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Let A and B be C^*-algebras and let $A \otimes_\alpha B$ be their C^*-tensor product with Turumaru's α-norm. The author has previously defined mappings R_φ: $A \otimes_\alpha B \rightarrow B$ and L_ψ: $A \otimes_\alpha B \rightarrow A$ via bounded linear functionals φ on A and ψ on B, as follows:

$$R_\varphi\left(\sum_{i=1}^{n} a_i \otimes b_i\right) = \sum_{i=1}^{n} < a_i, \varphi > b_i \, ,$$

$$L_\psi\left(\sum_{i=1}^{n} a_i \otimes b_i\right) = \sum_{i=1}^{n} < b_i, \psi > a_i \, ,$$

and has shown how the families $\{R_\varphi \mid \varphi \in A^*\}$ and $\{L_\psi \mid \psi \in B^*\}$ determine the structure of the tensor product of A and B. Moreover, in a joint paper with J. Hakeda the author also proved the existence of these kinds of mappings in tensor products of von Neumann algebras and gave some of their applications. Further applications of these mappings are shown in the present paper.

Theorem 2 says that the product $M \otimes N$ has property L if one of the factors M or N has property L. This answers a question of Sakai. It can be shown that the above families of mappings determine completely the tensor products of von Neumann algebras (Theorem 3). Theorem 4 shows that if π_1 and π_2 are projection of norm one from M_1 and N_1 to their subalgebras M_2 and N_2, then there exists, without assuming their σ-weak continuity, a projection of norm one π from $M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_1$ to $M_2 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_2$ such that $\pi(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$, where $a \in M_1$ and $b \in N_1$.

We always denote by $M \otimes N$ the tensor product of the von Neumann algebras M and N and by $M \otimes_\alpha N$ their tensor product as C^*-algebras. M^* means the conjugate space of M and M_* the predual of the von Neumann algebra M.

The following theorem is the basic result cited in the above introduction; it is a more precise version of Lemma 2.5 of [1]. We give the proof for the sake of completeness.

THEOREM 1. *Let M and N be von Neumann algebras and $M \otimes N$ their tensor product. Then for each $\varphi \in M_*$* (resp. $\psi \in N_*$) *there exists a σ-weakly continuous mapping R_φ: $M \otimes N \rightarrow N$* (resp. *$L_\psi$: $M \otimes N \rightarrow M$*) *satisfying the following conditions:*

(1) $$R_\varphi\Big(\sum_{i=1}^n a_i \otimes b_i\Big) = \sum_{i=1}^n <a_i, \varphi> b_i \,,$$

$$\Big(\text{resp. } L_\psi\Big(\sum_{i=1}^n a_i \otimes b_i\Big) = \sum_{i=1}^n <b_i, \psi> a_i\Big) \,.$$

(2) $R_\varphi((1 \otimes a)x(1 \otimes b)) = aR_\varphi(x)b$ *for* $x \in M \otimes N$,

(resp. $L_\psi((a \otimes 1)x(b \otimes 1)) = aL_\psi(x)b$.

(3) $<x, \varphi \otimes \psi> = <R_\varphi(x), \psi> = <L_\psi(x), \varphi>$ *for* $x \in M \otimes N$.

Moreover, the families of mappings $\{R_\varphi \mid \varphi \in M_*\}$ *and* $\{L_\psi \mid \psi \in N_*\}$ *are total in* $M \otimes N$.

Proof. Let $\varphi \otimes \psi$ be the product functional of φ and ψ which is σ-weakly continuous in $M \otimes N$. Put $f_{\varphi,x}(\psi) = < x, \varphi \otimes \psi >$ for $x \in M \otimes N$. Then $f_{\varphi,x}$ is clearly a bounded linear functional on N_* and as $(N_*)^* = N$ there exists a element $R_\varphi(x)$ in N such that

$$<x, \varphi \otimes \psi> = f_{\varphi,x}(\psi) = <R_\varphi(x), \psi> \,.$$

It is an easy verification by this definition that the mapping $R_\varphi: x \to R_\varphi(x)$ is a σ-weakly continuous linear mapping. Similarly, we get the mapping L_ψ and it is easily seen that assertion 3 holds.

