SUM THEOREM II. *Let X be a topological space and let $\mathscr{V}$ be a σ-locally finite cover of X, each element of which is elementary and belongs to Q. Then X is in Q.*

REMARK. The statement of Sum Theorem II for metrizable spaces generalizes a result by Stone [16, p. 365].

SUM THEOREM III. *Let X be a regular space and let $\mathscr{V}$ be a σ-locally finite open cover of X, each element of which is in Q and has compact boundary. Then X is in Q.*

Proof. By Sum Theorem I it suffices to show that the closure of each element of $\mathscr{V}$ belongs to Q. So let V be an arbitrary but fixed element of $\mathscr{V}$ and let $B = \bar{V} - V$. Let $\{W_i: 1 \leqq i \leqq n\}$ be a finite open collection in X covering B such that the closure of each W_i is contained in some element of $\mathscr{V}$. For $i = 1, \cdots, n$ let $F_i = \bar{W}_i \cap \bar{V}$ and let $F_0 = \bar{V} - \bigcup_{i=1}^n W_i$. Then $\{F_i: i = 0, \cdots, n\}$ is a finite closed cover of $\bar{V}$, each element of which is in Q, and so by (Σ) $\bar{V}$ is in Q.

REMARK. The statement of Sum Theorem III for metrizable spaces (paracompact spaces) generalizes a result by Stone [16] (Hanai—Okuyama [7]).

4. A Subset Theorem. Consider the following statement about a class Q of topological spaces.

(β): *Let X be a topological space such that every open subset of X belongs to Q. Then every subset of X belongs to Q.*

It is known that (β) holds when Q is the class of normal spaces [5], collectionwise normal spaces [9], or paracompact spaces [4], and it is easy to verify that (β) also holds for pointwise paracompact spaces.

Now let Q denote a class of topological spaces satisfying (Σ), (β), and which is hereditary with respect to closed subsets. We then have the following

SUBSET THEOREM. *Let X be a totally normal space such that X is in Q. Then every subset of X is in Q.*

Proof. Let V be a subset of X; since Q satisfies (β) we may assume that V is open. Since X is totally normal, $V = \cup \{V_\alpha : \alpha \text{ in } A\}$, where $\{V_\alpha\}$ is a locally finite collection in V and each V_α is an open F_σ subset of X. For each α in A let $V_\alpha = \bigcup_{i=1}^\infty F_{\alpha,i}$, where each $F_{\alpha,i}$ is a closed subset of X. By normality of X there is an open set $W_{\alpha,i}$ such that $F_{\alpha,i} \subseteqq W_{\alpha,i} \subseteqq \bar{W}_{\alpha,i} \subseteqq V_\alpha$. Let $\mathscr{W}_i = \{W_{\alpha,i} : \alpha \text{ in } A\}$ and let $\mathscr{W} = \bigcup_{i=1}^\infty \mathscr{W}_i$. Then $\mathscr{W}$ is a σ-locally finite open cover of V such that the closure of each element of $\mathscr{W}$ is in Q. Hence by Sum Theorem I V is in Q.

REMARK. From the Subset Theorem we obtain the result by Dowker [5] that every totally normal space is completely normal. (Let Q = normal spaces.) We also obtain two results by the author [9], namely that every totally normal collectionwise normal space (paracompact space) is hereditarily collectionwise normal (hereditarily paracompact). Finally we obtain the new result that every totally normal pointwise paracompact space is hereditarily pointwise paracompact.

5. Pointwise paracompact spaces. Two remarks are in order before beginning the proof that (Σ) holds for pointwise paracompact spaces. First, suppose that $\mathscr{V} = \{V_\alpha : 0 \leqq \alpha < \eta\}$ is a cover of a topological space X and $\mathscr{W}$ is a point finite open refinement of $\mathscr{V}$. Then one can obtain a point finite open cover $\{W_\alpha : 0 \leqq \alpha < \eta\}$ of X such that $W_\alpha \subseteqq V_\alpha$ for all α. Indeed, if we let $W_\alpha = \cup \{W \in \mathscr{W} : W \subseteqq V_\alpha, W \not\subseteqq V_\beta, \beta < \alpha\}$, then $\{W_\alpha : 0 \leqq \alpha < \eta\}$ does the trick. Second, suppose that $\mathscr{V}$ is a cover of a set X and p is a point of X. Then $\operatorname{ord}(\mathscr{V}, p) < \infty$ means that p is contained in at most a finite number of elements of $\mathscr{V}$.

