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HOMOMORPHISMS OF ANNIHILATOR BANACH ALGEBRAS, II

GREGORY F. BACHELIS

Let A be a semi-simple annihilator Banach algebra, and let ν be a homomorphism of A into a Banach algebra. In this paper it is shown that there exists a constant K and dense two-sided ideals containing the socle, I_L and I_R, such that $\|\nu(xy)\| \leqq K\|x\| \cdot \|y\|$ whenever $x \in I_L$ or $y \in I_R$. If A has a bounded left or right approximate identity, then ν is continuous on the socle. Thus if $A = L_1(G)$, where G is a compact topological group, then any homomorphism of A into a Banach algebra is continuous on the trigonometric polynomials.

In [1] we considered the problem of deducing continuity properties of a homomorphism ν from a semi-simple annihilator Banach algebra A into an arbitrary Banach algebra. The main theorem there (Theorem 5.1) had a conclusion more restrictive than the one stated above and required the additional hypothesis that $I \oplus \mathfrak{R}(I) = A$, for all closed two-sided ideals I, where $\mathfrak{R}(I) = \{x \mid Ix = (0)\}$. The main theorem of this paper applies when $A = L_p(G)$, $1 \leqq p < \infty$ or $C(G)$, where G is a compact topological group and multiplication is convolution, and when A is topologically-simple, whereas the earlier theorem did not.

Any terms not defined in this paper are those of Rickart's book [10]. For facts about annihilator algebras, the reader is referred to [4] or [10].

Given the left-right symmetry in the definition of annihilator algebras, it follows that, given any theorem about left (right) ideals, the corresponding theorem for right (left) ideals also holds. Specifically, this is the case for the theorems in [4, § 4] and [1, § 4]. We will make tacit use of this fact throughout this paper.

2. Structural lemmas.

In this section several lemmas are established which will be used later in proving the main result. Throughout this section, we assume that A is a semi-simple annihilator Banach algebra.

LEMMA 2.1. *If $\{x_1, \cdots, x_n\}$ is contained in the socle of A, then there exist idempotents e and f such that $x_i \in eAf$, $1 \leqq i \leqq n$.*

Proof. By [1, Corollary 4.9], for each i there exist idempotents e_i and f_i such that $x_i \in e_iA \cap Af_i \subset e_iAf_i$. By [1, Th. 4.8], there exist idempotents e and f such that $e_1A + \cdots + e_nA = eA$ and $Af_1 + \cdots + Af_n = Af$. Thus $x_i \in e_iAf_i = ee_iAf_if \subset eAf$, $1 \leqq i \leqq n$.

LEMMA 2.2. *Suppose A is topologically-simple, and e is a minimal idempotent in A. Then there exists a constant L such that:*

Given $f = f^2 \in A$ and $x \in eA$, there exists $g = g^2 \in A$ such that:

(1) $x(1 - f)g = x(1 - f) = xg$

(2) $fg = gf = 0$

(3) $||g|| \leqq (1 + ||f||)L$.

The corresponding statement holds for $x \in Ae$.

Proof. Let F_0 denote the bounded operators on Ae of finite rank. Then via the left regular representation, we may regard A algebraically as a subalgebra of the uniform closure of F_0 which contains F_0 (see [4], Ths. 9 and 10).

If $a \in eA$, $u \in Ae$, then $au = eaue = \lambda e = \phi_a(u)e$, and $a \to \phi_a$ defines an isomorphism and homeomorphism between eA and the bounded linear functionals on Ae [4, Th. 13]. Hence there exists a constant L such that $||a|| \leqq (L/2)||\phi_a||$ for all $a \in eA$.

Let $x \in eA$ and $f = f^2 \in A$. Then $x(1 - f) \in eA$, and e is minimal, so range$(x(1 - f))$ is one-dimensional. Let $M = (x(1 - f))^{-1}(0)$. Then M is a closed subspace of co-dimension one in Ae, so there exists a bounded linear functional β on Ae such that $||\beta|| = 1$ and $\beta^{-1}(0) = M$. Let $w \in Ae$ such that $||w|| \leqq 2$ and $\beta(w) = ||\beta|| = 1$. Now $w = (1 - f)w + fw$, and $fw \in (1 - f)^{-1}(0) \subset M$, so $\beta((1 - f)w) = \beta(w) = 1$.

Let $G(u) = \beta(u)(1 - f)w$, $u \in Ae$. Then G is a bounded operator on Ae with one-dimensional range and $G = G^2$, so there exists an idempotent $g \in A$ such that $gu = \beta(u)(1 - f)w$, $u \in Ae$. If $u \in Ae$, then $u - \beta(u)(1 - f)w \in \beta^{-1}(0) = M = (x(1 - f))^{-1}(0)$, so $x(1 - f)u = x(1 - f)\beta(u)(1 - f)w = x\beta(u)(1 - f)w = xgu$. Therefore $x(1 - f) = xg$. Thus $x(1 - f)g = xg^2 = xg = x(1 - f)$. This establishes (1).

To prove (2), we see that $(1 - f)w \in f^{-1}(0)$, so $fg = 0$, and range$(f) = (1 - f)^{-1}(0) \subset M = g^{-1}(0)$, so $gf = 0$.

To establish (3), let $h \in eA$ such that $\phi_h = \beta$. If $u \in Ae$, then

$$(1 - f)whu = (1 - f)w\beta(u)e = \beta(u)(1 - f)we = \beta(u)(1 - f)w = gu.$$

Therefore $(1 - f)wh = g$, so

$$||g|| \leqq ||h||(1 + ||f||)||w|| \leqq (L/2)(1 + ||f||)2 \leqq L(1 + ||f||) .$$

3. The ideals I_L and I_R. In this section we discuss the ideals which enter into the main theorem. Throughout this section, we assume that A is a Banach algebra and that ν is a homomorphism of A into a Banach algebra.

DEFINITION 3.1. Let $I_L = \{x \in A \mid y \to \nu(xy)$ is continuous on $A\}$ and let $I_R = \{x \in A \mid y \to \nu(yx)$ is continuous on $A\}$.

These sets were introduced by Stein, who shows they are two-sided ideals in A [11]. Another useful concept is that of the *separating ideal*, S, which is defined to be the set of $s \in \mathrm{cl}\,(\nu(A))$ such that $\inf_{x \in A}\{\|x\| + \|s - \nu(x)\|\} = 0$. The separating ideal was introduced in the form above by Yood [13]. It is a closed two-sided ideal in $\mathrm{cl}\,(\nu(A))$.

In [12], Stein notes that $I_L \subset \{x \in A \mid \nu(x)S = (0)\}$ and similarly for I_R. One actually has equality: For suppose $\nu(x)S = (0)$. If $x_n \to 0$ in A, then by [8, Lemma 2.1], $\nu(x_n) + S \to S$ in $\mathrm{cl}\,(\nu(A))/S$. Hence there exists $\{s_n\} \subset S$ such that $\nu(x_n) + s_n \to 0$. Thus $\nu(xx_n) = \nu(x)\nu(x_n) = \nu(x)(\nu(x_n) + s_n) \to 0$, so $x \in I_L$.

4. Homomorphisms of annihilator algebras. In this section we establish the main results of this paper. We will make frequent use of the "Main Boundedness Theorem" of Badé and Curtis.

THEOREM 4.1. *Suppose that A is a Banach algebra, and that ν is a homomorphism of A into a Banach algebra. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A such that $x_n y_m = 0$, $n \neq m$. Then*

$$\sup_n \frac{\|\nu(x_n y_n)\|}{\|x_n\|\,\|y_n\|} < \infty .$$

Proof. This is Theorem 3.1 of [5]. The statement there includes the unnecessary hypothesis that $y_n y_m = 0$, $n \neq m$.

Throughout the remainder of this section, A will denote a semi-simple annihilator Banach algebra with socle F, and ν will denote a homomorphism of A into a Banach algebra. We first prove:

LEMMA 4.2. *If A is topologically-simple, and e is a minimal idempotent in A, then $\nu \mid eA$ and $\nu \mid Ae$ are continuous.*

Proof. (For $\nu \mid eA$). Let L be as in Lemma 2.2. Suppose the conclusion fails. Choose $x_1 \in eA$ such that $\|\nu(x_1)\| > L\|x_1\|$. By Lemma 2.2, with $f = 0$, there exists $g_1 = g_1^2 \in A$ such that $\|g_1\| \leqq L$ and $x_1 g_1 = x_1$. Thus $\|\nu(x_1)\| > \|x_1\|\,\|g_1\|$.

Assume that elements $x_i \in eA$, $g_i \in A$ have been chosen such that $x_i g_i = x_i$,

$$g_i g_j = 0\ , \qquad\qquad i \neq j\ ,$$

and $\|\nu(x_i)\| > i\|x_i\|\,\|g_i\|$, $1 \leqq i, j \leqq n$.

Let $f = g_1 + \cdots + g_n$. Then $f = f^2$, $g_i f = g_i = f g_i$, and $x_i \in eAf$, $1 \leqq i \leqq n$. Since f can be expressed as the sum of minimal idempotents [1, Th. 4.5], eAf is finite-dimensional, so let K be the norm of $\nu \mid eAf$. Now choose $u \in eA$ such that $\|\nu(u)\| > (1+\|f\|)^2 L(n+1)\|u\| + K\|f\|\,\|u\|$. Then

$$\begin{aligned}\|\nu(u)\| &\leqq \|\nu(uf)\| + \|\nu(u(1-f))\| \\ &\leqq K\|u\|\,\|f\| + \|\nu(u(1-f))\|\ ,\end{aligned}$$

so

$$\begin{aligned}\|\nu(u(1-f))\| &> (1+\|f\|)^2 L(n+1)\|u\| \\ &\geqq (1+\|f\|)L(n+1)\|u(1-f)\|\ .\end{aligned}$$

Let $x_{n+1} = u(1-f) \in eA$. By Lemma 2.2, there exists $g_{n+1} = g_{n+1}^2 \in A$ such that $x_{n+1} g_{n+1} = x_{n+1}$, $g_{n+1} f = f g_{n+1} = 0$, and $\|g_{n+1}\| \leqq L(1 + \|f\|)$. Thus

$$g_{n+1} g_i = 0 = g_i g_{n+1}\ , \qquad\qquad 1 \leqq i \leqq n$$

and

$$\|\nu(x_{n+1})\| > (n+1)\|x_{n+1}\|\,\|g_{n+1}\|\ .$$

Thus by induction there exist sequences $\{x_n\}$, $\{g_n\}$ such that $x_n g_m = x_n g_n g_m = 0$, $n \neq m$ and $\|\nu(x_n g_n)\| > n\|x_n\|\,\|g_n\|$, which contradicts Theorem 4.1.

We now show that I_L and I_R are dense in A:

LEMMA 4.3. $F \subset I_L \cap I_R$.

Proof. If e is a minimal idempotent, then e is contained in a minimal-closed two-sided ideal M, M is a topologically-simple semi-simple annihilator Banach algebra, and $eM = eA$. The preceding lemma gives that $\nu \mid eA$ is continuous. Thus $x \to \nu(ex)$ is continuous on A, so $e \in I_L$. Hence I_L contains all the minimal idempotents of A. Since I_L is an ideal, this implies that $I_L \supset F$. Similarly, $I_R \supset F$.

LEMMA 4.4. *If* $\|\nu(xy)\| > r\|x\|\,\|y\|$, *and if* $x \in I_L$ *or* $y \in I_R$, *then there exist* $x_1, y_1 \in F$ *such that* $\|\nu(x_1 y_1)\| > r\|x_1\|\,\|y_1\|$.

Proof. Suppose $x \in I_L$. Since $w \to \nu(xw)$ is continuous on A and F is dense in A, there exists $y_1 \in F$ such that $\|\nu(x y_1)\| > r\|x\|\,\|y_1\|$. Now $y_1 \in I_R$, so there exists $x_1 \in F$ such that $\|\nu(x_1 y_1)\| > r\|x_1\|\,\|y_1\|$.

We can now prove the main theorem:

THEOREM 4.5. *Let A be a semi-simple annihilator Banach algebra, and let ν be a homomorphism of A into a Banach algebra. Then there exists a constant K such that*

$$\| \nu(xy) \| \leqq K \| x \| \| y \|$$

for all x and y in A such that $x \in I_L$ or $y \in I_R$.

Proof. In view of the preceding lemma (or by symmetry considerations), it is enough to show that there exists a K such that $\| \nu(xy) \| \leqq K \| x \| \| y \|$ whenever $x \in I_L$. Suppose this is not the case. By the preceding lemma, there exist x_1 and y_1 in F such that $\| \nu(x_1 y_1) \| > \| x_1 \| \| y_1 \|$.

Assume that elements $x_i, y_i \in F$ have been chosen such that

$$x_i y_j = 0 , \qquad i \neq j$$

and

$$\| \nu(x_i y_i) \| > i \| x_i \| \| y_i \| , \qquad 1 \leqq i, j \leqq n .$$

By Lemma 2.1, there exist idempotents e and f such that $\{x_1, \cdots, x_n, y_1, \cdots, y_n\} \subset eAf$. By [1, Th. 4.5], e and f are in F, and by Lemma 4.2, $F \subset I_L \cap I_R$. Now an idempotent is in I_L (I_R) if and only if the restriction of ν to the right (left) ideal it generates is continuous, so let L be the maximum of the norms of the continuous mappings $\nu \mid Ae$, $\nu \mid eA$, $\nu \mid Af$, $\nu \mid fA$, and let

$$K' = L^2(\| e \|^2 + \| f \|^2 + \| e \| \| f \|) .$$

If $x, y \in A$, then

$$\begin{aligned} \| \nu((x - xe)(y - fy)) \| &= \| \nu(xy) - \nu(xeey) - \nu(xffy) + \nu(xefy) \| \\ &\geqq \| \nu(xy) \| - \| \nu(xe) \| \| \nu(ey) \| \\ &\quad - \| \nu(xf) \| \| \nu(fy) \| - \| \nu(xe) \| \| \nu(fy) \| \\ &\geqq \| \nu(xy) \| - K' \| x \| \| y \| . \end{aligned}$$

By the preceding lemma, there exist $u, v \in F$ such that

$$\| \nu(uv) \| > \{(n + 1)(1 + \| e \|)(1 + \| f \|) + K'\} \| u \| \| v \| .$$

Let $x_{n+1} = u - ue$, $y_{n+1} = v - fv$. By the above, we have that

$$\begin{aligned} \| \nu(x_{n+1} y_{n+1}) \| &> (n + 1)(1 + \| e \|) \| u \| (1 + \| f \|) \| v \| \\ &\geqq (n + 1) \| x_{n+1} \| \| y_{n+1} \| . \end{aligned}$$

Also, $x_i y_{n+1} = x_i f(v - fv) = 0$, $x_{n+1} y_i = (u - ue)ey_i = 0$, $1 \leqq i \leqq n$, and x_{n+1}, $y_{n+1} \in F$.

Thus by induction there exist sequences $\{x_n\}$, $\{y_n\}$ such that $x_n y_m = 0$, $n \neq m$, and $\|\nu(x_n y_n)\| > n \|x_n\| \|y_n\|$, which contradicts Theorem 4.1.

REMARK 4.6. If $x \in I_L$, let $K(x)$ be the norm of the mapping $y \to \nu(xy)$. Then $\|\nu(xy)\| \leqq (K(x)/\|x\|)\|x\| \|y\|$, $y \in A$. The above theorem shows that $\{(K(x)/\|x\|) \mid x \in I_L\}$ is bounded.

The following corollary is an analog for annihilator algebras of a theorem by Badé and Curtis on homomorphisms of commutative, regular semi-simple Banach algebras [2, Th. 3.7]; it gives Theorem 5.1 of [1] as a special case.

COROLLARY 4.7. *Let A be a semi-simple annihilator Banach algebra, and let ν be a homomorphism of A into a Banach algebra. Then there exists a constant K such that.*

$$\|\nu(x)\| \leqq K \|x\| \|y\|$$

for all x and y in A such that $yx = x$ or $xy = x$.

Proof. If $yx = x$ or $xy = x$, then by [1, Corollary 4.12], $x \in F \subset I_L \cap I_R$.

DEFINITION 4.8. A Banach algebra A is said to have a bounded left (right) approximate identity if there exists a norm-bounded net $\{e_\alpha\} \subset A$ such that $e_\alpha x \to x$ $(xe_\alpha \to x)$ for all $x \in A$.

COROLLARY 4.9. *Let A be a semi-simple annihilator Banach algebra with a bounded left or right approximate identity, and let ν be a homomorphism of A into a Banach algebra. Then ν is continuous on the socle of A.*

Proof. Suppose that A has a bounded left approximate identity. Let $z \in F$. By Cohen's factorization theorem [6], there exists a constant L (independent of z) and elements x and y such that $z = xy$, $\|z - y\| \leqq \|z\|$, $\|x\| \leqq L$, and y is in the closed left ideal generated by z. By [1, Corollary 4.9], there exists an idempotent-generated left ideal, J, containing z. Since J is closed, we have $y \in J \subset F \subset I_R$. Thus if K is as in the above theorem, then

$$\|\nu(z)\| = \|\nu(xy)\| \leqq K \|x\| \|y\| \leqq KL(\|z\| + \|z - y\|) \leqq 2KL \|z\| .$$

We conclude this paper with several remarks:

REMARK 4.10. Let G be a compact topological group and let $A = L_p(G)$, $1 \leqq p < \infty$, or $C(G)$, with convolution for multiplication. Then Theorem 4.5 applies to A and the above corollary applies to $L_1(G)$. Here F is the set of trigonometric polynomials, that is, the set of linear combinations of component functions of strongly continuous irreducible unitary representations of G (see [10, p. 330]).

REMARK 4.11. If X is a reflexive Banach space, if F denotes the bounded operators on X of finite rank, and if $A \subset \mathfrak{B}(X)$ is a Banach algebra containing F as a dense subset, then Theorem 4.5 applies to A [10, pp. 102–104]. Here the socle of A is F.

If A is the uniform closure of F in $\mathfrak{B}(X)$, if A has a bounded left or right approximate identity, and if X has a continued bisection, then Johnson has shown that every homomorphism of A into a Banach algebra is actually continuous [8, Th. 3.5]. His theorem is stated for the algebra of compact operators on X (which may indeed always coincide with A), but his method of proof works equally well for A.[1]

REMARK 4.12. Although examples do exist of discontinuous homomorphisms of annihilator algebras (see [2, p. 597, p. 606] [3, p. 853], and [9]), it is still the case for these examples that $I_L = A$. One might conjecture that this is always true. As a small move in this direction, we show below that, in two special cases, I_L properly contains F the socle of A.

(1) Let $\mathfrak{M} = \{M_\lambda \mid \lambda \in \Lambda\}$ denote the minimal-closed two-sided ideals of A and suppose that $\mathfrak{M}$ forms an unconditional decomposition for A. Then $x \in A$ implies $x = \Sigma_\lambda x_\lambda$, where $x_\lambda \in M_\lambda$, and an equivalent Banach algebra norm for A is given by $|x| = \sup\{\|\sum_{\lambda \in \Lambda_1} x_\lambda\| : \Lambda_1$ is a finite subset of $\Lambda\}$. [1, pp. 231–232]. Thus $|x| = \sup_{\Lambda_1 \subset \Lambda} |\sum_{\lambda \in \Lambda_1} x_\lambda|$. For $\mathfrak{N} \subset \mathfrak{M}$, let $A(\mathfrak{N})$ denote those x in A whose summands are all in $\mathfrak{N}$. If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are disjoint subsets of $\mathfrak{M}$, then $A(\mathfrak{N}_1) \cdot A(\mathfrak{N}_2) = (0)$. If $x \in A(\mathfrak{N})$ and $x \notin I_L$, then given K there exists $y \in A$ such that $\|\nu(xy)\| > K|x||y|$. Since removing the summands of y that are not in members of $\mathfrak{N}$ does not increase its norm and does not affect xy, we may assume that $y \in A(\mathfrak{N})$ as well. Thus if $\{\mathfrak{N}_n\}_{n=1}^\infty$ is any sequence of disjoint subsets of $\mathfrak{M}$, then Theorem 4.1 implies that $A(\mathfrak{N}_n) \subset I_L$ for all but finitely many n.

If A is strongly semi-simple, we can say a bit more. In this case, each $M \in \mathfrak{M}$ is finite-dimensional [1, Proposition 4.7]. Let $\mathscr{P}(\mathfrak{M})$ denote the set of subsets of $\mathfrak{M}$, let $\mathscr{F}$ denote the set of finite subsets of $\mathfrak{M}$, and let $[\mathfrak{N}]$ denote an element of the Boolean algebra

[1] See "Added in proof."

$\mathscr{P}(\mathfrak{M})/\mathscr{F}$. If $A(\mathfrak{N}) \subset I_L$, and $\mathfrak{N}_1 \in [\mathfrak{N}]$, then $\mathfrak{N}_1 \cap \mathfrak{N} \in \mathscr{F}$, so $A(\mathfrak{N}_1) \subset I_L$. Thus $\sum_{\mathfrak{N}_1 \in [\mathfrak{N}]} A(\mathfrak{N}_1) \subset I_L$. (Here "$\Sigma$" denotes the algebraic sum. Note that $F = \sum_{\mathfrak{N} \in \mathscr{F}} A(\mathfrak{N})$.) Let $\mathscr{J} = \{[\mathfrak{N}] \in \mathscr{P}(\mathfrak{M})/\mathscr{F} \mid A(\mathfrak{N}) \subset I_L\}$. Then $\mathscr{J}$ is an ideal in $\mathscr{P}(\mathfrak{M})/\mathscr{F}$. If $[\mathfrak{N}] \neq \mathscr{F}$, then there exists $\mathscr{F} \neq [\mathfrak{N}_1] \leqq [\mathfrak{N}]$ such that $[\mathfrak{N}_1] \in \mathscr{J}$: Otherwise, we could find a pairwise disjoint family $\{\mathfrak{N}_n\}$, with $A(\mathfrak{N}_n) \not\subset I_L$ for any n, which would contradict Theorem 4.1. But this says that the annihilator of $\mathscr{J}$ is $\mathscr{F}$, and thus $\mathscr{J}$ corresponds to a dense open set in the dual space of $\mathscr{P}(\mathfrak{M})/\mathscr{F}$, $\beta(\mathfrak{M}) - \mathfrak{M}$, where $\mathfrak{M}$ has the discrete topology[2] (see [7], pp. 76, 84, and 88). Since dividing by $\mathscr{F}$ in effect "mods out the socle", we see in this case that I_L is significantly larger than F.

(2) Suppose that A has proper involution $x \rightarrow x^*$ and that $I \oplus \mathfrak{R}(I)^* = A$ for all closed left ideals I. Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be a maximal family of orthogonal hermitian idempotents. Then $x \in A$ implies $x = \Sigma_\lambda e_\lambda x = \Sigma_\lambda x e_\lambda$, and we may assume $\| x \| = \sup_{\Lambda_1 \subset \Lambda} \| \sum_{\lambda \in \Lambda_1} e_\lambda x \|$. [1, pp. 231-233]. For $\Lambda_1 \subset \Lambda$, let $A(\Lambda_1) = \{x \in A \mid x e_\lambda = 0, \lambda \notin \Lambda_1\}$. If $x \in A(\Lambda_1)$ and $\| \nu(xy) \| > K \| x \| \| y \|$, let $y_1 = \sum_{\lambda \in \Lambda_1} e_\lambda y$. Then $xy_1 = xy$, $\| y_1 \| \leqq \| y \|$, and $x' y_1 = 0$ if $x' \in A(\Lambda_2)$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Thus, given any sequence $\{\Lambda_n\}$ of disjoint subsets of Λ, Theorem 4.1 implies that $A(\Lambda_n) \subset I_L$ for all but finitely many n. (Of course there may exist $x \in F$ such that $x e_\lambda \neq 0$ for infinitely many λ, but clearly $A(\Lambda_1) \not\subset F$ if Λ_1 is infinite.) Since $\nu \mid A e_\lambda$ is continuous, remarks similar to those in the above paragraph can be made in this situation, with $\mathfrak{M}$ replaced by $\{A e_\lambda \mid \lambda \in \Lambda\}$.

Added in Proof. (continuation of Remark 4.11) If X is a Hilbert space and $A = \mathfrak{F}_1$, the algebra of trace class operators, or $\mathfrak{F}_2$, the algebra of Hilbert-Schmidt operators, then the methods of [8, Th. 3.3] can be adapted to show that $A^2 \subset I_L$. The statement in [8] that these methods imply continuity is in error. The following example (communicated to the author by Professor Johnson) illustrates this: If ν is a discontinuous linear functional on $\mathfrak{F}_2$ which vanishes on $\mathfrak{F}_2^2$ $(=\mathfrak{F}_1)$, then by defining zero multiplication in the complex numbers, one obtains a discontinuous homomorphism of $\mathfrak{F}_2$.

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AN EXPLICIT FORMULA FOR THE UNITS OF AN ALGEBRAIC NUMBER FIELD OF DEGREE $n \geqq 2$

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An infinite set of algebraic number fields is constructed; they are generated by a real algebraic irrational w, which is the root of an equation $f(w) = 0$ with integer rational coefficients of degree $n \geqq 2$. In such fields polynomials $P_s(w) = a_0 w^s + a_1 w^{s-1} + \cdots + a_{s-1} w + a_s$ and

$$Q_s(w) = b_0 w^s + b_1 w^{s-1} + \cdots + b_{s-1} w + b_s$$

$(s = 1, \cdots, n-1$; a_k, b_k rational integers) are selected so that the Jacobi-Perron algorithm of the $n-1$ numbers

$$P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$$

carried out in this decreasing order of the polynomials, and of the $n-1$ numbers

$$Q_1(w), Q_2(w), \cdots, Q_{n-1}(w)$$

carried out in this increasing order of the polynomials both become periodic.

It is further shown that $n-1$ different Modified Algorithms of Jacobi-Perron, each carried out with $n-1$ polynomials $P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$ yield periodicity. From each of these algorithms a unit of the field $K(w)$ is obtained by means of a formula proved by the authors is a previous paper.

It is proved that the equation $\mathbf{f}(x) = 0$ has n real roots when certain restrictions are put on its coefficients and that, under further restrictions, the polynomial $f(x)$ is irreducible in the field of rational numbers. In the field $K(w)$ $n-1$ different units are constructed in a most simple form as polynomials in w; it is proved in the Appendix that they are independent; the authors conjecture that these $n-1$ independent units are basic units in $K(w)$.

1. Algorithm of $n-1$ numbers. An ordered $(n-1)$-tuple

$$(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)}), \qquad (n \geqq 2) \tag{1}$$

of given numbers, real or complex, among whom there is at least one irrational, will be called a basic sequence; the infinitely many $(n-1)$-tuples

$$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}), \qquad (v = 0, 1, \cdots) \tag{2}$$

will be called supporting sequences. We shall denote by

$$A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)}) \tag{3}$$

the following algorithm connecting the components of the basic sequence with those of the supporting sequences:

$$\begin{aligned} a_k^{(v+1)} &= \frac{a_{k+1}^{(v)} - b_{k+1}^{(v)}}{a_1^{(v)} - b_1^{(v)}}\,, \quad (k = 1, \cdots, n-2\,;\ v = 0, 1, \cdots)\,; \\ a_{n-1}^{(v+1)} &= \frac{1}{a_1^{(v)} - b_1^{(v)}}\,;\ a_1^{(v)} \neq b_1^{(v)}\,; \qquad (v = 0, 1, \cdots)\,. \end{aligned} \tag{4}$$

The $(n-1)$-tuples $(a_1^{(v)}, a_2^{(v)}, \cdots, a_{n-11}^{(v)})$, $(v = 0, 1, \cdots)$ will be called generating sequences of the algorithm. $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ is called periodic, if there exist nonnegative integers s and natural numbers t such that

$$a_i^{(v+t)} = a_i^{(v)}\,, \quad (i = 1, \cdots, n-1\,;\ v = s, s+1, \cdots)\,. \tag{5}$$

Let be

$$\min s = S\,; \qquad \min t = T\,; \tag{6}$$

then the S supporting sequences

$$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)})\,, \qquad (v = 0, 1, \cdots, S-1) \tag{7}$$

are called the primitive preperiod of the algorithm and S is called the length of the preperiod; the T supporting sequences

$$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)})\,, \quad (v = S, S+1, \cdots, S+T-1) \tag{8}$$

are called the primitive period of the algorithm, T is called the length of the period; $S + T$ is called the length of the algorithm. If $S = 0$, the algorithm is called purely periodic.

Two crucial questions emerge from a first look at such an algorithm:

(a) can a formation law be defined by whose help the supporting sequences could be obtained from the basic sequences and the generating sequences?

(b) under what condition is $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ periodic; what is then the nature of the basic sequence and what is the corresponding formation law for the supporting sequences?

For $n = 3$ an algorithm $A(a_1^{(0)}, a_2^{(0)})$ was first introduced by Jacobi [17] and a profound theory of an algorithm of $n-1$ numbers for $n \geqq 2$ was later developed by Oskar Perron [18]; in honor of these great mathematicians the first author of this paper called $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ the algorithm of Jacobi-Perron; they both used the following formation law for the supporting sequences: let $a_i^{(v)}$ be the components of the generating sequences; then

$$b_i^{(v)} = [a_i^{(v)}] , \qquad (i = 1, \cdots, n-1 ; \ v = 0, 1, \cdots) \tag{9}$$

where $[x]$ denotes, as customary, the greatest integer not exceeding x. For $n = 2$ the algorithm of Jacobi-Perron becomes the usual Euclidean algorithm.

One of Perron's [18] most significant results is the following

THEOREM. *Let the supporting sequences $b_i^{(v)}$ $(i = 1, \cdots, n-1;$ $v = 0, 1, \cdots)$ be obtained from the basic sequence $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ of real numbers by the formation law* (9). *If the nonnegative integers $A_i^{(v)}$ are formed by the recursion formula*

$$\begin{cases} A_i^{(i)} = 1 ; \ A_i^{(v)} = 0 ; & (i \neq v ; \ i, v = 0, \cdots, n-1) \\ A_i^{(v+n)} = A_i^{(v)} + \sum_{j=1}^{n-1} b_j^{(v)} A_i^{(v+j)} , & (i = 0, \cdots, n-1, \ v = 0, 1, \cdots) \end{cases} \tag{10}$$

then $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ converges in the sense that

$$a_i^{(0)} = \lim_{v \to \infty} \frac{A_i^{(v)}}{A_0^{(v)}} . \qquad (i = 1, \cdots, n-1) . \tag{11}$$

Moreover, this theorem can be generalized, as was done by the First author ([8], [10], [11], [12[) in the following way:

Let the supporting sequences be obtained from the basic sequence by any formation law; if the $a_i^{(v)}$, $b_i^{(v)}$ are real numbers such that

$$\begin{cases} a_i^{(v)} > 0 ; \quad (i = 1, \cdots, n-1) \\ b_i^{(v)} \geqq 0 ; \quad (i = 1, \cdots, n-2) \quad 0 < b_{n-1}^{(v)} \leqq C ; \\ b_i^{(v)} / b_{n-1}^{(v)} \leqq C ; \ C \text{ a positive constant,} \qquad (v = 0, 1, \cdots) \end{cases} \tag{12}$$

and the numbers $A_i^{(v)}$ (here not necessary integers) are formed as in (10), then $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ converges in the sense of (11).

2. Previous results of the first author. Perron [18] has proved that if $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ becomes periodic then the $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ belong to an algebraic number field of degree $\leqq n$. However, he did not succeed to construct, in a general way, algebraic fields K and to select out of K such $n-1$ numbers whose algorithm would become periodic. This was achieved by the first author for an infinite set of algebraic number fields $K(w)$, w being a real irrational root of an algebraic equation $f(w) = 0$ with rational coefficients. In his papers ([1]-[7]) he used (9) for the formation law of the supporting sequences, thus operating with the algorithm of Jacobi-Perron, though heavy restrictions had to be imposed on the coefficients of $f(w)$ in order to achieve periodicity. The first author succeeded to remove these restrictions by introducing a new formation law that generalizes (9) and is defined in the following way:

The $a_i^{(0)}$ and, subsequently, the $a_i^{(v)}$, $(i = 1, \cdots, n-1;\ v = 0, 1, \cdots)$ being numbers of the field $K(w)$ have, generally, he form

$$a_i^{(v)} = a_i^{(v)}(w) , \qquad (i = 1, \cdots, n-1; v = 0, 1, \cdots) \tag{13}$$

as long as the $b_i^{(v)}$ are rationals. Let be

$$[w] = D ; \tag{14}$$

then the formation law of the supporting sequences is given by the formula

$$b_i^{(v)} = a_i^{(v)}(D) , \qquad (i, v \text{ as in } (14)) . \tag{15}$$

In previous papers of the authors the $a_i^{(0)}$ had the form

$$a_i^{(0)} = P_i(w) , \qquad (i = 1, \cdots, n-1) , \tag{16}$$

thus being polynomials in w with rational coefficients; now the second author of this paper asked the question, whether the algorithm of Jacobi-Perron or any other algorithm

$$A(P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w))$$

of polynomials of decreasing order would yield periodicity, too. This challenging problem could not be solved at first, with the exception of a very few numerical examples, w being a rather simple cubic irrational. Only recently the first author ([13], [14]) could give an affirmative answer. He achieved this by means of a highly complicated formation law for the supporting sequences. But while the new model works well for an infinite set of algebraic number fields $K(w)$; and though in certain cases it is identical with the Jacobi-Perron algorithm—its application does not, at least in this initial stage, seem to go beyond narrow limitations.

In this paper an algebraic number field $K(w)$ is constructed where w is a real algebraic irrational of highly complex nature; but just here it is possible to select polynomials in w such that the algorithms of Jacobi-Perron, viz. for the given $(n-1)$-tuples

$$(P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)) ,$$
$$(Q_{n-1}(w), Q_{n-2}(w), \cdots, Q_1(w))$$

both become periodic.