Next, take an element $\sum_{i=1}^n a_i \otimes b_i$; then

$$\Big\langle R_\varphi\Big(\sum_{i=1}^n a_i \otimes b_i\Big), \psi\Big\rangle = \Big\langle \sum_{i=1}^n a_i \otimes b_i, \varphi \otimes \psi\Big\rangle$$
$$= \sum_{i=1}^n <a_i, \varphi><b_i, \psi> = \Big\langle \sum_{i=1}^n <a_i, \varphi> b_i, \psi\Big\rangle$$
$$= \Big\langle \sum_{i=1}^n <b_i, \psi> a_i, \varphi\Big\rangle = \Big\langle L_\psi\Big(\sum_{i=1}^n a_i \otimes b_i\Big), \varphi\Big\rangle \,,$$

which implies 1. From these relations, we get

$$R_\varphi\Big((1 \otimes a)\sum_{i=1}^n x_i \otimes y_i(1 \otimes b)\Big) = R_\varphi\Big(\sum_{i=1}^n x_i \otimes ay_ib\Big)$$
$$= \sum_{i=1}^n <x_i, \varphi> ay_ib = a\Big(\sum_{i=1}^n <x_i, \varphi> y_i\Big)b = aR_\varphi\Big(\sum_{i=1}^n x_i \otimes y_i\Big)b \,,$$

and since R_φ is σ-weakly continuous, $R_\varphi((1 \otimes a)x(1 \otimes b)) = aR_\varphi(x)b$ for all $x \in M \otimes N$. The argument for L_ψ goes similarly.

Now, suppose $R_\varphi(x) = 0$ for all φ in M_*, then $<x, \varphi \otimes \psi> = <R_\varphi(x), \psi> = 0$ for $\varphi \in M_*$ and $\psi \in N_*$. Hence $<x, \sum_{i=1}^n \varphi_i \otimes \psi_i> = 0$ where $\varphi_i \in M_*$ and $\psi_i \in N_*$ $(i = 1, 2, \cdots, n)$. Since the family $\{\sum_{i=1}^n \varphi_i \otimes \psi_i \mid \varphi_i \in M_*, \psi_i \in N_*\}$ is dense in $(M \otimes N)_*$ (cf. [13]), we get $x = 0$. Similarly the family $\{L_\psi \mid \psi \in N_*\}$ is also total in $M \otimes N$. This completes the proof.

We notice that the families $\{R_\varphi \mid \varphi \in M_*$ and positive$\}$ and $\{M_\psi \mid \psi \in N_*$ and positive$\}$ are also total.

Recall that a factor M (on a separable Hilbert space) has the property L if there is a sequence $\{u_n\}$ of unitary elements in M such that σ-weak limit $u_n = 0$ and strong- limit $u_n^* a u_n =$ a for all $a \in M$.

Sakai proved that if one of the factors M or N is finite and has property L, then $M \otimes N$ has property L, and asked whether the restriction of finiteness could be dropped [4, Th. 6.4 and Remark 6.2]. Here we shall answer this question as an application of the above mappings.

THEOREM 2. *Let M and N be factors and suppose that M or N has property L; then $M \otimes N$ has property L.*

Proof. Suppose N has property L; then there is a sequence $\{u_n\}$ of unitary elements in N such that σ-weak limit $u_n = 0$ and strong-limit $u_n^* a u_n = a$ for all $a \in N$. Put $\tilde{u}_n = 1 \otimes u_n$, then $\{\tilde{u}_n\}$ is a sequence of unitary elements in $M \otimes N$ and for $\varphi \in M_*$ and $\psi \in N_*$,

$$\lim_n \langle \tilde{u}_n, \varphi \otimes \psi \rangle = \lim_n \langle 1, \varphi \rangle \langle u_n, \psi \rangle = 0 .$$

Hence, $\lim_n \langle \tilde{u}_n, \sum_{i=1}^m \varphi_i \otimes \psi_i \rangle = 0$ where $\varphi_i \in M_*$ and $\psi_i \in N_*$ $(i = 1, 2, \cdots, m)$. Since $\{\tilde{u}_n\}$ is uniformly bounded, this implies σ-weak $\text{limit}_n\, \tilde{u}_n = 0$.

