

Pacific Journal of Mathematics

ALGEBRAS FORMED BY THE ZORN VECTOR MATRIX

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In the Zorn vector matrix algebra the three dimensional vector algebra is replaced by a finite dimensional Lie algebra L over a field of characteristic not 2 equipped with an associative symmetric bilinear form (a, b) and having the property: $[a[bc]] = (a, c)b - (a, b)c, a, b, c \in L$. We determine all the alternative algebras \mathfrak{A} obtained in this way: If the bilinear form (a, b) on L is nondegenerate then \mathfrak{A} is the split Cayley algebra or a quaternion algebra. For a degenerate form (a, b) , \mathfrak{A} is a direct sum of its radical and a subalgebra which is either a quaternion or two dimensional separable algebra. As an immediate consequence of the first result we have shown that if the bilinear form on the Lie algebra L is nondegenerate then L is simple with dimension three or one.

Let Φ be a field of characteristic not two throughout this paper. Let A be an anti-commutative algebra over Φ with a symmetric bilinear form (a, b) which is associative, i.e., $(ac, b) = (a, cb), a, b, c \in A$, and we consider the set \mathfrak{A} of 2×2 vector matrices of the form:

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \alpha, \beta \in \Phi; a, b \in A.$$

\mathfrak{A} is a vector space Φ under the usual addition, $+$, and multiplication by scalars. A multiplication in \mathfrak{A} ([5] and [2]) is defined to be:

$$(1) \quad \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma - (a, d), & \alpha c + \delta a + bd \\ \gamma b + \beta d + ac, & \beta\delta - (b, c) \end{pmatrix}.$$

Then \mathfrak{A} is a flexible algebra over Φ in the sense that

$$(xy)x = x(yx), x, y \in \mathfrak{A}.$$

Furthermore \mathfrak{A} is an alternative algebra over Φ , i.e., $x^2y = x(xy)$ and $(yx)x = yx^2, x, y \in \mathfrak{A}$ if and only if the anti-commutative algebra A has the following property:

$$(2) \quad a(bc) = (a, c)b - (a, b)c, a, b, c \in A.$$

This is checked easily by a comparison of entries of vector matrices x^2y and $x(xy)$. We note that this property implies the Jacobi identity: $a(bc) + b(ca) + c(ab) = 0$ and A is a Lie algebra over the field Φ .

We shall determine all the alternative algebras over Φ which are constructed from the Lie algebras with (2) by the Zorn vector matrices. First we determine all the Lie algebras with (2) and let L be a finite

dimensional Lie algebra over Φ equipped with an associative symmetric bilinear form (a, b) and having the property (2). We return to writing $[a \ b]$ in place of ab . Set $L^\perp = \{a \in L \mid (a, b) = 0, b \in L\}$ the radical of the bilinear form. If the bilinear form (a, b) is nondegenerate, i.e., $L^\perp = 0$, it follows from (2) that L is a simple Lie algebra. On the other hand, if (a, b) is degenerate we have the following.

LEMMA. *If the bilinear form (a, b) is degenerate, then the Lie algebra L is nilpotent with $L^3 = 0$ or $L = \Phi u + L^\perp$ where L^\perp is a nonzero abelian ideal and $(ad \ u)^2|_{L^\perp} = \rho I, \rho = -(u, u) \neq 0$ in Φ .*

Proof. If $L^\perp = L$, the condition (2) implies $L^3 = 0$. In the rest of the proof we assume that $L^\perp \neq L$, and L^\perp is a nonzero proper ideal of L . There exists an element $u \neq 0$ in L which is not in L^\perp and satisfies $(u, u) \neq 0$. Let (y_1, y_2, \dots, y_m) be a basis for L^\perp .

$$(ad \ u)^2|_{L^\perp} = -(u, u)I$$

because we have $(ad \ u)^2 y_i = [u[u, y_i]] = -(u, u)y_i$ for all y_i . Since

$$\rho = -(u, u) \neq 0$$

in Φ , the mapping $ad \ u$ is nonsingular on L^\perp .

$$(ad \ u)[y_i, y_j] = (u, y_j)y_i - (u, y_i)y_j = 0$$

for all i, j imply $[y_i, y_j] = 0$ which means L^\perp abelian. Finally we show that L is the direct sum of two subspaces Φu and L^\perp . Let x be any element of L , not in L^\perp . $(ad \ u)[x, y_i] = -(u, x)y_i$ and set $\tau = -(u, x)$. Then $(ad \ u)ad(\tau u - \rho x)|_{L^\perp} = 0$. Since $ad \ u$ is nonsingular on L^\perp , $ad(\tau u - \rho x)|_{L^\perp} = 0$. We wish to show that $(y, \tau u - \rho x) = 0$ for any y of L , which is equivalent to saying that $x \in \Phi u + L^\perp$. Since $[\tau u - \rho x, y_i] = 0$ for all y_i of the basis for L^\perp , $0 = [y[\tau u - \rho x, y_i]] = -(y, \tau u - \rho x)y_i$. This has completed our proof.