3. The generating polynomial. We shall call the polynomial of degree $n \geqq 2$, viz.

$$\begin{aligned} &f(x) = (x - D)(x - D_1)(x - D_2) \cdots (x - D_{n-1}) - d\,; \\ (17) \qquad &D, D_i, d \text{ rational integers}\,;\ d \geqq 1\,; \\ &D > D_i\,;\ d \mid (D - D_i),\ (i = 1, \cdots, n-1)\,, \end{aligned}$$

a Generating Polynomial, to be denoted by *GP*.

In what follows we shall need two theorems regarding the roots of the *GP*.

THEOREM 1. *The GP has one and only one real root w in the open interval $(D, +\infty)$. This root lies in the open interval $(D, D+1)$.*

Proof. The two assertions are immediate consequences of the following three inequalities which follow from the conditions in (17):

$$\begin{aligned} f(D) &= -d < 0\,, \\ f'(x) &= (f(x) + d)\Big(\frac{1}{x-D} + \frac{1}{x-D_1} + \cdots + \frac{1}{x-D_{n-1}}\Big) > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x > D\,, \\ f(D+1) &= (D+1-D_1)(D+1-D_2)\cdots(D+1-D_{n-1}) - d \\ &\geqq (d+1)^{n-1} - d \geqq (d+1) - d = 1 > 0\,. \end{aligned}$$

THEOREM 2. *Let the integers D, D_i occuring in the GP satisfy, in addition to* (17), *the conditions*

$$D = D_0 > D_1 > \cdots > D_{n-1}\,, \tag{18}$$

and in the special case $d = 1$ moreover

$$\left\{\begin{array}{l} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4, \text{ for } n = 3\,; \\ \begin{cases} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \geqq 3 \text{ or} \\ D_0 - D_1, D_2 - D_3 \geqq 2, \text{ for } n = 4\,. \end{cases} \end{array}\right. \tag{19}$$

Then the GP has exactly n different real roots. Of these lie

1 *in the open interval $(D_0, +\infty)$, more exactly in the open interval $(D_0, D_0 + 1)$,*

2 *in each of the open intervals (D_{2i}, D_{2i-1}), more exactly* 1 *in the open left half,* 1 *in the open right half of these intervals with* $2 \leqq 2i \leqq n-1$, 1 *in the open interval $(-\infty, D_{n-1})$ if n is even.*

Proof. Since the total number of roots asserted in the latter three statements is exactly equal to the degree n of the *GP*, it suffices to prove the existence of at least 1, 2, 1 roots respectively within the indicated open intervals. For the first interval this has been done in Theorem 1. For the other intervals it suffices, besides the obvious facts

$$f(D_i) = -d < 0 \qquad (i = 0, 1, \cdots, n-1)$$

and

$$\lim_{x \to -\infty} f(x) = +\infty \quad \text{if} \quad n \text{ is even} ,$$

to verify the inequalities

$$f(c_i) > 0 \qquad (2 \leqq 2i \leqq n-1) ,$$

i.e.,

$$f(c_i) + d = (c_i - D_0)(c_i - D_1) \cdots (c_i - D_{n-1}) > d ,$$

with $2 \leqq 2i \leqq n-1$ and $c_i = (D_{2i-1} + D_{2i})/2$. Now according to (18)

$$\begin{aligned} c_i - D_j &< 0 \quad \text{for} \quad j = 0, 1, \cdots, 2i-1 , \\ c_i - D_j &> 0 \quad \text{for} \quad j = 2i, 2i+1, \cdots, n-1 , \end{aligned}$$

and as the j in the first line are in even number, certain at least

$$f(c_i) + d > 0 .$$

According to (17) and the obvious consequence $d \mid (D_i - D_j)$ one has more precisely

$$\begin{aligned} |c_i - D_j| &\geqq d + \frac{d}{2} = \frac{3}{2}d \quad \text{for} \quad j \neq 2i-1, 2i , \\ |c_i - D_j| &\geqq \frac{1}{2}d \quad \text{for} \quad j = 2i-1, 2i , \end{aligned}$$

and hence

$$f(c_i) + d \geqq (3d/2)^{n-2}(d/2)^2 = \frac{3^{n-2}}{2}(d/2)^{n-1}d .$$

Observing that $2 \leqq 2i \leqq n-1$ implies $n \geqq 3$, one obtains thus for $d \geqq 2$ the desired inequalities

$$f(c_i) + d \geqq 3d/2 > d .$$

In the special case $d = 1$ still more precise lower estimates are required, viz.,

$$|c_i - D_j| \geqq (2i-1-j)d + \frac{d}{2} = 2i-1-j+\frac{1}{2} \quad \text{for} \quad j = 0, 1, \cdots, 2i-1,$$

$$|c_i - D_j| \geqq (j-2i)d + \frac{d}{2} = j - 2i + \frac{1}{2} \quad \text{for} \quad j = 2i, \cdots, n-1.$$

The lower bounds have values from the sequence $1/2, 3/2, 5/2, \cdots$ For each relevant i two values $1/2$ and, if $n \geqq 5$, at least two values $3/2$ and one value $5/2$ occur. For $n \geqq 5$ therefore certainly

$$f(c_i) + 1 \geqq \left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right)^2 \left(\frac{5}{2}\right) > 1.$$

In the remaining cases $d = 1$ with $n = 3,4$ there is only one relevant i, viz., $i = 1$. One verifies easily that the desired inequality

$$f(c_1) + 1 > 1$$

is true under the conditions (19).

We shall now rearrange $f(x)$ in powers of $x - D$. We shall first prove the formula

$$\begin{aligned} f^{(k)}(x) &= k! \sum (x - D_{i_1}) \cdots (x - D_{i_{n-k}}), \\ &0 \leqq i_1 < i_2 < \cdots < i_{n-k} \leqq n-1, \\ &k = 1, \cdots, n-1. \end{aligned} \tag{20}$$

We shall denote

$$\begin{aligned} g(x) &= (x - D_0)(x - D_1) \cdots (x - D_{n-1}); \; f(x) = g(x) - d. \\ f'(x) &= g'(x) = g(x) \sum (1/(x - D_j)) \\ &= 1! \sum (x - D_{i_1})(x - D_{i_2}) \cdots (x - D_{i_{n-1}}) \\ &\qquad\qquad 0 \leqq i_1 < i_2 < \cdots < i_{n-1} \leqq n-1. \end{aligned} \tag{21}$$

Thus formula (20) is correct for $k = 1$. Let it be correct for $k = m$, namely

$$\begin{aligned} f^{(m)}(x) &= m! \sum (x - D_{i_1})(x - D_{i_2}) \cdots (x - D_{i_{n-m}}), \\ &\qquad 0 \leqq i_1 < i_2 < i_3 < \cdots < i_{n-m} \leqq n-1 \end{aligned}$$

or, in virtue of (21)

$$\left\{ \begin{aligned} f^{(m)}(x) &= m!\, g(x) \sum \frac{1}{(x - D_{j_1})(x - D_{j_2}) \cdots (x - D_{j_m})} \\ &\qquad 0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1. \end{aligned} \right. \tag{22}$$

Differentiating (22) we obtain

$$
\begin{aligned}
&\frac{1}{m!}f^{(m+1)}(x)\\
&= g'(x)\sum_{0\leqq j_1<j_2<j_3<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\\
&\quad + g(x)\Bigl(\sum_{0\leqq j_1<j_2<j_3<\cdots<j_m<n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\Bigr)'\\
&= g(x)\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum_{0\leqq j_1<j_2<j_3<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\\
&\quad - g(x)\sum_{r=1}^{m}\sum\frac{1}{(x-D_{j_1})\cdots(x-D_{j_{r-1}})(x-D_{j_r})^2(x-D_{j_{r+1}})\cdots(x-D_{j_m})}\,.
\end{aligned}
$$

But it is easily seen that

$$
\begin{aligned}
&\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum_{0\leqq j_1<j_2<j_3<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\\
&= \sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum_{s\neq j_1,\cdots,j_m;\ 0\leqq j_1<j_2<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\\
&\quad + \sum_{0\leqq j_1<j_2<j_3<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})\cdots(x-D_{j_{r-1}})(x-D_{j_r})^2(x-D_{j_{r+1}})\cdots(x-D_{j_m})}\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{m!}f^{(m+1)}(x)\\
&= g(x)\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum_{s\neq j_1,\cdots,j_m;\ 0\leqq j_1<j_2<j_3<\cdots<j_m\leqq n-1}\frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\\
&= (m+1)g(x)\sum_{0\leqq t_1<t_2<t_3<\cdots<t_{m+1}\leqq n-1}\frac{1}{(x-D_{t_1})(x-D_{t_2})\cdots(x-D_{t_{m+1}})}\,,
\end{aligned}
$$

$$
\begin{aligned}
f^{(m+1)}(x) &= (m+1)!\,g(x)\sum_{0\leqq t_1<t_2<t_3<\cdots<t_{m+1}\leqq n-1}\frac{1}{(x-D_{t_1})(x-D_{t_2})\cdots(x-D_{t_{m+1}})}\\
&= (m+1)!\sum_{0\leqq i_1<i_2<i_3<\cdots<i_{n-(m+1)}\leqq n-1}(x-D_{i_1})(x-D_{i_2})\cdots(x-D_{i_{n-(m+1)}})
\end{aligned}
$$

which proves formula (20).

From (20) we obtain for $x = D_0 = D$, taking into account that $D - D_{i_1} = 0$ for $i_1 = 0$

$$f^{(k)}(D) = k! \sum (D - D_{i_1})(D - D_{i_2}) \cdots (D - D_{i_{n-k}}) , \quad 1 \leqq i_1 < i_2 < \cdots < i_{n-k} \leqq n-1 , \quad k = 1, \cdots, n-1 . \tag{23}$$

From (17) we obtain

$$f(D) = -d ; \; f^{(n)}(D) = n! , \tag{23. a}$$

and, combining (23), (23. a) and using Taylor's formula for developing $f(x)$ in powers of $x - D$,

$$f(x) = (x - D)^n + \left(\sum_{s=1}^{n-1} k_s (x - D)^{n-s}\right) - d , \quad k_s = \sum (D - D_{i_1})(D - D_{i_2}) \cdots (D - D_{i_s}) , \quad 1 \leqq i_1 < i_2 < \cdots < i_s \leqq n - 1 . \tag{24}$$

4. Inequalities. In this chapter we shall prove the inequalities needed for carrying out the Algorithm of Jacobi-Perron with a basic sequence $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ chosen from the field $K(w)$.

We obtain from Theorem 1 and $D < w < D + 1$

$$[w] = D . \tag{25}$$

In the sequel we shall find the following notations useful

$$\begin{cases} P_{i,i} = P_i = w - D_i , & (i = 1, \cdots, n-1) \\ P_{i,k} = P_i P_{i+1} \cdots P_k ; & 1 \leqq i \leqq k \leqq n - 1 . \end{cases} \tag{26}$$

One of the basic inequalities needed in the following

$$\begin{cases} [(w - D) P_{i_1} P_{i_2} \cdots P_{i_k}] = 0 , \\ 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n - 2 . \end{cases} \tag{27}$$

To prove (27) we have to verify

$$0 < (w - D) P_{i_1} P_{i_2} \cdots P_{i_k} < 1 . \tag{28}$$

From (25), (26) we obtain

$$P_i = w - D_i > D - D_i > 0 .$$

Thus the left-hand inequality of (28) is proved. From (17) we obtain

$$w - D = \frac{d}{P_{1,n}} . \tag{29}$$

Therefore

$$\begin{aligned}(w - D)P_{i_1}P_{i_2} \cdots P_{i_k} &= dP_{i_1}P_{i_2} \cdots P_{i_k}/P_{1,n} \\ = d/P_{i_{k+1}}P_{i_{k+2}} \cdots P_{i_{n-1}} &< d/(D - D_{i_{k+1}})(D - D_{i_{k+2}}) \cdots (D - D_{i_{n-1}}) ;\end{aligned}$$

but, as was proved before, $D - D_{i_j} \geqq d$; $(j = 1, \cdots, n-1)$ therefore

$$(w - D)P_{i_1}P_{i_2} \cdots P_{i_k} < \frac{d}{d^{n-k-1}} \leqq \frac{d}{d} = 1 ,$$

which proves the right-hand inequality of (28).

From (27) we obtain easily, since $d \geqq 1$

$$\text{(30)} \qquad \begin{cases} [(w - D)P_{i_1}P_{i_2} \cdots P_{i_k}] = 0 ; \\ 1 \leqq i_1 < i_2 < \cdots i_k \leqq n - 2 . \end{cases}$$

We further obtain, in virtue of (25)

$$\text{(31)} \qquad [P_i] = D - D_i \qquad (i = 1, \cdots, n-1) .$$

From (31) we obtain, since $d \mid D - D_i$,

$$\text{(32)} \qquad [P_i/d] = (D - D_i)/d \qquad (i = 1, \cdots, n-1) .$$

5. Jacobi-perron algorithm for polynomials of decreasing order.

DEFINITION. An $(n-1)$ by $(n-1)$ matrix of the form

$$\text{(33)} \qquad \left\{ \begin{Vmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & 0 & A_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & A_{n-1} \end{Vmatrix} \right.$$

will be called a *fugue*; the last column vector

$$\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{matrix}$$

will be called the *generator* of the fugue.

THEOREM 3. *Let $f(x)$ be the GP from* (17) *and w its only real root in the open interval* $(D, D+1)$. *The Jacobi Perron Algorithm of the decreasing order polynomials*

$$
(34) \qquad \begin{cases} a_s^{(0)} = \dfrac{1}{d}(w - D)P_{1,1}P_{2+s,n-1}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(0)} = \dfrac{1}{d}(w - D)P_{1,1}\,, & \\ a_{n-1}^{(0)} = P_{1,1}\,, & \end{cases}
$$

is purely periodic and its primitive length is $T = n(n - 1$ *for* $d \neq 1$, *and* $T = n - 1$ *for* $d = 1$. *The period of length* $n(n - 1)$ *consist of* n *fugues. The generator of the first fugue has the form*

$$
(35) \qquad \begin{matrix} D - D_1 \\ D - D_2 \\ \vdots \\ D - D_{n-1}\,. \end{matrix}
$$

The generator of the $r + 1$ *-th fugue* $(r = 1, \cdots, n - 1)$ *has the form*

$$
(36) \qquad \begin{cases} D - D_1 \\ D - D_2 \\ \vdots \\ \dfrac{D - D_r}{d} \\ \vdots \\ D - D_{n-1} \end{cases} .
$$

The period of length $n - 1$ *consists of one fugue whose generator has the form* (35).

Proof. In the sequel we shall use the notation

$$
(37) \qquad \begin{cases} u\,;\ v = u(n - 1 +)\,v\,; & (u = 0, 1, \cdots;\ v = 0, 1, \cdots, n-2) \\ u\,;\ n - 1 = u + 1\,;\ 0\,. & \end{cases}
$$

Because of (26) the formula holds

$$
(38) \qquad P_{i,s}/P_{i,k} = 1/P_{s+1,k}\,; \qquad 1 \leqq i \leqq s < k \leqq n - 1\,.
$$

Since, from (17),

$$
(w - D)(w - D_1)(w - D_2) \cdots (w - D_{n-1}) - d = 0\,,
$$

we obtain

$$(39)\qquad \begin{cases} \dfrac{1}{d}(w-D)P_{1,1}P_{2+s,n-1} = \dfrac{1}{P_{2,1+s}}\,; & (s=1,\cdots,n-3) \\ \dfrac{1}{d}(w-D)P_{1,1} = \dfrac{1}{P_{2,n-1}}\,. \end{cases}$$

We shall substitute these values for $a_s^{(0)}$ in (34), so that

$$(40)\qquad \begin{cases} a_s^{(0)} = \dfrac{1}{P_{2,1+s}}\,; & (s=1,\cdots,n-2) \\ a_{n-1}^{(0)} = P_{1,1}\,. \end{cases}$$

We obtain from (34), in virtue of (30), (31)

$$(41)\qquad b_s^{(0)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(0)} = D - D_1\,.$$

We obtain from (31)

$$(42)\qquad \begin{cases} P_{i,i} - [P_{i,i}] = w - D\,; \\ \dfrac{P_{i,i}}{d} - \left[\dfrac{P_{i,i}}{d}\right] = \dfrac{w-D}{d} & (i=1,\cdots,n-1)\,. \end{cases}$$

From (40)—(42) we obtain

$$\begin{cases} a_s^{(0)} - b_s^{(0)} = \dfrac{1}{P_{2,1+s}}\,, & (s=1,\cdots,n-2) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = w - D\,; \\ a_1^{(0)} - b_1^{(0)} = \dfrac{1}{P_{2,2}}\,; \\ a_{1+s}^{(0)} - b_{1+s}^{(0)} = \dfrac{1}{P_{2,2+s}}\,; & (s=1,\cdots,n-3) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = w - D\,, \end{cases}$$

so that, in virtue of (4)

$$\begin{cases} a_s^{(1)} = \dfrac{P_{2,2}}{P_{2,2+s}}\,, & (s=1,\cdots,n-3) \\ a_{n-2}^{(1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(1)} = P_{2,2}\,. \end{cases}$$

From these formulas we obtain, in virtue of (40)

$$(43)\qquad \begin{cases} a_s^{(1)} = \dfrac{1}{P_{3,2+s}}\,, & (s=1,\cdots,n-3) \\ a_{n-2}^{(1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(1)} = P_{2,2} \end{cases}$$

Since

$$1/P_{3,2+s} = \frac{1}{d}(w - D)P_1P_2 ,$$

we obtain, from (43) and in virtue of (30), (27), (31)

$$b_s^{(1)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = D - D_2 ,$$

and from (43), (44), in virtue of (42)

$$\begin{cases} a_1^{(1)} - b_1^{(1)} = \dfrac{1}{P_{3,3}} , \\ a_{1+s}^{(1)} - b_{1+s}^{(1)} = \dfrac{1}{P_{3,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-2}^{(1)} - b_{n-2}^{(1)} = (w - D)P_{2,2} , \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = w - D . \end{cases} \tag{45}$$

Frow (45) we obtain, in virtue of (4) and (38)

$$\begin{cases} a_s^{(2)} = \dfrac{P_{3,3}}{P_{3,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = (w - D)P_{2,2}P_{3,3} , \\ a_{n-2}^{(2)} = (w - D)P_{3,3} , \\ a_{n-1}^{(2)} = P_{3,3} ; \end{cases}$$

$$\begin{cases} a_s^{(2)} = \dfrac{1}{P_{4,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = (w - D)P_{2,3} , \\ a_{n-2}^{(2)} = (w - D)P_{3,3} , \\ a_{n-1}^{(2)} = P_{3,3} . \end{cases} \tag{46}$$

We shall now prove the formula

$$\begin{cases} a_s^{(k)} = 1/P_{k+2,k+1+s} , & (s = 1, \cdots, n-k-2) \\ a_{n-k-2+i}^{(k)} = (w - D)P_{1+i,k+1} , \\ a_{n-1}^{(k)} = P_{k+1,k+1} , \\ k = 2, \cdots, n-3 . \end{cases} \tag{47}$$

Formula (47) is valid for $k = 2$ in virtue of (46). We shall prove its validity for $k + 1$. Since

$$1/P_{k+2,k+1+s} = \frac{1}{d}(w - D)P_{1,k+1} ,$$

we obtain from (47), in virtue of (30), (27), (31)

$$b_j^{(k)} = 0 ; \quad (j = 1, \cdots, n-2) \qquad b_{n-1}^{(k)} = D - D_{k+1} , \tag{48}$$

and from (47), (48), in virtue of (42)

$$\begin{cases} a_s^{(k)} - b_s^{(k)} = 1/P_{k+2,k+1+s}\,, & (s = 1, \cdots, n-k-2) \\ a_{n-k-2+i}^{(k)} - b_{n-k-2+i}^{(k)} = (w-D)P_{1+i,k+1}\,, & (i = 1, \cdots, k) \\ a_{n-1}^{(k)} - b_{n-1}^{(k)} = w - D\,; \end{cases}$$

$$\begin{cases} a_1^{(k)} - b_1^{(k)} = 1/P_{k+2,k+2}\,, \\ a_{1+s}^{(k)} - b_{1+s}^{(k)} = 1/P_{k+2,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-2+i}^{(k)} - b_{n-k-2+i}^{(k)} = (w-D)P_{1+i,k+1}\,, & (i = 1, \cdots, k) \\ a_{n-1}^{(k)} - b_{n-1}^{(k)} = w - D\,, \end{cases}$$

so that, in virtue of (4)

$$\begin{cases} a_s^{(k+1)} = P_{k+2,k+2}/P_{k+2,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-3+i}^{(k+1)} = (w-D)P_{1+i,k+1}P_{k+2,k+2}\,, & (i = 1, \cdots k) \\ a_{n-2}^{(k+1)} = (w-D)P_{k+2,k+2}\,, \\ a_{n-1}^{(k+1)} = P_{k+2,k+2}\,, \end{cases}$$

and, in virtue of (42),

$$\begin{cases} a_s^{(k+1)} = 1/P_{k+3,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-3+i}^{(k+1)} = (w-D)P_{1+i,k+2}\,, & (i = 1, \cdots, k+1) \\ a_{n-1}^{(k+1)} = P_{k+2,k+2}\,. \end{cases} \tag{49}$$

With (49) formula (47) is proved.

We now obtain from (47) for $k = n - 3$

$$\begin{cases} a_1^{(n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+i}^{(n-3)} = (w-D)P_{1+i,n-2}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-3)} = P_{n-2,n-2}\,. \end{cases} \tag{50}$$

From (50) we obtain, in virtue of (30), (27), (31)

$$b_s^{(n-3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3)} = D - D_{n-2}\,, \tag{51}$$

and from (50), (51), in virtue of (42)

$$\begin{cases} a_1^{(n-3)} - b_1^{(n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+i}^{(n-3)} - b_{1+i}^{(n-3)} = (w-D)P_{1+i,n-2}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-3)} - b_{n-1}^{(n-3)} = w - D\,. \end{cases} \tag{52}$$

From (52) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(n-2)} = (w-D)P_{1+i,n-2}P_{n-1,n-1}\,, & (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-2)} = (w-D)P_{n-1,n-1}\,, \\ a_{n-1}^{(n-2)} = P_{n-1,n-1}\,; \end{cases}$$

or

$$(53)\qquad \begin{cases} a_s^{(n-2)} = (w - D)P_{1+i,n-1}\,, & (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-2)} = P_{n-1,n-1}\,. \end{cases}$$

From (53) we obtain, in virtue of (27), (31),

$$(54)\qquad b_s^{(n-2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2)} = D - D_{n-1}\,,$$

and from (53), (54), in virtue of (42),

$$\begin{cases} a_s^{(n-2)} - b_s^{(n-2)} = (w - D)P_{1+s,n-1}\,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D\,; \end{cases}$$

$$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = (w - D)P_{2,n-1}\,, \\ a_{1+s}^{(n-2)} - b_{1+s}^{(n-2)} = (w - D)P_{2+s,n-1}\,, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D\,, \end{cases}$$

so that, in virtue of (4),

$$\begin{cases} a_s^{(n-1)} = P_{2+s,n-1}/P_{2,n-1}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(n-1)} = 1/P_{2,n-1}\,, \\ a_{n-1}^{(n-1)} = 1/(w - D)P_{2,n-1}\,; \end{cases}$$

but, from (39) we obtain

$$1/(w - D)P_{2,n-1} = P_{1,1}/d\,;$$

therefore,

$$\begin{cases} a_s^{(n-1)} = 1/P_{2,1+s}\,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-1)} = P_{1,1}/d\,; \end{cases}$$

thus, with the notation of (37),

$$(55)\qquad \begin{cases} a_s^{(1;0)} = 1/P_{2,1+s}\,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(1;0)} = P_{1,1}/d\,. \end{cases}$$

From (55) we obtain, in virtue of (30), (32), and since

$$1/P_{2,1+s} = \frac{1}{d}(w - D)P_{2+s,n-1}$$

$$(56)\qquad b_s^{(1;0)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;0)} = \frac{D - D_1}{d}\,,$$

and from (55), (56), in virtue of (42)

$$\begin{cases} a_s^{(1;0)} - b_s^{(1;0)} = 1/P_{2,1+s} \;, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = \dfrac{w - D}{d} \;; \end{cases}$$

or

$$\begin{cases} a_1^{(1;0)} - b_1^{(1;0)} = 1/P_{2,2} \;, \\ a_{1+s}^{(1;0)} - b_{1+s}^{(1;0)} = 1/P_{2,2+s} \;, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = \dfrac{w - D}{d} \;; \end{cases}$$

thus, in virtue of (4),

$$\begin{cases} a_s^{(1;1)} = P_{2,2}/P_{2,2+s} \;, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = (w - D)P_{2,2}/d \;, \\ a_{n-1}^{(1;1)} = P_{2,2} \;; \end{cases}$$

or

$$\begin{cases} a_s^{(1;1)} = 1/P_{3,2+s} \;, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = (w - D)P_{2,2}/d \;, \\ a_{n-1}^{(1;1)} = P_{2,2} \;. \end{cases} \tag{57}$$

From (57) we obtain, as before,

$$b_s^{(1;1)} = 0 \;; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;1)} = D - D_2 \;, \tag{58}$$

and from (57), (58), in virtue of (42),

$$\begin{cases} a_s^{(1;1)} - b_s^{(1;1)} = 1/P_{3,2+s} \;, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = (w - D)P_{2,2}/d \;, \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = w - D \;; \end{cases}$$

or

$$\begin{cases} a_1^{(1;1)} - b_1^{(1;1)} = 1/P_{3,3} \;, \\ a_{1+s}^{(1;1)} - b_{1+s}^{(1;1)} = 1/P_{3,3+s} \;, & (s = 1, \cdots, n-4) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = (w - D)P_{2,2}/d \;, \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = w - D \;. \end{cases} \tag{59}$$

From (59) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(1;2)} = P_{3,3}/P_{3,3+s} \;, & (s = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = (w - D)P_{2,2}P_{3,3}/d \;, \\ a_{n-2}^{(1;2)} = (w - D)P_{3,3} \;, \\ a_{n-1}^{(1;2)} = P_{3,3} \;, \end{cases}$$

or

$$
(60) \qquad \begin{cases} a_s^{(1;2)} = 1/P_{4,3+s} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = (w - D)P_{2,3}/d , & \\ a_{n-2}^{(1;2)} = (w-D)P_{3,3} , & \\ a_{n-1}^{(1;2)} = P_{3,3} . & \end{cases}
$$

We shall now prove the formula

$$
(61) \qquad \begin{cases} a_2^{(1;k)} = 1/P_{k+2,k+1+s} , & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d , & \\ a_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1} , & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} = P_{k+1,k+1} , & \\ k = 2, \cdots, n-3 . & \end{cases}
$$

Formula (61) is correct for $k = 2$, in virtue of (60). We shall prove by induction that it is correct for $k + 1$.

We obtain from (61), as before,

$$
(62) \qquad b_s^{(1;k)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(k)} = D - D_{k+1} ,
$$

and from (61), (62), in virtue of (42)

$$
\begin{cases} a_s^{(1;k)} - b_s^{(1;k)} = 1/P_{k+2,k+1+3} , & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(1;k)} - b_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d , & \\ a_{n-k-1+i}^{(1;k)} - b_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1} & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} - b_{n-1}^{(1;k)} = w - D ; & \end{cases}
$$

or

$$
(63) \qquad \begin{cases} a_1^{(1;k)} - b_1^{(1;k)} = 1/P_{k+2,k+2} , & \\ a_{1+s}^{(1;k)} - b_{1+s}^{(1;k)} = 1/P_{k+2,k+2+s} , & (s = 1, \cdots, n-k-3) \\ a_{n-k-1}^{(1;k)} - b_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d , & \\ a_{n-k-1+i}^{(1;k)} - b_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1} , & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} - b_{n-1}^{(1;k)} = w - D . & \end{cases}
$$

From (63) we obtain, in virtue of (4),

$$
\begin{cases} a_s^{(1;k+1)} = P_{k+2,k+2}/P_{k+2,k+2+s} , & (s = 1, \cdots, n-k-3) \\ a_{n-k-2}^{(1;k+1)} = (w-D)P_{2,k+1}P_{k+2,k+2}/d , & \\ a_{n-k-2+i}^{(1;k+1)} = (w - D)P_{2+i,k+1}P_{k+2,k+2} , & (i = 1, \cdots, k-1) \\ a_{n-2}^{(1;k+1)} = (w - D)P_{k+2,k+2} , & \\ a_{n-1}^{(1;k+1)} = P_{k+2,k+2} ; & \end{cases}
$$

or

$$(64)\quad \begin{cases} a_s^{(1;k+1)} = 1/P_{k+3,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-2}^{(1;k+1)} = (w - D)P_{2,k+2}\,, & \\ a_{n-k-2+i}^{(1;k+1)} = (w - D)P_{2+i,k+2}\,, & (i = 1, \cdots, k) \\ a_{n-1}^{(1;k+1)} = P_{k+2,k+2}\,. & \end{cases}$$

With (64) formula (61) is proved.

We now obtain from (61) for $k = n - 3$

$$(65)\quad \begin{cases} a_1^{(1;n-3)} = 1/P_{n-1,n-1}\,, & \\ a_2^{(1;n-3)} = (w - D)P_{2,n-2}/d\,, & \\ a_{2+i}^{(1;n-3)} = (w - D)P_{2+i,n-2}\,, & (i = 1, \cdots, n-4) \\ a_{n-1}^{(1;n-3)} = P_{n-2}\,. & \end{cases}$$

From (65) we obtain, as before,

$$(66)\qquad b_s^{(1;n-3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;n-3)} = D - D_{n-2}\,.$$

From (65), (66) we obtain as before

$$(67)\quad \begin{cases} a_1^{(1;n-3)} - b_1^{(1;n-3)} = 1/P_{n-1,n-1}\,, & \\ a_2^{(1;n-3)} - b_2^{(1;n-3)} = (w - D)P_{2,n-2}/d\,, & \\ a_{2+i}^{(1;n-3)} - b_{2+i}^{(1;n-3)} = (w - D)P_{2+i,n-2}\,, & (i = 1, \cdots, n-4) \\ a_{n-1}^{(1;n-3)} - b_{n-i}^{(1;n-3)} = w - D\,; & \end{cases}$$

and from (67), in virtue of (4),

$$\begin{cases} a_1^{(1;n-2)} = (w - D)P_{2,n-2}P_{n-1,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} = (w - D)P_{2+i,n-2}P_{n-1,n-1}\,, & (i = 1, \cdots, n-4) \\ a_{n-2}^{(1;n-2)} = (w - D)P_{n-1,n-1}\,, & \\ a_{n-1}^{(1;n-2)} = P_{n-1,n-1}\,; & \end{cases}$$

or

$$(68)\quad \begin{cases} a_1^{(1;n-2)} = (w - D)P_{2,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} = (w - D)P_{2+i,n-1}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(1;n-2)} = P_{n-1,n-1}\,. & \end{cases}$$

From (68) we obtain, as before,

$$(69)\qquad b_s^{(1;n-2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;n-2)} = D - D_{n-1}\,,$$

and from (68), (69), in virtue of (42)

$$(70)\quad \begin{cases} a_1^{(1;n-2)} - b_1^{(1;n-2)} = (w - D)P_{2,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} - b_{1+i}^{(1;n-2)} = (w - D)P_{2+i,n-1}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(1;n-2)} - b_{n-1}^{(1;n-2)} = w - D\,. & \end{cases}$$

From (70) we obtain, in virtue of (4) and (39)

$$\begin{cases} a_s^{(2;0)} = dP_{2+s,n-1}/P_{2,n-1} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;0)} = d/P_{2,n-1} , & \\ a_{n-1}^{(2;0)} = P_{1,1} ; & \end{cases}$$

or

$$\begin{cases} a_s^{(2;0)} = d/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(2;0)} = P_{1,1} . & \end{cases} \tag{71}$$

From (71) we obtain, as before,

$$b_s^{(2;0)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;0)} = D - D_1 , \tag{72}$$

and from (71), (72), in virtue of (42)

$$\begin{cases} a_s^{(2;0)} - b_s^{(2;0)} = d/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(2;0)} - b_{n-1}^{(2;0)} = w - D ; & \end{cases}$$

or

$$\begin{cases} a_1^{(2;0)} - b_1^{(2;0)} = d/P_{2,2} , & \\ a_{1+s}^{(2;0)} - b_{1+s}^{(2;0)} = d/P_{2,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(2;0)} - b_{n-1}^{(2;0)} = w - D . & \end{cases} \tag{73}$$

From (73) we obtain, in virtue of (4)

$$\begin{cases} a_s^{(2;1)} = P_{2,2}/P_{2,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} = (w - D)P_{2,2}/d , & \\ a_{n-1}^{(2;1)} = P_{2,2}/d ; & \end{cases}$$

or

$$\begin{cases} a_s^{(2;1)} = 1/P_{3,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} = (w - D)P_{2,2}/d , & \\ a_{n-1}^{(2;1)} = P_{2,2}/d ; & \end{cases} \tag{74}$$

and from (74), as before,

$$b_s^{(2;1)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;1)} = (D - D_2)/d . \tag{75}$$

From (74), (75) we obtain, in virtue of (42)

$$\begin{cases} a_s^{(2;1)} - b_s^{(2;1)} = 1/P_{3,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} - b_{n-2}^{(2;1)} = (w - D)P_{2,2}/d , & \\ a_{n-1}^{(2;1)} - b_{n-1}^{(2;1)} = (w - D)/d ; & \end{cases}$$

or

$$(76)\qquad \begin{cases} a_1^{(2;1)} - b_1^{(2;1)} = 1/P_{3,3}\ , & \\ a_{1+s}^{(2;1)} - b_{1+s}^{(2;1)} = 1/P_{3,3+s}\ , & (s = 1, \cdots, n-4) \\ a_{n-2}^{(2;1)} - b_{n-2}^{(2;1)} = (w - D)P_{2,2}/d\ , & \\ a_{n-1}^{(2;1)} - b_{n-1}^{(2;1)} = (w - D)/d\ . & \end{cases}$$

From (76) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(2;2)} = P_{3,3}/P_{3,3+s}\ , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} = (w - D)P_{2,2}P_{3,3}/d\ , & \\ a_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\ , & \\ a_{n-1}^{(2;2)} = P_{3,3}\ ; & \end{cases}$$

or

$$(77)\qquad \begin{cases} a_s^{(2;2)} = 1/P_{4,3+s}\ , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\ , & \\ a_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\ , & \\ a_{n-1}^{(2;2)} = P_{3,3}\ . & \end{cases}$$

From (77) we obtain, as before,

$$(78)\qquad b_s^{(2;2)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;2)} = D - D_3\ ,$$

and from (77), (78) and in virtue of (42),

$$\begin{cases} a_s^{(2;2)} - b_s^{(2;2)} = 1/P_{4,3+s}\ , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} - b_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\ , & \\ a_{n-2}^{(2;2)} - b_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\ , & \\ a_{n-1}^{(2;2)} - b_{n-1}^{(2;)} = w - D\ , & \end{cases}$$

or

$$(79)\qquad \begin{cases} a_1^{(2;2)} - b_1^{(2;2)} = 1/P_{4,4}\ , & \\ a_{1+s}^{(2;2)} - b_{1+s}^{(2;2)} = 1/P_{4,4+s}\ , & (s = 1, \cdots, n-5) \\ a_{n-3}^{(2;2)} - b_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\ , & \\ a_{n-2}^{(2;2)} - b_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\ , & \\ a_{n-1}^{(2;2)} - b_{n-1}^{(2;2)} = w - D\ . & \end{cases}$$

From (79) we obtain, in virtue of (4), and carrying out cancellation and multiplication as before,

$$(80)\qquad \begin{cases} a_s^{(2;3)} = 1/P_{5,4+s}\ , & (s = 1, \cdots, n-5) \\ a_{n-4}^{(2;3)} = (w - D)P_{2,4}/d\ , & \\ a_{n-3}^{(2;3)} = (w - D)P_{3,4}/d\ , & \\ a_{n-2}^{(2;3)} = (w - D)P_{4,4}\ , & \\ a_{n-1}^{(2;3)} = P_{4,4}\ . & \end{cases}$$

We shall now prove the formula

$$(81)\qquad \begin{cases} a_s^{(2;k)} = 1/P_{k+2,k+1+s}\ , & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(2;k)} = (w-D)P_{2,k+1}/d\ , & \\ a_{n-k}^{(2;k)} = (w-D)P_{3,k+1}/d\ , & \\ a_{n-k+i}^{(2;k)} = (w-D)P_{3+i,k+1}\ , & (i = 1, \cdots, k-2) \\ a_{n-1}^{(2;k)} = P_{k+1,k+1}\ ; & k = 3, \cdots, n-3\ . \end{cases}$$

The proof of (81) is by induction like that of formula (61) or (47). First we see that (81) is correct for $k = 3$; then we show that it is correct for $k+1$.