THEOREM 5.1. *Let X be a topological space and let $\mathscr{F} = \{F_\alpha : 0 \leqq \alpha < \eta\}$ be a locally finite closed cover of X, each element of which is pointwise paracompact. Then X is pointwise paracompact.*

Proof. Let $\mathscr{V}$ be an open cover of X. By replacing $\mathscr{V}$ by an open refinement if necessary, we may assume that each element of $\mathscr{V}$ intersects at most a finite number of elements of $\mathscr{F}$. (The reason for doing this will become apparent.) Thus, let $\mathscr{V} = \{V_\sigma : \sigma \text{ in } B\}$; we shall construct a point finite open cover $\mathscr{W} = \{W_\sigma : \sigma \text{ in } B\}$ of X such that $W_\sigma \subseteqq V_\sigma$, for all σ in B. The collection $\mathscr{W}$ is obtained by transfinite induction; for each ordinal α, $0 \leqq \alpha < \eta$, we construct an open cover $\mathscr{V}_\alpha = \{V_\sigma^\alpha : \sigma \text{ in } B\}$ of X and then take $\mathscr{W} = \{\bigcap_{\alpha<\eta} V_\sigma^\alpha : \sigma \text{ in } B\}$ as the desired point finite open refinement of $\mathscr{V}$.

To obtain $\mathscr{V}_0$, consider the collection $\{V_\sigma \cap F_0 : \sigma \text{ in } B\}$. This is an open cover of F_0 and so there is a point finite open (in F_0) collection

$\{U_\sigma : \sigma \text{ in } B\}$ covering F_0 such that $U_\sigma \subseteqq V_\sigma \cap F_0$, for all σ in B. For σ in B let

$$V_\sigma^0 = V^\sigma \cap [X - (F_0 - U_\sigma)]$$

and let $\mathscr{V}_0 = \{V_\sigma^0 : \sigma \text{ in } B\}$. It is easy to see that the collection $\mathscr{V}_0$ satisfies these properties.

(1)′ $\mathscr{V}_0$ is an open cover of X.

(2)′ For all σ in B, $V_\sigma^0 \subseteqq V_\sigma$.

(3)′ If $p \in F_0$ then $\operatorname{ord}(\mathscr{V}_0, p) < +\infty$.

(4)′ If $p \in V_\sigma$ and $p \notin F_0$ then $p \in V_\sigma^0$.

Now let α be a fixed ordinal, $1 \leqq \alpha < \eta$, and assume that for all $\beta < \alpha$ we have constructed a collection $\mathscr{V}_\beta = \{V_\sigma^\beta : \sigma \text{ in } B\}$ such that

(1) $\mathscr{V}_\beta$ is an open cover of X.

(2) For all σ in B, if $\gamma < \beta$ then $V_\sigma^\beta \subseteqq V_\sigma^\gamma$.

(3) If $p \in \bigcup_{\gamma \leqq \beta} F_\gamma$ then $\operatorname{ord}(\mathscr{V}_\beta, p) < +\infty$.

(4) If $p \in \bigcap_{\gamma < \beta} V_\sigma^\gamma$ and $p \notin F_\beta$ then $p \in V_\sigma^\beta$.

We now construct a collection $\mathscr{V}_\alpha = \{V_\sigma^\alpha : \sigma \text{ in } B\}$ such that (1)–(4) are satisfied. For each σ in B let $W_\sigma^\alpha = \bigcap_{\beta<\alpha} V_\sigma^\beta$ and let $\mathscr{W}_\alpha = \{W_\sigma^\alpha : \sigma \text{ in } B\}$. Suppose, for a moment, that $\mathscr{W}_\alpha$ is an open cover of X. We then obtain $\mathscr{V}_\alpha$ from $\mathscr{W}_\alpha$ in exactly the same way in which $\mathscr{V}_0$ was obtained from $\mathscr{V}$. (Thus, $W_\sigma^\alpha \cap F_\alpha : \sigma \text{ in } B\}$ is a open cover of F_α; proceed as above.) It is not difficult to show that the collection $\mathscr{V}_\alpha = \{V_\sigma^\alpha : \sigma \text{ in } B\}$ so constructed satisfies (1)–(4).

Now let us show that $\mathscr{W}_\alpha$ is an open cover of X. To prove that $\mathscr{W}_\alpha$ covers X, let p be an arbitrary point of X. Let β be the largest ordinal less than α such that F_β contains p. (Recall that $\mathscr{F}$ is point finite; if $p \notin F_\gamma$ for all $\gamma < \alpha$ let $\beta = 0$.) Now $\mathscr{V}_\beta$ is a cover of X so there is a σ in B such that V_σ^β contains p. It follows from (2) and (4) that W_σ^α contains p and so $\mathscr{W}_\alpha$ covers X. To prove that $\mathscr{W}_\alpha$ is an open cover let σ in B be fixed. Let β be the largest ordinal less than α such that $V_\sigma \cap F_\beta \neq \emptyset$. (Recall that $\mathscr{V}$ exhibits the local finiteness of $\mathscr{F}$; again, if $V_\sigma \cap F_\gamma = \emptyset$ for all $\gamma < \alpha$ let $\beta = 0$.) It follows from (2) and (4) that $W_\sigma^\alpha = V_\sigma^\beta$ and so W_σ^α is an open set.