Now we first take up the case the bilinear form (a, b) on the Lie algebra L is nondegenerate. It is known ([2]) that (a, b) on L is nondegenerate if and only if the algebra \mathfrak{A} constructed from L is simple. Since the alternative algebra \mathfrak{A} is simple, \mathfrak{A} is the split Cayley algebra or an associative algebra ([1]). We consider the latter case and follow Sagle's argument in [3]. Let

$$x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, y = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix}, z = \begin{pmatrix} \lambda & g \\ h & \mu \end{pmatrix}$$

be any elements of \mathfrak{A} . By a comparison of (1,1)-entries of $(xy)z =$

$x(yz)$ we have $([b d], h) = (a, [c g])$. Without loss of generality we may take $a = 0$ and we have $([b d], h) = 0$ for all $h \in L$. It follows from the nondegeneracy that $[b d] = 0$ for all b, d of L , i.e., $L^2 = 0$. From $0 = [a[b c]] = (a, c)b - (a, b)c$, we have $\dim L = 1$ and therefore \mathfrak{A} is a quaternion algebra. Hence we have the following

THEOREM 1. *Let L be a finite dimensional Lie algebra over a field Φ of characteristic $\neq 2$ equipped with an associative symmetric bilinear form (a, b) and having the property (2). If (a, b) is nondegenerate, then \mathfrak{A} is the split Cayley algebra or a quaternion algebra.*

A similar consideration to this theorem is given in [3]. As an immediate consequence of the theorem we have

COROLLARY. *Let L be as in Theorem 1. If the bilinear form (a, b) is nondegenerate L is simple with dimensionality three or one.*

Next we consider the remaining case, that is, (a, b) on L is degenerate. Let (u_1, u_2, \dots, u_n) be a basis for L over Φ and we set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$e_{12}^{(s)} = \begin{pmatrix} 0 & u_s \\ 0 & 0 \end{pmatrix}, \quad e_{21}^{(s)} = \begin{pmatrix} 0 & 0 \\ u_s & 0 \end{pmatrix}, \quad s = 1, 2, \dots, n.$$

These form a basis for the algebra \mathfrak{A} over Φ . Let $L = \Phi u + L^\perp$ be as in lemma and take the basis for L to be $u_1 = u$ and (u_2, \dots, u_n) a basis for the abelian ideal L^\perp . We have the following multiplication table for \mathfrak{A} :

$$e_i e_j = \delta_{ij} e_i,$$

$$e_i e_{ik}^{(s)} = e_{ik}^{(s)} e_k = e_{ik}^{(s)},$$

$$e_k e_{ik}^{(s)} e_i = e_{ik}^{(s)} e_i = 0,$$

$$e_{ik}^{(s)} e_{ki}^{(t)} = \begin{cases} \rho e_i & \text{if } (s, t) = (1, 1), \\ 0 & \text{otherwise,} \end{cases}$$

$$e_{ik}^{(s)} e_{ik}^{(t)} = -e_{ik}^{(t)} e_{ik}^{(s)} = \begin{cases} 0 & \text{if } s, t = 2, 3, \dots, n, \\ x_{ki} & \text{otherwise} \end{cases}$$

where $i, j, k = 1, 2; i \neq k; s, t = 1, 2, \dots, n$ and x_{ki} is a 2×2 vector matrix with 0 for all entries except for (k, i) -entry $[u_s u_i]$. The $e_{12}^{(s)}$ and $e_{21}^{(s)}$, $s = 2, 3, \dots, n$ are all properly nilpotent and therefore generate the radical \mathfrak{N} of \mathfrak{A} (Zorn Theorem 3.7 in [4]). It follows that $\mathfrak{A} = \mathfrak{Q} + \mathfrak{N}$ (direct sum) where \mathfrak{Q} is a quaternion subalgebra with basis

$(e_1, e_2, e_{12}^{(1)}, e_{21}^{(1)})$. We note that this quaternion subalgebra \mathfrak{S} is the same as one given in Theorem 1. Now we consider the remaining case: $L^\perp = L$ and L is nilpotent with $L^3 = 0$. Take a basis

$$(u_1, \dots, u_m, \dots, u_n)$$

for L such that (u_{m+1}, \dots, u_n) is a basis for the abelian ideal L^2 of L . We have

$$\begin{aligned} [u_i, u_j] &\in L^2, 1 \leq i, j \leq m \text{ and} \\ [u_i, u_j] &= 0 \text{ otherwise.} \end{aligned}$$

The multiplication table for \mathfrak{A} is as follows:

$$\begin{aligned} e_i e_j &= \delta_{ij} e_i, \\ e_i e_{ik}^{(s)} &= e_{ik}^{(s)} e_k = e_{ik}^{(s)}, \\ e_k e_{ik}^{(s)} &= e_{ik}^{(s)} e_i = 0, \\ e_{ik}^{(s)} e_{ki}^{(t)} &= -(u_s, u_t) e_i = 0, \\ e_{ik}^{(s)} e_{ik}^{(t)} &= x_{ki} \end{aligned}$$

where $i, j, k = 1, 2; i \neq k; s, t = 1, 2, \dots, n$ and x_{ki} is as before. The $e_{ik}^{(s)}, i \neq k, s = 1, 2, \dots, n$ are all properly nilpotent and generate the radical \mathfrak{N} of \mathfrak{A} . Hence \mathfrak{A} is a direct sum of \mathfrak{N} and a separable subalgebra $\mathfrak{O}e_1 + \mathfrak{O}e_2$. We have proved the following

THEOREM 2. *Let L be as in Theorem 1. If the bilinear form (a, b) is degenerate, then the algebra \mathfrak{A} constructed from L is a direct sum of its radical \mathfrak{N} and a subalgebra \mathfrak{S} where \mathfrak{S} is either a quaternion or 2-dimensional separable algebra.*

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Received November 1, 1968. This research was supported in part by the National Science Foundation under the grant AYE GY 3900.

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