A CONTINUOUS PARTIAL ORDER FOR PEANO CONTINUA

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A THEOREM OF R. J. KOCH STATES THAT A COMPACT CONTINUOUSLY PARTIALLY ORDERED SPACE WITH SOME NATURAL CONDITIONS ON THE PARTIAL ORDER IS ARCWISE CONNECTED. L. E. WARD, JR., HAS CONJECTURED THAT KOCH'S ARC THEOREM IMPLIES THE WELL-KNOWN THEOREM OF R. L. MOORE THAT A PEANO CONTINUUM IS ARCWISE CONNECTED. IN THIS PAPER WARD'S CONJECTURE IS PROVED.

1. Preliminaries. If $\Gamma$ is a partial order on a set $X$ we will write $x \leq y$ or $x \leq^\Gamma y$ for $(x, y) \in \Gamma$. We will let $L(a) = \{x: (x, a) \in \Gamma\}$. If $X$ is a topological space, then $\Gamma$ is a continuous partial order on $X$ provided the graph of $\Gamma$ is closed in $X \times X$. If $\Gamma$ is a continuous partial order on the space $X$, then $L(x)$ is a closed set for every $x \in X$. A zero of a continuously partially ordered space $X$ is an element $0$ such that $0 \in L(x)$ for all $x \in X$. An arc is a locally connected continuum with exactly two noncutpoints. A real arc is a separable arc. A Peano continuum is a locally connected metric continuum.

We will use the following statement of Koch's arc theorem.

**Theorem 1.** If $X$ is a compact continuously partially ordered space with zero such that $L(x)$ is connected for each $x \in X$, then $X$ is arcwise connected.

We will show that Peano continua admit such partial orders by proving the following:

**Theorem 2.** If $X$ is a compact connected locally connected metric space, then $X$ admits a continuous partial order with a zero such that $L(x)$ is connected for all $x \in X$.

The proof of this theorem will use some definitions and results due to R. H. Bing [1]. An $\varepsilon$-partition $\mathcal{P}$, of a subspace $K$ of a metric space $M$ is a finite set of closed subsets of $M$, each with diameter less than $\varepsilon$, the union of which is $K$, and such that the interiors in $M$ of all the elements of $\mathcal{P}$, are nonempty, connected, dense in the closed subset, and are pairwise disjoint. The subspace $K$ is partitionable if for each positive number $\varepsilon$, there exists an $\varepsilon$-partition of $K$.

**Lemma 1.** Let $M$ be a compact connected locally connected metric space. For each positive number $\varepsilon$ there exists an $\varepsilon$-partition $\mathcal{P}$, of
Bing proves this lemma in [1].

The proof of the Theorem 2 will follow in two parts. In the first part a relation $\mathcal{A}$ will be constructed on the Peano continuum $X$. The second part will be concerned with proving that $\mathcal{A}$ is the desired partial order on $X$. We will let $d$ denote the metric on $X$.

2. The construction of the relation $\mathcal{A}$. We will define inductively a sequence $\{\mathcal{F}(i)\}_{i=1}^{\infty}$ of finite partitions of $X$. With each partition we will associate a relation $\delta_i$. The set $\{\delta_i\}_{i=1}^{\infty}$ will be a nest of closed subsets of $X \times X$ and $\Delta = \cap \delta_i$ will be the desired partial order on $X$.

First choose an arbitrary element of $X$. Call this element 0. This will be the 0 of the partial order to be constructed on $X$.

We will now construct the relation $\delta_i$ as the first step of the induction.

Let $\mathcal{F}(1)$ be a finite partition on $X$ such that for $F \in \mathcal{F}(1)$, $\text{diam} (F) < 1/2$, and such that $F$ is partitionable. We will classify the elements of $\mathcal{F}(1)$ according to how "far away" they are from 0. Let $\mathcal{F}(1,0)$ be the set $\{F \in \mathcal{F}(1): 0 \in F\}$. If $\mathcal{F}(1, i)$ has been defined for $i = 1, 2, \ldots, t - 1$, let

$$ \mathcal{F}(1, t) = \{F \in \mathcal{F}(1) - \bigcup_{i=0}^{t-1} \mathcal{F}(1, i): F \cap (\bigcup \mathcal{F}(1, t - 1)) \neq \emptyset\} . $$

If $F$ is an element of $\mathcal{F}(1, t)$ we will say $F$ has order $t$. Because $\mathcal{F}(1)$ is a cover of the connected set $X$ with connected sets, there is a chain of elements of $\mathcal{F}(1)$ between any two points of $X$. That is, if $F$ is an element of $\mathcal{F}(1)$ then there exists some integer $t$ and a set $\{F_i\}_{i=0}^{t} \subset \mathcal{F}(1)$ such that $0 \in F_0$, $F = F_t$, and for $i, j \in \{0, 1, \ldots, t\}$ $F_i \cap F_j \neq \emptyset$ if and only if $|i - j| \leq 1$. This is the condition necessary for $F$ to have order $t$. Thus order is defined for all elements of $\mathcal{F}(1)$.

We now define sets $J(F)$, for $F \in \mathcal{F}(1)$, which will be in a sense "predecessors" of the elements of $F$. For $F \in \mathcal{F}(1, 0)$ let $J(F) = F$. If $J(F)$ has been defined for $F \in \mathcal{F}(1, t - 1)$ and if $F_i \in \mathcal{F}(1, t)$ let

$$ (2.1) \quad J(F_i) = F_i \cup \cup \{J(F): F \cap F_i \neq \emptyset, F \in \mathcal{F}(1, t - 1)\} . $$

We now define the relation $\delta_i$ on $X$ by defining for all $x \in X$ the set $L_i(x) = \{y: (y, x) \in \delta_i\}$. Set

$$ L_i(x) = \cup \{J(F): x \in F \in \mathcal{F}(1)\} . $$

The relation $\delta_i$ is reflexive but not anti-symmetric or transitive.
In order to define the relations $\delta_2, \ldots, \delta_n$, it will be useful to introduce some additional notation. Let $F$ be an arbitrary fixed element of $\mathcal{F}(1, t)$ for some nonnegative integer $t$. Let $\partial F$ denote the boundary of $F$. For $t = 0$, let $\mathcal{C}_*(F) = \{0\}$, and for $t > 0$, let

$$\mathcal{C}_*(F) = \{E \in \mathcal{F}(1, t - 1): E \cap F \neq \emptyset\}.$$  \hfill (3.1)