Next, take an arbitrary x in $M \otimes N$, then for $\varphi \in M_*$ and $\psi \in N_*$ we get by Theorem 1

$$\begin{aligned} &\lim_n \langle \tilde{u}_n^* x \tilde{u}_n, \varphi \otimes \psi \rangle = \lim_n \langle R_\varphi(\tilde{u}_n^* x \tilde{u}_n), \psi \rangle \\ &= \lim_n \langle u_n^* R_\varphi(x) u_n, \psi \rangle = \langle R_\varphi(x), \psi \rangle = \langle x, \varphi \otimes \psi \rangle . \end{aligned}$$

Hence $\lim_n \langle \tilde{u}_n^* x \tilde{u}_n, \sum_{i=1}^m \varphi_i \otimes \psi_i \rangle = \langle x, \sum_{i=1}^m \varphi_i \otimes \psi_i \rangle$ where $\varphi_i \in M_*$ and $\psi_i \in N_*$ $(i = 1, 2, \cdots, m)$. Since $\{\tilde{u}_n^* x \tilde{u}_n\}$ is uniformly bounded, this implies σ-weak $\text{limit}_n\, \tilde{u}_n^* x \tilde{u}_n = x$.

Let φ be a normal positive functional on $M \otimes N$, then

$$\begin{aligned} &\langle (\tilde{u}_n^* x \tilde{u}_n - x)^* (\tilde{u}_n^* x \tilde{u}_n - x), \varphi \rangle \\ &= \langle \tilde{u}_n^* x^* x \tilde{u}_n - \tilde{u}_n^* x^* \tilde{u}_n x - x^* \tilde{u}_n^* x \tilde{u}_n + x^* x, \varphi \rangle \\ &= \langle \tilde{u}_n^* x^* x \tilde{u}_n, \varphi \rangle - \langle \tilde{u}_n^* x^* \tilde{u}_n x, \varphi \rangle - \langle x^* \tilde{u}_n^* x \tilde{u}_n, \varphi \rangle \\ &\quad + \langle x^* x, \varphi \rangle \Rightarrow \langle x^* x, \varphi \rangle - \langle x^* x, \varphi \rangle - \langle x^* x, \varphi \rangle \\ &\quad + \langle x^* x, \varphi \rangle = 0 . \end{aligned}$$

That is, strongest-$\text{limit}_n\, \tilde{u}_n^* x \tilde{u}_n = x$. and strong-$\text{limit}_n\, \tilde{u}_n^* x \tilde{u}_n = x$ for all $x \in M \otimes N$. Hence $M \otimes N$ has property L.

Let M and N act on H and K. Denote by $\mathscr{B}(H)$ the algebra of all bounded linear operators on a Hilbert space H. Then $M \otimes N$ is naturally considered as the subalgebra of $\mathscr{B}(H) \otimes \mathscr{B}(K) = \mathscr{B}(H \otimes K)$ and as is easily seen the mappings R_φ and L_ψ in $M \otimes N$