We now obtain from (81) for $k = n-3$

$$(82)\qquad \begin{cases} a_1^{(2;n-3)} = 1/P_{n-1,n-1}\ , & \\ a_2^{(2;n-3)} = (w-D)P_{2,n-2}/d\ , & \\ a_3^{(2;n-3)} = (w-D)P_{3,n-2}/d\ , & \\ a_{3+i}^{(2;n-3)} = (w-D)P_{3+i,n-2}\ , & (i = 1, \cdots, n-5) \\ a_{n-1}^{(2;n-3)} = P_{n-2,n-2}\ ; & \end{cases}$$

and from (82), as before,

$$(83)\qquad b_s^{(2;n-3)} = 0\ ;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;n-3)} = D - D_{n-2}\ .$$

From (82), (83) we obtain, in virtue of (42),

$$(83)\qquad \begin{cases} a_1^{(2;n-3)} - b_1^{(2;n-3)} = 1/P_{n-1,n-1}\ , \\ a_2^{(2;n-3)} - b_2^{(2;n-3)} = (w-D)P_{2,n-2}/d\ , \\ a_3^{(2;n-3)} - b_3^{(2;n-3)} = (w-D)P_{3,n-2}/d\ , \\ a_{3+i}^{(2;n-3)} - b_{3+i}^{(2;n-3)} = (w-D)P_{3+i,n-2}\ , \\ a_{n-1}^{(2;n-3)} - b_{n-1}^{(2;n-3)} = w - D\ . \end{cases}$$

From (83) we obtain, in virtue of (4) and carrying out multiplication as before,

$$(84)\qquad \begin{cases} a_1^{(2;n-2)} = (w-D)P_{2,n-1}/d\ , & \\ a_2^{(2;n-2)} = (w-D)P_{3,n-1}/d\ , & \\ a_{2+i}^{(2;n-2)} = (w-D)P_{3+i,n-1}\ , & (i = 1, \cdots, n-4) \\ a_{n-1}^{(2;n-2)} = P_{n-1,n-1}\ . & \end{cases}$$

From (84) we obtain, as before,

$$(85)\qquad b_s^{(2;n-2)} = 0\ ;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;n-2)} = D - D_{n-1}\ ,$$

and from (84), (85), in virtue of (42),

$$(85a)\quad \begin{cases} a_1^{(2;n-2)} - b_1^{(2;n-2)} = (w-D)P_{2,n-1}/d\,, \\ a_2^{(2;n-2)} - b_2^{(2;n-2)} = (w-D)P_{3,n-1}/d\,, \\ a_{2+i}^{(2;n-2)} - b_{2+i}^{(2;n-2)} = (w-D)P_{3+i,n-1}\,, & (i=1,\cdots,n-4) \\ a_{n-1}^{(2;n-2)} - b_{n-1}^{(2;n-2)} = w-D\,. \end{cases}$$

From (85a) we obtain, in virtue of (4),

$$\begin{cases} a_1^{(3;0)} = P_{3,n-1}/P_{2,n-1}\,, \\ a_{1+i}^{(3;0)} = dP_{3+i,n-1}/P_{2,n-1}\,, & (i=1,\cdots,n-4) \\ a_{n-2}^{(3;0)} = d/P_{2,n-1}\,, \\ a_{n-1}^{(3;0)} = d/(w-D)P_{2,n-1}\,; \end{cases}$$

or, after carrying out the necessary cancellation and multiplication

$$(86)\quad \begin{cases} a_1^{(3;0)} = 1/P_{2,2}\,, \\ a_{1+i}^{(3;0)} = d/P_{2,2+i}\,, & (i=1,\cdots,n-3) \\ a_{n-1}^{(3;0)} = P_{1,1}\,. \end{cases}$$

From (86) we obtain, as before,

$$(87)\qquad b_s^{(3;0)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(3;0)} = D - D_1\,,$$

and from (86), (87), in virtue of (42)

$$(88)\quad \begin{cases} a_1^{(3;0)} - b_1^{(3;0)} = 1/P_{2,2}\,, \\ a_{1+i}^{(3;0)} - b_{1+i}^{(3;0)} = d/P_{2,2+i}\,, & (i=1,\cdots,n-3) \\ a_{n-1}^{(3;0)} - b_{n-1}^{(3;0)} = w-D\,. \end{cases}$$

From (88) we obtain, in virtue of (4), and carrying out the necessary cancellation

$$(89)\quad \begin{cases} a_i^{(3;1)} = d/P_{3,2+i}\,, & (i=1,\cdots,n-3) \\ a_{n-2}^{(3;1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(3;1)} = P_{2,2}\,; \end{cases}$$

and from (89), as before,

$$(90)\qquad b_s^{(3;1)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(3;1)} = D - D_2\,.$$

From (89), (90) we obtain, in virtue of (42),

$$(91)\quad \begin{cases} a_1^{(3;1)} - b_1^{(3;1)} = d/P_{3,3}\,, \\ a_{1+i}^{(3;1)} - b_{1+i}^{(3;1)} = d/P_{3,3+i}\,, & (i=1,\cdots,n-4) \\ a_{n-2}^{(3;1)} - b_{n-2}^{(3;1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(3;1)} - b_{n-1}^{(3;1)} = w-D\,. \end{cases}$$

From (91) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(92)\quad \begin{cases} a_i^{(3;2)} = 1/P_{4,3+i}\,, & (i = 1, \cdots, n-4) \\ a_{n-3}^{(3;2)} = (w-D)P_{2,3}/d\,, \\ a_{n-2}^{(3;2)} = (w-D)P_{3,3}/d\,, \\ a_{n-1}^{(3;2)} = P_{3,3}/d\,; \end{cases}$$

and from (92), as before,

$$(93)\qquad b_s^{(3;2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;2)} = \frac{D - D_3}{d}\,.$$

From (92), (93) we obtain, in virtue of (42)

$$(93a)\quad \begin{cases} a_1^{(3;2)} - b_1^{(3;2)} = 1/P_{4,4}\,, \\ a_{1+i}^{(3;2)} - b_{1+i}^{(3;2)} = 1/P_{4,4+i}\,, & (i = 1, \cdots, n-5) \\ a_{n-3}^{(3;2)} - b_{n-3}^{(3;2)} = (w-D)P_{2,3}/d\,, \quad a_{n-2}^{(3;2)} - b_{n-2}^{(3;2)} = \dfrac{(w-D)P_{3,3}}{d}\,, \\ a_{n-1}^{(3;2)} - b_{n-1}^{(3;2)} = (w-D)/d\,. \end{cases}$$

From (93a) we obtain, in virtue of (4),

$$(94)\quad \begin{cases} a_i^{(3;3)} = 1/P_{5,4+i}\,, & (i = 1, \cdots, n-5) \\ a_{n-4}^{(3;3)} = (w-D)P_{2,4}/d\,, \\ a_{n-3}^{(3;3)} = (w-D)P_{3,4}/d\,, \\ a_{n-2}^{(3;3)} = (w-D)P_{4,4}/d\,, \\ a_{n-1}^{(3;3)} = P_{4,4}\,. \end{cases}$$

From (94) we obtain, as before,

$$(95)\qquad b_s^{(3;3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;3)} = D - D_4\,,$$

and from (94), (95), in virtue of (42),

$$(96)\quad \begin{cases} a_1^{(3;3)} - b_1^{(3;3)} = 1/P_{5,5}\,, \\ a_{1+i}^{(3;3)} - b_{1+i}^{(3;3)} = 1/P_{5,5+i}\,, & (i = 1, \cdots, n-6) \\ a_{n-4}^{(3;3)} - b_{n-4}^{(3;3)} = (w-D)P_{2,4}/d\,, \\ a_{n-3}^{(3;3)} - b_{n-3}^{(3;3)} = (w-D)P_{3,4}/d\,, \\ a_{n-2}^{(3;3)} - b_{n-2}^{(3;3)} = (w-D)P_{4,4}/d\,, \\ a_{n-1}^{(3;3)} - b_{n-1}^{(3;3)} = w - D\,. \end{cases}$$

From (96) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(97)\qquad \begin{cases} a_i^{(3;4)} = 1/P_{6,5+i}\ , & (i = 1, \cdots, n-6) \\ a_{n-5}^{(3;4)} = (w - D)P_{2,5}/d\ , & \\ a_{n-4}^{(3;4)} = (w - D)P_{3,5}/d\ , & \\ a_{n-3}^{(3;4)} = (w - D)P_{4,5}/d\ , & \\ a_{n-2}^{(3;4)} = (w - D)P_{5,5}\ , & \\ a_{n-1}^{(3;4)} = P_{5,5}\ . & \end{cases}$$

We shall now prove the formula

$$(98)\qquad \begin{cases} a_i^{(3;k)} = 1/P_{k+2,k+1+i}\ , & (i = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(3;k)} = (w - D)P_{2,k+1}/d\ , & \\ a_{n-k}^{(3;k)} = (w - D)P_{3,k+1}/d\ , & \\ a_{n-k+1}^{(3;k)} = (w - D)P_{4,k+1}/d\ , & \\ a_{n-k+1+s}^{(3;k)} = (w - D)P_{4+s,k+1}\ , & (s = 1, \cdots, k-3) \\ a_{n-1}^{(3;k)} = P_{k+1,k+1}\ , & \\ k = 4, \cdots, n-3\ . & \end{cases}$$

Formula (98) is correct for $k = 4$ because of (97). We then prove as before, that it is correct for $k + 1$, so that (98) is verified. We obtain from (98), as before,

$$(99)\qquad b_s^{(3;k)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;k)} = D - D_{k+1}\ ,$$

and again from (98), for $k = n - 3$,

$$(100)\qquad \begin{cases} a_1^{(3;n-3)} = 1/P_{n-1,n-1}\ , & \\ a_{1+i}^{(3;n-3)} = (w - D)P_{1+i,n-2}/d\ , & (i = 1, 2, 3) \\ a_{4+s}^{(3;n-3)} = (w - D)P_{4+s,n-2}\ , & (s = 1, \cdots, n-6) \\ a_{n-1}^{(3;n-3)} = P_{n-2,n-2}\ . & \end{cases}$$

From (100) we obtain, as before,

$$(101)\qquad b_s^{(3;n-3)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;n-3)} = D - D_{n-2}\ ,$$

and from (100), (101), in virtue of (42)

$$(102)\qquad \begin{cases} a_1^{(3;n-3)} - b_1^{(3;n-3)} = 1/P_{n-1,n-1}\ , & \\ a_{1+i}^{(3;n-3)} - b_{1+i}^{(3;n-3)} = (w - D)P_{1+i,n-2}/d\ , & (i = 1, 2, 3) \\ a_{4+s}^{(2;n-3)} - b_{4+s}^{(3;n-3)} = (w - D)P_{4+s,n-2}\ , & (s = 1, \cdots, n-6) \\ a_{n-1}^{(3;n-3)} - b_{n-1}^{(3;n-3)} = w - D\ . & \end{cases}$$

From (102) we obtain in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(103)\qquad \begin{cases} a_i^{(3;n-2)} = (w - D)P_{1+i,n-1}/d\ , & (i = 1, 2, 3) \\ a_{3+s}^{(3;n-2)} = (w - D)P_{4+s,n-1}\ , & (s = 1, \cdots, n - 5) \\ a_{n-1}^{(3;n-2)} = P_{n-1,n-1}\ , & \end{cases}$$

and from (103), as before,

$$(104)\qquad b_s^{(3;n-2)} = 0\ ;\quad (s = 1, \cdots, n - 2) \qquad b_{n-1}^{(3;n-2)} = D - D_{n-1}\ .$$

From (103), (104) we obtain, in virtue of (42),

$$(105)\qquad \begin{cases} a_1^{(3;n-2)} - b_1^{(3;n-2)} = (w - D)P_{2,n-1}/d\ , & \\ a_2^{(3;n-2)} - b_2^{(3;n-2)} = (w - D)P_{3,n-1}/d\ , & \\ a_3^{(3;n-2)} - b_3^{(3;n-2)} = (w - D)P_{4,n-1}/d\ , & \\ a_{3+s}^{(3;n-2)} - b_{3+s}^{(3;n-2)} = (w - D)P_{4+s,n-1}\ , & (s = 1, \cdots, n - 5) \\ a_{n-1}^{(3;n-2)} - b_{n-1}^{(3;n-2)} = w - D\ , & \end{cases}$$

and from (105), in virtue of (4),

$$(106)\qquad \begin{cases} a_1^{(4;0)} = 1/P_{2,2}\ , & \\ a_2^{(4;0)} = 1/P_{2,3}\ , & \\ a_{2+s}^{(4;0)} = d/P_{2,3+s}\ , & (s = 1, \cdots, n - 4) \\ a_{n-1}^{(4;0)} = P_{1,1}\ . & \end{cases}$$

The reader will easily verify, on ground of previous formulas, that the $4(n - 1)$ supporting sequences

$$b_1^{(i;k)}, b_{2i}^{(i;k)}, \cdots, b_{n-1}^{(i;k)} \qquad (k = 0, \cdots, n - 2;\ i = 0, 1, 2, 3)$$

generate the first four fugues whose form is that as demanded by Theorem 3.

The complete proof of Theorem 3 is based on the following

LEMMA 1. *Let the generating sequence*

$$a_s^{(k;0)} \qquad (s = 1, \cdots, n - 1;\ k = 3, \cdots, n - 2)$$

have the form

$$(107)\qquad \begin{cases} a_i^{(k;0)} = 1/P_{2,1+i}\ , & (i = 1, \cdots, k - 2) \\ a_{k-2+s}^{(k,0)} = d/P_{2,k-1+s}\ , & (s = 1, \cdots, n - k) \\ a_{n-1}^{(k;0)} = P_{1,1}\ ; & \end{cases}$$

then the $n - 1$ *supporting sequences*

$$b_1^{(k;0)}, b_2^{(k;0)}, \cdots, b_{n-1}^{(k;0)}\ , \qquad (i = 0, \cdots, n - 2)$$

generate a fugue which has the form of the $k + 1$ *-th fugue as demanded* *by Theorem* 3, *and the generating sequence* $a_s^{(k+1;0)}$ $(s = 1, \cdots, n-1)$ *has the form of* (107), *where* k *is to be substituted by* $k+1$.

Proof. In virtue of formula (86), the generating sequence

$$a_1^{(k;0)}, a_2^{(k;0)}, \cdots, a_{n-1}^{(k;0)}$$

has the form as in (107) for $k = 3$. The $n-1$ supporting sequences $b_1(3; 0), b_2(3; 0), \cdots, b(3; 0)$ form the fourth fugue of the period as demanded by Theorem 3. The generating sequence

$$a_1^{(k+1;0)}, a_2^{(k+1;0)}, \cdots, a_{n-1}^{(k+1;0)}$$

too, has the form as in (107) for $k = 3$, in virtue of formula (106). Thus the lemma is correct for $k = 3$. Let it be correct for $k = m$. That means that the n $-$ 1 supporting sequences

$$b_1^{(m;i)}, b_2^{(m;i)}, \cdots, b_{n-1}^{(m;i)}, \qquad (i = 0, \cdots, n-2)$$

form the $m+1$ -th fugue as demanded by Theorem 3, and that the generating sequence

$$a_1^{(m+1;0)}, a_2^{(m+1;0)}, \cdots, a_{n-1}^{(m+1;0)}$$

has the form

$$\tag{108} \begin{cases} a_i^{(m+1;0)} = 1/P_{2,1+i}, & (i = 1, \cdots, m-1) \\ a_{m-1+s}^{(m+1;0)} = d/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-1}^{(m+1;0)} = P_{1,1}. \end{cases}$$

From (108) we obtain, as before,

$$\tag{109} b_s^{(m+1;0)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;0)} = D - D_1$$

and from (108), (109), in virtue of (42),

$$\tag{110} \begin{cases} a_1^{(m+1;0)} - b_1^{(m+1;0)} = 1/P_{2,2}, \\ a_{1+i}^{(m+1;0)} - b_{1+i}^{(m+1;0)} = 1/P_{2,2+i}, & (i = 1, \cdots, m-2) \\ a_{m-1+s}^{(m+1;0)} - b_{m-1+s}^{(m+1;0)} = d/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-1}^{(m+1;0)} - b_{n-1}^{(m+1;0)} = w - D. \end{cases}$$

From (110) we obtain, in virtue of (4)

$$\begin{cases} a_i^{(m+1,1)} = P_{2,2}/P_{2,2+i}, & (i = 1, \cdots, m-2) \\ a_{m-2+s}^{(m+1;1)} = dP_{2,2}/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-2}^{(m+1;1)} = (w-D)P_{2,2}, \\ a_{n-1}^{(m+1;1)} = P_{2,2}\,; \end{cases}$$

or

$$
(111) \qquad \begin{cases} a_i^{(m+1;1)} = 1/P_{3,2+i} , & (i = 1, \cdots, m-2) \\ a_{m-2+s}^{(m+1;1)} = d/P_{3,m+s} , & (s = 1, \cdots, n-m-1) \\ a_{n-2}^{(m+1;1)} = (w - D)P_{2,2} , & \\ a_{n-1}^{(m+1;1)} = P_{2,2} . & \end{cases}
$$

We shall now prove the formula

$$
(112) \qquad \begin{cases} a_i^{(m+1;t)} = 1/P_{t+2,t+1+i} , & (i = 1, \cdots, m-t-1) \\ a_{m-t-1+s}^{(m+1;t)} = d/P_{t+2,m+s} , & (s = 1, \cdots, n-m-1) \\ a_{n-t-2+j}^{(m+1;t)} = (w - D)P_{1+j,t+1} , & (j = 1, \cdots, t) \\ a_{n-1}^{(m+1;t)} = P_{t+1,t+1} , & \\ t = 1, \cdots, m-2 . & \end{cases}
$$

Formula (112) is correct for $t = 1$, in virtue of formula (111). We shall prove that, being correct for t, it is correct for $t+1$. From (112) we obtain, as before,

$$
(113) \qquad b_s^{(m+1;t)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;t)} = D - D_{t+1} ,
$$

and from (112), (113), in virtue of (42)

$$
(114) \qquad \begin{cases} a_1^{(m+1;t)} - b_1^{(m+1;t)} = 1/P_{t+2,t+2} , & \\ a_{1+i}^{(m+1;t)} - b_{1+i}^{(m+1;t)} = 1/P_{t+2,t+2+i} , & (i = 1, \cdots, m-t-2) \\ a_{m-t-1+s}^{(m+1;t)} - b_{m-t-1+s}^{(m+1;t)} = d/P_{t+2;m+s} , & (s = 1, \cdots, n-m-1) \\ a_{n-t-2+j}^{(m+1;t)} - b_{n-t-2+j}^{(m+1,t)} = (w - D)P_{1+j,t+1} , & (j = 1, \cdots, t) \\ a_{n-1}^{(m+1;t)} - b_{n-1}^{(m+1,t)} = w - D . & \end{cases}
$$

From (114) we obtain, in virtue of (4)

$$
(115) \qquad \begin{cases} a_i^{(m+1,t+1)} = P_{t+2,t+2}/P_{t+2,t+2+i} , & (i = 1, \cdots, m-t-2) \\ a_{m-t-2+s}^{(m+1;t+1)} = dP_{t+2,t+2}/P_{t+2,m+S} , & (s = 1, \cdots, n-m-1) \\ a_{n-t-3+j}^{(m+1;t+1)} = (w - D)P_{1+j,t+1}P_{t+2,t+2} , & (j = 1, \cdots, t) \\ a_{n-2}^{(m+1;t+1)} = (w - D)P_{t+2,t+2} , & \\ a_{n-1}^{(m+1;t+1)} = P_{t+2,t+2} , & \end{cases}
$$

and from (115), carrying out the necessary cancellation and multiplication

$$
(116) \qquad \begin{cases} a_i^{(m+1;t+1)} = 1/P_{t+3,t+2+i} , & (i = 1, \cdots, m-t-2) \\ a_{m-t-2+s}^{(m+1;t+1)} = d/P_{t+3,m+s} , & (s = 1, \cdots, n-m-1) \\ a_{n-t-3+j}^{(m+1;t+1)} = (w - D)P_{1+j,t+2} & (j = 1, \cdots, t+1) \\ a_{n-1}^{(m+1;t+1)} = P_{t+2,t+2} . & \end{cases}
$$

With (116) formula (112) is proved. We now obtain from (112), for $t = m-2$,

$$(117)\quad \begin{cases} a_1^{(m+1;m-2)} = 1/P_{m,m}\,, & \\ a_{1+s}^{(m+1;m-2)} = d/P_{m,m+s}\,, & (s = 1, \cdots, n-m-1) \\ a_{n-m+j}^{(m+1;m-2)} = (w-D)P_{1+j,m-1}\,, & (j = 1, \cdots, m-2) \\ a_{n-1}^{(m+1;m-2)} = P_{m-1,m-1}\,; & \end{cases}$$

and from (117), as before,

$$(118)\quad b_s^{(m+1;m-2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-2}^{(m+1;m-2)} = D - D_{m-1}\,.$$

From (117), (118) we obtain, in view of (42)

$$(119)\quad \begin{cases} a_1^{(m+1;m-2)} - b_1^{(m+1;m-2)} = 1/P_{m,m}\,, & \\ a_{1+s}^{(m+1;m-2)} - b_{1+s}^{(m+1;m-2)} = d/P_{m,m+s}\,, & (s = 1, \cdots, n-m-1) \\ a_{n-m+j}^{(m+1;m-2)} - b_{n-m+j}^{(m+1;m-2)} = (w-D)P_{1+j,m-1}\,, & (j = 1, \cdots, m-2) \\ a_{n-1}^{(m+1;m-2)} - b_{n-1}^{(m+1;m-2)} = w - D\,; & \end{cases}$$

and from (119), in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(120)\quad \begin{cases} a_s^{(m+1;m-1)} = d/P_{m+1,m+s}\,, & (s = 1, \cdots, n-m-1) \\ a_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\,, & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} = P_{m,m}\,. & \end{cases}$$

From (120) we obtain, as before,

$$(121)\quad b_s^{(m+1;m-1)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1,m-1)} = D - D_m\,,$$

and from (120), (121), in virtue of (42)

$$\begin{cases} a_s^{(m+1;m-1)} - b_s^{(m+1;m-1)} = d/P_{m+1,m+s}\,, & (s = 1, \cdots, n-m-1) \\ a_{n-m-1+j}^{(m+1;m-1)} - b_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\,, & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} - b_{n-1}^{(m+1;m-1)} = w - D\,; & \end{cases}$$

or

$$(122)\quad \begin{cases} a_1^{(m+1;m-1)} - b_1^{(m+1;m-1)} = d/P_{m+1,m+1}\,, & \\ a_{1+s}^{(m+1;m-1)} - b_{1+s}^{(m+1;m-1)} = d/P_{m+1,m+1+s}\,, & (s = 1, \cdots, n-m-2) \\ a_{n-m-1+j}^{(m+1;m-1)} - b_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\,, & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} - b_{n-1}^{(m+1;m-1)} = w - D\,. & \end{cases}$$

From (122) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication

$$(123)\quad \begin{cases} a_s^{(m+1;m)} = 1/P_{m+2,m+1+s}\,, & (s = 1, \cdots, n-m-2) \\ a_{n-m-1+j}^{(m+1;m)} = (w-D)P_{1+j,m+1}/d\,, & (j = 1, \cdots, m) \\ a_{n-1}^{(m+1;m)} = P_{m+1,m+1}/d\,. & \end{cases}$$

From (123) we obtain, as before,

$$(124) \quad b_s^{(m+1;m)} = 0 \; ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;m)} = \frac{D - D_{m+1}}{d} \, ,$$

and from (123), (124), in virtue of (42),

$$(125) \quad \begin{cases} a_1^{(m+1;m)} - b_1^{(m+1;m)} = 1/P_{m+2,m+2} \, , \\ a_{1+s}^{(m+1;m)} - b_{1+s}^{(m+1;m)} = 1/P_{m+2,m+2+s} \, , & (s = 1, \cdots, n-m-3) \\ a_{n-m-2+j}^{(m+1;m)} - b_{n-m-2+j}^{(m+1;m)} = (w-D)P_{1+j,m+1}/d \, , & (j = 1, \cdots, m) \\ a_{n-1}^{(m+1;m)} - b_{n-1}^{(m+1;m)} = (w-D)/d \, . \end{cases}$$

From (125) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication,

$$(126) \quad \begin{cases} a_s^{(m+1;m+1)} = 1/P_{m+3,m+2+s} \, , & (s = 1, \cdots, n-m-3) \\ a_{n-m-3+j}^{(m+1;m+1)} = (w-D)P_{1+j,m+2}/d \, , & (j = 1, \cdots, m+1) \\ a_{n-1}^{(m+1;m+1)} = P_{m+2,m+2} \, . \end{cases}$$

From (126) we obtain, as before,

$$(127) \quad b_s^{(m+1;m+1)} = 0 \; ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;m+1)} = D - D_{m+2} \, ,$$

and from (126), (127), in virtue of (42)

$$(128) \quad \begin{cases} a_1^{(m+1;m+1)} - b_1^{(m+1;m+1)} = 1/P_{m+3,m+3} \, , \\ a_{1+s}^{(m+1;m+1)} - b_{1+s}^{(m+1;m+1)} = 1/P_{m+3,m+3+s} \, , & (s = 1, \cdots, n-m-4) \\ a_{n-m-3+j}^{(m+1;m+1)} - b_{n-m-3+j}^{(m+1;m+1)} = (w-D)P_{1+j,m+2}/d \, , & (j = 1, \cdots, m+1) \\ a_{n-1}^{(m+1;m+1)} - b_{n-1}^{(m+1;m+1)} = w - D \, . \end{cases}$$

From (128) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(129) \quad \begin{cases} a_s^{(m+1;m+2)} = 1/P_{m+4,m+3+s} \, , & (s = 1, \cdots, n-m-4) \\ a_{n-m-4+j}^{(m+1;m+2)} = (w-D)P_{1+j,m+3}/d \, , & (j = 1, \cdots, m+1) \\ a_{n-2}^{(m+1;m+2)} = (w-D)P_{m+3,m+3} \, , \\ a_{n-1}^{(m+1;m+2)} = P_{m+3,m+3} \, . \end{cases}$$

We shall now prove the formula

$$(130) \quad \begin{cases} a_s^{(m+1;m+k)} = 1/P_{m+k+2,m+k+1+s} \, , & (s = 1, \cdots, n-m-k-2) \\ a_{n-m-k-2+j}^{(m+1;m+k)} = (w-D)P_{1+j,m+k+1}/d \, , & (j = 1, \cdots, m+1) \\ a_{n-k-1+t}^{(m+1;m+k)} = (w-D)P_{m+2+t,m+k+1} \, , & (t = 1, \cdots, k-1) \\ a_{n-1}^{(m+1;m+k)} = P_{m+k+1,m+k+1} \, , \\ k = 2, \cdots, n-m-3 \, . \end{cases}$$

Formula (130) is correct for $k = 2$, in virtue of (129). Presuming it is correct for k, we shall prove its correctness for $k + 1$.

From (130) we obtain, as before,

$$(131)\qquad b_s^{(m+1;m+k)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;m+k)} = D - D_{m+k+1}$$

and from (130), (131), in virtue of (42)

$$(132)\qquad \begin{cases} a_1^{(m+1;m+k)} - b_1^{(m+1;m+k)} = 1/P_{m+k+2,m+k+2}\,, \\ a_{1+s}^{(m+1;m+k)} - b_{1+s}^{(m+1;m+k)} = 1/P_{m+k+2,m+k+2+s}\,, \ (s=1,\cdots,n-m-k-3) \\ a_{n-m-k-2+j}^{(m+1;m+k)} - b_{n-m-k-2+j}^{(m+1;m+k)} = (w-D)P_{1+j,m+k+1}/d\,, \ \ (j=1,\cdots,m+1) \\ a_{n-k-1+t}^{(m+1;m+k)} - b_{n-k-1+t}^{(m+1;m+k)} = (w-D)P_{m+2+t,m+k+1}\,, \ \ (t=1,\cdots,k-1) \\ a_{n-1}^{(m+1;m+k)} - b_{n-1}^{(m+1;m+k)} = w - D\,. \end{cases}$$

From (132) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication

$$\begin{cases} a_s^{(m+1;m+k+1)} = 1/P_{m+k+3,m+k+2+s}\,, & (s = 1, \cdots, n-m-k-3) \\ a_{n-m-k-3+j}^{(m+1;m+k+1)} = (w-D)P_{1+j,m+k+2}/d\,, & (j = 1, \cdots, m+1) \\ a_{n-k-2+t}^{(m+1;m+k+1)} = (w-D)P_{m+2+t,m+k+2}\,, & (t = 1, \cdots, k) \\ a_{n-1}^{(m+1;m+k+1)} = P_{m+k+2,m+k+2}\,, & \end{cases}$$

which is formula (130) with k being replaced by $k+1$; this proves formula (130).

We now obtain from (130) for $k = n - m - 3$

$$(134)\qquad \begin{cases} a_1^{(m+1;n-3)} = 1/P_{n-1;n-1}\,, & \\ a_{1+j}^{(m+1;n-3)} = (w-D)P_{1+j,n-2}/d\,, & (j = 1, \cdots, m+1) \\ a_{m+2+t}^{(m+1;n-3)} = (w-D)P_{m+2+t,n-2}\,, & (t = 1, \cdots, n-m-4) \\ a_{n-1}^{(m+1;n-3)} = P_{n-2,n-2}\,, & \end{cases}$$

and from (134), as before,

$$(135)\qquad b_s^{(m+1;n-3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;n-3)} = D - D_{n-2}\,,$$

From (134), (135) we obtain, in virtue of (42),

$$(136)\qquad \begin{cases} a_1^{(m+1;n-3)} - b_1^{(m+1;n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+j}^{(m+1;n-3)} - b_{1+j}^{(m+1;n-3)} = (w-D)P_{1+j,n-2}/d\,, \ \ (j = 1, \cdots, m+1) \\ a_{m+2+t}^{(m+1;n-3)} - b_{m+2+t}^{(m+1;n-3)} = (w-D)P_{m+2+t,n-2}\,, \ (t = 1, \cdots, n-m-4) \\ a_{n-1}^{(m+1;n-3)} - b_{n-1}^{(m+1;n-3)} = w - D\,, \end{cases}$$

and from (136), in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(137) \quad \begin{cases} a_j^{(m+1;n-2)} = (w-D)P_{1+j,n-1}/d\,, & (j=1,\cdots,m+1) \\ a_{m+1+t}^{(m+1;n-2)} = (w-D)P_{m+2+t,n-1}\,, & (t=1,\cdots,n-m-3) \\ a_{n-1}^{(m+1;n-2)} = P_{n-1,n-1}\,. \end{cases}$$

From (137) we obtain, as before,

$$(138) \quad b_s^{(m+1;n-2)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(m+1;n-2)} = D - D_{n-1}\,,$$

and from (137), (138), in virtue of (4),

$$(139) \quad \begin{cases} a_s^{(m+1;n-2)} - b_1^{(m+1;n-2)} = (w-D)P_{2,n-1}/d\,, & \\ a_{1+j}^{(m+1;n-2)} - b_{1+j}^{(m+1;n-2)} = (w-D)P_{2+j,n-1}/d\,, & (j=1,\cdots,m) \\ a_{m+1+t}^{(m+1;n-2)} - b_{m+1+t}^{(m+1;n-2)} = (w-D)P_{m+2+t,n-1}\,, & (t=1,\cdots,n-m-3) \\ a_{n-1}^{(m+1,n-2)} - b_{n-1}^{(m+1;n-2)} = w-D\,. \end{cases}$$

From (139) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(140) \quad \begin{cases} a_j^{(m+2;0)} = 1/P_{2,1+j}\,, & (j=1,\cdots,m) \\ a_{m+1+t}^{(m+2;0)} = d/P_{2,m+1+t}\,, & (t=1,\cdots,n-m-2) \\ a_{n-1}^{(m+2;0)} = P_{1,1}\,. \end{cases}$$

According to formula (109) (one line of the period), formula (113) ($m-2$ lines of the period), formula (121) (one line of the period), formula (124) (one line of the period), formula (127) (one line of the period), (131), ($n-m-4$ lines of the period) and formula (138) (one line of the period-totally $1+m-2+1+1+1+n-m-4+1=n-1$) the $m-2$ -th fugue has the form as demanded by Theorem 3. Since (140) is formula (107) for $k=m+2$, the Lemma 1 is completely proved.