Notice that $\mathcal{C}_*(F)$ is not empty by (1) since $F \in \mathcal{F}(1, t)$. Let

$$\partial_*(F) = F \cap [\cup \mathcal{C}_*(F)].$$  \hfill (4.1)

Except for the case when $t = 0$ and $\partial_*(F) = \{0\}$, $\partial_*(F)$ is that part of the boundary of $F$ which is also part of the boundary of sets of order $t - 1$. Let

$$\mathcal{E}(F) = \{E \in \mathcal{F}(1): E \neq F \text{ and } E \cap F - \partial_*(F) \neq \emptyset\}.$$  \hfill (5.1)

That is, $\mathcal{E}(F)$ is the set of elements of $\mathcal{F}(1)$, other than $F$ itself and the sets of order $t - 1$, whose intersection with $F - \partial_*(F)$ is not empty. Note that the elements of $\mathcal{E}(F)$ either have order $t$ or order $t + 1$. Let $\mathcal{E}^*(F)$ be the set $\{E \in \mathcal{E}(F): F \in \mathcal{E}^*(E)\}$ and let

$$\partial^*F = \cup\{F \cap E: E \in \mathcal{E}^*(F)\}. \hfill (6.1)$$

Then $\mathcal{E}^*(F)$ is the set of sets in $\mathcal{F}(1)$ which have order $t + 1$ and have a nonempty intersection with $F$. The sets $\mathcal{E}(F)$ and $\mathcal{E}^*(F)$ may be empty. For $E \in \mathcal{E}(F) \cup \mathcal{E}^*(F)$ let $\partial_*(F) = E \cap F$.

If $\mathcal{E}^*(F)$ is not empty, let $\rho(F)$ be $d(\partial_*(F), \partial^*F)$. Thus $\rho(F)$ is the infimum of the distances between the points of $F$ which are also in the sets of order $t - 1$ and those points of $F$ which are also in sets of order $t + 1$. This distance is positive since, by (1), for each $E \in \mathcal{E}^*(F), \partial_*(F)$ and $\partial^*F$ are disjoint closed sets. If $\mathcal{E}^*(F)$ is empty, let $\rho(F)$ be $\text{diam}(F)$.

The remainder of the construction of $\delta_2$ generalizes directly to the construction of $\delta_n$. Thus we will assume that $\mathcal{F}(n)$, a partition of $X$, and the sets $\mathcal{F}(n, t)$ have been defined for $t = 0, 1, \ldots$, and that for $F \in \mathcal{F}(n), \partial_*(F), \mathcal{E}_*(F), \mathcal{E}(F), \mathcal{E}^*(F), \partial^*F$ and $\rho(F)$ have been defined and that for each $E \in \mathcal{E}(F) \cup \mathcal{E}^*(F), \partial_*(F)$ has been defined. We will now define some special subsets of each $F \in \mathcal{F}(n)$ which we will use to define the relation $\delta_n$.

In order for the final relation $\Delta$ to be transitive it will be necessary that the elements of $\partial F - (\partial_*(F) \cup \partial^*F)$ have no successors in the relation $\Delta$. To this end we want to find for each $E \in \mathcal{E}(F) \cup \mathcal{E}^*(F)$ a partitionable subset of $F$ which contains $\partial_*(F)$ and $\partial^*F$ but contains no points of $\partial F$ which are not "close" to $\partial_*(F)$ or $\partial^*F$. We use the following lemma.
Lemma 2. Let $\varepsilon > 0$ and let $F$ be a partitionable compact subset of a metric space $X$ such that the interior of $F$ is connected and locally connected. Let $B_0$ be either a nonempty closed subset of $\partial F$ or a point in the interior of $F$. Let $\{B_i\}_{i=0}^m$ be a finite set of nonempty closed subsets of $\partial F$, such that $\bigcup_{i=0}^m B_i \supseteq \partial F$. Then there exists a set $\{C_i\}_{i=0}^m$ of partitionable subsets of $F$ such that for $i = 0, 1, \ldots, m$ $C_i$ is closed, $(\text{int } C_i) \cup B_i \cup B_0$ is connected, $B_i \subset C_i$ and if $x \in \partial F \cap C_i$ then either $d(x, B_i) < \varepsilon$ or $d(x, B_0) < \varepsilon$. Further $C_0 \subset C_i$, $i = 0, 1, \ldots, m$ and $F = \bigcup_{i=0}^m C_i$.

Proof. By Lemma 1, $F$ is partitionable so let $\mathcal{P}(F)$ be a partition of $F$ such that for $P \in \mathcal{P}(F)$, $\text{diam } (P) < \varepsilon/2$ and $P$ is partitionable.

For $x \in \text{int } F$ let $U_x$ be a connected open set containing $x$ whose closure misses $\partial F$. Let $\mathcal{U} = \{U_x : x \in \text{int } F\}$. For each $P \in \mathcal{P}(F)$ choose $x(P) \in \text{int } P \cap \text{int } F$ and let

$$Q = \{x(P) : P \in \mathcal{P}(F)\} \cup \{P \in \mathcal{P}(F) : P \cap \partial F = \emptyset\}.$$ 

Let $\mathcal{U}_1$ be a finite cover of the closed set $Q$ by elements of $\mathcal{U}$. We can write $\mathcal{U}_1 = \{U_i\}_{i=1}^k$. Now fix some element $P_0 \in \mathcal{P}(F)$ such that $P_0 \cap B_0 \neq \emptyset$. The interior of $F$ is connected by the connected open sets of $\mathcal{U}$, so that for each $U_i \in \mathcal{U}_1$ there exists $\{U_{ij}\}_{j=0}^{k(i)} \subset \mathcal{U}$ such that $x(P_0) \in U_{i0}$, $U_{ik(i)} = U_i$ and $U_{ij} \cap U_{kl} \neq \emptyset$ if and only if $|j - l| \leq 1$. That is, there is a finite chain of sets of $\mathcal{U}$ connecting each element of $\mathcal{U}_1$ with $x(P_0)$. Let

$$\mathcal{U}' = \{U_{ij} : i = 0, \ldots, k ; j = 0, \ldots, k(i)\} \cup \{P \in \mathcal{P}(F) : P \cap B_0 \neq \emptyset\}.$$ 