are nothing but the restrictions of those mappings $R_{\tilde{\varphi}}(\tilde{\varphi} \in \mathscr{B}(H)_*)$ and $L_{\tilde{\psi}}(\tilde{\psi} \in \mathscr{B}(K)_*)$ in $\mathscr{B}(H) \otimes \mathscr{B}(K)$ where $\tilde{\varphi}$ and $\tilde{\psi}$ are extensions of φ and ψ. Now the following question naturally arises. Let $\mathfrak{A}$ be a von Neumann algebra on $H \otimes K$ and suppose $\mathfrak{A}$ satisfies the following condition: $R_\varphi(\mathfrak{A}) \subset N$ for all $\varphi \in \mathscr{B}(H)_*$ and $L_\psi(\mathfrak{A}) \subset M$ for all $\psi \in \mathscr{B}(K)_*$, then what is the relation between $\mathfrak{A}$ and $M \otimes N$? All we know is that $\mathfrak{A}$ is contained in $M \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N$.[1] In fact, let $x \in \mathfrak{A}$ and take an arbitrary element $a \in N'$ then we get

$$R_\varphi((1 \otimes a)x) = aR_\varphi(x) = R_\varphi(x)a = R_\varphi(x(1 \otimes a))$$

for all $\varphi \in \mathscr{B}(H)_*$. Hence $(1 \otimes a)x = x(1 \otimes a)$ and $x \in (1 \otimes N')' = \mathscr{B}(H) \otimes N$. Similarly $x \in (M' \otimes 1)' = M \otimes \mathscr{B}(K)$. Thus

$$\mathfrak{A} \subset M \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N .$$

Now let us consider the situation described in Theorem 1. Putting $\tilde{R}_\varphi(x) = 1 \otimes R_\varphi(x)$ and $\tilde{L}_\psi(x) L_\psi(x) \otimes 1$ we see that for commuting subalgebras $M \otimes 1$ and $1 \otimes N$ which generate $M \otimes N$ there are sufficiently many σ-weakly continuous $M \otimes 1-$module (resp. $1 \otimes N-$module) linear mappings from $M \otimes N$ to $M \otimes 1$ (resp. $1 \otimes N$) which induce σ-weakly continuous functional on each component algebra. We shall show that this situation completely determines the tensor product structure of von Neumann algebras. Namely

THEOREM 3. *Let $\mathfrak{A}$ be a von Neumann algebra and M and N be subalgebras satisfying the following conditions*:

(1) $\mathfrak{A} = R(M, N)$, *i.e., M and N generate $\mathfrak{A}$*,

(2) *M and N commute with each other*,

(3) *There is a total family of σ-weakly continuous $N -$ module mappings $\{R_\alpha \mid \alpha \in I\}$ from $\mathfrak{A}$ to N such that $R_\alpha(a) = \lambda_a^\alpha 1$ for $a \in M$ where λ_a^α is a complex number associated with a.*

Then $\mathfrak{A}$ is isomorphic to $M \otimes N$.

Proof. Take an element $\sum_{i=1}^n a_i b_i$ where $a_i \in M$ and $b_i \in N$ $(i = 1, 2, \cdots, n)$. We assert that the mapping

$$\Phi\colon \sum_{i=1}^n a_i b_i \longrightarrow \sum_{i=1}^n a_i \otimes b_i$$

is well defined and one-to-one. So, let $\sum_{i=1}^n a_i b_i = 0$. We may assume that $\{b_i \mid i = 1, 2, \cdots, n\}$ are linearly independent. Then, from the relation

[1] According to the recent result [8] by Tomita about the general standard form of von Neumann algebras, the commutation theorem in the tensor products of von Neumann algebras follows as the corollary. Hence $\mathfrak{A} \subset M \otimes N$, i.e., $M \otimes N$ is the largest von Neumann algebra having M and N as its components in H and K. A similar remark should also be added to Theorem 4.

$$R_\alpha\Big(\sum_{i=1}^n a_i b_i\Big) = \sum_{i=1}^n R_\alpha(a_i)b_i = \sum_{i=1}^n \lambda_{a_i}^\alpha b_i = 0 ,$$

we get $\lambda_{a_i}^\alpha = 0$ for $i = 1, 2, \cdots, n$ and $\alpha \in I$. Therefore $R_\alpha(a_i) = 0$ for all $\alpha \in I$ and this means that $a_i = 0$ for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^n a_i \otimes b_i = 0$. Since the fact that $\sum_{i=1}^n a_i \otimes b_i = 0$ implie $\sum_{i=1}^n a_i b_i = 0$, the above result shows that Φ is a well defined one-to-one mapping. Therefore the C^*-algebra C^* (M, N) generated by M and N is isomorphic to the C^*-tensor product of M and N with the compatible C^*-norm β defined by

$$\Big\|\sum_{i=1}^n a_i \otimes b_i\Big\|_\beta = \Big\|\sum_{i=1}^n a_i b_i\Big\| \qquad \text{(cf. [7])} .$$