In view of the Lemma 1 we obtain that the $(n-5)(n-1)$ lines

$$b_1^{(k;0)},\, b_2^{(k,0)},\, \cdots,\, b_{n-1}^{(k;0}\,, \qquad (k=4,\cdots,n-2)$$

form $n-5$ fugues, beginning with the fifth fugue, as demanded by Theorem 3; we further obtain, applying the lemma for $k=n-2$, $k+1=n-1$, that the generating sequence $a_i^{(n-1;0)}$, $(i=1,\cdots,n-1)$ has the form, following (108)

$$(141) \quad \begin{cases} a_i^{(n-1;0)} = 1/P_{2,1+i}\,, & (i=1,\cdots,n-3) \\ a_{n-2}^{(n-1;0)} = d/P_{2,n-1}\,, & \\ a_{n-1}^{(n-1;0)} = P_{1,1}\,. \end{cases}$$

From (141) we obtain, as before,

(142) $\quad b_s^{(n-1;0)} = 0 \;; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;0)} = D - D_1 ,$

and from (141), (142), in virtue of (42)

$$(143) \qquad \begin{cases} a_1^{(n-1;0)} - b_1^{(n-1;0)} = 1/P_{2,2} , \\ a_{1+i}^{(n-1;0)} - b_{1+i}^{(n-1;0)} = 1/P_{2,2+i} , & (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = d/P_{2,n-1} , \\ a_{n-1}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = w - D . \end{cases}$$

From (143) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(144) \qquad \begin{cases} a_i^{(n-1;1)} = 1/P_{3,2+i} , \\ a_{n-3}^{(n-1;1)} = d/P_{3,n-1} , \\ a_{n-2}^{(n-1;1)} = (w - D)P_{2,2} , \\ a_{n-1}^{(n-1;1)} = P_{2,2} . \end{cases}$$

We shall now prove the formula

$$(145) \qquad \begin{cases} a_i^{(n-1;k)} = 1/P_{k+2,k+1+i} , & (i = 1, \cdots, n-k-3) \\ a_{n-k-2}^{(n-1;k)} = d/\, P_{k+2,n-1} , \\ a_{n-k-2+s}^{(n-1;k)} = (w - D)P_{1+s,1+k} , & (s = 1, \cdots, k) \\ a_{n-1}^{(n-1;k)} = P_{k+1,k+1} , \\ k = 1, \cdots, n-4 . \end{cases}$$

In virtue of (144) formula (145) is correct for $k = 1$. We prove, by completely analogous methods used to prove previous, similar formulae that it is correct for $k + 1$, thus verifying its correctness. We now obtain from (145), as before,

(146) $\quad b_s^{(n-1;k)} = 0 \;; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;k)} = D - D_{k+1} ,$

and again from (145), for $k = n - 4$,

$$(147) \qquad \begin{cases} a_1^{(n-1,n-4)} = 1/P_{n-2,n-2} , \\ a_2^{(n-1;n-4)} = d/P_{n-2,n-1} , \\ a_{2+s}^{(n-1;n-4)} = (w - D)P_{1+s,n-3} , & (s = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-4)} = P_{n-3,n-3} . \end{cases}$$

From (147) and (146) (for $k = n - 4$) we obtain, in virtue of (42),

$$(148) \qquad \begin{cases} a_1^{(n-1;n-4)} - b_1^{(n-1;n-4)} = 1/P_{n-2,n-2} , \\ a_2^{(n-1;n-4)} - b_2^{(n-1;n-4)} = d/P_{n-2,n-1} , \\ a_{2+s}^{(n-1;n-4)} - b_{2+s}^{(n-1;n-4)} = (w - D)P_{1+s,n-3} , & (s = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-4)} - b_{n-1}^{(n-1;n-4)} = w - D , \end{cases}$$

and from (148), in virtue of (4), and carrying out the necessary cancellation and multiplication,

$$\text{(149)} \qquad \begin{cases} a_1^{(n-1;n-3)} = d/P_{n-1,n-1} , \\ a_{1+s}^{(n-1;n-3)} = (w-D)P_{1+s,n-2} , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-3)} = P_{n-2,n-2} . \end{cases}$$

From (149) we obtain, as before,

$$\text{(150)} \qquad b_s^{(n-1;n-3)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-3)} = D - D_{n-2} ,$$

and from (149), (150), in virtue of (42)

$$\text{(151)} \qquad \begin{cases} a_1^{(n-1;n-3)} - b_1^{(n-1;n-3)} = d/P_{n-1,n-1} , \\ a_{1+s}^{(n-1;n-3)} - b_{1+s}^{(n-1;n-3)} = (w-D)P_{1+s,n-2} , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-3)} - b_{n-1}^{(n-1;n-3)} = w - D . \end{cases}$$

From (151) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$\text{(152)} \qquad \begin{cases} a_s^{(n-1;n-2)} = (w-D)P_{1+s,n-1}/d , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-1;n-2)} = P_{n-1,n-1}/d , \end{cases}$$

and from (152), as before,

$$\text{(153)} \qquad b_s^{(n-1,n-2)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-2)} = \frac{D - D_{n-1}}{d} .$$

From (152), (153) we obtain, in virtue of (42),

$$\text{(154)} \qquad \begin{cases} a_1^{(n-1;n-2)} - b_1^{(n-1;n-2)} = (w-D)P_{2,n-1}/d , \\ a_{1+s}^{(n-1;n-2)} - b_{1-s}^{(n-1;n-2)} = (w-D)P_{2+s,n-1}/d , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} - b_{n-1}^{(n-1;n-2)} = (w-D)/d , \end{cases}$$

and from (154), in virtue of (4),

$$\begin{cases} a_s^{(n;0)} = P_{2+s,n-1}/P_{2,n-1} = 1/P_{2,1+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(n;0)} = 1/P_{2,n-1} , \\ a_{n-1}^{(n;0)} = d/(w-D)P_{2,n-1} = (w-D)P_{1,n-1}/(w-D)P_{2,n-1} = P_{1,1} . \end{cases}$$

Thus

$$\text{(155)} \qquad \begin{cases} a_s^{(n;0)} = 1/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n;0)} = P_{1,1} . \end{cases}$$

Comparing (40) with (155) we see that

$$a_s^{(n;0)} = a_s^{(0)} , \qquad (s = 1, \cdots, n-1)$$

i.e.,

$$a_s^{(n(n-1))} = a_s^{(0)} \qquad (s = 1, \cdots, n-1) \tag{156}$$

which proves that, in case $d \neq 1$, the Jacobi-Perron Algorithm of the basic sequence $a_s^{(0)}$ $(s = 1, \cdots, n-1)$ from (34) is purely periodic and its length $T = (n-1)n$. Since, in virtue of (142), (146), (150), (153) the $n-1$ supporting sequences

$$b_1^{(n-1;k)}, b_2^{(n-1;k)}, \cdots, b_{n-1}^{(n-1;k)} \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which is the n -th fugue of the period, we see that this last fugue, together with the $4 + (n-5) = n-1$ preceding ones form the n fugues of the period, as demanded by Theorem 3.

In case $d = 1$, we obtain from (55)

$$a_s^{(1;0)} = 1/P_{2,1+s} , \quad (s = 1, \cdots, n-2) ; \qquad a_{n-1}^{(1;0)} = P_{1,1} , \tag{157}$$

so that, comparing (157) with (40), we obtain

$$a_s^{(n-1)} = a_s^{(0)} , \qquad (s = 1, \cdots, n-1) \tag{158}$$

so that the length of the period is here $T = n-1$; from (41), (44), (48) (54) we obtain that in the case $d = 1$ the period has the form as demanded by Theorem 3.

The reader should note that proving case $d \neq 1$ we presumed $n \geqq 6$. The special cases $n = 2, 3, 4, 5$ are proved analogously.

We shall now give a few numeric examples. Let the generating polynomial be

$$f(x) = x^5 - 15x^4 + 54x^3 - 3 = 0 ,$$

which can be easily rearranged into

$$f(x) = (x-9)(x-6)x^3 - 3 = 0$$

and has the form (17) with

$$D = 9, \ D_1 = 6, \ D_2 = D_3 = D_4 = 0 ; \ d = 3 ;$$
$$9 < w < 10 ; \ (w-9)(w-6)w^3 - 3 = 0 .$$

The Jabobi-Perron Algorithm of the basic sequence

$$\frac{(w-9)(w-6)w^2}{3}, \frac{(w-9)(w-6)w}{3}, \frac{(w-9)(w-6)}{3}, w-6$$

or

$$\frac{w^4-15w^3+54w^2}{3}, \frac{w^3-15w^2+54w}{3}, \frac{w^2-15w+54}{3}, w-6$$

is purely periodic with period length $T = 20$. The period has the form

$$\begin{array}{cccc}
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\[4pt]
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\[4pt]
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\[4pt]
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\[4pt]
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 3 .
\end{array}$$

Let the generating polynomial be

$$f(x) = x^6 - 3x^5 - 5x^4 + 15x^3 + 4x^2 - 12x - 1 = 0 ,$$

which is easily rearranged into

$$f(x) = (x-3)(x-2)(x-1)x(x+1)(x+2) - 1 = 0$$

and has the form (17) with

$$D = 3,\ D_1 = 2,\ D_2 = 1,\ D_3 = 0,\ D_4 = -1,\ D_5 = -2\,;\ d = 1\,;$$
$$3 < w < 4\,,$$
$$(w-3)(w-2)(w-1)(w+1)(w+2)w - 1 = 0\,.$$

The Jacobi-Perron algorithm of the basic sequence

$$\begin{aligned}
a_1^{(0)} &= (w-3)(w-2)w(w+1)(w+2) &&= w^5 - 2w^4 - 7w^3 + 8w^2 + 12w\,,\\
a_2^{(0)} &= (w-3)(w-2)(w+1)(w+2) &&= w^4 - 2w^3 - 7w^2 + 8w + 12\,,\\
a_3^{(0)} &= (w-3)(w-2)(w+2) &&= w^3 - 3w^2 - 4w + 12\,,\\
a_4^{(0)} &= (w-3)(w-2) &&= w^2 - 5w + 6\,,\\
a_5^{(0)} &= w-2 &&= w-2\,,
\end{aligned}$$

is purely periodic and the period length is $T = 5$. The period has the form

$$\begin{matrix}
0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 0 & 3\\
0 & 0 & 0 & 0 & 4\\
0 & 0 & 0 & 0 & 5\,.
\end{matrix}$$

Let the generating polynomial be

$$f(x) = x^3 - 16x - 2 = 0\,,$$

which is easily rearranged into

$$f(x) = (x-4)x(x+4) - 2 = 0$$

and has the form (17) with

$$D = 4,\ D_1 = 0,\ D_2 = -4\,;\ d = 2\,;$$
$$4 < w < 5\,,$$
$$(w-4)(w+4)w - 2 = 0\,.$$

The Jacobi-Perron algorithm of the basic sequence

$$\frac{(w-4)w}{2} = \frac{w^2-4w}{2}\,,\ w$$

is purely periodic and the period length $T = 6$; the period has the form

$$\begin{matrix}
0 & 4\\
0 & 8\\
0 & 2\\
0 & 8\\
0 & 4\\
0 & 4\,.
\end{matrix}$$

6. The Jacobi-perron algorithm for polynomial of increasing order. In this section we shall show that, by imposing further conditions on the coefficients of the GP from (17), one can select increasing order polynomials from the algebraic number field $K(w)$ generated by $f(w) = 0$, $D < w < D + 1$, such that their Jacobi-Perron algorithm is purely periodic. This result is stated in

THEOREM 4. *Let the coefficients of the GP in addition to* (17) *fulfil the inequalities $D - D_i \geqq 2d(n-1)$, i.e., altogether*

$$\begin{aligned} &D,\ D_i,\ d \text{ rational integers}\ ;\ d \geqq 1\ ;\ n \geqq 2\ ; \\ &d \mid (D - D_i)\ ;\ D - D_i \geqq 2d(n-1)\ ; \qquad (i = 1, 2, \cdots, n-1)\ ; \end{aligned} \tag{159}$$

Let w be the only real root in the open interval $(D\ ;\ D+1)$. Then the Jacobi-Perron algorithm of the basic sequence

$$\begin{aligned} &a_i^{(0)}(w) = \textstyle\sum_{j=0}^{i} k_s (w - D)^{i-s}\ , \qquad (s = 1, \cdots, n-1)\ ;\ k_s = 1\ ; \\ &k_s = \textstyle\sum (D - D_{j_1})(D - D_{j_2}) \cdots (D - D_{j_s})\ , \quad (s = 1, \cdots, n-1)\ , \\ &1 \leqq j_1 < j_2 < \cdots < j_s \leqq n-1 \end{aligned} \tag{160}$$

is purely periodic and its length $T = n$ for $d > 1$, and $T = 1$ for $d = 1$. The period has the form

$$\begin{aligned} &b_i^{(0)} = k_i \qquad\qquad (i = 1, \cdots, n-1)\ ; \\ &\begin{cases} b_i^{(s)} = k_i & (i = 1, \cdots, n-1-s)\ , \\ b_i^{(s)} = k_i/d & (i = n-s, \cdots, n-1\ ;\ s = 1, \cdots, n-2)\ ; \end{cases} \\ &b_i^{(n-1)} = k_i/d \qquad\qquad (i = 1, \cdots, n-1)\ ; \\ &d > 1\ . \end{aligned} \tag{161}$$

$$b_i^{(0)} = k_i \qquad (i = 1, \cdots, n-1)\ ;\ d = 1\ . \tag{161a}$$

Proof. This is essentially based on the simple formula

$$[a_i^{(0)}(w)] = k_i \qquad (i = 1, \cdots, n-1)\ . \tag{162}$$

Since, as will be proved later, w is irrational under the conditions (159), we have to verify the two inequalities

$$k_i < a_i^{(0)}(w) < k_i + 1 \qquad (i = 1, \cdots, n-1)\ , \tag{163}$$

or, in virtue of (160)

$$0 < (w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D) < 1\ . \tag{164}$$

The left-hand inequality of (164) follows from $w > D$ and $k_i > 0$. We shall prove the right-hand inequality

$$(w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D) < 1\ . \tag{165}$$

Since $0 < w - D < 1$, we obtain $(w - D)^i \leqq w - D$, and we shall prove, since

$$\begin{aligned}&(w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D)\\ &\leqq (w - D) + k_1(w - D) + \cdots + k_{i-1}(w - D)\,,\end{aligned}$$

$$(w - D)(1 + k_1 + k_2 + \cdots + k_{i-1}) < 1\,. \tag{166}$$

From $w > D$, $(w - D)(w - D_1) \cdots (w - D_{n-1}) - d = 0$, we obtain

$$\begin{aligned}w - D &= d/((w - D_1)(w - D_2) \cdots (w - D_{n-1}))\\ &< d/((D - D_1(D - D_2) \cdots (D - D_{n-1}))\,.\end{aligned} \tag{167}$$

We shall now prove the inequality

$$k_s(w - D) < 2^{-(n-1-s)}\,, \qquad (s = 1, \cdots, n-2)\,. \tag{168}$$

Let the D_i be arranged in nondecreasing order, so that

$$D - D_1 \geqq D - D_2 \geqq \cdots \geqq D - D_{n-1}\,. \tag{169}$$

In virtue of (169), and taking into account the values of k_s from (160) we obtain

$$\begin{aligned}&k_s(w - D) \leqq (w - D) \sum (D - D_1)(D - D_2) \cdots (D - D_s)\\ &= \binom{n-1}{s}(w - D)(D - D_1)(D - D_2) \cdots (D - D_s)\\ &< \frac{\binom{n-1}{s}(D - D_1)(D - D_2) \cdots (D - D_s)d}{(D - D_1)(D - D_2) \cdots (D - D_{n-1})}\,,\end{aligned}$$

in virtue of (17). Therefore

$$k_s(w - D) < \frac{\binom{n-1}{s}d}{(D - D_{s+1})(D - D_{s+2}) \cdots (D - D_{n-1})}\,. \tag{170}$$

But $D - D_i \geqq 2d(n-1)$; therefore we obtain from (170)

$$\begin{aligned}&k_s(w - D) < \frac{\binom{n-1}{s}d}{(2d(n-1))^{n-s-1}}\\ &= \frac{\binom{n-1}{s}}{2^{n-s-1}(n-1)^{n-s-1}d^{n-2-s}} \leqq \frac{\binom{n-1}{s}}{2^{n-s-1}(n-1)^{n-s-1}} = \frac{\binom{n-1}{n-s-1}}{2^{n-s-1}(n-1)^{n-s-1}}\\ &= \frac{1}{2^{n-s-1}} \cdot \frac{n-1}{n-1} \cdot \frac{n-2}{2(n-1)} \cdot \cdots \cdot \frac{s+1}{(n-s-1)(n-1)} \leqq \frac{1}{2^{n-s-1}}\,,\end{aligned}$$

which proves formula (168).

We further obtain from (167)

$$w - D < \frac{d}{(2d(n-1))^{n-1}} = \frac{1}{2^{n-1}(n-1)^{n-1}d^{n-2}} \leqq \frac{1}{2^{n-1}} .$$

In virtue of this result and of (168), we now obtain from (166)

$$(w - D)(1 + k_1 + \cdots + k_{i-1}) < \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^{n-i}}$$

$$\leqq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2} = 1 - \frac{1}{2^{n-1}} < 1 .$$

Thus (162) is proved.

In virtue of (163), we obtain the inequalities

$$\frac{k_i}{d} < \frac{a_i^{(0)}(w)}{d} < \frac{k_i + 1}{d} \leqq \frac{k_i}{d} + 1 ,$$

so that

$$\left[\frac{a_i(w)}{d}\right] = \frac{k_i}{d} . \tag{171}$$

(162) and (171) provide the key to our proof of Theorem 4. The further course of the proof is similar to methods used in previous papers ([10], [12]) and we shall, therefore, give here only a very general outline of same. Denoting in the sequel

$$a_i^{(0)}(w) = a_i^{(0)} , \qquad (i = 1, \cdots, n-1) \tag{173}$$

we obtain from (160), (162)

$$a_{i+1}^{(0)} = (w - D)a_i^{(0)} + k_{i+1} , \qquad (i = 0, \cdots, n-2)a_0^{(0)} = 1 ,$$

$$a_{i+1}^{(0)} - b_{i+1}^{(0)} = a_{i+1}^{(0)} - k_{i+1} ,$$

$$a_{i+1}^{(0)} - b_{i+1}^{(0)} = (w - D)a_i^{(0)} \qquad (i = 0, \cdots, n-2) . \tag{174}$$

We further obtain from (24), for $f(w) = 0$,

$$(w - D)^n + k_1(w - D)^{n-1} + k_2(w - D)^{n-2} + \cdots + k_{n-1}(w - D) - d = 0 ,$$

$$\frac{1}{w - D} = \frac{(w - D)^{n-1} + k_1(w - D)^{n-2} + \cdots + k_{n-1}}{d} = \frac{a_{n-1}^{(0)}}{d} ;$$

since, from (174), $a_1^{(0)} - b_1^{(0)} = w - D$, we obtain

$$\frac{1}{a_1^{(0)} - b_1^{(0)}} = \frac{a_{n-1}^{(0)}}{d} . \tag{175}$$

We shall now carry out the Jacobi-Perron algorithm of the basic sequence (160) and obtain from (162)

$$b_s^{(0)} = k_s\,, \qquad (s = 1, \cdots, n-1) \tag{176}$$

and from (174), (175), in virtue of (4)

$$\begin{cases} a_1^{(0)} - b_1^{(0)} = w - D\,, \\ a_{i+1}^{(0)} - b_{i+1}^{(0)} = (w - D)a_i^{(0)}\,, & (i = 1, \cdots, n-2) \end{cases}$$

$$\begin{cases} a_i^{(1)} = a_i^{(0)}\,, & (i = 1, \cdots, n-2) \\ a_{n-1}^{(1)} = a_{n-1}^{(0)}/d\,. \end{cases} \tag{177}$$

From (177) we obtain, in virtue of (162), (171)

$$b_s^{(1)} = k_s\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = k_{n-1}/d\,, \tag{178}$$

and from (177), (178), in virtue of (174), (175)

$$\begin{cases} a_1^{(1)} - b_1^{(1)} = w - D\,, \\ a_{1+i}^{(1)} - b_{1+i}^{(1)} = (w - D)a_i^{(0)}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = (w - D)a_{n-2}^{(0)}/d\,, \end{cases}$$

$$\begin{cases} a_i^{(2)} = a_i^{(0)}\,, & (i = 1, \cdots, n-3) \\ a_{n-2}^{(2)} = a_{n-2}^{(0)}/d\,, \\ a_{n-1}^{(2)} = a_{n-1}^{(0)}/d\,. \end{cases} \tag{178a}$$

It will now be easy to prove formula

$$\begin{cases} a_i^{(s)} = a_i^{(0)}\,, & (i = 1, \cdots, n-s-1) \\ a_{n-s-1+j}^{(s)} = a_{n-s-1+j}^{(s)}/d\,, & (j = 1, \cdots, s) \\ s = 1, \cdots, n-2\,. \end{cases} \tag{179}$$

Formula (179) is correct for $s = 1, 2$ in virtue of formulas (177), and (178a). It is then presumed that it is correct for $s = m$ and proved that it is correct for $s = m + 1$.

We now obtain from (179), in virtue of (162), (171)

$$\begin{aligned} & b_i^{(s)} = k_i\,; && (i = 1, \cdots, n-s-1) \\ & b_{n-s-1+j}^{(s)} = \frac{k_{n-s-1+j}}{d}\,; && (j = 1, \cdots, s) \\ & s = 1, \cdots, n-2\,. \end{aligned} \tag{180}$$

We further obtain from (179), (180) for $s = n - 2$

$$\begin{aligned} & a_1^{(n-2)} = a_1^{(0)}\,; \quad a_{1+j}^{(n-2)} = a_{1+j}^{(0)}/d\,; && (j = 1, \cdots, n-2) \\ & b_1^{(n-2)} = k_1\,; \quad b_{1+j}^{(n-2)} = k_{1+j}/d\,, && (j = 1, \cdots, n-2) \end{aligned}$$

so that, in virtue of (174), (175), (4)

$$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = w - D\,, \\ a_{1+j}^{(n-2)} - b_{1+j}^{(n-2)} = (w - D)a_j^{(0)}/d\,, \qquad (j = 1, \cdots, n-2) \end{cases}$$

$$a_i^{(n-1)} = a_i^{(0)}/d \qquad (i = 1, \cdots, n-1)\,. \tag{181}$$

From (181) we obtain, in virtue of (171)

$$b_i^{(n-1)} = k_i/d\,, \qquad (i = 1, \cdots, n-1) \tag{182}$$

and from (181), (182), in virtue of (174)

$$\begin{cases} a_1^{(n-1)} - b_1^{(n-1)} = (w - D)/d\,, \\ a_{1+i}^{(n-1)} - b_{1+i}^{(n-1)} = (w - D)a_i^{(0)}/d\,, \qquad (i = 1, \cdots, n-2) \end{cases}$$

so that, in virtue of (4) and (175)

$$a_i^{(n)} = a_i^{(0)}\,, \qquad (i = 1, \cdots, n-1) \tag{183}$$

which proves that the Jacobi-Perron algorithm of the basic sequence $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ is purely periodic and its length $T = n$ for $d > 1$. We further obtain from (177), for $d = 1$,

$$a_i^{(1)} = a_i^{(0)}\,, \qquad (i = 1, \cdots, n-1)$$

so that in this case the Jacobi-Perron algorithm is purely periodic and its length $T = 1$.

From (176), (180), (182) we conclude that the period of the algorithm has the form as demanded by Theorem 4, for $d \geqq 1$.

We shall take up the numeric examples of § 5 to illustrate Theorem 4.

1. $\qquad f(x) = x^5 - 15x^4 + 54x^3 - 3 = (x - 9)(x - 6)x^3 - 3 = 0\,.$

Developing $f(x)$ in powers of $x - 9$ we obtain

$$\begin{aligned} f(x) = (x - 9)^5 + 30(x - 9)^4 + 324(x - 9)^3 \\ + 1458(x - 9)^2 + 2187(x - 9) - 3 = 0\,. \end{aligned}$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w - 9) + 30 = w + 21\,; \\ a_2^{(0)} &= (w - 9)^2 + 30(w - 9) + 324 = w^2 + 12w + 135\,; \\ a_3^{(0)} &= (w - 9)^3 + 30(w - 9)^2 + 324(w - 9) + 1458 \\ &= w^3 + 3w^2 + 27w + 243\,; \\ a_4^{(0)} &= (w - 9)^4 + 30(w - 9)^3 + 324(w - 9)^2 + 1458(w - 9) + 2187 \\ &= w^4 - 6w^3\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

$$\begin{array}{cccc} 30 & 324 & 1458 & 2187 \\ 30 & 324 & 1458 & 729 \\ 30 & 324 & 486 & 729 \\ 30 & 108 & 486 & 729 \\ 10 & 108 & 486 & 729\,. \end{array}$$

2. $$\begin{aligned} f(x) &= x^6 - 3x^5 - 5x^4 + 15x^3 + 4x^2 - 12x - 1 \\ &= (x-3)(x-2)(x-1)x(x+1)(x+2) - 1 = 0\,. \end{aligned}$$

Developing $f(x)$ in powers of $x - 3$ we obtain

$$\begin{aligned} f(x) = (x-3)^6 + 15(x-3)^5 + 85(x-3)^4 + 225(x-3)^3 \\ + 274(x-3)^2 + 120(x-3) - 1 = 0\,. \end{aligned}$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w-3) + 15 = w + 12\,; \\ a_2^{(0)} &= (w-3)^2 + 15(w-3) + 85 = w^2 + 9w + 49\,; \\ a_3^{(0)} &= (w-3)^3 + 15(w-3)^2 + 85(w-3) + 225 \\ &= w^3 + 6w^2 + 22w + 78\,; \\ a_4^{(0)} &= (w-3)^4 + 15(w-3)^3 + 85(w-3)^2 + 225(w-3) + 274 \\ &= w^4 + 3w^3 + 94w^2 - 258w + 40\,; \\ a_5^{(0)} &= (w-3)^5 + 15(w-3)^4 + 85(w-3)^3 + 225(w-3)^2 \\ &\quad + 274(w-3) + 120 = w^5 - 5w^3 + 4w\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

$$15 \quad 85 \quad 225 \quad 274 \quad 120\,.$$

3. $$f(x) = x^3 - 16x - 2 = (x-4)x(x+4) - 2 = 0\,.$$

Developing $f(x)$ in powers of $x - 4$ we obtain

$$f(x) = (x-4)^3 + 12(x-4)^2 + 32(x-4) - 2 = 0\,.$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w-4) + 12 = w + 8\,; \\ a_2^{(0)} &= (w-4)^2 + 12(w-4) + 32 = w^2 + 4w\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

12	32
12	16
6	16 .

We shall now return to formula (11) in order to calculate w and obtain for Theorem 3 :

$$a_{n-1}^{(0)} = w - D_1 = \lim_{v\to\infty} (A_{n-1}^{(v)}/A_0^{(v)}) \ ,$$

for Theorem 4 :

$$a_1^{(0)} = w - D + k_1 = \lim_{v\to\infty} (A_1^{(v)}/A_0^{(v)}) \ ,$$

where the $A_0^{(v)}$, $A_{n-1}^{(v)}$ from Theorem 3 are not the same as $A_0^{(v)}$, $A_1^{(v)}$ from Theorem 4. Yet, as the first author has proved, there are always indices v_3 for the $A_i^{(v)}$ from Theorem 3 and indices v_4 for the $A_i^{(v)}$ from Theorem 4 such that

$$A_i^{(v_3)} = A_i^{(v_4)} \ .$$

7. **Units of the field $K(w)$.** Let the coefficients of the GP

$$f(x) = (x - D_0)(x - D_1) \cdots (x - D_{n-1}) - d$$

now fulfil the conditions (17), (18), (19) from Theorems 1, 2 and the supplementary inequalities from Theorem 3, i.e., altogether

$$\begin{gathered} D_i,\ d \text{ rational integers}\ ;\ d \geqq 1\ ;\ n \geqq 2\ ; \\ D_0 > D_1 > \cdots > D_{n-1}\ ;\ d \mid (D_0 - D_i)\ , \\ D_0 - D_i \geqq 2d(n-1)\ ,\ (i = 1, \cdots n-1)\ ; \\ \text{and in the special case } d = 1 \text{ moreover} \\ D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4 \text{ for } n = 3\ , \\ D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \ngeqq 3 \text{ or} \\ D_0 - D_1,\ D_2 - D_3 > 2 \text{ for } n = 4\ , \end{gathered} \tag{184}$$

and let be

$$f(w) = (w - D_0)(w - D_1) \cdots (w - D_{n-1}) - d = 0\ ; \qquad D_0 < w < D_0 + 1\ . \tag{185}$$

Perron [18] has proved the following important theorem :

If the supporting sequences of the Jacobi-Perron algorithm fulfil the conditions

$$b_{n-1}^{(v)} \geqq n + b_1^{(v)} + b_2^{(v)} + \cdots b_{n-2}^{(v)}\ , \qquad (v = 0, 1, \cdots) \tag{186}$$

then $f(w)$ is irreducible in the rational number field.

We shall apply Theorem 3. Here

$$b_1^{(v)} = b_2^{(v)} = \cdots = b_{n-2}^{(v)} = 0 \; ; \; b_{n-1}^{(v)} = D_0 - D_i \text{ or } \frac{D_0 - D_i}{d} \; .$$

In order to verify (186), we thus have to prove $D_0 - D_i \geqq nd$. But in virtue of (184) we have, indeed,

$$D_0 - D_i \geqq 2d(n-1) \geqq nd, \text{ since } n \geqq 2 \; , \quad (i = 1, \cdots, n-1) \; .$$

Thus $f(w)$ is irreducible in the field of rational numbers, which is true already under the conditions (159), and w, as well as the other roots of $f(x)$ are algebraic irrationals of degree n. Thus, in virtue of Theorem 2 and the conditions (184), $f(x)$ has n different real roots which are all algebraic irrationals of degree n. According to the famous Dirichlet theorem, the exact number of (independent) basic units of the field $K(w)$ is $N = r_1 + r_2 - 1$, where

r_1 is the number of real roots of $f(x)$,

r_2 is the number of pairs of conjugate complex roots of $f(x)$.

In our case $r_1 = n$; $r_2 = 0$, so that $N = n - 1$. We shall now prove

THEOREM 5. *Under the conditions* (184) *the n algebraic irrationals*

$$e_k = \frac{(w - D_k)^n}{d} \, , \qquad (k = 0, 1, \cdots, n-1) \tag{188}$$

are n different units of the field $K(w)$.

That the numbers (188) are all different follows from $D_i \neq D_j$, $(i \neq j \, ; i, j = 0, 1, \cdots, n-1)$. We further note that one of the numbers (188), for instance

$$e_{n-1} = (w - D_{n-1})^n / d$$

can be expressed by the other $n - 1$ numbers. We obtain from (185)

$$d/(w - D_{n-1}) = (w - D_0)(w - D_1) \cdots (w - D_{n-2}) \; ,$$
$$d^n/(w - D_{n-1})^{+n} = (w - D_0)^n (w - D_1)^n \cdots (w - D_{n-2})^n \; ,$$

and from this

$$\frac{d}{(w - D_{n-1})^n} = \frac{(w - D_0)^n}{d} \cdot \frac{(w - D_1)^n}{d} \cdots \frac{(w - D_{n-2})^n}{d} \; ,$$

so that

$$e_{n-1}^{-1} = e_0 e_1 \cdots e_{n-2} \,. \tag{189}$$

There is a simple algebraic method to prove that the e_k are all units (see the Appendix by H. Hasse); for this purpose, in view of (189), it suffices to show that the e_k are algebraic integers. This, however, does not disclose the more organic connection between a unit of a field and the periodic algorithm of a basis of the field; after a unit of a field has been found by some device, it is easy to verify that it is one, indeed. The problem of calculating a unit in a quadratic field $K(\sqrt{m})$ is entirely solved by developing $\sqrt{m}$ in a periodic continuous fraction by Euclid's algorithm.