Note that $\bigcup \mathcal{U}'$ is a connected subset of $F$ and that if $x \in \text{Cl}(\bigcup \mathcal{U}')$ and $d(x, B_0) > \varepsilon/2$, then $x \in \partial F$. This is because the boundary of each element of $\mathcal{U}$ misses the boundary of $F$, so that if $x$ were in $\partial F$, $x$ would be an element of $P$ for some $P \in \mathcal{P}(F)$ such that $P \cap B_0 \neq \emptyset$ and we have that $\text{diam } (P) < \varepsilon/2$. Also note that

$$F \subset (\bigcup \mathcal{U}') \cup \{P \in \mathcal{P}(F) : P \cap \partial F \neq \emptyset\}$$

since $\mathcal{U}_1 \subset \mathcal{U}'$ and $\mathcal{U}_1$ is a cover of $\bigcup \{P \in \mathcal{P}(F) : P \cap \partial F = \emptyset\}$.

Now consider $\mathcal{U}_2 = \{U \in \mathcal{U}' : \text{Cl } U \cap \partial F = \emptyset\}$. Let

$$\nu(F) = \min \{\varepsilon/2, \min \{d(\text{Cl } U, \partial F) : U \in \mathcal{U}_2\}\}.$$ 

For each $P \in \mathcal{P}(F)$ let $\mathcal{E}(F, P)$ be a partition of $P$ such that if

$$F' \in \mathcal{E}(F, P), \text{ then } \text{diam } (F') < \frac{\nu(F)}{4}$$

and $F'$ is partitionable. Let
We are ready now to define the sets $C_i, i = 0, 1, \cdots, m$. The set $C_0$ will meet $\partial F$ only "close" to $B_0$ and $C_i, i = 1, \cdots, m$ will meet $\partial F$ only "close" to $B_i$ or $B_0$. Let $D = [(\cup \mathcal{X}') - \partial F] \cup B_0$. The set $D$ is a connected subset of $(\text{int } F) \cup B_0$. Let

$$C_0 = \bigcup \{F' \in \mathcal{X}(F): F' \cap D \neq \emptyset \}.$$ 

Because $D$ is connected and covered by $\mathcal{X}(F)$, $C_0$ is a closed and connected subset of $F$. Also, if $x \in C_0 \cap \partial F$, then $d(x, B_0) < \varepsilon$, for if $x \in C_0 - B_0$ then $x \in F' \in \mathcal{X}(F)$ such that $F' \cap D \neq \emptyset$. Consequently there exists a $U \in \mathcal{X}'$ such that $F' \cap U \neq \emptyset$. It then follows that if $x$ were in $\partial F$ then, by definition of $\nu(F)$, $U = P$ for some $P \in \mathcal{P}(F)$ such that $P \cap B_0 \neq \emptyset$, and

$$d(x, B_0) \leq \text{diam } F' + \text{diam } P < \frac{\nu(F)}{4} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{2} < \varepsilon.$$

If we let $C'_0 = [(\text{int } F) \cap C_0] \cup B_0$, then $C'_0$ is connected because $C_0$ contains $D$ and

$$C'_0 = \bigcup \{[F' \cap (\text{int } F)] \cup [F' \cap B_0]: F' \subset C_0\},$$

which is a union of connected sets which cover $D$ and each of which has nonempty intersection with $D$.

Now let

$$C_i = C_0 \cup \bigcup \{\bigcup \mathcal{X}(F, P): P \in \mathcal{P}(F), P \cap B_i \neq \emptyset \}.$$ 

We see that $C_i$ is a closed subset of $F$ and it is connected because $C_0$ and each $P \in \mathcal{P}(F)$ is connected and $x(P) \in P \cap C_0$. Let $C'_i = [(\text{int } F) \cap C_i] \cup B_i \cup B_0$. Then $C'_i$ is a connected subset of $F$, for

$$C'_i = C'_0 \cup \bigcup \{[P \cap \text{int } F] \cup [B_i \cap P]: P \in \mathcal{P}(F), P \cap B_i \neq \emptyset \},$$

and $C'_0$ and $[P \cap \text{int } F] \cup [P \cap B_i]$ are connected and $x(P) \in C'_0 \cap P \cap \text{int } F$ for each $P \in \mathcal{P}(F)$.

Further note that if $x \in C_i \cap \partial F$, then either $d(x, B_i) < \varepsilon$ or $d(x, B_0) < \varepsilon$. Also $F$ is a subset of $\bigcup_{i=0}^{m} C_i$.

This completes the proof of Lemma 2.

To apply this lemma to the theorem we let $\varepsilon = \rho(F)/3, B_0 = \partial \ast F$ and\

$$[B_i]_{i=1}^{m(F)} = \{[F^* E]: E \in \mathcal{X}(F) \cup \mathcal{X}^*(F)\}.$$ 

Thus for $F \in \mathcal{T}(n)$ we get sets $C_i, i = 0, 1, \cdots, m(F)$ satisfying the conditions of the lemma. For clarity we will sometimes write $C(F)$ for $C_0(F)$ and use $C(F, E)$ for $C_i(F)$ when $B_i = \partial \ast F$ for $E \in \mathcal{X}(F) \cup \mathcal{X}^*(F)$. We will also use $C'(F)$ for $C'_0$ and $C'(F, E)$ for $C'_i$.

We will now define the relation $\delta_n$ on $X$. First we inductively
define sets $J(F)$ and $J(F, E)$ for each $F \in \mathcal{T}(n)$, $E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)$. The elements of $J(F)$ and $J(F, E)$ will, in a sense, be "predecessors" of the elements of $C(F)$ and $C(F, E)$ respectively.