Next consider the functional $<a, \varphi_\alpha> = \lambda_a^\alpha$ on $a \in M$ for a mapping R_α. One easily sees that this is a σ-weakly continuous linear functional on M, i.e., $\varphi_\alpha \in M_*$. Now, for $\psi \in N_*$, we get

$$\Big\langle \sum_{i=1}^n a_i b_i, {}^tR_\alpha(\psi) \Big\rangle = \sum_{i=1}^n < R_\alpha(a_i b_i), \psi >$$
$$= \sum_{i=1}^n < R_\alpha(a_i) b_i, \psi > = \sum_{i=1}^n < a_i, \varphi_\alpha >< b_i, \psi >$$
$$= \Big\langle \sum_{i=1}^n a_i \otimes b_i, \varphi_\alpha \otimes \psi \Big\rangle .$$

Hence for $x \in C^*(M, N)$

$$< R_\alpha(x), \psi > = < x, {}^tR_\alpha(\psi) > = < \Phi(x), \varphi_\alpha \otimes \psi > \cdots (*) .$$

Therefore if $<\Phi(x), \varphi_\alpha \otimes \psi> = 0$ for all φ_α and $\psi \in N_*$, then $R_\alpha(x) = 0$ for all $\alpha \in I$ and $x = 0$. That is, $\Phi(x) = 0$. Thus in $M \otimes_\beta N$ the family of all product functionals $\varphi \otimes \psi$ ($\varphi \in M^*$, $\psi \in N^*$) is total, hence the norm β must coincide with Turumaru's α-norm and $C^*(M, N) \cong M \otimes_\alpha N$. (see [7, Th. 2]).

Let $V =$ Linear span of $\{{}^tR_\alpha(\psi)\colon \alpha \in I, \psi \in N_*\}$. Since $\{R_\alpha \mid \alpha \in I\}$ is total in $\mathfrak{A}$, V is uniformly dense in $\mathfrak{A}_*$. On the other hand, let $V' =$ linear span of $\{\varphi_\alpha \otimes \psi \mid \alpha \in I, \ \psi \in N_*\}$, then V' is also uniformly dense in $(M \otimes N)_*$ and by the equality (*) we get ${}^t\Phi[V' \mid M \otimes_\alpha N] = V \mid C^*(M, N)$ where $V' \mid M \otimes_\alpha N$ and $V \mid C^*(M, N)$ are the restrictions of elements in V' and V to $M \otimes_\alpha N$ and $C^*(M, N)$ respectively. Now by Kaplansky's density theorem, V and V' are isometric to $V \mid C^*(M, N)$ and $V' \mid M \otimes_\alpha N$, so that ${}^t\Phi$ induces the isometry between V' and V, hence the isometry ρ between $(M \otimes N)_*$ and $\mathfrak{A}_*$. It is not difficult to see that ${}^t\rho$ is the extended isomorphism of Φ between $\mathfrak{A}$ and $M \otimes N$. This completes the proof.

In the above theorem, the case where $\mathfrak{A}$ is a finite factor is due to Nakamura [2] and the case where $\mathfrak{A}$ is a (general) factor is proved in Takesaki [6].

Our next result is somewhat different from those treated above and is included essentially in Lemma 2.3 and in the proof of Theorem 3.2 of the author's joint work [1] with Hakeda. However, it may be useful to reformulate these results in the simple form shown below. We give its proof for completeness.