In a joint paper with Helmut Hasse [16] it was proved that in the case of a periodic Jacobi-Perron algorithm carried out on a basis $w, w^2, \cdots, w^{n-1}$ of an algebraic field $K(w)$, $w = (D^n + d)^{1/n}$; d, D natural numbers, $d \mid D$, a unit of the field is given by the formula

$$e^{-1} = a_{n-1}^{(S)} a_{n-1}^{(S+1)} \cdots a_{n-1}^{(S+T-1)} \,, \tag{190}$$

where S and T (see (6)) denote the length of the preperiod and the period of the algorithm respectively.[1]

Turning to Theorem 3, we obtain $S = 0$, $T = n(n-1)$ for $d \neq 1$, and formula (190) takes the form

$$e^{-1} = \prod_{v=0}^{n(n-1)-1} a_{n-1}^{(v)} = \prod_{i=0}^{n-1} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} \,. \tag{191}$$

Following up the various stages of the proof of Theorem 3, one can easily varify the relations

$$\prod_{k=0}^{n-2} a_{n-1}^{(k)} = P_{1,1} P_{2,2} \cdots P_{n-1,n-1} \,, \tag{192}$$

$$\prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)-k} = d^{-1} P_{1,1} P_{2,2} \cdots P_{n-1,n-1}, \qquad (i = 1, \cdots n-1) \,. \tag{193}$$

In virtue of (192), (193) we obtain from (191)

$$e^{-1} = d^{-(n-1)} (P_{1,1} P_{2,2} \cdots P_{n-1,n-1})^n \,. \tag{194}$$

From (39) we obtain

$$P_{1,1} P_{2,2} \cdots P_{n-1,n-1} = \frac{d}{w - D} \,, \tag{195}$$

and from (194), (195)

$$e^{-1} = \frac{d}{(w-D)^n} \,, \quad e = \frac{(w-D)^n}{d} \,, \tag{196}$$

[1] Formula (190) holds for any algebraic irrational w.

which proves Theorem 5 for $k = 0$, since $D = D_0$. Yet it is rather complicated to prove the remaining statement of Theorem 5, namely that the other e_k $(k = 1, \cdots, n-2)$ are units of $K(w)$ which can be derived from a periodic algorithm like e_0. We say deliberately periodic algorithm and not periodic Jacobi-Perron algorithm, which has its good reasons in the following observation: if one reads the author's joint paper with Professor Helmut Hasse carefully enough, he will soon realize that in order to prove formula (190) two presumptions are necessary—first that the numbers $b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}$ $(v = 0, 1, \cdots)$ be all integers; second that the algorithm be periodic, while the formation law by which the $b_i^{(v)}$ are derived from the $a_i^{(v)}$ is altogether not essential. In this chapter we shall define a new formation law for the $b_i^{(v)}$ and obtain, on ground of it, a periodic algorithm for $n-1$ polynomials chosen from the field $K(w)$. In this algorithm the $b_i^{(v)}$ will all be rational integers so that formula (190) can be applied. These results are laid down in Theorem 6. Before we state this theorem, we shall explain the new formation law for the $b_i^{(v)}$ and introduce, to this end, a few more notations.

DEFINITION. Let w be the only real root in the open interval $(D_0, D_0 + 1)$ of equation (185), so that

$$(w - D_0)(w - D_1) \cdots (w - D_{n-1}) - d = 0 .$$

Let the elements of the basic sequence of an algorithm G be polynomials in w with rational coefficients, i.e.,

$$a_s^{(0)} = a_s^{(0)}(w) = \textstyle\sum_{i=0}^{s} C_i w^{s-i} \qquad (s = 1, \cdots, n-1)\,; \tag{197}$$

if the $b_s^{(v)}$ $(s = 1, \cdots, n-1;\ v = 0, 1, \cdots)$ are rationals, then, in virtue of (4), the $a_s^{(v)}$, too, are polynomials in w with rational coefficients for all s, v, i.e.,

$$a_s^{(v)} = a_s^{(v)}(w) = \textstyle\sum_{i=0}^{s} C_i^{(v)} w^{s-i} \quad (s = 1, \cdots, n-1\,;\ v = 0, 1, \cdots) \tag{198}$$

G is called the Modified Algorithm of Jacobi-Perron, if the $b_i^{(v)}$ are obtained from the $a_i^{(v)}$ by the formation law

$$b_s^{(v)} = a_s^{(v)}(D_k) \qquad (s, v \text{ as in (198)})\,. \tag{199}$$

Here D_k is one of the numbers $D_0, D_1, \cdots, D_{n-1}$; D_k remains the same during the process of G.

We shall now introduce the following notations

$$
(200)\quad \begin{aligned}
&R_{i,i} = w - D_{i,i}\,;\ D_{i,i} \text{ any of the numbers } D_0, \cdots, D_{n-1}\,;\\
&R_{i,i} \neq R_{j,j} \text{ for } i \neq j\,;\\
&R_{i,j} = R_{i,i}R_{i+1,i+1}\cdots R_{j,j}\,, \qquad (0 \leqq i \leqq j \leqq n-1).
\end{aligned}
$$

From (185) and (200) we obtain

$$
(201)\quad \begin{aligned}
R_{0,n-1} &= d\,;\\
1/R_{i,j} &= R_{0,i-1}R_{j+1,n-1}/d\,, && (0 < i \leqq j < n-1)\\
1/R_{0,j} &= R_{j+1,n-1}/d\,, && (0 \leqq j < n-1)\\
1/R_{i,n-1} &= R_{0,i-1}/d\,, && (0 < i \leqq n-1)\,.
\end{aligned}
$$

We are now able to state

THEOREM 6. *Under the conditions* (186) *let*

$$
(202)\qquad R_{1,1},\, R_{2,2},\, \cdots,\, R_{n-2,n-2}
$$

be any $n-2$ *of the* $n-1$ *polynomials*

$$
(203)\qquad P_{0,0},\, \cdots,\, P_{k-1,k-1},\, P_{k+1,k+1},\, \cdots\, P_{n-1,n-1}\,; \quad (k = 1, \cdots, n-2)
$$

then the Modified Algorithm of Jacobi-Perron of the basis

$$
(204)\qquad a_i^{(0)} = R_{1,n-1-i}P_{k,k}\,; \quad (i = 1, \cdots, n-2) \qquad a_{n-1}^{(0)} = R_{1,1}
$$

is purely periodic; the length of the period is $T = n(n-1)$ *for* $d > 1$ *and* $T = n-1$ *for* $d = 1$. *The period of length* $T = n-1$ *consists of one fugue; its generator has the form*

$$
(205)\qquad \begin{cases}
D_k - D_{1,1}\\
D_k - D_{n-1,n-1}\\
D_k - D_{n-2,n-2}\\
\cdots\cdots\cdots\cdots\\
D_k - D_{2,2}\,.
\end{cases}
$$

The period of length $n(n-1)$ *consists of* n *fugues; the generator of the first fugue has the form*

$$
(206a)\qquad \begin{cases}
D_k - D_{1,1}\\
(D_k - D_{n-1,n-1})d^{-1}\\
D_k - D_{n-2,n-2}\\
D_k - D_{n-3,n-3}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
D_k - D_{2,2}
\end{cases}
$$

The generator of the i *-th fugue* $(i = 2, \cdots, n-3)$ *has the form*

$$(206b)\qquad \begin{cases} D_k - D_{1,1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ D_k - D_{n-(i-1),n-(i-1)} \\ (D_k - D_{n-i,n-i})d^{-1} \\ D_k - D_{n-(i+1),n-(i+1)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ D_k - D_{2,2}\ . \end{cases}$$

The generator of the $n-2$ *-th fugue has the form*

$$(206c)\qquad \begin{cases} D_k - D_{1,1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots\cdots \\ D_k - D_{3,3} \\ (D_k - D_{2,2})d^{-1}\ . \end{cases}$$

The generator of the $n-1$ *-th fugue has the form* (205) ; *the generator of the* n *-th fugue has the form*

$$(206d)\qquad \begin{cases} (D_k - D_{1,1})d^{-1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots\cdots \\ D_k - D_{2,2}\ . \end{cases}$$

The reader should note that the generators (205) and (206a)-(206d) consist of rational integers only. The differences $D_k - D_{i,i}$ $(i = 1, \cdots, n-1)$ are algebraic sums of natural numbers ; and since $d \mid D_k$, $d \mid D_{i,i}$, so is $d \mid D_k - D_{i,i}$. One further notes that these generators contain no zeros, since $P_{k,k} \neq R_{i,i}$ and therefore $D_k \neq D_{i,i}$, $(i = 1, \cdots, n-1)$.

Proof of Theorem 6. We first make the following observation : since, in virtue of (202), (203), we can have either

$$P_{k,k} = R_{0,0} \text{ or } P_{k,k} = R_{n-1,n-1}$$

we shall choose

$$(207)\qquad P_{k,k} = R_{0,0}\ .$$

We shall now carry out the Modified Algorithm of Jacobi-Perron for the basic sequence (204). We obtain from (204), since every factor $a_i^{(0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$b_i^{(0)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(0)} = D_k - D_{1,1} \, . \tag{208}$$

From (204), (208) we obtain, since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$

$$\begin{cases} a_1^{(0)} - b_1^{(0)} = R_{1,n-2}P_{k,k} \, , \\ a_{1+i}^{(0)} - b_{1+i}^{(0)} = R_{1,n-2-i}P_{k,k} \, , & (i = 1, \cdots, n-3) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = P_{k,k} \, , \end{cases} \tag{209}$$

and from (209), in virtue of (4) and (201), (207)

$$\begin{cases} a_i^{(1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d \, , & (i = 1, \cdots, n-3) \\ a_{n-2}^{(1)} = R_{n-1,n-1}P_{k,k}/d \, , \\ a_{n-1}^{(1)} = R_{n-1,n-1}/d \, . \end{cases} \tag{210}$$

From (210) we obtain, since every $a_i^{(1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$b_i^{(1)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = (D_k - D_{n-1,n-1})d^{-1} \, , \tag{211}$$

and from (210), (211), since

$$(R_{n-1,n-1}/d) - (D_k - D_{n-1,n-1})d^{-1} = (w - D_k)d^{-1} = P_{k,k}d^{-1} \, ,$$

$$\begin{cases} a_1^{(1)} - b_1^{(1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}d^{-1} \, , \\ a_{1+i}^{(1)} - b_{1+i}^{(1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}d^{-1} \, , & (i = 1, \cdots, n-4) \\ a_{n-2}^{(1)} - b_{n-2}^{(1)} = R_{n-1,n-1}P_{k,k}d^{-1} \, , \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = P_{k,k}d^{-1} \, . \end{cases} \tag{211}$$

From (211) we obtain, in virtue of (4) and (201), (207),

$$\begin{cases} a_i^{(2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d \, , & (i = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = R_{n-2,n-1}P_{k,k}/d \, , \\ a_{n-2}^{(2)} = R_{n-2,n-2}P_{k,k}/d \, , \\ a_{n-1}^{(2)} = R_{n-2,n-2} \, . \end{cases} \tag{212}$$

From (212) we obtain, since every $a_i^{(2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$b_i^{(2)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(2)} = D_k - D_{n-2,n-2} \, , \tag{213}$$

and from (212), (213), since $R_{n-2,n-2} - (D_k - D_{n-2,n-2}) = w - D_k = P_{k,k}$

$$
(214)\quad \begin{cases}
a_1^{(2)} - b_1^{(2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d\ , \\
a_{1+i}^{(2)} - b_{1+i}^{(2)} = R_{1,n-4-i}R_{n-2,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-5) \\
a_{n-3}^{(2)} - b_{n-3}^{(2)} = R_{n-2,n-1}P_{k,k}/d\ , \\
a_{n-2}^{(2)} - b_{n-2}^{(2)} = R_{n-2,n-2}P_{k,k}/d\ , \\
a_{n-1}^{(2)} - b_{n-1}^{(2)} = P_{k,k}
\end{cases}
$$

From (214) we obtain, in virtue of (4) and (201), (207)

$$
(215)\quad \begin{cases}
a_i^{(3)} = R_{1,n-4-i}R_{n-3,n-1}P_{k,k}/d\ , \qquad (i = 1, \cdots, n-5) \\
a_{n-4}^{(3)} = R_{n-3,n-1}P_{k,k}/d\ , \\
a_{n-3}^{(3)} = R_{n-3,n-2}P_{k,k}/d\ , \\
a_{n-2}^{(3)} = R_{n-3,n-3}P_{k,k}\ , \\
a_{n-1}^{(3)} = R_{n-3,n-3}\ .
\end{cases}
$$

We shall now prove the formula

$$
(216)\quad \begin{cases}
a_i^{(t)} = R_{1,n-1-t-i}R_{n-t,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-2-t) \\
a_{n-1-t}^{(t)} = R_{n-t,n-1}P_{k,k}/d\ , \\
a_{n-t}^{(t)} = R_{n-t,n-2}P_{k,k}/d\ , \\
a_{n-t+j}^{(t)} = R_{n-t,n-2-j}P_{k,k}\ , \qquad (j = 1, \cdots, t-2) \\
a_{n-1}^{(t)} = R_{n-t,n-t}\ , \\
t = 3, \cdots, n-t\ .
\end{cases}
$$

Formula (216) is correct for $t = 3$, in virtue of formula (215). Let it be correct for $t = m$ $(m = 3, \cdots, n-4)$. From (216) we obtain, for $t = m$, since every $a_i^{(m)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$
(217)\qquad b_i^{(m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m)} = D_k - D_{n-m,n-m}\ ,
$$

and from (216) (for $t = m$) and (217), since

$$
R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}
$$

$$
(218)\quad \begin{cases}
a_1^{(m)} - b_1^{(m)} = R_{1,n-2-m}R_{n-m,n-1}P_{k,k}/d\ , \\
a_{1+i}^{(m)} - b_{1+i}^{(m)} = R_{1,n-2-m-i}R_{n-m,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-3-m) \\
a_{n-1-m}^{(m)} - b_{n-1-m}^{(m)} = R_{n-m,n-1}P_{k,k}/d\ , \\
a_{n-m}^{(m)} - b_{n-m}^{(m)} = R_{n-m,n-2}P_{k,k}/d\ , \\
a_{n-m+j}^{(m)} - b_{n-m+j}^{(m)} = R_{n-m,n-2-j}P_{k,k}\ , \qquad (j = 1, \cdots, m-2) \\
a_{n-1}^{(m)} - b_{n-1}^{(m)} = P_{k,k}\ .
\end{cases}
$$

From (218) we obtain, in virtue of (4) and (201), (207)

$$(219)\quad \begin{cases} a_i^{(m+1)} = R_{1,n-2-m-i}R_{n-m-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-3-m) \\ a_{n-2-m}^{(m+1)} = R_{n-m-1,n-1}P_{k,k}/d\ , & \\ a_{n-1-m}^{(m+1)} = R_{n-m-1,n-2}P_{k,k}/d\ , & \\ a_{n-1-mj}^{(m+1)} = R_{n-m-1,n-2-j}P_{k,k}\ , & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1)} = R_{n-m-1,n-m-1}\ . & \end{cases}$$

But (219) is formula (216) for $t = m + 1$, which completes the proof of this formula. We now obtain from (216) for $t = n - 3$

$$(220)\quad \begin{cases} a_1^{(n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\ , & \\ a_2^{(n-3)} = R_{3,n-1}P_{k,k}/d\ , & \\ a_3^{(n-3)} = R_{3,n-2}P_{k,k}/d\ , & \\ a_{3+j}^{(n-3)} = R_{3,n-2-j}P_{k,k}\ , & (j = 1, \cdots, n-5) \\ a_{n-1}^{(n-3)} = R_{3,3}\ . & \end{cases}$$

From (220) we obtain, since every $a_i^{(n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(221)\qquad b_i^{(n-3)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3)} = D_k - D_{3,3}$$

and from (220), (221), since $P_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$

$$(222)\quad \begin{cases} a_1^{(n-3)} - b_1^{(n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\ , & \\ a_2^{(n-3)} - b_2^{(n-3)} = R_{3,n-1}P_{k,k}/d\ , & \\ a_3^{(n-3)} - b_3^{(n-3)} = R_{3,n-2}P_{k,k}/d\ , & \\ a_{3+j}^{(n-3)} - b_{3+j}^{(n-3)} = R_{3,n-2-j}P_{k,k}\ , & (j = 1, \cdots, n-5) \\ a_{n-1}^{(n-3)} - b_{n-1}^{(n-3)} = P_{k,k}\ . & \end{cases}$$

From (222) we obtain, in virtue of (4) and (201), (207),

$$(223)\quad \begin{cases} a_1^{(n-2)} = R_{2,n-1}P_{k,k}/d;\ a_2^{(n-2)} = R_{2,n-2}P_{k,k}/d\ , \\ a_{2+j}^{(n-2)} = R_{2,n-2-j}P_{k,k},\ (j = 1, \cdots, n-4),\ a_{n-1}^{(n-2)} = R_{2,2}\ , \end{cases}$$

and from (223), since every $a_i^{(n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199);

$$(224)\qquad b_i^{(n-2)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2)} = D_k - D_{2,2}\ .$$

From (223), (224) we obtain, since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(225)\quad \begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = R_{2,n-1}P_{k,k}/d\ , & \\ a_2^{(n-2)} - b_2^{(n-2)} = R_{2,n-2}P_{k,k}/d\ , & \\ a_{2+j}^{(n-2)} - b_{2+j}^{(n-2)} = R_{2,n-2-j}P_{k,k}\ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = P_{k,k}\ , & \end{cases}$$

and from (225), in virtue of (4) and (201), (207)

$$(226)\quad \begin{cases} a_1^{(1;0)} = R_{1,n-2}P_{k,k}/d\ , \\ a_{1+j}^{(1;0)} = R_{1,n-2-j}P_{k,k}\ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} = R_{1,1}\ . \end{cases}$$

Here we are making use of the notation (37) $u; v = u(n-1) + v$. In virtue of formulae (208), (211), (213), (217), (224) the first $n-1$ supporting sequences of the algorithm form a fugue which has the form of the first fugue as demanded by Theorem 6.

From (226) we obtain, since every $a_i^{(1;0)}$ $(i = 1, \cdots, n-2)$ has the factor $P_{k,k}$, and in virtue of (199)

$$(227)\qquad b_i^{(1;0)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;0)} = D_k - D_{1,1}\ ,$$

and from (226), (227), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(228)\quad \begin{cases} a_1^{(1;0)} - b_1^{(1;0)} = R_{1,n-2}P_{k,k}/d\ , \\ a_{1+j}^{(1;0)} - b_{1+j}^{(1;0)} = R_{1,n-2-j}P_{k,k}\ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = P_{k,k}\ . \end{cases}$$

From (228) we obtain, in virtue of (4) and (201), (207),

$$(229)\quad \begin{cases} a_j^{(1;1)} = R_{1,n-2-j}R_{n-1,n-1}P_{k,k}\ , & (j = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = R_{n-1,n-1}P_{k,k}\ , \\ a_{n-1}^{(1;1)} = R_{n-1,n-1}\ , \end{cases}$$

and from (229), since every $a_i^{(1;1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(230)\qquad b_i^{(1;1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;1)} = D_k - D_{n-1,n-1}\ .$$

From (229), (230) we obtain, since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$

$$(231)\quad \begin{cases} a_1^{(1;1)} - b_1^{(1;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}\ , \\ a_{1+j}^{(1;1)} - b_{1+j}^{(1;1)} = R_{1,n-3-j}R_{n-1,n-1}P_{k,k}\ , & (j = 1, \cdots, n-4) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = R_{n-1,n-1}P_{k,k}\ , \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = P_{k,k}\ , \end{cases}$$

and from (231), in virtue of (4) and (201), (207)

$$(232)\quad \begin{cases} a_j^{(1;2)} = R_{1,n-3-j}R_{n-2,n-1}P_{k,k}/d\ , & (j = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = R_{n-2,n-1}P_{k,k}/d\ , \\ a_{n-2}^{(1;2)} = R_{n-2,n-2}P_{k,k}/d\ , \\ a_{n-1}^{(1;2)} = R_{n-2,n-2}/d\ . \end{cases}$$

From (232) we obtain, since every $a_i^{(1;2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199) ,

$$b_i^{(1;2)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;2)} = (D_k - D_{n-2,n-2})/d \ , \tag{233}$$

and from (232), (233), since

$$(R_{n-2,n-2}/d) - ((D_k - D_{n-2,n-2})/d) = (w - D_k)/d = P_{k,k}/d \ ,$$

$$\begin{cases} a_1^{(1;2)} - b_1^{(1;2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d \ , \\ a_{1+j}^{(1;2)} - b_{1+j}^{(1;2)} = R_{1,n-4-j}R_{n-2,n-1}P_{k,k}/d \ , \quad (j = 1, \cdots, n-5) \\ a_{n-3}^{(1;2)} - b_{n-3}^{(1;2)} = R_{n-2,n-1}P_{k,k}/d \ , \\ a_{n-2}^{(1;2)} - b_{n-2}^{(1;2)} = R_{n-2,n-2}P_{k,k}/d \ , \\ a_{n-1}^{(1;2)} - b_{n-1}^{(1;2)} = P_{k,k}/d \ . \end{cases} \tag{234}$$

From (234) we obtain, in virtue of (4) and (201), (207) ,

$$\begin{cases} a_j^{(1;3)} = R_{1,n-4-j}R_{n-3,n-1}P_{k,k}/d \ , & (j = 1, \cdots, n-5) \\ a_{n-5+s}^{(1;3)} = R_{n-3,n-s}P_{k,k}/d \ , & (s = 1, \cdots, 3) \\ a_{n-1}^{(1;3)} = R_{n-3,n-3} \ . \end{cases} \tag{235}$$

From (235) we obtain, since every $a_i^{(1;3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$b_i^{(1;3)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;3)} = D_k - D_{n-3,n-3} \ , \tag{236}$$

and from (235), (236), since $R_{n-3,n-3} - (D_k - D_{n-3,n-3}) = w - D_k = P_{k,k}$

$$\begin{cases} a_1^{(1;3)} - b_1^{(1;3)} = R_{1,n-5}R_{n-3,n-1}P_{k,k}/d \ , \\ a_{1+j}^{(1;3)} - b_{1+j}^{(1;3)} = R_{1,n-5-j}R_{n-3,n-1}P_{k,k}/d \ , \quad (j = 1, \cdots, n-6) \\ a_{n-5+s}^{(1;3)} - b_{n-5+s}^{(1;3)} = R_{n-3,n-s}P_{k,k}/d \ , \qquad (s = 1, 2, 3) \\ a_{n-1}^{(1;3)} - b_{n-1}^{(1;3)} = P_{k,k} \ . \end{cases} \tag{237}$$

From (237) we obtain, in virtue of (4) and (201), (207),

$$\begin{cases} a_j^{(1;4)} = R_{1,n-5-j}R_{n-4,n-1}P_{k,k}/d \ , & (j = 1, \cdots, n-6) \\ a_{n-6+s}^{(1;4)} = R_{n-4,n-s}P_{k,k}/d \ , & (s = 1, 2, 3) \\ a_{n-2}^{(1;4)} = R_{n-4,n-4}P_{k,k} \ , \\ a_{n-1}^{(1;4)} = R_{n-4,n-4} \ . \end{cases} \tag{238}$$

We shall now prove the formula

$$\begin{cases} a_j^{(1,t)} = R_{1,n-1-t-j}R_{n-t,n-1}P_{k,k}/d \ , & (j = 1, \cdots, n-t-2) \\ a_{n-t-2+s}^{(1;t)} = R_{n-t,n-s}P_{k,k}/d \ , & (s = 1, 2, 3) \\ a_{n-t+1+u}^{(1;t)} = R_{n-t,n-3-u}P_{k,k} \ , & (u = 1, \cdots, t-3) \\ a_{n-1}^{(1;t)} = R_{n-t,n-t} \ , \\ t = 4, \cdots, n-3 \ . \end{cases} \tag{239}$$

Formula (239) is correct for $t = 4$, in virtue of (238). We presume (239) is correct for $m \geqq 4$, i.e.,

$$
(240)\quad \begin{cases} a_j^{(1;m)} = R_{1,n-1-m-j}R_{n-m,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-m-2) \\ a_{n-m-2+s}^{(1;m)} = R_{n-m,n-s}P_{k,k}/d\,, & (s = 1, 2, 3) \\ a_{n-m+1+u}^{(1;m)} = R_{n-m,n-3-u}P_{k,k}\,, & (u = 1, \cdots, m-3) \\ a_{n-1}^{(1;m)} = R_{n-m,n-m}\,. \end{cases}
$$

From (240) we obtain, since every $a_i^{(1;m)}$ contains the factor $P_{k,k}$ $(i = 1, \cdots, n-2)$, and in virtue of (199)

$$
(241)\quad b_i^{(1;m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;m)} = D_k - D_{n-m,n-m}\,,
$$

and from (240), (241), since

$$
R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}\,,
$$

$$
(242)\quad \begin{cases} a_1^{(1;m)} - b_1^{(1;m)} = R_{1,n-m-2}R_{n-m,n-1}P_{k,k}/d\,, \\ a_{1+j}^{(1;m)} - b_{1+j}^{(1;m)} = R_{1,n-m-2-j}R_{n-m,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-m-3) \\ a_{n-m-2+s}^{(1;m)} - b_{n-m-2+s}^{(1;m)} = R_{n-m,n-s}P_{k,k}/d & (s = 1, 2, 3) \\ a_{n-m+1+u}^{(1;m)} - b_{n-m+1+u}^{(1;m)} = R_{n-m,n-3-u}P_{k,k}\,, & (u = 1, \cdots, m-3) \\ a_{n-1}^{(1;m)} - b_{n-1}^{(1;m)} = P_{k,k}\,. \end{cases}
$$

From (242) we obtain, in virtue of (4) and (201), (207),

$$
(243)\quad \begin{cases} a_j^{(1;m+1)} = R_{1,n-m-2-j}R_{n-m,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-m-3) \\ a_{n-m-3+s}^{(1;m+1)} = R_{n-m-1,n-s}P_{k,k}/d\,, & (s = 1, 2, 3) \\ a_{n-m+u}^{(1;m+1)} = R_{n-m-1,n-3-u}P_{k,k}\,, & (u = 1, \cdots, m-2) \\ a_{n-1}^{(1;m+1)} = R_{n-m-1,n-m-1}\,. \end{cases}
$$

Substituting $m + 1$ for t in formula (239) we obtain formula (243) which completes the proof of (239).

From (239) we now obtain for $t = n - 3$,

$$
(244)\quad \begin{cases} a_1^{(1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, \\ a_{1+s}^{(1;n-3)} = R_{3,n-s}P_{k,k}/d\,, & (s = 1, 2, 3) \\ a_{4+u}^{(1;n-3)} = R_{3,n-3-u}P_{k,k}\,, & (u = 1, \cdots, n-6) \\ a_{n-1}^{(1;n-3)} = R_{3,3}\,, \end{cases}
$$

and from (244), since every $a_i^{(1;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$
(246)\quad \begin{cases} a_1^{(1;n-3)} - b_1^{(1;n-3)} = R_{1,1}R_{3,n-1}R_{k,k}/d\,, \\ a_{1+s}^{(1;n-3)} - b_{1+s}^{(1;n-3)} = R_{3,n-s}P_{k,k}/d\,, & (s = 1, 2, 3) \\ a_{4+u}^{(1;n-3)} - b_{4+u}^{(1;n-3)} = R_{3,n-3-u}P_{k,k}\,, & (u = 1, \cdots, n-6) \\ a_{n-1}^{(1;n-3)} - b_{n-1}^{(1;n-3)} = P_{k,k}\,. \end{cases}
$$

From (246) we obtain, in virtue of (4) and (201), (207)

$$(247)\qquad \begin{cases} a_1^{(1;n-2)} = R_{2,n-1}P_{k,k}/d , \\ a_2^{(1;n-2)} = R_{2,n-2}P_{k,k}/d , \\ a_3^{(1;n-2)} = R_{2,n-3}P_{k,k}/d , \\ a_{3+u}^{(1;n-2)} = R_{2,n-3-u}P_{k,k} , & (u = 1, \cdots, n-5) \\ a_{n-1}^{(1;n-2)} = R_{2,2} . \end{cases}$$

From (247) we obtain, since every $a_i^{(1;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(248)\qquad b_i^{(1;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;n-2)} = D_k - D_{2,2} ,$$

and from (247), (248), since

$$R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$$

$$(249)\qquad \begin{cases} a_1^{(1;n-2)} - b_1^{(1;n-2)} = R_{2,n-1}P_{k,k}/d , \\ a_2^{(1;n-2)} - b_2^{(1;n-2)} = R_{2,n-2}P_{k,k}/d , \\ a_3^{(1;n-2)} - b_3^{(1;n-2)} = R_{2,n-3}P_{k,k}/d , \\ a_{3+i}^{(1;n-2)} - b_{3+i}^{(1;n-2)} = R_{2,n-3-i}P_{k,k} , \\ a_{n-1}^{(1;n-2)} - b_{n-1}^{(1;n-2)} = P_{k,k} . \end{cases}$$

From (249) we obtain, in virtue of (4) and (201), (207) ,

$$(250)\qquad \begin{cases} a_1^{(2;0)} = R_{1,n-2}P_{k,k}/d , \\ a_2^{(2;0)} = R_{1,n-3}P_{k,k}/d , \\ a_{2+i}^{(2;0)} = R_{1,n-3-i}P_{k,k} , & (i = 1, \cdots, n-4) \\ a_{n-1}^{(2;0)} = R_{1,1} . \end{cases}$$

In virtue of formulae (227), (230), (233), (236), (241), (248), the $n-1$ supporting sequences, starting with the n -th sequence of the algorithm, form a fugue which has the form of the second fugue as demanded by Theorem 6.

The proof of Theorem 6 is essentially based on the following

LEMMA 2. *If the generating sequence*

$$a_i^{(t;0)}; (i = 1, \cdots, n-1;\ t = 1, \cdots, n-4)$$

has the form

$$(251)\qquad \begin{cases} a_i^{(t;0)} = R_{1,n-1-i}P_{k,k}/d , & (i = 1, \cdots, t) \\ a_{t+j}^{(t;0)} = R_{1,n-1-t-j}P_{k,k} , & (j = 1, \cdots, n-t-2) \\ a_{n-1}^{(t;0)} = R_{1,1} , \end{cases}$$

then the $n-1$ supporting sequences

$$b_1^{(t;s)}, b_2^{(t;s)}, \cdots, b_{n-1}^{(t;s)} \qquad (s = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the $t+1$ *-th fugue as demanded by Theorem* 6, *and the generating sequence*

$$a_1^{(t+1;0)}, a_2^{(t+1;0)}, \cdots, a_{n-1}^{(t+1;0)}$$

has the form (251) *where* t *is to be replaced by* $t+1$.