For $F \in \mathcal{T}(n, 0)$, let $J(F) = C(F)$ and for $E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)$ let $J(F, E) = C(F, E) \cup J(F)$. Then suppose $J(F)$ and $J(F, E)$ have been defined for all $F \in \mathcal{T}(n, t - 1)$, $E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)$. Let $F$ be an element of $\mathcal{T}(n, t)$. Define

$$J(F) = C(F) \cup \{J(F_*, F) : F_* \in \mathcal{C}_*(F)\}$$

and let

$$J(F, E) = C(F, E) \cup J(F) \text{ for } E \in \mathcal{C}(F) \cup \mathcal{C}^*(F).$$

Thus we can define $J(F)$ and $J(F, E)$ for all

$$F \in \mathcal{T}(n), E \in \mathcal{C}(F) \cup \mathcal{C}^*(F).$$

The sets $J(F)$ and $J(F, E)$ are each closed since they are a finite union of closed sets. Also $J(F, E)$ is connected if $J(F)$ is connected since for each $E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)$, $C(F, E)$ contains $C(F)$. But $J(F)$ is connected since if $F_* \in \mathcal{C}_*(F)$ then for each $P \in \mathcal{T}(F_*)$ such that

$$P \cap F \neq \emptyset, P \cap \partial F \cap C(F_*, F') \neq \emptyset.$$

Thus $C(F)$ is not separated from $C(F_*, F)$ for any $F_* \in \mathcal{C}_*(F)$.

We will let $L_n(x) = \{y : (y, x) \in \delta_n\}$ and define $\delta_n$ by defining the sets $L_n(x)$ for all $x \in X$. Let $x \in X$ and $F \in \mathcal{T}(n)$. If $x \notin F$, let $K_F(x) = \emptyset$. If $x \in F$ and $x \in C(F)$, let $K_F(x) = J(F)$. If $x \in F$ and $x \in \bigcup \{C(F, E) : E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)\} - C(F)$ there exists some $E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)$ such that $x \in C(F, E)$, so let

$$K_F(x) = \bigcup \{J(F, E) : x \in C(F, E), E \in \mathcal{C}(F) \cup \mathcal{C}^*(F)\}.$$

Then let

$$L_n(x) = \bigcup \{K_F(x) : F \in \mathcal{T}(n)\}.$$

Then $L_n(x)$ is closed and connected for each $x$, for it is a nonempty finite union of closed sets, and the nonempty sets comprising that union are each connected and contain $x$.

The relation $\delta_n$ is closed because

$$\delta_n = \bigcup \{C(F) \times C(F') : F \in \mathcal{T}(n)\}$$

$$\cup \bigcup \{C(F', E') \times C(F, E) : F \in \mathcal{T}(n),$$

$$E \in \mathcal{C}(F) \cup \mathcal{C}^*(F), C(F', E') \subset J(F, E)\}$$

which is a finite union of products of closed sets.

To complete the induction we will assume $\delta_n$ has been defined and we will define the sets, $\mathcal{T}(n + 1)$, $\mathcal{T}(n + 1, t), t = 0, 1, \ldots$, and for
each \( F \in \mathcal{F}(n+1) \) we must define \( \delta_\ast F, \varepsilon_\ast(F), \varepsilon(F), \varepsilon_\ast(F), \delta_\ast F, \rho(F) \) and for each \( E \in \mathcal{C}(F) \cup \varepsilon_\ast(F), \partial E \).

First let \( \mathcal{F}(n+1) = \bigcup \{ \mathcal{C}(F); F \in \mathcal{F}(n) \} \) where \( \mathcal{C}(F) \) is as defined in (5).

As in the initial induction step, we will assign to each \( F \in \mathcal{F}(n+1) \) an order which will, in a sense, classify the sets of \( \mathcal{F}(n+1) \) according to how “far away” they are from 0. But since we want to assure that \( \delta_{n+1} \subset \delta_n \), or, what is the same thing, \( L_{n+1}(x) \subset L_n(x) \) for all \( x \in X \), we must take more care in defining the order of an element \( F \) of \( \mathcal{F}(n+1) \). Because each \( F \in \mathcal{F}(n+1) \) is contained in an unique element of \( \mathcal{F}(n) \), the “predecessors” of the elements of \( F \) must be contained in the set of “predecessors” of that unique element of \( \mathcal{F}(n) \) which contains it.

We will partition \( \mathcal{F}(n+1) \) into the sets \( \mathcal{F}(n+1, t) \), \( t = 0, 1, \ldots \), and if \( F \in \mathcal{F}(n+1, t) \) we will say \( F \) has order \( t \). First we let

\[
\mathcal{F}(n+1, 0) = \{ F \in \mathcal{F}(n+1); 0 \in F \}.
\]

Let \( F_n \) be an element of \( \mathcal{F}(n, s-1) \) and suppose that order has been defined for the elements of some subset of \( \mathcal{C}(F_n) \) which contains at least \( \{ F \in \mathcal{C}(F_n); F \cap \partial F_n \neq \emptyset \} \). Let \( F \) be an element of \( \mathcal{C}(F_n) \) such that \( F \cap C'(F_n) \neq \emptyset \) and such that order has not yet been defined for \( F \). We will let \( F \) be an element of \( \mathcal{F}(n+1, t) \) and say \( F \) has order \( t \) if \( t \) is the smallest positive integer such that there exists \( F_\ast \in \mathcal{C}(F_n) \) such that \( F_\ast \subset C(F_n) \), \( F_\ast \) has order \( t - 1 \), and \( F_\ast \cap F \cap \text{int } F_n \neq \emptyset \). Let

\[
(3.2) \quad \mathcal{C}_\ast(F) = \{ F_\ast \in \mathcal{C}(F_n); F_\ast \in \mathcal{F}(n+1, t-1), F_\ast \subset C(F_n) \}
\]

and \( F_\ast \cap F \cap \text{int } F_n \neq \emptyset \).