THEOREM 4. *Let M_1 and N_1 be von Neumann algebras on H and K and M_2 and N_2 be their von Neumann subalgebras respectively. Suppose there are projections of norm one π_1 and π_2 from M_1 to M_2 and from N_1 to N_2. Then there is a projection of norm one*

$$\pi\colon M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_1 \rightarrow M_2 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_2$$

such that $\pi(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$ where $a \in M_1$ and $b \in N_1$.

It is known that in the above case there is a unique projection of norm one $\pi_1 \otimes \pi_2$ from $M_1 \otimes_\alpha N_1$ to $M_2 \otimes_\alpha N_2$ such that $\pi_1 \otimes \pi_2(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$, and if π_1 and π_2 are σ-weakly continuous it can be also shown that we can extend the above $\pi_1 \otimes \pi_2$ to the σ-weak continuous projection of norm one from $M_1 \otimes N_1$ to $M_2 \otimes N_2$ which is a posteriori unique (cf. [10]). However, it is the crucial point of the above theorem that even if we lack the condition of σ-weak continuity of π_1 and π_2 we get still the extension of $\pi_1 \otimes \pi_2$ to the algebra $M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_1$.

Proof of the Theorem. Let $\{e_i \mid i \in I\}$ be the family of orthogonal minimal projections in $\mathscr{B}(K)$ corresponding to the orthogonal basis in K. Put $\tilde{e}_i = 1 \otimes e_i$, $e_J = \sum_{i \in J} e_i$ and $\tilde{e}_J = 1 \otimes e_J = \sum_{i \in J} \tilde{e}_i$ where J is a finite subset of I. Then

$$\tilde{e}_J M_1 \otimes \mathscr{B}(K) \tilde{e}_J = M_1 \otimes e_J \mathscr{B}(K) e_J = M_1 \otimes_\alpha e_J \mathscr{B}(K) e_J$$

(the last equality holds, since $e_J \mathscr{B}(K) e_J$ is a finite dimensional algebra). Let π_J be the projection of norm one from $M_1 \otimes_\alpha e_J \mathscr{B}(K) e_J$ to $M_2 \otimes_\alpha e_J \mathscr{B}(K) e_J$ defined by $\pi_J(a \otimes b) = \pi_1(a) \otimes b$ where $a \in M_1$ and $b \in e_J \mathscr{B}(K) e_J$ (cf. [10, Th. 1]) and put $\pi_J^1(x) = \pi_J(\tilde{e}_J x \tilde{e}_J)$ for $x \in M_1 \otimes \mathscr{B}(K)$. Then $\{\pi_J^1(x) \mid J$ is a finite subset of $I\}$ is a family of elements in $M_2 \otimes \mathscr{B}(K)$ bounded by $\| x \|$. Put $\pi^1(x) = \mathrm{Lim}_J \pi_J^1(x)$ (operator Banach limit in the sense of Schwartz [5] with respect to the subsets J). By the property of the operator Banach limit shown in [5], we have

$$\pi^1(x) \in M_2 \otimes \mathscr{B}(K) \text{ and } \| \pi^1(x) \| \leqq \| x \| .$$

Moreover, for $x \in M_2 \otimes \mathscr{B}(K)$, $x = \sigma$-weak $\text{limit}_J\, \tilde{e}_J x \tilde{e}_J$ implies

$$\begin{aligned}\text{Lim}_J\, \pi_J^1(x) &= \text{Lim}_J\, \pi_J(\tilde{e}_J x \tilde{e}_J) = \text{Lim}_J\, \tilde{e}_J x \tilde{e}_J \\ &= \sigma\text{-weak limit } \tilde{e}_J x \tilde{e}_J = x\,.\end{aligned}$$