Proof. The Lemma 2 is correct for $t = 1$, as can be easily verified by the formulae (226), (250) and the remark following formula (250). We shall presume that the Lemma 2 is correct for $t = m - 1$ ($m \geqq 2$) and shall prove its correctness for $t + 1 = m$. We obtain from (251), on ground of the second statement of the Lemma 2 (viz. for $t + 1 = m$)

$$\text{(252)} \qquad \begin{cases} a_i^{(m;0)} = R_{1,n-1-i}P_{k,k}/d\,, & (i = 1, \cdots, m) \\ a_{m+j}^{(m;0)} = R_{1,n-1-m-j}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-1}^{(m;0)} = R_{1,1}\,. \end{cases}$$

From (252) we obtain, since every $a_i^{(m;0)}$ ($i = 1, \cdots, n-2$) contains the factor $P_{k,k}$, and in virtue of (199)

$$\text{(253)} \qquad b_i^{(m;0)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;0)} = D_k - D_{1,1}\,,$$

and from (252), (253), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$\text{(254)} \qquad \begin{cases} a_1^{(m;0)} - b_1^{(m;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+i}^{(m;0)} - b_{1+i}^{(m;0)} = R_{1,n-2-i}P_{k,k}/d\,, & (i = 1, \cdots, m-1) \\ a_{m+j}^{(m;0)} - b_{m+j}^{(m;0)} = R_{1,n-1-m-j}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-1}^{(m;0)} - b_{n-1}^{(m;0)} = P_{k,k}\,. \end{cases}$$

From (254) we obtain, in virtue of (4) and (201), (207),

$$\text{(255)} \qquad \begin{cases} a_i^{(m,1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-1) \\ a_{m-1+j}^{(m;1)} = R_{1,n-1-m-j}R_{n-1,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-2}^{(m;1)} = R_{n-1,n-1}P_{k,k}\,, \\ a_{n-1}^{(m;1)} = R_{n-1,n-1}\,. \end{cases}$$

From (255) we obtain, since every $a_i^{(m;1)}$ ($i = 1, \cdots, n-2$) contains the factor $P_{k,k}$, and in virtue of (199),

$$\text{(256)} \qquad b_i^{(m;1)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;1)} = D_k - D_{n-1,n-1}\,,$$

and from (255), (256), since

$$R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}\,,$$

$$(257)\quad \begin{cases} a_1^{(m;1)} - b_1^{(m;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d\ , \\ a_{1+i}^{(m;1)} - b_{1+i}^{(m;1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-2) \\ a_{m-1+j}^{(m;1)} - b_{m-1+j}^{(m;1)} = R_{1,n-1-m-j}R_{n-1,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-2}^{(m;1)} - b_{n-2}^{(m;1)} = R_{n-1,n-1}P_{k,k}\ , \\ a_{n-1}^{(m;1)} - b_{n-1}^{(m;1)} = P_{k,k}\ . \end{cases}$$

From (257) we obtain, in virtue of (4) and (201), (207),

$$(258)\quad \begin{cases} a_i^{(m;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-2) \\ a_{m-2+j}^{(m;2)} = R_{1,n-1-m-j}R_{n-2,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-3}^{(m;2)} = R_{n-2,n-1}P_{k,k}\ , \\ a_{n-2}^{(m;3)} = R_{n-2,n-2}P_{k,k}\ , \\ a_{n-1}^{(m;3)} = R_{n-2,n-2}\ . \end{cases}$$

We shall now prove the formula

$$(259)\quad \begin{cases} a_i^{(m;t)} = R_{1,n-1-t-i}R_{n-t,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-t) \\ a_{m-t+j}^{(m;t)} = R_{1,n-1-m-j}R_{n-t,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-t-2+u}^{(m;t)} = R_{n-t,n-u}P_{k,k}\ , & (u = 1, \cdots, t) \\ a_{n-1}^{(m;t)} = R_{n-t,n-t}\ , \\ t = 1, \cdots, m-1\ . \end{cases}$$

Formula (259) is correct for $t = 1, 2$, in virtue of formulae (255), (258). Let it be correct for $t = s \geqq 2$, i.e.,

$$(260)\quad \begin{cases} a_i^{(m;s)} = R_{1,n-1-s-i}R_{n-s,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-s) \\ a_{m-s+j}^{(m;s)} = R_{1,n-1-m-j}R_{n-s,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-s-2+u}^{(m;s)} = R_{n-s,n-u}P_{k,k}\ , & (u = 1, \cdots, s) \\ a_{n-1}^{(m;s)} = R_{n-s,n-s}\ . \end{cases}$$

From (260) we obtain, since every $a_i^{(m;s)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(261)\qquad b_i^{(m;s)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;s)} = D_k - D_{n-s,n-s}\ ,$$

and from (260), (261), since $R_{n-s,n-s} - (D_k - D_{n-s,n-s}) = w - D_k = P_{k,k}$,

$$(262)\quad \begin{cases} a_1^{(m;s)} - b_1^{(m;s)} = R_{1,n-2-s}R_{n-s,n-1}P_{k,k}/d\ , \\ a_{1+i}^{(m;s)} - b_{1+i}^{(m;s)} = R_{1,n-2-s-i}R_{n-s,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-s-1) \\ a_{m-s+j}^{(m;s)} - b_{m-s+j}^{(m;s)} = R_{1,n-1-m-j}R_{n-s,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-s-2+u}^{(m;s)} - b_{n-s-2+u}^{(m;s)} = R_{n-s,n-u}P_{k,k}\ , & (u = 1, \cdots, s) \\ a_{n-1}^{(m;s)} - b_{n-1}^{(m;s)} = P_{k,k}\ . \end{cases}$$

From (262) we obtain, in virtue of (4) and (201), (207)

$$(263)\quad \begin{cases} a_i^{(m;s+1)} = R_{1,n-2-s-i}R_{n-1-s,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-s-1) \\ a_{m-1-s+j}^{(m;s+1)} = R_{1,n-1-m-j}R_{n-1-s,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-s-3+u}^{(m;s+1)} = R_{n-1-s,n-u}P_{k,k}\ , & (u = 1, \cdots, s+1) \\ a_{n-1}^{(m;s+1)} = R_{n-1-s,n-1-s}\ . \end{cases}$$

But (263) is formula (260) where s is to be replaced by $s+1$; this completes the proof of formula (259),

We now obtain from (259), for $t = m-1$,

$$(264)\quad \begin{cases} a_1^{(m;m-1)} = R_{1,n-m-1}R_{n-m+1,n-1}P_{k,k}/d\ , \\ a_{1+j}^{(m;m-1)} = R_{1,n-1-m-j}R_{n-m+1,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-m-1+u}^{(m;m-1)} = R_{n-m+1,n-u}P_{k,k}\ , & (u = 1, \cdots, m-1) \\ a_{n-1}^{(m;m-1)} = R_{n-m+1,n-m+1}\ . \end{cases}$$

From (264) we obtain, since every $a_i^{(m;m-1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(265)\quad b_i^{(m;m-1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m-1)} = D_k - D_{n-m+1,n-m+1}\ ,$$

and from (264), (265), since

$$R_{n-m+1,n-m+1} - (D_k - D_{n-m+1,n-m+1}) = w - D_k = P_{k,k}\ ,$$

$$(266)\quad \begin{cases} a_1^{(m;m-1)} - b_1^{(m;m-1)} = R_{1,n-m-1}R_{n-m+1,n-1}P_{k,k}/d\ , \\ a_{1+j}^{(m;m-1)} - b_{1+j}^{(m;m-1)} = R_{1,n-1-m-j}R_{n-m+1,n-1}P_{k,k}\ , \\ & (j = 1, \cdots, n-m-2) \\ a_{n-m-1+u}^{(m;m-1)} - b_{n-m-1+u}^{(m;m-1)} = R_{n-m+1,n-u}P_{k,k}\ , & (u = 1, \cdots, m-1) \\ a_{n-1}^{(m;m-1)} - b_{n-1}^{(m;m-1)} = P_{k,k}\ . \end{cases}$$

From (266) we obtain, in virtue of (4) and (201), (207),

$$(267)\quad \begin{cases} a_j^{(m;m)} = R_{1,n-1-m-j}R_{n-m,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-m-2+u}^{(m;m)} = R_{n-m,n-u}P_{k,k}\ , & (u = 1, \cdots, m) \\ a_{n-1}^{(m;m)} = R_{n-m,n-m}\ . \end{cases}$$

From (267) we obtain, since every $a_i^{(m;m)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(268)\quad b_i^{(m;m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m)} = D_k - D_{n-m,n-m}\ ,$$

and from (267), (268), since $R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}$,

$$(269)\quad \begin{cases} a_1^{(m;m)} - b_1^{(m;m)} = R_{1,n-2-m}R_{n-m,n-1}P_{k,k}\ , & \\ a_{1+j}^{(m;m)} - b_{1+j}^{(m;m)} = R_{1,n-2-m-j}R_{n-m,n-1}P_{k,k}\ , & (j=1,\cdots,n-m-3) \\ a_{n-m-2+u}^{(m;m)} - {}_{n-m-2+u}^{(m;m)} = R_{n-m,n-u}P_{k,k}\ , & (u=1,\cdots,m) \\ a_{n-1}^{(m;m)} - b_{n-1}^{(m;m)} = P_{k,k}\ . & \end{cases}$$

From (269) we obtain, in virtue of (4) and (201), (207) ,

$$(270)\quad \begin{cases} a_j^{(m;m+1)} = R_{1,n-2-m-j}R_{n-m-1,n-1}P_{k,k}/d\ , & (j=1,\cdots,n-m-3) \\ a_{n-m-3+u}^{(m;m+1)} = R_{n-m-1,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+1) \\ a_{n-1}^{(m;m+1)} = R_{n-m-1,n-m-1}/d\ , & \end{cases}$$

and from (270), since every $a_i^{(m;m+1)}$ $(i=1,\cdots,n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(271)\quad b_i^{(m;m+1)} = 0;\ (i=1,\cdots,n-2)\quad b_{n-1}^{(m;m+1)} = (D_k - D_{n-m-1,n-m-1})/d\ .$$

From (270), (271) we obtain, since

$$(R_{n-m-1,n-m-1}/d) - ((D_k - D_{n-m-1,n-m-1})/d) = (w-D_k)/d = P_{k,k}/d\ ,$$

$$(272)\quad \begin{cases} a_1^{(m;m+1)} - b_1^{(m;m+1)} = R_{1,n-3-m}R_{n-1-m,n-1}P_{k,k}/d\ , & \\ a_{1+j}^{(m;m+1)} - b_{1+j}^{(m;m+1)} = R_{1,n-3-m-j}R_{n-1-m,n-1}P_{k,k}/d\ , & \\ & (j=1,\cdots,n-m-4) \\ a_{n-3-m+u}^{(m;m+1)} - b_{n-3-m+u}^{(m;m+1)} = R_{n-m-1,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+1) \\ a_{n-1}^{(m;m+1)} - b_{n-1}^{(m;m+1)} = P_{k,k}/d\ , & \end{cases}$$

and from (272), in virtue of (4) and (201), (207)

$$(273)\quad \begin{cases} a_j^{(m;m+2)} = R_{1,n-3-m-j}R_{n-2-m,n-1}P_{k,k}/d\ , & (j=1,\cdots,n-m-4) \\ a_{n-m-4+u}^{(m;m+2)} = R_{n-m-2,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{n-1}^{(m;m+2)} = R_{n-m-2,n-m-2}\ . & \end{cases}$$

From (273) we obtain, since every $a_i^{(m,m+2)}$ $(i=1,\cdots,n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(274)\quad b_i^{(m;m+2)} = 0;\ (i=1,\cdots,n-2)\qquad b_{n-1}^{(m;m+2)} = D_k - D_{n-m-2,n-m-2}\ ,$$

and from (273), (274), since

$$R_{n-m-2,n-m-2} - (D_k - D_{n-m-2,n-m-2}) = w - D_k = P_{k,k}\ ,$$

$$(275)\quad \begin{cases} a_1^{(m;m+2)} - b_1^{(m;m+2)} = R_{1,n-m-4}R_{n-m-2,n-1}P_{k,k}/d\ , & \\ a_{1+j}^{(m;m+2)} - b_{1+j}^{(m;m+2)} = R_{1,n-m-4-j}R_{n-m-2,n-1}P_{k,k}/d\ , & \\ & (j=1,\cdots,n-m-5) \\ a_{n-m-4+u}^{(m;m+?)} - b_{n-m-4+u}^{(m;m+2)} = R_{n-m-2,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{n-1}^{(m;m+2)} - b_{n-1}^{(m;m+2)} = P_{k,k}\ . & \end{cases}$$

From (275) we obtain, in virtue of (4) and (201), (207),

$$(276)\quad \begin{cases} a_j^{(m;m+3)} = R_{1,n-m-4-j}R_{n-m-3,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-m-5) \\ a_{n-m-5+u}^{(m;m+3)} = R_{n-m-3,n-u}P_{k,k}/d\,, & (u = 1, \cdots, m+2) \\ a_{n-2}^{(m;m+3)} = R_{n-m-3,n-m-3}P_{k,k}\,, & \\ a_{n-1}^{(m;m+3)} = R_{n-m-3,n-m-3}\,. & \end{cases}$$

We shall now prove the formula

$$(277)\quad \begin{cases} a_j^{(m;m+t)} = R_{1,n-m-1-t-j}R_{n-m-t,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-m-2-t) \\ a_{n-m-2-t+u}^{(m;m+t)} = R_{n-m-t,n-u}P_{k,k}/d\,, & (u = 1, \cdots, m+2) \\ a_{n-t+i}^{(m;m+t)} = R_{n-m-t,n-m-2-i}P_{k,k}\,, & (i = 1, \cdots, t-2) \\ a_{n-1}^{(m;m+t)} = R_{n-m-t,n-m-t}\,, & \\ t = 3, \cdots, n-m-3 & \end{cases}$$

Formula (277) is correct for $t = 3$, in virtue of (276). As before, (277) is proved by induction.

We now obtain, from (277), since every $a_i^{(m;m+t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(278)\quad b_i^{(m;m+t)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m+t)} = D_k - D_{n-m-t,n-m-t}\,.$$

We further obtain from (278), for $t = n - m - 3$

$$(279)\quad \begin{cases} a_1^{(m;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, & \\ a_{1+u}^{(m;n-3)} = R_{3,n-u}P_{k,k}/d\,, & (u = 1, \cdots, m+2) \\ a_{m+3+i}^{(m;n-3)} = R_{3,n-m-2-i}P_{k,k}\,, & (i = 1, \cdots, n-m-5) \\ a_{n-1}^{(m;n-3)} = R_{3,3}\,. & \end{cases}$$

From (279) we obtain, since every $a_i^{(m;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(280)\quad b_i^{(m;n-3)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;n-3} = D_k - D_{3,3}\,,$$

and from (279), (280), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$,

$$(281)\quad \begin{cases} a_1^{(m;n-3)} - b_1^{(m;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, & \\ a_{1+u}^{(m;n-3)} - b_{1+u}^{(m;n-3)} = R_{3,n-u}P_{k,k}/d\,, & (u = 1, \cdots, m+2) \\ a_{m+3+i}^{(m;n-3)} - b_{m+3+i}^{(m;n-2)} = R_{3,n-m-2-i}P_{k,k}\,, & (i = 1, \cdots, n-m-5) \\ a_{n-1}^{(m;n-3)} - b_{n-1}^{(m;n-3)} = P_{k,k} & \end{cases}$$

From (281) we obtain, in virtue of (4) and (201), (207) ,

$$(282)\quad \begin{cases} a_i^{(m;n-2)} = R_{2,n-i}P_{k,k}/d\,, & (i = 1, \cdots, m+2) \\ a_{m+2+j}^{(m;n-2)} = R_{2,n-m-2-j}P_{k,k}\,, & (j = 1, \cdots, n-m-4) \\ a_{n-1}^{(m;n-2)} = R_{2,2}\,. & \end{cases}$$

From (282) we obtain, since every $a_i^{(m;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(283) \qquad b_i^{(m;n-2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;n-2)} = D_k - D_{2,2} ,$$

and from (282), (283), since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(284) \quad \begin{cases} a_1^{(m;n-2)} - b_1^{(m;-2)} = R_{2,n-1}P_{k,k}/d , \\ a_{1+i}^{(m;n-2)} - b_{1+i}^{(m;n-2)} = R_{2,n-1-i}P_{k,k}/d , & (i = 1, \cdots, m+1) \\ a_{m+2+j}^{(m;n-2)} - b_{m+2+j}^{(m;n-2)} = R_{2,n-m-2-j}P_{k,k} , & (j = 1, \cdots, n-m-4) \\ a_{n-1}^{(m;n-2)} - b_{n-1}^{(m;n-2)} = P_{k,k} . \end{cases}$$

From (284) we obtain, in virtue of (4) and (201), (207)

$$(285) \quad \begin{cases} a_i^{(m+1;0)} = R_{1,n-1-i}P_{k,k}/d , & (i = 1, \cdots, m+1) \\ a_{m+1+j}^{(m+1;0)} = R_{1,n-m-2-j}P_{k,k} , & (j = 1, \cdots, n-m-3) \\ a_{n-1}^{(m+1;0)} = R_{1,1} . \end{cases}$$

We note that formula (285) is obtained from formula (252) replacing in the latter m by $m+1$.
We further note that the $n-1$ supporting sequences

$$b_1^{(m;s)}, b_2^{(m;s)}, \cdots, b_{n-1}^{(m;s)} \qquad (s = 0, 1, \cdots, n-2)$$

generate a fugue which has the form of the $m+1$ -th fugue, as demanded by Theorem 6. Thus the Lemma 2 is completely proved.

We now obtain, on ground of the lemma, and in virtue of formula (251) for $t = n-3$, since (251) is correct for $t+1$, too,

$$(286) \quad \begin{cases} a_1^{(n-3;0)} = R_{1,n-1-i}P_{k,k}/d , & (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-3;0)} = R_{1,1}P_{k,k} , \\ a_{n-1}^{(n-3;0)} = R_{1,1} . \end{cases}$$

from (286) we obtain, since every $a_i^{(n-3;0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$ and in virtue of (199)

$$(287) \qquad b_i^{(n-3;0)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;0)} = D_k - D_{1,1} ,$$

and from (286), (287), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(288) \quad \begin{cases} a_1^{(n-3;0)} - b_1^{(n-3;0)} = R_{1,n-2}P_{k,k}/d , \\ a_{1+i}^{(n-3;0)} - b_{1+i}^{(n-3;0)} = R_{1,n-2-i}P_{k,k}/d , & (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-3;0)} - b_{n-2}^{(n-3;0)} = R_{1,1}P_{k,k} , \\ a_{n-1}^{(n-3;0)} - b_{n-1}^{(n-3;0)} = P_{k,k} . \end{cases}$$

From (288) we obtain, in virtue of (4) and (201), (207),

$$(289)\quad \begin{cases} a_i^{(n-3;1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-4) \\ a_{n-3}^{(n-3;1)} = R_{1,1}R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;1)} = R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-1}^{(n-3;1)} = R_{n-1,n-1}\ , & \end{cases}$$

and from (289), since every $a_i^{(n-3;1)}$ contains the factor $P_{k,k}$ $(i = 1, \cdots, n-2)$, and in virtue of (199)

$$(290)\quad b_i^{(n-3;1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;1)} = D_k - D_{n-1,n-1}\ .$$

From (289), (290) we obtain, since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$

$$(291)\quad \begin{cases} a_1^{(n-3;1)} - b_1^{(n-3;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d\ , & \\ a_{1+i}^{(n-3;1)} - b_{1+i}^{(n-3;1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-5) \\ a_{n-3}^{(n-3;1)} - b_{n-3}^{(n-3;1)} = R_{1,1}R_{n-1n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;1)} - b_{n-2}^{(n-3;1)} = R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-1}^{(n-3;1)} - b_{n-1}^{(n-3;1)} = P_{k,k}\ , & \end{cases}$$

and from (291), in virtue of (4) and (201), (207),

$$(292)\quad \begin{cases} a_i^{(n-3;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-5) \\ a_{n-4}^{(n-3;2)} = R_{1,1}R_{n-2,n-1}P_{k,k};\ a_{n-3}^{(n-3;2)} = R_{n-2,n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;2)} = R_{n-2,n-2}P_{k,k};\ a_{n-1}^{(n-3;2)} = R_{n-2,n-2}\ . & \end{cases}$$

It is now easy to prove the formula

$$(293)\quad \begin{cases} a_i^{(n-3;t)} = R_{1,n-t-1-i}R_{n-t,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-t-3) \\ a_{n-t-2}^{(n-3;t)} = R_{1,1}R_{n-t,n-1}P_{k,k}\ , & \\ a_{n-t-2+j}^{(n-3;t)} = R_{n-t,n-j}P_{k,k}\ , & (j = 1, \cdots, t) \\ a_{n-1}^{(n-3;t)} = R_{n-t,n-t}\ , & \\ t = 1, \cdots, n-4\ . & \end{cases}$$

Formula (293) is true for $t = 1, 2$, in virtue of (289), (292). It is then presumed that (293) is true for $m \geqq 1$ and proved, as before, that it is correct for $m + 1$, too, which completes the proof of (293). From (293) we obtain, since every $a_i^{(n-3;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(294)\quad b_i^{(n-3;t)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;t)} = D_k - D_{n-t,n-t}\ ,$$

and further for $t = n - 4$,

$$(295)\quad \begin{cases} a_1^{(n-3;n-4)} = R_{1,2}R_{4,n-1}P_{k,k}/d\ , & \\ a_2^{(n-3,n-4)} = R_{1,1}R_{4,n-1}P_{k,k}\ , & \\ a_{2+j}^{(n-3;n-4)} = R_{4,n-j}P_{k,k}\ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-3;n-4)} = R_{4,4}\ . & \end{cases}$$

From (295) we obtain, since every $a_i^{(n-3;n-4)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(296) $\quad b_i^{(n-3,n-4)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-4)} = D_k - D_{4,4}$,

and from (295), (296), since $R_{4,4} - (D_k - D_{4,4}) = w - D_k = P_{k,k}$,

$$(297) \qquad \begin{cases} a_1^{(n-3;n-4)} - b_1^{(n-3;n-4)} = R_{1,2}R_{4,n-1}P_{k,k}/d \ , \\ a_2^{(n-3;n-4)} - b_2^{(n-3;n-4)} = R_{1,1}R_{4,n-1}P_{k,k} \ , \\ a_{2+j}^{(n-3;n-4)} - b_{2+j}^{(n-3;n-4)} = R_{4,n-j}P_{k,k} \ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-3;n-4)} - b_{n-1}^{(n-3;n-4)} = P_{k,k} \ . \end{cases}$$

From (297) we obtain, in virtue of (4) and (201), (207),

$$(298) \qquad \begin{cases} a_1^{(n-3;n-3)} = R_{1,1}R_{3,n-1}P_{k,k} \ , \\ a_{1+j}^{(n-3;n-3)} = R_{3,n-j}P_{k,k} \ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-3)} = R_{3,3} \ . \end{cases}$$

From (298) we obtain, since every $a_i^{(n-3;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(299) $\quad b_i^{(n-3;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-3)} = D_k - D_{3,3}$,

and from (298), (299), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$,

$$(299a) \qquad \begin{cases} a_1^{(n-3;n-3)} - b_1^{(n-3;n-3)} = R_{1,1}R_{3,n-1}P_{k,k} \ , \\ a_{1+j}^{(n-3;n-3)} - b_{1+j}^{(n-3;n-3)} = R_{3,n-j}P_{k,k} \ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-3)} - b_{n-1}^{(n-3;n-3)} = P_{k,k} \ . \end{cases}$$

From (299a) we obtain, in virtue of (4) and (201), (207) ,

$$(300) \qquad \begin{cases} a_j^{(n-3;n-2)} = R_{2,n-j}P_{k,k}/d \ , & (j = 1, \cdots, n-2) \\ a_{n-1}^{(n-3;n-2)} = R_{2,2}/d \ , \end{cases}$$

and from (300), since every $a_j^{(n-3;n-2)}$ $(j = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(301) $\quad b_i^{(n-3;n-2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-2)} = (D_k - D_{2,2})/d$.

From (300), (301) we obtain, since $(R_{2,2}/d) - ((D_k - D_{2,2})/d) = P_{k,k}/d$,

$$(302) \qquad \begin{cases} a_1^{(n-3;n-2)} - b_1^{(n-3;n-2)} = R_{2,n-1}P_{k,k}/d \ , \\ a_{1+j}^{(n-3;n-2)} - b_{1+j}^{(n-3;n-2)} = R_{2,n-1-j}P_{k,k}/d \ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-2)} - b_{n-1}^{(n-3;n-2)} = P_{k,k}/d \ , \end{cases}$$

and from (302), in virtue of (4) and (201), (207),

$$(303) \qquad \begin{cases} a_j^{(n-2;0)} = R_{1,n-1-j}P_{k,k}/d \ , & (j = 1, \cdots, n-2) \\ a_{n-1}^{(n-2;0)} = R_{1,1} \ . \end{cases}$$

Formulae (287), (290), (294), (299), (301) show that the $n-1$ supporting sequences

$$b_1^{(n-3;k)}, b_2^{(n-3;k)}, \cdots, b_{n-1}^{(n-3;k)} \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the $n-2$ -th fugue as demanded by Theorem 6.

From (303) we obtain, since every $a_j^{(n-2;0)}$ $(j = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(304) \qquad b_i^{(n-2;0)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;0)} = D_k - D_{1,1} ,$$

and from (303), (304), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(305) \qquad \begin{cases} a_1^{(n-2;0)} - b_1^{(n-2;0)} = R_{1,n-2}P_{k,k}/d , \\ a_{1+j}^{(n-2;0)} - b_{1+j}^{(n-2;0)} = R_{1,n-2-j}P_{k,k}/d , \qquad (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;0)} - b_{n-1}^{(n-2;0)} = P_{k,k} . \end{cases}$$

From (305) we obtain, in virtue of (4) and (107), (108) ,

$$(306) \qquad \begin{cases} a_j^{(n-2;1)} = R_{1,n-2-j}R_{n-1,n-1}P_{k,k}/d , \qquad (j = 1, \cdots, n-3) \\ a_{n-2}^{(n-2;1)} = R_{n-1,n-1}P_{k,k} , \\ a_{n-1}^{(n-2;1)} = R_{n-1,n-1} . \end{cases}$$

It is now easy to prove the formula

$$(307) \qquad \begin{cases} a_j^{(n-2;t)} = R_{1,n-1-t-j}R_{n-t,n-1}P_{k,k}/d , \qquad (j = 1, \cdots, n-2-t) \\ a_{n-2-t+i}^{(n-2;t)} = R_{n-t,n-i}P_{k,k} , \qquad (i = 1, \cdots, t) \\ a_{n-1}^{(n-2;t)} = R_{n-t,n-t} , \\ t = 1, \cdots, n-3 . \end{cases}$$

Formula (307) is correct for $t = 1$, in virtue of formula (306), (307) is then proved by induction.
From (307) we obtain, since every $a_i^{(n-2;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(308) \qquad b_i^{(n-2;t)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;t)} = D_k - D_{n-t,n-t} ,$$

and further from (307), for $t = n-3$,

$$(309) \qquad \begin{cases} a_1^{(n-2;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d , \\ a_{1+i}^{(n-2;n-3)} = R_{3,n-i}P_{k,k} , \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-3)} = R_{3,3} . \end{cases}$$

From (309) we obtain, since every $a_i^{(n-2;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$ and in virtue of (199),

$$(310) \qquad b_i^{(n-2;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;n-3)} = D_k - D_{3,3} ,$$

and from (309) (310), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$,

$$(311)\qquad \begin{cases} a_1^{(n-2;n-3)} - b_1^{(n-2;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(n-2;n-3)} - b_{1+i}^{(n-2;n-3)} = R_{3,n-i}P_{k,k}\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-3)} - b_{n-1}^{(n-2;n-3)} = P_{k,k}\,. \end{cases}$$

From (311) we obtain, in virtue of (4) and (201), (207),

$$(312)\qquad \begin{cases} a_i^{(n-2;n-2)} = R_{2,n-i}P_{k,k}\,, \qquad (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-2;n-2)} = R_{2,2}\,, \end{cases}$$

and from (312), since every $a_i^{(n-2,n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(313)\qquad b_i^{(n-2;n-2)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;n-2)} = D_k - D_{2,2}\,.$$

From (312), (313) we obtain, since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(314)\qquad \begin{cases} a_1^{(n-2;n-2)} - b_1^{(n-2;n-2)} = R_{2,n-1}P_{k,k}\,, \\ a_{1+i}^{(n-2;n-2)} - b_{1+i}^{(n-2;n-2)} = R_{2,n-1-i}P_{k,k}\,, \quad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-2)} - b_{n-1}^{(n-2;n-2)} = P_{k,k}\,, \end{cases}$$

and from (314), in virtue of (4) and (201), (207),

$$(315)\qquad \begin{cases} a_i^{(n-1;0)} = R_{1,n-1-i}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-1;0)} = R_{1,1}/d\,. \end{cases}$$

Formula (304), (308), (313) show that the $n-1$ supporting sequences $b_i^{(n-2,k)}$ $(i = 1, \cdots, n-1;\ k = 0, 1, \cdots, n-2)$ form a fugue which has the form as demanded by Theorem 6.

From (315) we obtain, since every $a_i^{(n-1;0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(316)\qquad b_i^{(n-1;0)} = 0,\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;0)} = (D_k - D_{1,1})/d\,,$$

and from (315), (316), since $(R_{1,1}/d) - ((D_k - D_{1,1})/d) = (w - D_k)/d = P_{k,k}/d$,

$$(317)\qquad \begin{cases} a_1^{(n-1;0)} - b_1^{(n-1;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+i}^{(n-1;0)} - b_{1+i}^{(n-1;0)} = R_{1,n-2-i}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = P_{k,k}/d\,. \end{cases}$$

From (317) we obtain, in virtue of (4) and (201), (207),

$$(318)\qquad \begin{cases} a_i^{(n-1;1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-1;1)} = R_{n-1,n-1}P_{k,k}/d\,, \\ a_{n-1}^{(n-1;1)} = R_{n-1,n-1}\,. \end{cases}$$

From (318) we obtain, since every $a_i^{(n-1;1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(319) $\quad b_i^{(n-1;1)} = 0, (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;1)} = D_k - D_{n-1,n-1}$,

and from (318), (319), since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$,

$$(320) \quad \begin{cases} a_1^{(n-1;1)} - b_1^{(n-1;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d \ , \\ a_{1+i}^{(n-1;1)} - b_{1+i}^{(n-1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d \ , \qquad (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-1;1)} - b_{n-2}^{(n-1;1)} = R_{n-1,n-1}P_{k,k}/d \ , \\ a_{n-1}^{(n-1;1)} - b_{n-1}^{(n-1;1)} = P_{k,k} \ . \end{cases}$$

From (320) we obtain, in virtue of (4) and (201), (207),

$$(321) \quad \begin{cases} a_i^{(n-1;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d \ , \qquad (i = 1, \cdots, n-4) \\ a_{n-3}^{(n-1;2)} = R_{n-2,n-1}P_{k,k}/d \ , \\ a_{n-3}^{(n-1;2)} = R_{n-2,n-2}P_{k,k} \ , \\ a_{n-1}^{(n-1;2)} = R_{n-2,n-2} \ , \end{cases}$$

and from (321), since every $a_i^{(n-1;2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(322) $\quad b_i^{(n-1;2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;2)} = D_k - D_{n-2,n-2}$.

From (321), (322) we obtain, since $R_{n-2,n-2} - (D_k - D_{n-2,n-2}) = w - D_k = P_{k,k}$,

$$(323) \quad \begin{cases} a_1^{(n-1;2)} - b_1^{(n-1,2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d \ , \\ a_{1+i}^{(n-1;2)} - b_{1+i}^{(n-1;2)} = R_{1,n-4-i}R_{n-2,n-1}P_{k,k}/d \ , \qquad (i = 1, \cdots, n-5) \\ a_{n-3}^{(n-1;2)} - b_{n-3}^{(n-1;2)} = R_{n-2,n-1}P_{k,k}/d \ , \\ a_{n-2}^{(n-1;2)} - b_{n-2}^{(n-1;2)} = R_{n-2,n-2}P_{k,k} \ , \\ a_{n-1}^{(n-1;2)} - b_{n-1}^{(n-1;2)} = P_{k,k} \ , \end{cases}$$

and from (323), in virtue of (4) and (201), (207),

$$(324) \quad \begin{cases} a_i^{(n-1;3)} = R_{1,n-4-i}R_{n-3,n-1}P_{k,k}/d \ , \qquad (i = 1, \cdots, n-5) \\ a_{n-4}^{(n-1;3)} = R_{n-3,n-1}P_{k,k}/d \ , \\ a_{n-3}^{(n-1;3)} = R_{n-3,n-2}P_{k,k}; \ a_{n-2}^{(n-1;3)} = R_{n-3,n-3}P_{k,k} \ ; \\ a_{n-1}^{(n-1;3)} = R_{n-3,n-3} \ . \end{cases}$$

It is now easy to prove the formula

$$(325) \quad \begin{cases} a_i^{(n-1;t)} = R_{1,n-t-i}R_{n-t,n-1}P_{k,k}/d \ , \quad (i = 1, \cdots, n-2-t) \\ a_{n-1-t}^{(n-1;t)} = R_{n-t,n-1}P_{k,k}/d \ , \\ a_{n-1-t+j}^{(n-1;t} = R_{n-t,n-1-j}P_{k,k} \ , \qquad (j = 1, \cdots, t-1) \\ a_{n-1}^{(n-1;t)} = R_{n-t,n-t} \ , \\ t = 3, \cdots, n-3 \ . \end{cases}$$

Formula (325) is correct for $t = 3$, in virtue of formula (324), (325) is then proved by induction.

From (325) we obtain, since every $a_i^{(n-1;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(326) \qquad b_i^{(n-1;t)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;t)} = D_k - D_{n-t,n-t} \ ,$$

and further from (325), for $t = n - 3$,

$$(326a) \qquad \begin{cases} a_1^{(n-1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d \ , \\ a_2^{(n-1;n-3)} = R_{3,n-1}P_{k,k}/d \ , \\ a_{2+j}^{(n-1;n-3)} = R_{3,n-1-j}P_{k,k} \ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-3)} = R_{3,3} \ . \end{cases}$$

From (326a) we obtain, since every $a_i^{(n-1;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(327) \qquad b_i^{(n-1;n-3)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-3)} = D_k - D_{3,3} \ ,$$

and from (326), (327), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = D_{k,k}$,

$$(328) \qquad \begin{cases} a_1^{(n-1\cdot n-3)} - b_1^{(n-1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d \ , \\ a_2^{(n-1;n-3)} - b_2^{(n-1;n-3)} = R_{3,n-1}P_{k,k}/d \ , \\ a_{2+j}^{(n-1;n-3)} - b_{2+j}^{(n-1;n-3)} = R_{3,n-1-j}P_{k,k} \ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-1,n-3)} - b_{n-1}^{(n-1,n-3)} = P_{k,k} \ . \end{cases}$$

From (328) we obtain, in virtue of (4) and (201), (207),

$$(329) \qquad \begin{cases} a_1^{(n-1;n-2)} = R_{2,n-1}P_{k,k}/d \ , \\ a_{1+i}^{(n-1;n-2)} = R_{2,n-1-j}P_{k,k} \ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} = R_{2,2} \ . \end{cases}$$

From (329) we obtain, since every $a_i^{(n-1;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(330) \qquad b_i^{(n-1;n-2)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-2)} = D_k - D_{2,2} \ ,$$

and from (329), (330), since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(331) \qquad \begin{cases} a_1^{(n-1;n-2)} - b_1^{(n-1;n-2)} = R_{2,n-1}P_{k,k}/d \ , \\ a_{1+i}^{(n-1;n-2)} - b_{1+i}^{(n-1;n-2)} = R_{2,n-1-j}P_{k,k} \ , & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} - b_{n-1}^{(n-1;n-2)} = P_{k,k} \ . \end{cases}$$

From (331) we obtain, in virtue of (4) and (201), (207),

$$(332) \qquad \begin{cases} a_i^{(n;0)} = R_{1,n-1-j}P_{k,k} \ , & (j = 1, \cdots, n-2) \\ a_{n-i}^{(n;0)} = R_{1,1} \ . \end{cases}$$

Comparing formula (332) with formula (204), we obtain

$$a_i^{(0)} = a_i^{(n;0)} = a_i^{(n(n-1))} , \qquad (i = 1, \cdots, n-1) , \tag{333}$$

so that the Modified Algorithm of Jacobi-Perron for the basic sequence (204) is indeed purely periodic with length of period $T = n(n-1)$ for $d > 1$.

For $d = 1$ we obtain, comparing formula (226) with (204),

$$a_i^{(0)} = a_i^{(1;0)} = a_i^{(n-1)} , \qquad (i = 1, \cdots, n-1) , \tag{334}$$

so that in this case the Algorithm is purely periodic with length of period $T = n - 1$.

Formulae (316), (319), (322), (326), (327), (330) show that the $n - 1$ supporting sequences

$$b_1^{(n-1.k)}, b_2^{(n-1;k)}, \cdots, b_{n-1}^{(n-1;k)} , \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the n -th fugue as demanded by Theorem 6. Thus, for $d > 1$, and from what was proved before, the $n(n-1)$ supporting sequences of the Modified Algorithm of Jacobi-Perron form n fugues of the form (206a)—(206d). In case $d = 1$, they all have the form (205). By this Theorem 6 is completely proved.

The reader should note the necessity to presume $n > n_0$, (n_0 a constant) while carrying out the proof of Theorem 6. The cases $n = 2, \cdots, n_0$ are easily proved separately by the same mothods used for the proof of Theorem 6.