Notice that this is enough to define order for all \( F \in \mathcal{C}(F_n) \) such that \( F \cap C'(F_n) \neq \emptyset \), since \( C'(F_n) \) is connected and covered by the connected sets \( [F \cap \text{int } F_n] \cup [F' \cap \partial F_n] \). Now suppose \( F \in \mathcal{C}(F_n) \) but \( F \cap C'(F_n) = \emptyset \). Then \( F \subset P \) for some unique \( P \in \mathcal{P}(F_n) \), where \( \mathcal{P}(F_n) \) is as defined in the proof of Lemma 2. Let \( F \) be an element of \( \mathcal{F}(n+1, t) \) and say \( F \) has order \( t \) if \( t \) is the smallest positive integer such that there exists some \( F_\ast \in \mathcal{F}(n+1, t-1) \) such that \( F_\ast \subset P \) and \( F_\ast \cap F \cap \text{int } P \neq \emptyset \). Let

\[
(3.3) \quad \mathcal{C}_\ast(F) = \{ F_\ast \in \mathcal{C}(F_n); F_\ast \in \mathcal{F}(n+1, t-1), F_\ast \subset P \}
\]

and \( F_\ast \cap F \cap \text{int } P \neq \emptyset \).

This is enough to define order for all \( F \in \mathcal{C}(F_n) \) since for each \( P \in \mathcal{P}(F) \), \( \text{int } P \) is connected and covered by the connected sets \( F \cap \text{int } P \) for \( F \in \mathcal{C}(F_n, P) \) and \( P \cap C'(F_n) \neq \emptyset \).
Suppose order has been defined for all $F \in \mathcal{F}(F_n)$ where $F_n \in \mathcal{F}(n, s - 1)$. Let $F_{n,s}$ be an element of $\mathcal{F}(n, s)$ and let $F$ be an element of $\mathcal{E}(F_{n,s})$ such that $F \cap \partial*F_{n,s} \neq \emptyset$. We will let $F'$ be an element of $\mathcal{F}(n + 1, t)$ and say $F'$ has order $t$ if $t$ is the smallest positive integer such that there exists $F* \in \mathcal{F}(n + 1, t - 1)$ such that

$$F* \subseteq F_{n,s} \in \mathcal{F}(n, s), \quad F_{n,s} \in \mathcal{E}_*(F_{n,s})$$

and $F \cap F* \cap \partial*F_{n,s} \neq \emptyset$. Let

$$\mathcal{E}_*(F') = \{F* \in \mathcal{F}(n + 1, t - 1): F* \subseteq F_{n,s} \in \mathcal{F}(n, s - 1), \quad F_{n,s} \in \mathcal{E}_*(F_{n,s}), \quad F \cap F* \cap \partial*F_{n,s} \neq \emptyset\}.$$

With this we have defined a unique order for each $F \in \mathcal{F}(n + 1)$ and we have $\mathcal{F}(n + 1) = \bigcup_t \mathcal{F}(n + 1, t)$.

Now let $F'$ be an element of $\mathcal{F}(n + 1)$ and suppose $F \cap F' \subseteq \mathcal{F}(n)$. As mentioned earlier, in order to make the relation $\Delta$ a transitive order, it will be necessary that the elements of $\partial F_n - (\partial*F_n \cup \partial*F_{n})$ have no successors. To ensure that this happens since $\partial*F'$ will have successors in the relation $\partial_{n+1}$, we must define $\partial*F'$ for $F \in \mathcal{E}(F_n)$ so that

$$\partial*F \cap [\partial F_n - (\partial*F_n \cup \partial*F_{n})] = \emptyset,$$

when $F \cap \partial*F_n = \emptyset$. Also, if $F \cap C(F_n) = \emptyset$ and $F \subseteq P \in \mathcal{P}(F_n)$, we want $\partial*F \cap \partial P = \emptyset$.

We do this as follows. If $F \in \mathcal{E}(F_n)$ and $F \cap F_{n,s} \neq \emptyset$, set

$$\partial*F = F \cap \partial*F_n.$$

If $F \cap \partial*F_n = \emptyset$, but $F \cap C'(F_n) \neq \emptyset$, for each $E \in \mathcal{E}_*(F)$ choose $p(F, E) \in F \cap E \cap \text{int } F_n$. Let

$$T(F_n) = \{p(F, E): F \in \mathcal{E}_*(F_n), \quad F \cap \partial*F_n = \emptyset, \quad F \cap C'(F_n) \neq \emptyset, \quad E \in \mathcal{E}_*(F)\}.$$

Since $T(F_n)$ is a finite set it is a closed subset of $\text{int } F_n$. Because $F_n$ is normal, we can find $S(F_n)$, an open subset of $F_n$ such that

$$\text{Cl}(S(F_n)) \cap T(F_n) = \emptyset \quad \text{and} \quad \partial F_n \subset S(F_n).$$

Then for $F \in \mathcal{E}_*(F_n)$ such that $F \cap \partial*F_n = \emptyset$ and $F \cap C'(F_n) \neq \emptyset$, set

$$\partial*F = \{F \cap (\cup \mathcal{E}_*(F)) \} - S(F_n).$$

Since $\mathcal{E}_*(F) \neq \emptyset$ and for $E \in \mathcal{E}_*(F)$, $p(F, E) \in S(F_n)$, it follows that $\partial*F$ is a nonempty closed subset of $\partial F$ and $\partial*F \cap \{\partial F_n - \partial*F_n\} = \emptyset$. Similarly for $F \in \mathcal{E}_*(F_n)$ such that $F \cap C'(F_n) = \emptyset$, we know that $F \subseteq P$ for some unique $P \subseteq \mathcal{P}(F_n)$. Now for each $F \subseteq P$ such that $F \cap C'(F_n) = \emptyset$ and each $E \in \mathcal{E}_*(F)$, we can choose $p(F, E)$ to be an
element of $F \cap E \cap \text{int } P$. Let

$$T(F_n, p) = \{p(F, E) : F \in S(F_n, P), F \cap C'(F_n) = \emptyset, \text{ and } E \in \mathcal{E}(F)\}.$$ 

Since $T(F_n, P)$ is a finite set it is a closed subset of $\text{int } P$. Therefore we can find an open set $S(F_n, P)$ such that $\partial P \subset S(F_n, P)$ and

$$\text{Cl}(S(F_n, P)) \cap T(F_n, P) = \emptyset.$$ 

Now for each $F \subset P$ such that $F \cap C'(F_n) = \emptyset$, set

$$(4.4) \quad \partial* F = [F \cap (\cup \mathcal{E}(F))] - S(F_n, P).$$

It follows that $\partial* F \cap \partial P = \emptyset$ and $\partial* F$ is not empty.