To obtain the final collection $\mathscr{W}$, let $W_\sigma = \bigcap_{\alpha<\eta} V_\sigma^\alpha$ and let $\mathscr{W} = \{W_\sigma : \sigma \text{ in } B\}$. To prove that $\mathscr{W}$ is an open cover of X, proceed as in the above paragraph. Clearly $\mathscr{W}$ refines $\mathscr{V}$, and it follows from (3) that $\mathscr{W}$ is point finite. This completes the proof that X is pointwise paracompact.

6. Collectionwise normal spaces. The proof that (Σ) holds for collectionwise normal spaces is a special case of a result by Morita [13]. In this section we give an alternate (and must simpler) proof of this special case.

THEOREM (*Morita*). *Let X be a topological space and let $\{F_\alpha$: α in $A\}$ be a locally finite closed cover of X such that each F_α is collectionwise normal. Then X is collectionwise normal.*

Proof. Let $\{H_\sigma: \sigma \text{ in } B\}$ be a discrete collection of closed sets in X. We shall construct a mutually disjoint open collection $\{V_\sigma: \sigma \text{ in } B\}$ such that H_σ is contained in V_σ, for all σ in B. For fixed α $\{F_\alpha \cap H_\sigma: \sigma \text{ in } B\}$ is a discrete collection of closed sets in F_α so there exists a mutually disjoint collection $\{V_{\alpha,\sigma}: \sigma \text{ in } B\}$, where each $V_{\alpha,\sigma}$ is an open subset of F_α containing $F_\alpha \cap H_\sigma$. For each σ in B let K_σ be the closed set $\cup \{(F_\alpha - V_{\sigma,\alpha}): \alpha \text{ in } A\}$ and let $V_\sigma = X - K_\sigma$. Then, as is easily checked, H_σ is contained in V_σ for all σ in B and $\{V_\sigma: \sigma \text{ in } B\}$ is a mutually disjoint open collection.

REMARK. As observed in Example 2, a normal space which is a countable union of closed metrizable spaces need not be collectionwise normal. However, every collectionwise normal space which is a countable union of closed paracompact spaces is paracompact. For, such a space is clearly F_σ-screenable and thus paracompact [10]. See [16, p. 363] for related remarks about metrizable spaces.

REFERENCES

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175-186.
2. C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1-16.
3. J. G. Ceder, *Some généralizations of metric spaces*, Pacific J. Math. **11** (1961), 105-126.
4. J. Dieudonné, *Une generalisation des espaces compacts*, J. Math. Pures Appl. **23** (1944), 65-76.
5. C. H. Dowker, *Inductive dimension of completely normal spaces*, Quart. J. Math. (1953), 267-281.
6. L. Gillman and M. Jerison, *Rings of continuous functions*, 1960.
7. S. Hanai and A. Okuyama, *On paracompactness of topological spaces*, Proc. Japan Acad. **36** (1960), 466-469.
8. R. W. Heath, *Screenability, pointwise paracompactness, and metrization of Moore spaces*, Canad. J. Math. **16** (1964), 763-770.
9. R. E. Hodel, *Total normality and the hereditary property*, Proc. Amer. Math. Soc. **17** (1966), 462-465.
10. L. F. McAuley, *A note on complete collectionwise normality and paracompactness*, Proc. Amer. Math. Soc. **9** (1958), 796-799.
11. E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831-838.
12. ———, *Point finite and locally finite coverings*, Canad. J. Math. **7** (1955). 275-279.
13. K. Morita, *On spaces having the weak topology with respect to closed coverings*, Proc. Japan Acad. **29** (1953). 537-543.
14. J. Nagata, *On a necessary and sufficient condition of metrizability*, J. Inst. Polytech. Osaka City Univ. (A) **1** (1950), 93-100.
15. Yu. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Doklady Akad. Nauk. SSSR. N. S. **77** (1951), 197-200.

16. A. H. Stone, *Metrizability of unions of spaces*, Proc. Amer. Math. Soc. **10** (1959), 361–366.

Received February 16, 1968. This research was supported in part by the National Science Foundation, Grant Gp-5919.

DUKE UNIVERSITY
DURHAM, NORTH CAROLINA

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 30, No. 1 September, 1969