We shall now find units of the field $K(w)$ by means of the Modified Algorithm of Jacobi-Perron.

As Hasse and I have proved in our paper [16], a unit e of the field $K(w)$ is obtained from a periodic Jacobi-Perron Algorithm by means of formula (190), viz.

$$e^{-1} = \prod_{v=S}^{S+T-1} a_{n-1}^{(v)} ,$$

where S and T denote, as before, the lengths of the pre-period and period of the periodic Jacobi-Perron algorithm respectively.

It is one of the most striking and basic properties of *any* periodic algorithm G with integral supporting sequences

$$b_i^{(v)} , \qquad (i = 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

$b_i^{(v)}$ rational integers, that formula (190) holds for this general case of the G. The proof of this statement is not too complicated and follows exactly the lines of the methods used in [16], though certain additional results are necessary (see, for example, my paper [12]).

We then obtain from (190), since in our case again $S = 0$, $T = n(n-1)$ for $d > 1$, as in (191),

$$e_k^{-1} = \prod_{v=0}^{n(n-1)-1} a_{n-1}^{(v)} = \prod_{i=0}^{n-1} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} .$$

Now it is not difficult to verify, following up the various stages of the proof of the Modified algorithm of Jacobi-Perron, that the relations hold

$$(335) \qquad \begin{cases} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} = R_{1,n-1}/d , & (i = 0, 1, \cdots, n-3, n-1) \\ \prod_{k=0}^{n-2} a_{n-1}^{((n-2)(n-1)+k)} = R_{1,n-1} . \end{cases}$$

We thus obtain from (191), in virtue of (335),

$$(336) \qquad e_k^{-1} = (R_{1,n-1})^n/d^{n-1} .$$

From (201) we obtain $1/R_{1,n-1} = R_{0,0}/d$, and, since $R_{0,0} = R_{k,k}$,

$$(337) \qquad R_{1,n-1} = d/P_{k,k} .$$

From (336), (337) we now obtain

$$e_k^{-1} = d/(P_{k,k})^n ,$$

or

$$(338) \qquad e_k = \frac{(w - D_k)^n}{d} \qquad (k = 1, \cdots, n-1) ,$$

so that with (196), (338) Theorem 5. is now completely proved by means of the Modified Algorithm of Jacobi-Perron, since (338) includes the case $d = 1$, too.

The $n-1$ units $e_0, e_1, \cdots, e_{n-2}$ are all different, since $D_k > D_{k+1}$ $(k = 0, 1, \cdots, n-2)$. It is proved below that they are independent (see the Appendix by Hasse) in the sense that there cannot exist an equation of the form

$$e_0^{a_0} e_1^{a_1} \cdots e_{n-2}^{a_{n-2}} = 1 ,$$

where the $a_0, a_1, \cdots, a_{n-2}$ are rational integers not all equal zero.

Concluding we shall illustrate (338) by a numeric example. Let the GP be a fourth degree polynomial

$f(x) = (x-10)(x-6)(x-2)(x+4) - 2 = 0$;
$f(w) = 0$; $10 < w < 11$;
$D_0 = 10$; $D_1 = 6$; $D_2 = 2$; $D_3 = -4$; $d = 2$;
w is a fourth degree irrational.

We obtain from $f(w) = 0$:

$$w^4 - 14w^3 + 20w^2 + 248w - 482 = 0 ,$$
$$w^4 = 14w^3 - 20w^2 - 248w + 482 .$$

Thus

$$(w - 6)^4 = -10w^3 + 196w^2 - 1112w + 1778 ;$$
$$(w - 2)^4 = 6w^3 + 4w^2 - 280w + 498 ;$$
$$(w + 4)^4 = 30w^3 + 76w^2 + 8w + 738 .$$

Substituting these values in (338) we obtain the independent units

$$e_1 = 5w^3 - 98w^2 + 556w - 889 ;$$
$$e_2 = 3w^3 + 2w^2 - 140w + 249 ;$$
$$e_3 = 15w^3 + 38w^2 + 4w + 369 .$$

Appendix. (By Helmut HASSE, at present Honolulu (Hawaii)). In § 7 of this paper L. Bernstein, by applying a modified Jacobi-Perron algorithm to suitable bases of a certain type of totally real algebraic number-fields K of degree $n \geqq 2$, obtained a system of n algebraic units in K with product 1. I shall prove here under slightly stronger conditions that every $n - 1$ of these units are independent.

The fields K in question are generated by a root w of a polynomial of type

$$f(x) = \prod_{v=0}^{n-1} (x - D_v) - d , \tag{1}$$

where the D_v and d are rational integers, $d \geqq 1$, satisfying the conditions (184), viz.

$$D_0 > D_1 > \cdots > D_{n-1} , \tag{2}$$

$$D_v \equiv D_0 \operatorname{mod.} d , \tag{3}$$

$$D_0 - D_v \geqq 2d(n - 1) , \qquad (v = 1, \cdots, n - 1) , \tag{4}$$

and in the special case $d = 1$ moreover the inequalities (19), viz.

$$\begin{cases} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4 \text{ for } n = 3 , \\ D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \geqq 3 \text{ or} \\ \qquad D_0 - D_1, D_2 - D_3 \geqq 2 \text{ for } n = 4 . \end{cases} \tag{5}$$

In addition to these conditions I shall have to presuppose the inequalities

$$D_{2k-1} - D_{2k} \geqq 2 \qquad (2 \leqq 2k \leqq n - 1) \tag{6}$$

to be satisfied in the special case $d = 1$.

I shall prove

THEOREM. *Let w be a root of a polynomial of type* (1) *whose coefficients satisfy the conditions* (2), (3), (4), (5), (6). *Then the n algebraic numbers*

$$e_m = \frac{(w - D_m)^n}{d} \qquad (m = 0, 1, \cdots, n-1)$$

are algebraic units with product

$$\prod_{m=0}^{n-1} e_m = 1 ,$$

and every $n - 1$ of them are independent.

Proof. (a) By (3)

$$(w - D_m)^n \equiv \prod_{v=0}^{n-1} (w - D_v) \bmod. d ,$$

and by (1)

$$\prod_{v=0}^{n-1} (w - D_v) = f(w) + d = d .$$

Hence

$$(w - D_m)^n \equiv 0 \bmod. d ,$$

so that the e_m are algebraic integers.

(b) By (1) their product

$$\prod_{m=0}^{n-1} e_m = \prod_{m=0}^{n-1} d^{-1}(w - D_m)^n = \frac{(f(w) + d)^n}{d^n} = \frac{d^n}{d^n} = 1 .$$

Hence the e_m are algebraic units.

(c) According to Theorem 2, the generating polynomial $f(x)$ has n different real roots

$$w^{(0)} > w^{(1)} > \cdots > w^{(n-1)}$$

(each of which may take the place of the above w), and the relative position of these roots between and outside of the sequence (2) is such that, for every fixed v, in virtue of the congruences (3)

$$| w^{(v)} - D_m | > \begin{cases} d \text{ for all } m \neq v \text{ except possibly one} \\ \dfrac{1}{2}d \text{ for the possible exception } m \neq v . \end{cases}$$

The possible exception occurs for one of the two D_m which include $w^{(v)}$ (so far $v > 0$ and for even n also $v < n - 1$), and hence only for $n \geqq 3$ (since for $n = 2$ both roots $w^{(0)}, w^{(1)}$ are excluded by D_0, D_1). From these inequalities it follows that the units

$$e_m^{(v)} = \frac{(w^{(v)} - D_m)^n}{d}$$

for every fixed v satisfy the inequalities

$$|\, e_m^{(v)} \,| > \left\{\begin{array}{l} d^{n-1}/d = d^{n-2} \text{ for all } m \neq v \text{ except possibly one} \\ \dfrac{1}{2}d^{n-1}/d = \dfrac{1}{2}d^{n-2} \text{ for the possible exception } m \neq v \end{array}\right\} .$$

Since the exception does not occur for $n = 2$, and since in virtue of the presupposition (6) the factor 1/2 may be dropped in the special case $d = 1$, these inequalities imply throughout

$$|\, e_m^{(v)} \,| > 1 \quad \text{for} \quad m \neq v \,.$$

On the strength of the product relation then necessarily

$$|\, e_v^{(v)} \,| < 1 \,.$$

Now the polynomial $f(x)$ is irreducible, as Bernstein derived at the beginning of § 7 from Theorem 3. under the conditions (4). Hence for each fixed m the $e_m^{(v)}$ are the algebraic conjugates of e_m. Hence by a well-known theorem of Minkowski[1] the latter inequalities imply that for any fixed pair m_0, v_0 the determinant

$$|\log |\, e_m^{(v)} \,|\,|_{m \neq m_0, v \neq v_0} \neq 0 \,.$$

From this it follows that every $n - 1$ of the n units e_m are independent.

Note. In spite of this very simple theory of the unit system e_m, Bernstein's more lengthly subordination of these units under a modified Jacobi-Perron algorithm by means of Theorem 6. seems to me still to be of importance. "The more organic connection between a unit in a field K and a periodic algorithm of a basis of K", as Bernstein put it after Theorem 5, may be essential for attacking the important question whether those units are fundamental units of a ring (Dedekind order) in K. An answer to this question may lead to lower estimates of the class number h of K.[2]

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[2] See H. Hasse, Zahlentheorie, 2. Aufl., Berlin 1963; 28, **2**, Hilfsatz.

[3] Compare for this: H. Hasse, Über mehrklassige, aber eingeschlechtige reelquadratische Zahlkoerper, Elem. d. Math. **20** (1965), 49–59.

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BEST CONSTANTS IN A CLASS OF INTEGRAL INEQUALITIES

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In this paper a method is developed for determining best constants in inequalities of the following form:

$$\int_a^b |y|^p |y^{(n)}|^q w(x)dx \leqq K\left\{\int_a^b |y^{(n)}|^r m(x)dx\right\}^{(p+q)/r},$$

where $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0$ and $y^{(n-1)}$ is absolutely continuous.

It is first shown that for a certain class of m and w, equality can be attained in the inequality. Applying variational techniques reduces the determination of the best constant to a nonlinear eigenvalue problem for an integral operator. If m and w are sufficiently smooth this reduces further to a boundary value problem for a differential equation. The method is illustrated by determining the best constants in case (a, b) is a finite interval, $m(x) \equiv w(x) \equiv 1$, and $n = 1$.

A number of special cases of the inequality have been studied but usually without obtaining best constants. An exception to this is the case $n = 1, q = 0, p = r$ which was studied very thoroughly by Beesack [1], who gave a direct method for determining best constants. The method of [1] was modified by Boyd and Wong [5] to apply to the case $n = 1, q = 1, r = p + 1$. Recently Beesack and Das [2] obtained constants for the case $n = 1, r = p + q$ but these were not in general best possible.

We shall state our result only for $n = 1$ although it will be clear that the analogous result for $n > 1$ is valid. In our closing remarks we indicate a number of other inequalities to which the method of this paper applies.

1. **Preliminaries.** Throughout we assume that p, q, r, a, b are real numbers satisfying $p > 0, r > 1, 0 \leqq q < r$ and $-\infty \leqq a < b \leqq \infty$. The functions m and w are measurable and positive almost everywhere. We write $d\mu(x) = m(x)dx$ and

$$||f||_s = \left\{\int_a^b |f|^s d\mu\right\}^{1/s} \quad \text{for} \quad 0 < s < \infty .$$

The space L_m^s is the set of functions with $||f||_s < \infty$, with the usual identification. We shall use the notation $f_n \rightarrow f$ if $||f_n - f||_s \rightarrow 0$, and if $s \geqq 1$ so L_m^s is a Banach space, we write $f_n \xrightarrow{w} f$ for weak convergence in L_m^s. We denote the dual of L_m^s by $L_m^{s'}$ so for $s > 1$, $s' = s/(s-1)$.

We shall consider integral operators of the type

$$Tf(x) = \int_a^b k(x, t)f(t)d\mu(t) , \tag{1}$$

where $k(x, t) \geqq 0$ a.e. A function f is in the domain of T if

$$T|f|(x) < \infty \quad \text{a.e.}$$

For Theorem 1, the operator T becomes

$$T_1 f(x) = w(x)^{1/p} m(x)^{-1/p} \int_a^x f(t)dt , \tag{2}$$

so that $k(x, t) = w(x)^{1/p} m(x)^{-1/p} m(t)^{-1} \chi_{[a,x]}(t)$. A necessary and sufficient condition for the domain of T_1 to contain L_m^r is that

$$\int_a^x m(t)^{-1(r-1)} dt < \infty \quad \text{for} \quad a \leqq x < b .$$

This follows from Hölder's inequality and its converse.

If T maps $L_m^r \to L_m^s$, where $s = pr/(r - q)$, with norm $\| T \| < \infty$, then we can define the functional J on L_m^r by

$$J(f) = \int_a^b | Tf |^p | f |^q d\mu . \tag{3}$$

It then follows from Hölder's inequality that

$$J(f) \leqq \| T \|^p \| f \|_r^{p+q} . \tag{4}$$

2. Main results.

THEOREM 1. *Suppose that* $w, m \in C^1(a, b)$, *that* $w(x) > 0$ a.e. *and* $m(x) > 0$ *for* $a < x < b$, *that* $p > 0, r > 1, 0 \leqq q < r$, *and that the operator* T_1 *defined by* (2) *is compact from* $L_m^r \to L_m^s (s = pr/(r - q))$. *Then the following eigenvalue problem* (P) *has solutions* (y, λ) *with* $y \in C^2(a, b)$ *and* $y(x) > 0, y'(x) > 0$ *in* (a, b).

$$(P)\begin{cases} \text{(i)} & \dfrac{d}{dx}(r\lambda y'^{r-1}m - qy^p y'^{q-1}w) + py^{p-1}y'^q w = 0 \\ \text{(ii)} & \lim_{x\to a} y(x) = 0 \textit{ and } \lim_{x\to b}(r\lambda y'^{r-1}m - qy^p y'^{q-1}w) = 0 \\ \text{(iii)} & \| y' \|_r = 1 . \end{cases}$$

There is a largest value λ *such that* (P) *has a solution and if* λ^* *denotes this value, then for any* $f \in L_m^r$,

$$\int_a^b \left| \int_a^x f \right|^p | f |^q w(x)dx \leqq \frac{r\lambda^*}{p + q} \left\{ \int_a^b | f |^r m(x)dx \right\}^{(p+q)/r} . \tag{5}$$

Equality holds in (5) *if and only if* $f = cy'$ a.e. *where* y *is a solution* *of* (P) *corresponding to* $\lambda = \lambda^*$, *and* c *is any constant.*

The proof will require two lemmas which we state in reasonable generality.

LEMMA 1. *Suppose that* $p > 0, r > 1, 0 \leqq q < r$, *and that* T, *as defined by* (1) *is a compact operator from* $L_m^r \to L_m^s$, $(s = pr/(r-q))$. *Let* J *be defined by* (3), *and*

$$K^* = \sup\{J(f): \|f\|_r \leqq 1\} . \tag{6}$$

Then, there is an element $f_0 \in L_m^r$ *with* $\|f_0\|_r = 1$ *such that* $J(f_0) = K^*$.

Proof. Since $J(f) < J(|f|)$ unless f is of constant sign a.e., we can restrict consideration in (6) to $f \geqq 0$. Let $\{f_n\} \in L_m^r$ be a sequence with $f_n \geqq 0, \|f_n\| \leqq 1$ such that $J(f_n) \to K^*$. We begin by assuming $q > 0$ so that $1 < r/q < \infty$. By the weak sequential compactness of the unit balls of L_m^r and $L_m^{r/q}$ ([7], p. 68), and by the compactness of T, we may assume that there are functions

$$f \in L_m^r,\ h \in L_m^{r/q},\ g \in L_m^s$$

such that $f_n \xrightarrow{w} f$, $f_n^q \xrightarrow{w} h$, $Tf_n \to g$ in the appropriate spaces; clearly $Tf = g$. Furthermore, by the uniform convexity of L_m^r and $L_m^{r/q}$ we may assume that f_n and f_n^q are strongly $(C, 1)$-summable to their weak limits ([7], p. 462), so that

$$\hat{f}_n = n^{-1}\sum_{k=1}^{n} f_k \longrightarrow f \quad \text{and} \quad \hat{h}_n = n^{-1}\sum_{k=1}^{n} f_k^q \longrightarrow h .$$

Now, we have

$$J(f_n) - \int_a^b g^p h\,d\mu = \int_a^b ((Tf_n)^p - g^p)f_n^q + \int_a^b g^p(f_n^q - h) . \tag{7}$$

Now, since $f_n^q \xrightarrow{w} h$ in $L_m^{r/q}$ and since $g^p \in L_m^{s/p} = L_m^{(r/q)'}$, the second integral in the right member of (7) tends to zero as $n \to \infty$. To show that the first integral tends to zero we consider separately $0 \leqq p < 1$ and $1 \leqq p < \infty$. If $0 \leqq p < 1$, we use the inequality $|A^p - B^p| \leqq |A - B|^p$ for $A \geqq 0, B \geqq 0$ to obtain

$$\begin{aligned} \left|\int_a^b (Tf_n)^p f_n^q - \int_a^b g^p f_n^q\right| &\leqq \int_a^b |Tf_n - g|^p f_n^q \\ &\leqq \|Tf_n - g\|_s^p \|f_n\|_r^q \leqq \|Tf_n - g\|_s^p . \end{aligned} \tag{8}$$

The second step follows from Hölder's inequality with exponents $s/p = r/(r-q)$ and r/q. The final term in (8) tends to zero since $Tf_n \to g$ in L_m^s.

In case $1 \leqq p < \infty$, we consider instead

$$\left|\left\{\int_a^b (Tf_n)^p f_n^q\right\}^{1/p} - \left\{\int_a^b g^p f_n^q\right\}^{1/p}\right| \leqq \left\{\int_a^b |Tf_n - g|^p f_n^q\right\}^{1/p} \tag{9}$$

by Minkowski's inequality. As in (8), the right member of (9) tends to zero. Thus, if $A_n = \int_a^b (Tf_n)^p f_n^q$ and $B_n = \int_a^b g^p f_n^q$, we have that

$$|A_n^{1/p} - B_n^{1/p}| \longrightarrow 0 .$$

But $\{A_n\}$ and $\{B_n\}$ are bounded sequences ($A_n = J(f_n) \leqq \|T\|^p$ by (4), $B_n \leqq \|g\|_s^p \leqq \|T\|^p$ by Hölder's inequality and $Tf_n \to g$), and thus $|A_n - B_n| \leqq p\,|A_n^{1/p} - B_n^{1/p}| \cdot \|T\|^{p-1}$ shows that $A_n - B_n \to 0$ as required. Hence, we have

$$K^* = \int_a^b g^p h d\mu . \tag{10}$$

In case $q = 0$, (10) also holds with $h \equiv 1$, by a similar argument.

Now we show the existence of f_0 for which $J(f_0) = K^*$. The cases $0 \leqq q < 1$ and $1 \leqq q < r$ are considered separately. If $0 \leqq q < 1$, define $f_0 = f$. Since $\varphi(t) = t^q$ is concave, we have

$$\hat{f}_n^q = (n^{-1} \sum_1^n f_k)^q \geqq n^{-1} \sum_1^n f_k^q = \hat{h}_n . \tag{11}$$

Now, since $\hat{f}_n \to f_0$ in L_m^r, we have

$$\begin{aligned} \left|\int_a^b g^p f_0^q - \int_a^b g^p \hat{f}_n^q\right| &\leqq \int_a^b g^p |f_0^q - \hat{f}_n^q| \\ &\leqq \int_a^b g^p |f_0 - \hat{f}_n|^q \leqq \|g\|_s^p \|f_0 - \hat{f}_n\|_r^q \longrightarrow 0 . \end{aligned} \tag{12}$$

Similarly, $\int_a^b g^p \hat{h}_n \to \int_a^b g^p h$. Thus, combining (10), (11) and (12) we obtain

$$\begin{aligned} J(f_0) &= \int_a^b g^p f_0^q = \lim \int_a^b g^p \hat{f}_n^q \\ &\geqq \lim \int_a^b g^p \hat{h}_n = \int_a^b g^p h = K^* . \end{aligned} \tag{13}$$

However $\|f_0\|_r \leqq 1$ so $J(f_0) \leqq K^*$ and hence (13) implies $J(f_0) = K^*$ from which it is clear that $\|f_0\|_r = 1$.

In case $1 \leqq q < r$, let $f_0 = h^{1/q}$. Now, instead of (11), we have $\hat{f}_n^q \leqq \hat{h}_n$. Since $\|\hat{h}_n - h\|_{r/q} \to 0$, and since $|\hat{h}_n^{1/q} - h^{1/q}| \leqq |\hat{h}_n - h|^{1/q}$ we have

$$\begin{aligned} \|\hat{h}_n^{1/q} - h^{1/q}\|_r &= \left\{\int_a^b |\hat{h}_n^{1/q} - h^{1/q}|^r\right\}^{1/r} \\ &\leqq \left\{\int_a^b |\hat{h}_n - h|^{r/q}\right\}^{1/r} = \|\hat{h}_n - h\|_{r/q}^{1/q} . \end{aligned}$$

Thus $\hat{h}_n^{1/q} \to h^{1/q} = f_0$ in L_m^r and since T is continuous, $T\hat{h}_n^{1/q} \to Tf_0$ in L_m^r. However $\hat{f}_n \leqq \hat{h}_n^{1/q}$ and $k(x, t) \geqq 0$ a.e. so $T\hat{f}_n \leqq T\hat{h}_n^{1/q}$, a.e. and thus $Tf = g \leqq Tf_0$ a.e. Thus (10) implies $K^* \leqq J(f_0)$, which again means that $J(f_0) = K^*$ and $\|f_0\|_r = 1$.

REMARK. A simple sufficient condition for T to be compact from $L_m^r \to L_m^s$ is that k have finite (r', s)-double norm. That is

$$\text{(14)} \qquad |||T||| = \left\{\int_a^b \left[\int_a^b k(x, t)^{r'} d\mu(t)\right]^{s/r'} d\mu(x)\right\}^{1/s} < \infty$$

(see [9], p. 319; the proof there applies even if $0 < s < 1$).

Using (4), we see that $K^* \leqq \|T\|^p \leqq |||T|||^p$ so (14) also supplies an upper bound for K^* (rarely the best).

For the operator T_1 given by (2) one may calculate that

$$\text{(15)} \qquad |||T_1|||^s = \int_a^b w(x)^{r/(r-q)} m(x)^{-q/(r-q)} \left[\int_a^x m(t)^{-1/(r-1)} dt\right]^{s/r'} dx\ .$$

In the paper of Beesack and Das [2], the following inequality is proved: If $pq > 0$, $p + q > 1$, $y(a) = 0$ and y is absolutely continuous, then

$$\text{(16)} \qquad \int_a^b |y|^p |y'|^q w(x) dx \leqq K_1(b, p, q) \int_a^b |y'|^{p+q} m(x) dx\ ,$$

where $K_1(b, p, q)$ is explicitly given. The constant $K_1(b, p, q)$ equals the best constant K^* if and only if for some $c \geqq 0$

$$\text{(17)} \qquad w(x) = cm(x)^{(q-1)/(r-1)} \left(\int_a^x m(t)^{-1/(r-1)} dt\right)^{p(1-q)/q} (r = p + q)\ .$$

The constant $K_1(b, p, q)$ given there is in fact equal to $(q/r)^{q/r} |||T_1|||^p$, so, unless (17) holds we have

$$\text{(18)} \qquad K^* < K_1(b, p, q) < |||T_1|||^p\ .$$

LEMMA 2. *Suppose that T is given by* (1), *and that $k(x, t) > 0$ for almost all (x, t) with $a \leqq t \leqq x \leqq b$. Let $p > 0, r > 1, 0 \leqq q < r$, and suppose T is a bounded operator from $L_m^r \to L_m^s$. Let J be defined by* (3), *K^* by* (6). *Let f satisfy $\|f\|_r = 1$ and $J(f) = K^*$. Then f is of constant sign* a.e. *and*

(a) $f \neq 0$ a.e.

(b) *f satisfies* a.e. *the equation*

$$\text{(19)} \quad r\lambda f^{r-1}(x) - q(Tf)^p(x) f^{q-1}(x) - p\int_a^b k(t, x)(Tf)^{p-1}(t) f^q(t) d\mu(t) = 0\ ,$$

where $\lambda = \lambda^* = K^*(p + q)/r$. Furthermore λ^* is the largest value of

λ *for which* (19) *has a solution* f *with* $\|f\|_r = 1$.

Proof. (a) We have seen that f is of constant sign a.e. so we assume $f \geqq 0$ a.e. Let $E = \{x: f(x) = 0\}$; we must show that E is a null set. First choose a function $h \in L^r_m$ such that $h(x) \geqq 0$, and $h(x) > 0$ if and only if $x \in E$. Such h exist: if $\mu(E) < \infty$, take $h = \chi_E$, while if $\mu(E) = \infty$, let

$$E_n = E \cap [-n, n] \cap \{x: m(x) \leqq n\} ,$$

so $\mu(E_n) < \infty$, and define $h = \sum \gamma_n \chi_{E_n}$ where $\{\gamma_n\}$ is chosen so $\gamma_n > 0$ and $\sum \gamma_n^r \mu(E_n) < \infty$.

For $\varepsilon > 0$, define $f_\varepsilon = f + \varepsilon h$, and let $F = Tf$, $F_\varepsilon = Tf_\varepsilon$, $H = Th$. since $J(f)/\|f\|_r^{p+q}$ is maximal, we have

$$\begin{aligned} 0 &\leqq J(f_\varepsilon) - J(f) \leqq (\|f_\varepsilon\|_r^{p+q} - 1)J(f) \\ &= \{(1 + \varepsilon^r \|h\|_r^r)^{(p+q)/r} - 1\}J(f) \\ &= \varepsilon^r \|h\|_r^r \gamma \xi^{\gamma-1} J(f), \text{ (where } \gamma = (p+q)/r, \text{ and } 1 < \xi < 1 + \varepsilon^r \|h\|_r^r \\ &= 0(\varepsilon^r) \text{ as } \varepsilon \downarrow 0. \end{aligned} \tag{20}$$

First assume that $q > 0$, so if $CE = [a, b]\backslash E$, we may write

$$J(f_\varepsilon) - J(f) = \varepsilon^q \int_E F_\varepsilon^p h^q + \int_{CE} (F_\varepsilon^p - F^p) f^q . \tag{21}$$

From (20) and (21) we immediately deduce that

$$0 \leqq \int_E F^p h^q \leqq \int_E F_\varepsilon^p h^q = 0(\varepsilon^{r-q}) \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 .$$

Thus, $F(x) = 0$ a.e. on E so $k(x, t) = 0$ a.e. on $E \times CE$.

Next, we note that $F(x) > 0$ a.e. on CE, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$. Thus, for almost all x in CE, we have $(d/d\varepsilon)F_\varepsilon^p f^q = pF_\varepsilon^{p-1} H f^q$. Hence, if $0 < \varepsilon < \varepsilon_0$ we have.

$$pF^{p-1} H f^q \leqq \varepsilon^{-1}(F_\varepsilon^p - F^p) f^q \text{ a.e. on } CE, p \geqq 1 \tag{22}$$

$$pF_{\varepsilon_0}^{p-1} H f^q \leqq \varepsilon^{-1}(F_\varepsilon^p - F^p) f^q \text{ a.e. on } CE, 0 < p < 1 . \tag{23}$$

Thus, if $p \geqq 1$, (20), (21) and (22) imply that

$$0 \leqq \int_{CE} pF^{p-1} H f^q \leqq \varepsilon^{-1} \int_{CE} (F_\varepsilon^p - F^p) f^q = 0(\varepsilon^{r-1}) \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 . \tag{24}$$

Thus, since $F(x) \neq 0$, we have $H(x) = 0$ a.e. on CE. A similar argument using (20), (21) and (23) proves $H(x) = 0$ a.e. on CE, if $0 < p < 1$. Thus $k(x, t) = 0$ a.e. on $CE \times E$, and hence on $(E \times CE) \cup (CE \times E)$. But, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$, the last sentence implies that $ExCE$ has plane measure zero and so either $\mu(E) = 0$ or $\mu(CE) = 0$. However, $\mu(CE) = 0$ implies that $f = 0$ a.e. contradicting $J(f) = K^* \neq 0$.

Thus $\mu(E) = 0$ as required.

In case $q = 0$, (21) no longer holds. In this case, let

$$A = \{x: F(x) = 0\}$$

so $k(x, t) = 0$ a.e. on $A \times CE$. Clearly $\mu(A \cap CE) = 0$, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$. Instead of (21) we have

$$J(f_\varepsilon) - J(f) = \varepsilon^p \int_A H^p + \int_{CA} (F_\varepsilon^p - F^p) . \tag{25}$$

Proceeding as in (24), we use the second integral in (25) together with (20) to show that $H(x) = 0$ a.e. on CA, so $k(x, t) = 0$ a.e. on $CA \times E$. Now if $B = CA \cap E$ has $\mu(B) > 0$, we would have $k(x, t) = 0$ a.e. on $B \times B$ with contradicts $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$ and thus $\mu(B) = \mu(E \backslash A) = 0$. We already have shown that $\mu(A \backslash E) = 0$. Thus $k(x, t) = 0$ a.e. on $(A \times CE) \cup (CA \times E)$ means $k(x, t) = 0$ a.e. on $(E \times CE) \cup (CE \times E)$, which leads to a contradiction as before. (We note that if $p < r$, a simpler argument is available using the first integral in (25).)

(b) Consider the functional

$$I(f) = \lambda \| f \|_r^r - J(f) = \int_a^b [\lambda f^r - (Tf)^p f^q] d\mu .$$

We shall show that if $J(f) = K^*$, and if $|h| \leqq f$, then for $\lambda = \lambda^* = K^*(p + q)/r$, we have

$$\delta I(f; h) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(I(f + \varepsilon h) - I(f)) = 0 . \tag{26}$$

First, suppose that $|h| \leqq f$ and that $|\varepsilon| \leqq 1/2$. Now define $A(\varepsilon) = J(f + \varepsilon h)$ and $B(\varepsilon) = \| f + \varepsilon h \|_r^r$. Then A and B are differentiable at $\varepsilon = 0$, and

$$A'(0) = \int_a^b (pF^{p-1} f^q H + qF^p f^{q-1} h) d\mu \tag{27}$$

$$B'(0) = \int_a^b r f^{r-1} h d\mu . \tag{28}$$

To see this, note that $(d/d\varepsilon) F_\varepsilon^p f_\varepsilon^q = pF_\varepsilon^{p-1} H f_\varepsilon^q + qF_\varepsilon^p f_\varepsilon^{q-1} h$ a.e. since $f > 0$ a.e. by (a), and $F > 0$ a.e. since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$, and thus $f_\varepsilon > 0$ a.e., $F_\varepsilon > 0$ a.e. for $|\varepsilon| \leqq \frac{1}{2}$ and $|h| \leqq f$.

But, we have

$$\int_a^b F^{p-1} H f^q \leqq \left\{ \int_a^b F^s \right\}^{(p-1)/s} \left\{ \int_a^b H^s \right\}^{1/s} \left\{ \int_a^b f^r \right\}^{q/r} \tag{29}$$

by Hölder's general inequality with exponents $s/(p-1)$, s and r/q. Similarly, one shows $F^p f^{q-1} h$ is integrable. And, for $|\varepsilon| \leqq \frac{1}{2}$ one may bound $|(d/d\varepsilon) F_\varepsilon^p f_\varepsilon^q|$ in terms of $F^{p-1} H f^q$ and $F^p f^{q-1} h$. For example, if $p \geqq 1, q \geqq 1$, one has

$$(30)\qquad \left|\frac{d}{d\varepsilon}F_\varepsilon^p f_\varepsilon^q\right| \leqq \left(\frac{3}{2}\right)^{p+q-1}(pF^{p-1}Hf^q + qF^pf^{q-1}h) \quad \text{a.e.}$$

with similar bounds if $0 < p < 1$ or $0 \leqq q < 1$. Thus, Lebesgue's dominated convergence theorem gives (27). A similar argument gives (28).

By assumption $J(f)/||f||_r^{p+q} = A(0)/B(0)^{(p+q)/r}$ is maximal and hence

$$\frac{d}{d\varepsilon}(A(\varepsilon)B(\varepsilon)^{-(p+q)/r})\Big|_{\varepsilon=0} = 0 .$$

Differentiating and using $A(0) = K^*$ and $B(0) = 1$, we obtain

$$(30)\qquad A'(0) - K^*((p+q)/r)B'(0) = 0$$

or if we write $\lambda^* = K^*(p+q)/r$, we obtain

$$(31)\qquad \int_a^b (r\lambda^* f^{r-1}h - pF^{p-1}f^qH - qF^pf^{q-1}h)d\mu = 0 .$$

By Fubini's theorem we have

$$(32)\qquad \begin{aligned}\int_a^b F^{p-1}f^qH &= \int_a^b F^{p-1}(x)f^q(x)\left(\int_a^b k(x, t)h(t)d\mu(t)\right)d\mu(x)\\ &= \int_a^b h(t)\left(\int_a^b k(x, t)F^{p-1}(x)f^q(x)d\mu(x)\right)d\mu(t) .\end{aligned}$$

Thus, if we write T' for the operator with kernel $k(t, x)$ we have from (31) and (32)

$$(33)\qquad \begin{aligned}0 &= \int_a^b h(x)\{r\lambda^* f^{r-1} - q(Tf)^pf^{q-1} - pT'((Tf)^{p-1}f^q)\}d\mu(x)\\ &= \int_a h(x)G(x)d\mu(x) .\end{aligned}$$

To obtain (19) set $h(x) = f(x)\operatorname{sgn} G(x)$ in (33) and use the fact that $f(x) \neq 0$ a.e.

To see that λ^* is the largest value of λ for which a solution to (19) is possible with $||f||_r = 1$, note that if (19) holds then (33) and hence (31) hold for any $|h| \leqq f$ with λ in place of λ^*. Thus, setting $h = f$ in (31) (with λ for λ^*), we obtain $r\lambda||f||_r^r - (p+q)J(f) = 0$, and thus $\lambda = (p+q)J(f)/r \leqq (p+q)K^*/r = \lambda^*$.