For all $F \in \mathcal{F}(n + 1)$ let

$$\mathcal{E}*(F) = \{E \in \mathcal{F}(n + 1) : F \in \mathcal{E}(E)\}$$

and let

$$\partial* F = \cup \{\partial* E \cap F : E \in \mathcal{E}*(F)\}.$$ 

If $E \in \mathcal{E}*(F)$ let

$$\partial* F = \partial* E \cap F.$$ 

Let

$$\mathcal{E}(F) = \{E \in \mathcal{F}(n + 1) : E \neq F, (E \cap F) - (\partial* F \cup \partial* F) \neq \emptyset\}$$

and for $E \in \mathcal{E}(F)$ let

$$\partial* F = \text{Cl}[(E \cap F) - (\partial* F \cup \partial* F)].$$ 

If $\mathcal{E}*(F) \neq \emptyset$, let $\rho(F) = d(\partial* F, \partial* F)$ and if $\mathcal{E}*(F) = \emptyset$, let $\rho(F) = \text{diam } F$. If $F \cap \partial* F_n = \emptyset$ but $F \cap C'(F_n) \neq \emptyset$, let $\rho(F) = d(\partial* F, \partial F_n)$. If $F \cap C'(F_n) = \emptyset$, and $F \subset P \in \mathcal{P}(F_n)$, let $\rho(F) = d(\partial* F, \partial P)$; otherwise let $\rho(F) = \text{diam } F$. Finally let

$$(9) \quad \rho(F) = \min \{\rho, \rho(F)\}.$$ 

This completes the definitions necessary to define $\partial_n$ for all positive integers $n$.

We now define a relation $\Delta$ on $X$ by letting $\Delta = \bigcap_{n=1}^{\infty} \partial_n$. It remains to show that $\Delta$ is a partial order satisfying Theorem 2.

3. The relation $\Delta$ satisfies the hypotheses of Koch's Arc Theorem. The relation $\Delta$ is continuous on $X$ since $\Delta = \bigcap_{n=1}^{\infty} \partial_n$ and we have shown in (7) that each $\partial_n$ is closed in $X \times X$. Also 0 is a zero for $\Delta$ since $0 \in L(x)$ for all $x \in X$. We must further show that $L(x)$, the set of predecessors of each $x \in X$ under the relation $\Delta$, is a
connected set. To do this it is enough to show that \( L_{n+1}(x) \subseteq L_n(x) \) for each \( x \in X \), since then the set \( \{L_n(x)\}_{n=1}^{\infty} \) will be a nest of continua and \( L(x) = \bigcap_{n=1}^{\infty} L_n(x) \) will be a continuum and thus be connected.

Because \( L_{n+1}(x) \), (6), is a union of sets of the forms \( J(F) \) and \( J(F, E) \) where \( F \in \mathcal{F}(n + 1) \), \( E \in \mathcal{E}(F) \cup \mathcal{E}^*(F) \) and \( x \in C(F) \) or \( x \in C(F, E) \) to prove \( L_{n+1}(x) \subseteq L_n(x) \), it is sufficient to show that if \( x \in F \in \mathcal{F}(n + 1) \) and \( F \subseteq F' \in \mathcal{F}(n) \), then \( F' \cup J(F) \) is a subset of either \( J(F') \) or \( J(F', E') \) for some \( E' \in \mathcal{E}(F') \cup \mathcal{E}^*(F') \). This proof is omitted but is a straightforward induction on \( t \) when

\[
x \in F \subseteq F' \in \mathcal{F}(n, t)
\]

using definitions (2.1-2) of \( J(F) \) and (3.1-4) of \( \mathcal{E}_*(F) \).

It is clear that \( \Delta \) is reflexive. That \( \Delta \) is a partial order on \( X \) will be established by the following lemmas.

**Lemma 3.** Let \( F_1 \) and \( F_2 \) be distinct elements of \( \mathcal{F}(n) \) and let \( x \) be an element of \( \partial F_1 - (\partial_* F_1 \cup \partial^* F_1) \). Then \( x \) is an element of \( \partial F_2 - (\partial_* F_2 \cup \partial^* F_2) \).

**Proof.** We will proceed by induction on \( n \). Suppose \( n = 1 \), and that \( F_1 \) is an element of \( \mathcal{F}(1, t) \). Then the order of \( F_1 \) is either \( t - 1, t \) or \( t + 1 \), using (1) since \( F_1 \cap F_2 \neq \emptyset \). If \( F_2 \in \mathcal{F}(1, t - 1) \) then by (4.1) \( x \in \partial_* F_1 \) and if \( F_2 \in \mathcal{F}(1, t + 1) \) by (4.1) \( x \in \partial^* F_1 \), and both of these situations contradict the hypothesis. Thus \( F_2 \in \mathcal{F}(1, t) \). Suppose \( x \in \partial_* F_2 \). Then there exists a set \( F_3 \in \mathcal{F}(1, t - 1) \) such that \( x \in F_3 \cap F_1 \). But also \( x \in F_1 \) so \( x \in F_1 \cap F_3 \subset \partial_* F_1 \) which is a contradiction. Similarly, if \( x \in \partial^* F_2 \) there exists a set \( F_3 \in \mathcal{F}(1, t + 1) \) such that \( x \in F_3 \cap F_2 \), so \( x \in F_1 \cap F_3 \subset \partial* F_1 \) and we get another contradiction.