Therefore π^1 is a projection of norm one from $M_1 \otimes \mathscr{B}(K)$ to $M_2 \otimes \mathscr{B}(K)$. Take an element $a \otimes b \in M_1 \otimes \mathscr{B}(K)$. We have

$$\begin{aligned}\pi^1(a \otimes b) &= \text{Lim}_J\, \pi_J(a \otimes e_J b e_J) = \text{Lim}_J\, \pi_1(a) \otimes e_J b e_J \\ &= \sigma\text{-weak limit}_J\, \pi_1(a) \otimes e_J b e_J = \pi_1(a) \otimes b\,.\end{aligned}$$

Similarly we get a projection of norm one π^2 from $\mathscr{B}(H) \otimes N_1$ to $\mathscr{B}(H) \otimes N_2$ such that $\pi^2(a \otimes b) = a \otimes \pi_2(b)$.

Now take an element $x \in M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_1$. For arbitrary $y \in M_1' \otimes 1$ we get, by Theorem 1 in Tomiyama [9],

$$y\pi^2(x) = \pi^2(yx) = \pi^2(xy) = \pi^2(x)y$$

because $M_1' \otimes 1 \subset \mathscr{B}(H) \otimes N_2$ and $(M_1' \otimes 1)' = M_1 \otimes \mathscr{B}(K)$. Hence $\pi^2(x) \in M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_2$. Therefore, put $\pi(x) = \pi^1\pi^2(x)$ and take an element $y \in 1 \otimes N_2'$. Since $1 \otimes N_2' \subset M_2 \otimes \mathscr{B}(K)$ we get again by [9, Th. 1]

$$y\pi(x) = y\pi^1\pi^2(x) = \pi^1(y\pi^2(x)) = \pi^1(\pi^2(x)y) = \pi(x)y\,.$$

Thus $\pi(x) \in M_2 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_2$ and it is clear that this mapping π is a projection of norm one from $M_1 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_1$ to $M_2 \otimes \mathscr{B}(K) \cap \mathscr{B}(H) \otimes N_2$. Finally for $a \otimes b \in M_1 \otimes N_1$ we have

$$\pi(a \otimes b) = \pi^1\pi^2(a \otimes b) = \pi^1(a \otimes \pi_2(b)) = \pi_1(a) \otimes \pi_2(b)\,.$$

This completes the proof.

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Pacific Journal of Mathematics

Vol. 30, No. 1 September, 1969

William Wells Adams, *Simultaneous diophantine approximations and cubic irrationals* 1

Heinz Bauer and Herbert Stanley Bear, Jr., *The part metric in convex sets* 15

L. Carlitz, *A note on exponential sums* 35

Vasily Cateforis, *On regular self-injective rings* 39

Franz Harpain and Maurice Sion, *A representation theorem for measures on infinite dimensional spaces* 47

Richard Earl Hodel, *Sum theorems for topological spaces* 59

Carl Groos Jockusch, Jr. and Thomas Graham McLaughlin, *Countable retracing functions and* $\Pi_2{}^0$ *predicates* 67

Bjarni Jónsson and George Stephen Monk, *Representations of primary Arguesian lattices* 95

Virginia E. Walsh Knight, *A continuous partial order for Peano continua* 141

Kjeld Laursen, *Ideal structure in generalized group algebras* 155

G. S. Monk, *Desargues' law and the representation of primary lattices* 175

Hussain Sayid Nur, *Singular perturbation of linear partial differential equation with constant coefficients* 187

Richard Paul Osborne and J. L. Stern, *Covering manifolds with cells* 201

Keith Lowell Phillips and Mitchell Herbert Taibleson, *Singular integrals in several variables over a local field* 209

James Reaves Smith, *Local domains with topologically T-nilpotent radical* 233

Donald Platte Squier, *Elliptic differential equations with discontinuous coefficients* 247

Tae-il Suh, *Algebras formed by the Zorn vector matrix* 255

Earl J. Taft, *Ideals in admissible algebras* 259

Jun Tomiyama, *On the tensor products of von Neumann algebras* 263

David Bertram Wales, *Uniqueness of the graph of a rank three group* 271

Charles Robert Warner and Robert James Whitley, *A characterization of regular maximal ideals* 277