We now suppose the lemma is true for \( n = 1, 2, \ldots, k - 1 \). Let \( F_1 \) and \( F_2 \) be distinct elements of \( \mathcal{F}(k) \) and suppose \( F_1 \subset T_1 \in \mathcal{F}(k-1) \) and \( F_2 \subset T_2 \in \mathcal{F}(k-1) \). By (4.1-4) we have for \( i = 1, 2 \)

\[
(10) \quad \partial_* F_i \cup \partial^* F_i \subset (\text{int } T_i) \cup \partial_* T_i \cup \partial^* T_i.
\]

So \( x \in \partial_* F_1 \cup \partial^* F_1 \) implies by (4.2) that \( x \in \partial_* T_1 \cup \partial^* T_1 \).

Now if \( T_1 \neq T_2 \), \( x \in T_1 \cap T_2 \) implies \( x \in \partial T_1 \cap \partial T_2 \). From the induction hypothesis \( x \in \partial T_1 - (\partial_* T_1 \cup \partial^* T_2) \). Therefore by (10) \( x \in \partial F_2 - (\partial_* F_2 \cup \partial^* F_2) \).

If, however, both \( F_1 \) and \( F_2 \) are subsets of \( T_i \), we will consider first the case when \( x \in C(T_i) \). If \( x \in S(T_i) \), where \( S(T_i) \) is as defined in (8) then by (4.3) \( x \in \partial_* F \cup \partial^* F \) for any \( F \in \mathcal{E}(T_i) \) such that \( F \cap C(T_i) \neq \emptyset \). In particular \( x \in \partial_* F_2 \cup \partial^* F_2 \). If \( x \in S(T_i) \), the argument that \( x \in \partial_* F_2 \cup \partial^* F_2 \) is analogous to the situation when \( n = 1 \).

The final case when \( x \in C(T_i) \) follows by a similar argument using that either \( F_1 \subset P_1 \) and \( F_2 \subset P_2 \) where \( P_1 \neq P_2 \) and \( P_1 \) and \( P_2 \) are in
\( \mathcal{P}(T_i) \); or \( F_i \cup F_2 \subset P \) for some \( P \in \mathcal{P}(T_i) \) and that
\[
\partial_* F_i \cup \partial* F_i \subset [P - S(T_1, P)] \cup \partial* T_i
\]
for \( i = 1, 2 \).

**Note.** It follows from Lemma 3 that for \( x \in (\text{int } F_i) \cup \partial_* F_i \cup \partial* F_i \) where \( F_i \in \mathcal{F}(n, t) \) and if \( x \in F_2 \) for some \( F_2 \in \mathcal{F}(n) \), \( F_2 \neq F_1 \), then \( x \in \partial_* F_2 \cup \partial* F_2 \). Further if \( x \in \partial_* F_1 \) then \( F_2 \in \mathcal{F}(n, t - 1) \cup \mathcal{F}(n, t) \), and if \( x \in \partial* F_1 \), then \( F_2 \in \mathcal{F}(n, t + 1) \).

**Lemma 4.** Let \( F \in \mathcal{F}(n) \) and \( x \in \partial F - (\partial_* F \cup \partial* F) \). Let \( m \) be an integer such that there exists \( E \in \mathcal{F}(m) \) such that \( x \in E \subset F \) and \( E \cap (\partial_* F \cup \partial* F) = \emptyset \). Then \( C(E, E^*) \cap \partial F = \emptyset \) for all \( E^* \in \mathcal{E}^*(E) \).

**Proof.** Let \( F_m = E \) and \( F_n = F \). Then there exists \( \{F_i\}_{i=1}^m \) such that \( F_i \in \mathcal{F}(i) \) and \( F_{i+1} \subset F_i \). Let \( l \) be the greatest integer such that \( m > l \geq n \) and \( \partial_* E - \partial_* F_i \neq \emptyset \) and let \( k \) be the greatest integer such that \( \partial_* E - \partial* F_k \neq \emptyset \). Then \( m > l \geq n \) and \( m > k \geq n \). Without loss of generality suppose \( l \geq k \). Since \( \partial_* E \subset \partial_* F_{l+1} \),
\[
d(\partial_* E, \partial F_l) \geq d(\partial_* F_{l+1}, \partial F_l)
\]
From (8) \( \partial F_i \subset S(F_i) \) and by (4.3) and (4.4) \( \partial_* F_{l+1} \subset F_i \setminus S(F_i) \). Therefore
\[
d(\partial_* F_{l+1}, \partial F_l) = \rho(F_l) \geq \rho(F_i)
\]
Thus \( d(\partial_* E, \partial F_i) \geq \rho(F_i) \). Similarly
\[
d(\partial_* E, \partial F_k) \geq d(\partial_* F_k+1, \partial F_k) \geq \rho(F_k)
\]
Also
\[
d(\partial* E, \partial F) \geq d(\partial* E, \partial F_k) \text{ and } d(\partial* E, \partial F) \geq d(\partial* E, \partial F_i).
\]
Thus
\[
d(\partial* E \cup \partial* E, \partial F) \geq \min \{d(\partial* E, \partial F_k), d(\partial* E, \partial F_i)\}
\geq \min \{\rho(F_k), \rho(F_i)\} = \rho(F_i) \geq \rho(F_{m-l}).
\]
From Lemma 2 if \( x \in \partial E \) and
\[
d(x, \partial_* E \cup \partial* E) > \frac{\rho(F_{m-l})}{3}
\]
then \( x \in C(E, E^*) \) for any \( E^* \in \mathcal{E}^*(E) \). Thus \( \partial F \cap C(E, E^*) = \emptyset \) for any \( E^* \in \mathcal{E}^*(E) \).

**Lemma 5.** Let \( x \in \partial F - (\partial* F \cup \partial* F) \) for \( F \in \mathcal{F}(n) \). Then \( x \) has no successors other than itself in the relation \( \Delta \).
Proof. Assume \( y \geq x \) and \( y \neq x \). Choose \( m > n \) such that \( d(x, y) > 2^{-m} \) and such that \( d(x, \partial^*_x F \cup \partial^*F) > 2^{-m} \). Then since for \( F' \in \mathcal{F}(m) \) we have \( \text{diam } F' < 2^{-m} \), \( x \) and \( y \) are not both elements of any one \( F' \in \mathcal{F}(m) \). Also, if
\[
x \in F' \in \mathcal{F}(m), \quad \text{then } F' \cap (\partial^*_x F \cup \partial^*F) = \emptyset,
\]
so that \( m \) satisfies the conditions of Lemma 5. However, since \( x \in L_m(y) \) and \( x \) is in no element of \( \mathcal{F}(m) \) containing \( y \), by (6) \( x \in C(F', F^*) \) for some
\[
F' \in \mathcal{F}(m) \text{ and } F^* \in \mathcal{C}^*(F').
\]
But by Lemma 4, \( C(F', F^*) \cap \partial F = \emptyset \). This is a contradiction and proves that such a \( y \) cannot exist.

In the next lemma we will use the following notation. If \( x \in \text{int } F \) for some \( F \in \mathcal{F}(n, t) \), set \( q_n(x) = t \). If \( x \in \partial^*_x F \) for some \( F \in \mathcal{F}(n, t) \), set \( q_n(x) = t - 1 \). By the note after Lemma 3, \( q_n(x) \) is well-defined and single valued for all \( x \in (\text{int } F) \cup \partial^*_x F \cup \partial^*F \) where \( F \in \mathcal{F}(n) \).

**Lemma 6.** The relation \( \Delta \) is anti-symmetric.

**Proof.** Assume there exist \( x \) and \( y \) in \( X \) such that \( x \neq y \) and \( y \leq x \). Choose \( n \) such that \( d(x, y) > 2^{-n+1} \). Then, since \( x \in L_n(y) \), there exists some \( F_1 \in \mathcal{F}(n) \) such that \( y \in F_1 \in \mathcal{F}(n, t) \) and \( x \in J(F_1) \). By Lemma 5, \( y \in (\text{int } F_1) \cup \partial^*_x F_1 \cup \partial^*F_1 \), so \( q_n(y) \) is defined and \( q_n(y) \leq t - 1 \). Now because \( d(x, y) > 2^{-n+1} \), \( x \in F_2 \in \mathcal{F}(n) \) where \( F_2 \in \mathcal{F}(n, s) \) and \( s < t - 1 \). Also by Lemma 5, \( x \in \text{int } F_2 \cup \partial^*_x F_2 \cup \partial^*F_2 \), so \( q_n(x) \) is defined and \( q_n(x) \leq s < t - 1 \). It follows that \( q_n(y) > q_n(x) \). But by a symmetric argument since \( y \in L_n(x) \), it can be shown that \( q_n(x) > q_n(y) \). This contradiction proves that \( \Delta \) is anti-symmetric.

**Lemma 7.** The relation \( \Delta \) is transitive.

**Proof.** Let \( x, y \) and \( z \) be elements of \( X \) such that \( x \leq y \) and \( y \leq z \). We will show \( x \leq z \). We can assume \( x < y \) and \( y < z \). Choose \( n \) such that \( \min \{d(x, y), d(y, z), d(x, z)\} > 2^{-n+1} \). It is enough to show \( x \in L_n(z) \) since we have shown \( L_{n-1}(z) \supset L_n(z) \). Since \( y \in L_n(z) \), \( y \in F_y \) for some \( F_y \in \mathcal{F}(n, t) \) where \( y \in J(F_y, E') \subset L_n(z) \) for some \( E' \in \mathcal{C}^*(F_y) \). By Lemma 4, \( y \in \text{int } F_y \cup \partial^*_x F_y \cup \partial^*F_y \). If \( y \in \text{int } F_y \) then since \( x \in L_n(y), x \in J(F_y) \subset J(F_y, E') \subset L_n(z) \).

If \( y \in \text{int } F_y \) then either \( y \in \partial^*_x F_y \) or \( y \in \partial^*F_y \). We will consider the case when \( y \in \partial^*_x F_y \). The argument is similar when \( y \in \partial^*F_y \). By the note after Lemma 3 if \( y \in F \in \mathcal{F}(n) \), then \( F \in \mathcal{F}(n, t) \cup \mathcal{F}(n, t - 1) \).
If \( y \in F_* \in \mathcal{F}(n, t - 1) \) where \( x \in J(F_*) \) then \( x \in J(F_*, F_y) \subset L_n(z) \). If we assume this is not the case then \( x \in J(F_*, F_y) \) for any \( F_* \in \mathcal{C}_*(F_y) \). Let \( \mathcal{A} = \{ F_* \in \mathcal{F}(n, t - 1) : x \in J(F_*, F) \text{ for some } F \in \mathcal{F}(n, t) \} \) such that \( y \in F \). The set \( \mathcal{A} \) is not empty since \( x \in L_n(y) \). Let

\[
    r = \min \{ d(y, F_*): F_* \in \mathcal{A} \}.
\]

Since \( y \in F_* \) for any \( F_* \in \mathcal{A} \), \( r > 0 \). Choose \( m > n \) such that \( r > 2^{-m} \). Now because \( x \in L_m(y) \subset L_n(y) \) there exists a set \( T \in \mathcal{F}(m) \) such that \( y \in T \) and \( x \in J(T) \). Either \( T \subset F \) for some \( F \in \mathcal{F}(n, t) \) or \( T \subset F_* \) for some \( F_* \in \mathcal{F}(n, t - 1) \). However if

\[
    T \subset F_* \in \mathcal{F}(n, t - 1), x \in J(T) \subset J(F_*, F_y)
\]

which contradicts our assumption. Thus there exists \( F \in \mathcal{F}(n, t) \) such that \( T \subset F \). Now by (2.2) \( x \in J(T) \subset T \cup \cup \{ J(T_*, T) : T_* \cap T \neq \emptyset, T_* \in \mathcal{C}_*(T) \text{ and } T_* \subset F_* \text{ for some } F_* \in \mathcal{F}(n, t - 1) \} \). By the choice of \( m \) and \( r \), \( F_* \in \mathcal{A} \). But \( x \in J(T) \subset J(F_*, F) \) implies that \( F_* \in \mathcal{A} \). This contradiction says that \( x \in J(F_*, F_y) \) for some \( F_* \in \mathcal{C}_*(F_y) \) and thus \( x \in J(F_y) \subset L_n(z) \). This completes the proof that \( \mathcal{A} \) is transitive.

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