REMARK. Part (a) of Lemma 2 may be strengthened by allowing k to vanish on more extensive sets. However, the precise condition that is needed to insure $f \neq 0$ a.e. depends on the relationship of p, q and r. For example, if $q > 0$ and $p < r$, and if there are no sets E with $\mu(E) > 0$ and $\mu(CE) > 0$ such that k vanishes on $(E \times CE) \cup (CE \times E)$ then for f as in Lemma 2, one has $f \neq 0$ a.e.

Proof of Theorem 1. By Lemma 1, $\sup\{J(f): \|f\|_r \leqq 1\} = K^* < \infty$, and there is an $f \geqq 0$ with $\|f\|_r = 1$ and $J(f) = K^*$. Since $m(x) > 0$ and $w(x) > 0$ a.e., Lemma 2 applies and we have $f \neq 0$ a.e. in $[a, b]$, and f satisfies

$$(34) \qquad \lambda r f^{r-1}(x)m(x) - qF(x)^p f(x)^{q-1}w(x) - p\int_x^b F^{p-1}f^q w = 0 \text{ a.e.}$$

where $F(x) = \int_a^x f(t)dt$.

We claim that by modifying f on a set of measure zero, we will have $f \in C^1(a, b)$, $f(x) \neq 0$ in (a, b) and f will satisfy (34) everywhere. To see this, rewrite (34) as

$$(35) \qquad f^{r-1} - A(x)f^{q-1} = B(x) \text{ a.e.}$$

where $A(x) \geqq 0$, and $B(x) > 0$ for all $x \in (a, b)$.

Consider the equation $\zeta^{r-1} - \xi\zeta^{q-1} = \eta$. For $\eta > 0$, $\xi \geqq 0$ this has a unique positive solution $\zeta = \varphi(\xi, \eta)$ which can be extended to be C^∞ on an open region containing the set $\{(\xi, \eta): \xi \geqq 0, \eta \geqq 0, \xi + \eta > 0\}$. To see this, consider the function $\psi(\zeta) = \zeta^{r-1} - \xi\zeta^{q-1}$ for fixed ξ, r and q. First suppose $q \geqq 1$, and $\xi > 0$, then $\psi'(\zeta)$ has a single positive zero $\zeta_0 = \zeta_0(\xi)$, and ψ decreases from $\psi(0) = 0$ to $\psi(\zeta_0) < 0$ and is strictly increasing on $[\zeta_0, \infty)$ to $+\infty$. Thus $\psi(\zeta) = \eta$ has a unique solution for $\eta > 0$ which we denote $\varphi(\xi, \eta)$. We define $\varphi(\xi, \eta)$ for $\xi > 0$ and $0 \geqq \eta > \psi(\zeta_0(\xi))$ to be that solution of $\eta = \psi(\zeta)$ with $\zeta > \zeta_0$. If $q \geqq 1$ and $\xi \leqq 0$, then ψ is strictly increasing from $\psi(0) = 0$, hence $\psi(\zeta) = \eta$ has a unique solution for $\eta \geqq 0$. Thus, for $q \geqq 1$, $\psi(\xi, \eta)$ is defined on an open set containing $Q = \{(\xi, \eta): \xi \geqq 0, \eta \geqq 0, \zeta + \eta > 0\}$, and since $\psi'(\varphi(\xi, \eta)) > 0$, the implicit function theorem shows that $\varphi \in C^\infty$. To show that $\varphi(\xi, \eta) \to \varphi(0, 0) = 0$ as $(\xi, \eta) \to (0, 0)$ in Q, we note that if $0 \leqq \xi \leqq \delta$, $0 \leqq \eta \leqq \delta$ and $\zeta_1 = \alpha\delta^{1/(r-1)}$ with $\alpha = 2^{1/(r-q)}$, then

$$\psi(\zeta_1) \geqq \alpha^{r-1}\delta - \delta\alpha^{q-1}\delta^{(q-1)/(r-1)} \geqq \alpha^{q-1}(\alpha^{r-q} - 1)\delta \geqq \delta \text{ (if } \delta \leqq 1) .$$

Thus $\varphi(\xi, \eta) \leqq \alpha\delta^{1/(r-1)}$ for $0 \leqq \xi \leqq \delta$, $0 \leqq \eta \leqq \delta$ proving the assertion.

If $0 \leqq q < 1$ and $\xi > 0$, then ψ is strictly increasing from $-\infty$ to ∞ on $(0, \infty)$ so $\psi(\zeta) = \eta$ has a unique solution for all η. If $0 \leqq q < 1$ and $\xi < 0$, then $\psi(\zeta) \to \infty$ as $\zeta \to 0+$ or $\zeta \to \infty$, and ψ has a minimum at a point ζ_0 where $\psi(\zeta_0) = \gamma|\xi|^{(r-1)/(r-q)}$ and $\gamma > 0$. If $\xi = 0$, $\psi(\xi) = \eta$ has a unique solution for $\eta \geqq 0$. Again we have $\varphi \in C^\infty$ on an open set containing Q and that $\varphi(\xi, \eta) \to 0$ as $(\xi, \eta) \to (0, 0)$ in Q.

Now, from (35), by modifying f on a null set, we have

$$(36) \qquad f(x) = \varphi(A(x), B(x)) \quad \text{for all } x \in (a, b) .$$

If $w, m \in C^1$ then A, B are absolutely continuous so (36) shows that f

is absolutely continuous. But then $F \in C^1$ so in fact, $A, B \in C^1$ and (36) shows that $f \in C^1$. That $f(x) \neq 0$ for $x \in (a, b)$ follows immediately from (36).

Now, defining $y = F$ and differentiating (34) once gives (P) (i). The conditions (ii) and (iii) are apparent from (34). The problem (P) thus has solutions for $\lambda = K^*(p+q)/r$. To identify the largest eigenvalue of (P) as $K^*(p+q)/r$, we note that a solution of (P) gives a solution of (34) and by Lemma 2 the largest eigenvalue of (34) is $K^*(p+q)/r$.

The inequality (5) and the statement concerning equality are now obvious.

REMARK. If $m(x) > 0$ and $w(x) > 0$ for all $x \in [a, b]$, and if $q > 0$, then $A(x) > 0$ unless $x = a$ and $B(x) > 0$ unless $x = b$. Hence equation (36) shows that $f(x) > 0$ for all $x \in [a, b]$; and $f \in C^1[a, b]$. We also note that if $\lim_{x \to b} A(x)$ is finite and $\lim_{x \to a} B(x)$ is finite then $f(a) < \infty$ and $f(b) < \infty$. This will be used in § 3.

3. Some inequalities on a finite interval. As an application of Theorem 1, we obtain the best constants in case (a, b) is a finite interval and $m(x) \equiv w(x) \equiv 1$. We immediately consider

$$\int_0^1 |y|^p |y'|^q dx \leqq K(p, q, r)\left\{\int_0^1 |y'|^r dx\right\}^{(p+q)/r} \tag{37}$$

where y is absolutely continuous and $y(0) = 0$.

Some special cases of (37) are known. The case $q = 0, p = r = 2k$ (k a positive integer) is inequality 256 of [8], which was derived there by classical variational methods using the Weierstrass sufficient condition. This case was handled by elementary methods in [3]. Opial's inequality is the case $p = q = 1, r = 2$. If $q = 1, r = p + 1$, the best constant can be obtained by Hölder's inequality (see [5], for example). The case $r = p + q$ was considered in [6] but the best constant was found only when $q = 1$ or $r = 1$.

Note that if $q > r$, there is no inequality of the form (37), since for $y(x) = 1 - (1 - x)^{1-\gamma}, q^{-1} < \gamma < r^{-1}$, the left member of (37) is infinite while $\| y' \|_r < \infty$. The case $p = 0$ is simply Hölder's inequality with $K(0, q, r) = 1$.

THEOREM 2. *For $r \geqq 1, p > 0, 0 \leqq q \leqq r$, the inequality* (37) *is valid with a finite constant $K(p, q, r)$. The best such constant is given by the following expressions*

(a) *if $p > 0, r > 1, 0 \leqq q < r$, then*

$$K(p, q, r) = \frac{(r-q)p^p}{(r-1)(p+q)} \beta^{p+q-r} I(p, q, r)^{-p} \tag{38}$$

where

$$\beta = \left\{\frac{p(r-1)+(r-q)}{(r-1)(p+q)}\right\}^{1/r}$$

and

$$I(p, q, r) = \int_0^1 \left\{1 + \frac{r(q-1)}{r-q}\, t\right\}^{-(q+p+rp)/rp} \{1 + (q-1)t\} t^{1/p-1} dt\ .$$

(b) *If* $r = 1$, *then*

$$K(p, q, 1) = \begin{cases} q^q(p+q)^{-q}, & q > 0 \\ 1 & , q = 0\ . \end{cases}$$

(c) *If* $q = r$, *then*

$$K(p, r, r) = \frac{rp^p}{p+r}\left(\frac{p}{p+r}\right)^{p/r} B\left(\frac{1}{r} + 1, \frac{1}{p}\right)^{-p} . \tag{39}$$

If $r = 1, q = 0$, *there is strict inequality for all* $y \not\equiv 0$ *while in all other cases there is equality only for multiples of a single function* $y(p, q, r, x)$ *which is in* $C^\infty(0, 1)$, *and is concave if* $0 \leqq q < 1$, *convex if* $q > 1$, *linear if* $q = 1$.

For special cases of (a), (28) reduces to a simpler form. First, if $r = p + q$, we have

$$K(p, q, p+q) = q(p+q)^{p-1}\{pL(p, q) + q\}^{-p},\ q \neq 0 \tag{40}$$

where

$$L(p, q) = \int_0^1 \frac{ds}{1 - ks^p},\ k = \frac{(p+q)(q-1)}{(p+q-1)q}\ .$$

In particular,

$$K(1, q, 1+q) = \left\{1 + \frac{2q}{q^2-1}\log q\right\}^{-1},\ q \neq 0, 1\ .$$

If $q = 0$, and $r > 1$, we have

$$\begin{aligned} K(p, 0, r) &= A(p, r)^p \\ A(p, r) &= (r')^{1/p} p^{1/r'} (p + r')^{1/r - 1/p} B(1/p, 1/r')^{-1}\ , \end{aligned}$$

where $r' = r/(r-1)$. Note that $A(p, r)$ is the norm of the mapping $T: L^r \to L^p$ where here $Tf(x) = \int_0^x f(t)dt$. By (4), if $\|f\|_r = 1$ we have $J(f) \leqq \|T\|^p$, where $\|T\|$ is the norm of T as a mapping from $L^r_m \to L^s_m$, $(s = pr/(r-q))$, and so we always have $K(p, q, r) \leqq A(s, r)^p$.

We also note that in (38), if $q \neq 1$, one may make the replacement

$$(r-q)I(p,q,r)^{-p} = r^{p+1}(q-1)I_1(p,q,r)^{-p}\ , \tag{41}$$

where

$$I_1(p,q,r) = \pm\,(r-q)\int_0^{\pm T} t^{1/p}(1\mp t)^{\gamma-1}dt + r\int_0^{\pm T} t^{(1/p)-1}(1\mp t)^{\gamma}dt\ ,$$

where $T_1 = 1 - [(r-q)/q(r-1)]$, $\gamma = (p+q-r)/rp$, and the upper sign is used with $q > 1$, the lower sign with $q < 1$.

Proof. In case (a), Theorem 1 applies since certainly $|||\,T_1\,||| < \infty$. We seek solutions of the problem (P). We first observe that by the remark at the end of § 2, we have $y \in C^2[a, b]$ and $0 < y'(0) < \infty$, $0 < y'(1) < \infty$ except in case $q = 0$ when we have $y'(1) = 0$. To see this note that the functions A and B which appear in (35) are here just

$$A(x) = q(\lambda r)^{-1}y(x)^p,\ B(x) = p(\lambda r)^{-1}\int_x^1 y(t)^{p-1}y'(t)^q dt\ .$$

But $y(1) = \int_0^1 y'(t)dt \leqq \|\,y'\,\|_r < \infty$, so $A(1) < \infty$, and

$$\int_0^1 y^{p-1}y'^q \leqq \left\{\int_0^1 y^{r(p-1)/(r-q)}\right\}^{(r-q)/r}\|\,y'\,\|_r^q$$

which shows that $B(0) < \infty$.

Notice that equation (i) of (P) has the integrating factor y' from which we obtain

$$(r-1)\lambda y'^r - (q-1)y^p y'^q = \alpha\lambda\ , \tag{42}$$

where α is a constant which is evaluated by using $\|\,y'\,\|_r = 1$ and $J(y') = r\lambda/(p+q)$. Thus we have

$$\alpha = (r-1) - \frac{(q-1)r}{p+q} = \frac{pr-p-q+r}{p+q} > 0\ . \tag{43}$$

Solving (42) for $y = \lambda^{1/p}G(y')$ and differentiating leads to a variables separable equation for y', and if we write $z = y'$ we have, for $q \neq 1$

$$\begin{aligned} dx = \frac{\lambda^{1/p}}{p}&\left\{\frac{(r-1)z^{r-q} - \alpha z^{-q}}{q-1}\right\}^{(1/p)-1} \\ &\times\left\{\frac{r-1)(r-q)z^{r-q-2} + \alpha q z^{-q-2}}{q-1}\right\}dz\ . \end{aligned} \tag{44}$$

To obtain boundary conditions, we use (42) and (ii) and thus, since $z(0) \neq 0$ and $z(1) \neq 0$ for $q \neq 0$, we obtain

$$z(0)^r = \alpha/(r-1) \quad \text{and} \quad z(1)^r = \alpha q/(r-q) \ . \tag{45}$$

We now integrate (44) from $x = 0$ to $x = 1$ using (45) and make the change of variables (for $q \neq 1$)

$$t = \frac{r-q}{r\alpha}\cdot\frac{(r-1)z^r - \alpha}{q-1}$$

which leads to equation (38).

For $q = 1$, we note that $I(p, 1, r) = p$, $\beta = 1$, and so (38) gives $K(p, 1, r) = (p+1)^{-1}$ which is the correct result by our earlier remarks. In the equation $y = \lambda^{1/p}G(y')$, G is increasing if $q > 1$ and decreasing if $q < 1$. Thus, since y is increasing, we must have y' increasing if $q > 1$ and decreasing if $q < 1$. The solution to problem (P) with $\lambda = \lambda^*$ can be seen to be unique in the following way. We know that a solution of (P) must satisfy $y = \lambda^{1/p}G(y')$ and thus also $y' = \lambda^{1/p}G'(y')y''$, and hence y' satisfies

$$\lambda^{1/p}\int_{y'(0)}^{y'(x)} G'(z)\,\frac{dz}{z} = x \ . \tag{46}$$

But, for $q \neq 1$, $G'(z)$ does not change sign on the interval from $y'(0)$ to $y'(1)$ so (46) has a unique solution for $y'(x)$, and hence (P) has a unique solution when $\lambda = \lambda^*$.

To obtain the alternate expression for $(r-q)I(p, q, r)^{-p}$ given in (41), we make the change of variable $t = 1 - \alpha(r-1)^{-1}z^{-r}$ in (44).

To obtain the formula (40), we make the following change of variables in (38)

$$1 + \frac{r(q-1)}{r-q}t = \left(1 - \frac{r(q-1)}{q(r-1)}s^p\right)^{-1} = R(s)^{-1} \ .$$

Then $t = (p/q(r-1))s^pR(s)^{-1}$, and $t = 0, 1$ correspond to $s = 0, 1$ and one has

$$\begin{aligned} I(p, q, r) &= \text{const}\left\{\int_0^1 R(s)^{(q/rp)+(1/r)+1-(1/p)-1}(pR(s)^{-1} + q)ds\right\} \\ &= \text{const}\,\{pL(p, q) + q\}, \text{since } r = p + q \ . \end{aligned} \tag{47}$$

The formula (40) can be obtained in a more direct way by making the substitution $u = (q/r\lambda)^{1/p}(y/y')$ in equation (i), where we assume $q \neq 0$ so $y'(x) > 0$ for $x \in [a, b]$. Then the conditions (ii) give $u(0) = 0$, $u(1) = 1$, and equation (i) reduces to

$$(r-1)\left\{1 - \frac{r(q-1)}{q(r-1)}u^p\right\} = p\left(\frac{r\lambda}{q}\right)^{1/p}u' \qquad (r = p + q) \ . \tag{48}$$

Separating variables and integrating gives (40).

For case (b), let $z(x) = \int_0^x |y'|dt$, and then if $q > 0$

$$\begin{aligned}\int_0^1 |y|^p |y'|^q dx &\leqq \int_0^1 (z^{p/q}z')^q dx \\ &\leqq \left\{\int_0^1 dx\right\}^{1/q}\left\{\int_0^1 z^{p/q}z' dx\right\}^q = \left(\frac{q}{p+q}\right)^q z^{p+q} ,\end{aligned}$$

using Hölder's inequality with exponents $1/q > 1$ and $1/(1-q)$. Equality holds only if $z^{p/q}z'$ is constant, and $y = z$ which means $y(x) = cx^{q/(p+q)}$. For $q = 0$, we have

$$\int_0^1 |y|^p dx \leqq \int_0^1 |z|^p dx \leqq z(1)^p = \int_0^1 |z'(x)|dx .$$

Equality holds only if $y = z$, and $z(x) = z(1)$ for all x, so $y(x) = z(x) \equiv 0$.

For case (c), we let $q \to r-$ in formula (38), using the equation (41) to evaluate $\lim (r-q)I(p, q, r)^{-p}$. This shows that the best constant is given by (39), because if $y' \in L^r$ and $q < r$, then

$$\int_0^1 |y|^p |y'|^q \to \int_0^1 |y|^p |y'|^r$$

by dominated convergence. To handle the case of equality we cannot apply Lemma 2 directly since the proof of Lemma 2(a) used $r > q$. However, if there is an f with $J(f) = K(p, r, r) = K^*$ then we know that $f \geqq 0$ a.e. Now referring to the proof of Lemma 2(a), since $r > 1$ we do have (24) which proves that if $E = \{x: f(x) = 0\}$, then $k(x, t) = 0$ a.e. on $CE \times E$. This means that

$$(CE \times E) \cap \{(x, t): 0 \leqq t \leqq x \leqq 1\}$$

is a set of measure zero. This implies that E differs from an interval $[c, 1]$ by a set of measure zero. To see this, let

$$c = \sup\{x \leqq 1: [0, x] \cap E \quad \text{is of measure zero}\} ,$$

and let $d = \inf\{x \geqq 0: [x, 1] \cap CE$ is of measure zero$\}$. Clearly $d \geqq c$. But, if $d > c$, and $e = (d + c)/2$, then $[c, e] \cap E$ and $[e, d] \cap CE$ have positive measure; but then $CE \times E$ intersects $\{(x, t): 0 \leqq t \leqq x \leqq 1\}$ in a set of positive measure which is absurd.

However if equality held for such an f, we would have (writing $f(x) = y'(x)$),

$$\int_0^c |y|^p |y'|^r dx = K^*\left\{\int_0^c |y'|^r dx\right\}^{(p+r)/r} . \tag{49}$$

Define $z(t) = y(ct)$ so $z'(t) = cy'(ct)$, and from (49) we obtain

$$\int_0^1 |z|^p |z'|^r dx = K^* c^{-p(r-1)/r} \left\{\int_0^1 |z'|^r dx\right\}^{(p+r)/r} . \tag{50}$$

But, if $c < 1$, then $K^* c^{-p(r-1)/r} > K^*$ contradicting the maximality of K^*. Thus $c = 1$, so $f(x) = y'(x) > 0$ a.e. on $[0, 1]$.

Still proceeding on the assumption that there exists an f with $J(f) = K^*$ we have shown $f(x) > 0$ a.e. on $[0, 1]$, so the proof of Lemma 2(b) is valid and f satisfies

$$\lambda r f^{r-1}(x) - rF(x)^p f^{r-1}(x) - p\int_x^1 F^{p-1} f^r dt = 0 \text{ a.e.} \tag{51}$$

where $F(x) = y(x) = \int_0^x f(t)dt$, and $\lambda = (p + q)K^*/r$.

If x is any point where $f(x) > 0$ and (51) holds then (51) shows that $\lambda f^{r-1}(x) > F^p(x) f^{r-1}(x)$, so $F^p(x) < \lambda$ a.e. But F is strictly increasing so $F(x)^p < \lambda$ for $0 \leqq x < 1$. Now we can solve (51) for f and obtain

$$f(x) = \varphi(A(x), B(x)) \quad \text{for almost all} \quad x \in [0, 1) . \tag{52}$$

where $\varphi(\xi, \eta) = (\eta/(1 - \xi))^{1/(r-1)}$, $A(x) = \lambda^{-1}F(x)^p < 1$ for $0 \leqq x < 1$, and $B(x) = p(r\lambda)^{-1}\int_x^1 F^{p-1} f^r dt > 0$ for $0 \leqq x \leqq 1$. Now we proceed as in the proof of Theorem 1. If we modify f on a null set so that it satisfies (52) everywhere then we obtain $f \in C^1[0, 1)$, and $f(x) > 0$ for $0 \leqq x < 1$. Thus we see that if $J(f) = K^*$ and $y' = f$ then y must be a solution of the problem (P), with $\lambda = \lambda^* = (p + r)K^*/r$. But a solution of (P) must be a solution of (42) (with $q = r$) which is

$$\lambda y'^r - y^p y'^r = p\lambda/(p + r) . \tag{53}$$

However if (53) has a solution then it must also satisfy

$$\int_0^{y(x)} (\lambda - u^p)^{1/r} du = (p\lambda/(p + r))^{1/r} x . \tag{54}$$

To see that (54) has a unique solution for $0 \leqq x \leqq 1$, we note that

$$\int_0^{\lambda^{1/p}} (\lambda - u^p)^{1/r} du = p^{-1}\lambda^{1/p} B\left(\frac{1}{r} + 1, \frac{1}{p}\right) = (p\lambda/(p + r))^{1/r} , \tag{55}$$

using the formula for $K(p, r, r) = K^*$ and $\lambda = (p + r)K^*/r$. Since $\lambda - u^p > 0$ for $0 \leqq u < \lambda^{1/p}$, (54) has a unique solution $y = y(x)$ which is strictly increasing and has $y(0) = 0, y(1) = \lambda^{1/p}$. To complete the proof we must show that y in fact satisfies (i), (ii) and (iii) of (P). By the implicit function theorem $y \in C^2(a, b)$, and differentiating (54) twice shows that y satisfies (i). Clearly $y(0) = 0$. For the other part of (ii), we note that for $0 \leqq x < 1$, we have

$$(\lambda - y^p(x))^{1/r} y'(x) = (p\lambda/(p + r))^{1/r} ,$$

and since $y^p(x) \to \lambda$ as $x \to 1-$, we have $y'(x) \to \infty$ as $x \to 1-$. But this means that

$$(\lambda - y^p(x))y'(x)^{r-1} = (p\lambda/(p+r))y'(x)^{-1} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow 1 . \tag{56}$$

To verify that $\| y' \|_r = 1$, let us first introduce the function g by

$$g(t) = (p\lambda)^{-1/r}(p+r)^{1/r}\int_0^t (\lambda - u^p)^{1/r} du , \tag{57}$$

so $g(y(x)) = x$ for $x \in [0, 1]$ and hence $y(g(t)) = t$ for $t \in [0, \lambda^{1/p}]$. Now

$$\begin{aligned}\int_0^1 y'(x)^r dx &= \frac{p\lambda}{p+r}\int_0^1 \frac{dx}{\lambda - y^p(x)} \\ &= \left(\frac{p\lambda}{p+r}\right)^{1-(1/r)}\int_0^{\lambda^{1/p}} (\lambda - t^p)^{(1/r)-1} dt ,\end{aligned} \tag{58}$$

where we use the change of variable $x = g(t)$. Now using the formula for λ, we obtain

$$\int_0^1 y'(x)^r dx = \frac{p}{p+r} B\left(\frac{1}{r}, \frac{1}{p}\right) / B\left(\frac{1}{r} + 1, \frac{1}{p}\right) = 1 . \tag{59}$$

REMARKS. (1) As was mentioned above, the method of this paper applies to inequalities of the form (1) with $n > 1$. In this case T becomes

$$w(x)^{1/p} m(x)^{-1/p} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt .$$

A discussion of the special case $p = q = 1, r = 2$ will be found in [4] where, for $m(x) \equiv w(x) \equiv 1$, $[a, b] = [0, 1]$, the best constant is shown to be asymptotic to $1/4n!$

(2) The method is equally applicable to inequalities in which the function y is restricted by other boundary conditions. For example, if $[a, b]$ is a finite interval we may treat

$$\int_a^b y^p y''^q w(x) dx \leqq K\left\{\int_a^b y''^r m(x) dx\right\}^{(p+q)/r}$$
$$y(a) = y(b) = 0 .$$

In this case, if f is a given function in L_m^r, the boundary value problem $y'' = -f$, $y(a) = y(b) = 0$ has a solution $y(x) = \int_a^b G(x, t) f(t) dt$, where $G(x, t) \geqq 0$ a.e. Hence our lemmas apply.

(3) When Theorem 1 is specialized to the situation studied by Beesack in [1] ($q = 0, p = r$), the results are not as general as his. This is because we can effectively handle only those inequalities where

we can insure in advance that equality is possible. There is some compensation in the fact that the existence of solutions to the Euler-Lagrange equations (P) is a conclusion of our theorem rather than a hypothesis as in [1] and [5].

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AN EMBEDDING THEOREM FOR LATTICE-ORDERED FIELDS

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In this paper we develop a method for constructing lattice-ordered fields ("$\mathscr{L}$-fields") which are not totally ordered ("*o*-fields") and hence are not *f*-rings. We show that many of these fields admit a Hahn type embedding into a field of formal power series with real coefficients. In order to establish such an embedding we make use of the valuation theory for abelian $\mathscr{L}$-groups and prove the "well known" fact that each *o*-field can be embedded in an *o*-field of formal power series.

Let G be an $\mathscr{L}$-field that contains n disjoint elements, but not $n+1$ such elements. An element $0 < s \in G$ is *special* if there is a unique $\mathscr{L}$-ideal of $(G, +)$ that is maximal without containing s. We show that the set S of special elements of G form a multiplicative group if and only if $S \neq \emptyset$ and $s^{-1} > 0$ for each $s \in S$. If this is the case, then there is a natural mapping of S onto the set Γ of all values of the elements of G. Thus Γ is a *po*-group and if, in addition, Γ is torsion free, then there exists an $\mathscr{L}$-isomorphism of G into the $\mathscr{L}$-field $V(\Gamma, R)$ of all functions v of Γ into the real field R whose support $\{\gamma \in \Gamma \mid v(\gamma) \neq 0\}$ satisfies the ascending chain condition. If G is an *o*-field, then the above hypotheses are satisfied and hence the embedding theorem for *o*-fields is a special case of our embedding theorem. The authors wish to thank the referee for many constructive suggestions.

NOTATION. If S is a subset of a group G, then $[S]$ will denote the subgroup of G that is generated by S. If G is a *po*-group, then G^+ will denote the set $\{g \in G \mid g \geqq 0\}$ of positive elements. A *disjoint* subset of an $\mathscr{L}$-group G is a set S of strictly positive elements such that $a \wedge b = 0$ for all pairs $a, b \in S$.

2. A method for constructing lattice-ordered rings.

A *po*-set Γ is called a *root system* if for each $\gamma \in \Gamma$, the set $\{\alpha \in \Gamma \mid \alpha \geqq \gamma\}$ is totally ordered. A nonvoid subset Δ of a root system Γ is called a *W-set* if it is the join of a finite number of inversely well ordered subsets of Γ, and an *I*-set if it is infinite and trivially ordered or well ordered with order type ω. In [2] it is shown that Δ is a W-set if and only if Δ does not contain an I-set; while in [10] five other conditions are derived which are equivalent to Δ not containing an I-set.

If Γ is a root system and if $v\colon \Gamma \to R$ is a function into the real field R, then the support of v is defined as $\operatorname{supp} v = \{\gamma \in \Gamma \mid v(\gamma) \neq 0\}$. The set $V = V(\Gamma, R)$ of all v whose support satisfies the ascending chain condition (A.C.C.) is a *po*-group if one defines v to be positive if $v(\gamma) > 0$ for each maximal element γ in $\operatorname{supp} v$. Such a $v(\gamma)$ will be referred to as a *maximal component* of v. In [5] it is shown that V is an $\mathscr{L}$-group for an arbitray *po*-set Γ if and only if Γ is a root system. For a root system Γ

$$W = W(\Gamma, R) = \{v \in V(\Gamma, R) \mid \operatorname{supp} v \text{ is a } W\text{-set}\}$$

is an $\mathscr{L}$-subgroup of V.

Now suppose that the root system Γ is also a *strictly po-semigroup*:

$$\alpha < \beta \to \alpha + \gamma < \beta + \gamma \quad \text{and} \quad \gamma + \alpha < \gamma + \beta$$

for all $\alpha, \beta, \gamma \in \Gamma$. For $u, v \in W$ define $uv \in W$ by

$$(uv)(\gamma) = \sum_{\alpha+\beta=\gamma} u(\alpha)v(\beta) .$$

Then W is a ring (see [2], p. 76, or [10], p. 333). If $0 < u, v \in W$, then $0 < uv$ and so W is an $\mathscr{L}$-ring and also a real vector lattice. If Γ is an o-group, then $V = W$ is a totally ordered division ring (see [8], p. 137]). Throughout, a "field" is always commutative while a "division ring" is not necessarily commutative.

In § 6, there are two examples of strictly *po*-semigroups which are root systems and hence can be used to construct $\mathscr{L}$-rings. Although it does not appear likely that all such semigroups can be reasonably characterized, the next lemma completely characterizes all *po*-groups which are also root systems.

LEMMA 2.1. *Suppose that a group Γ has a totally ordered subgroup H with positive cone H^+. If $H^+ \lhd \Gamma$, then Γ with this positive cone H^+ is a po-group and a root system. Conversely, each po-group that is a root system is of this form.*

Proof. Clearly, Γ is just the join of disjoint totally ordered cosets and so in this partial order Γ becomes a root system and a *po*-group. Conversely, suppose that Γ is a *po*-group and a root system. Let $[\Gamma^+]$ be the subgroup of Γ generated by its positive cone Γ^+. Then $H = [\Gamma^+] \lhd \Gamma$ is a directed *po*-group. If H were not an o-group, then there would exist $\alpha, \beta, \gamma \in H$ such that $\alpha \geqq \beta, \alpha \geqq \gamma$ and such that β and γ are not comparable (notation $\beta \,\|\, \gamma$). But then $-\beta \,\|\, -\gamma$, and $-\beta, -\gamma \in \{\delta \in \Gamma \mid \delta \geqq -\alpha\}$ which contradicts the fact that Γ is a root system.

Now let Γ be a *po*-group and a root system and suppose that $H = [\Gamma^+] \lhd \Gamma$ is the unique totally ordered normal subgroup such that Γ is the disjoint union of totally ordered cosets of H. It is well known that if Γ is abelian and torsion free then the given partial order can be extended in a not necessarily unique way to yield a totally ordered group. The latter may fail for nonabelian groups. However, if Γ is torsion free with $H \lhd \Gamma$ and Γ/H finite, then the given total order on H can be extended uniquely to a total group order on Γ (see [14], p. 326). The hypothesis that Γ/H is finite can in fact be weakened to require merely that any finite set of elements of Γ/H generate a finite subgroup (see [14], p. 325).

PROPOSITION 2.2. *Suppose that Γ is a torsion free po-group, and $H = [\Gamma^+]$ is a totally ordered subgroup with Γ/H finite. Then $W(\Gamma, R) = V(\Gamma, R)$ is a lattice ordered division ring. Moreover, the lattice order of $V(\Gamma, R)$ can be extended to a total ring order on $V(\Gamma, R)$.*

Proof. Let Γ_1 be the totally ordered group having the same underlying set of elements as Γ given by the unique extension of the partial order of Γ to a total one. As has already been remarked ([8], p. 137), $V(\Gamma_1, R)$ is a totally ordered division ring. Since the support of $v \in V(\Gamma, R)$ is the join of a finite number of inversely well ordered sets in Γ, when $\operatorname{supp} v$ is viewed as a subset of Γ_1, it will satisfy the A.C.C. Thus $v \in V(\Gamma_1, R)$ and $V(\Gamma, R) \subseteqq V(\Gamma_1, R)$. Clearly, $V(\Gamma_1, R) \subseteqq V(\Gamma, R)$. Since $V(\Gamma, R) = V(\Gamma_1, R)$ as sets, the lattice order of $V(\Gamma, R)$ can be extended to a total order.

COROLLARY. *In the previous proposition $V(\Gamma, R)$ satisfies the following three conditions*:

(i) *$V(\Gamma, R)$ contains n pairwise disjoint elements but not $n + 1$ such elements.*

(ii) *If $0 < v \in V(\Gamma, R)$ has just one maximal component (such a v is called special), then so does its inverse. All the special elements form a multiplicative group.*

(iii) *The multiplicative group of special elements is torsion free.*

In § 4, we show that, conversely, an $\mathscr{L}$-field with these three properties can be embedded in $V(\Gamma, R)$.

3. Special elements in an $\mathscr{L}$-ring. In order to obtain an embedding theorem for an $\mathscr{L}$-field G, we assume that the special elements in G form a multiplicative group. In this section we investigate what this hypothesis means. In particular, we show that such special elements behave like elements in f-rings in that they distribute over joins and intersections.

Let G be an abelian $\mathscr{L}$-group. A convex subgroup of G which is also sublattice is called an *$\mathscr{L}$-ideal*. An $\mathscr{L}$-ideal L of G is called *regular* if it is maximal with respect to not containing some element $g \in G$. If this is the case, then G/L is an o-group (see [4] or [5]) and hence there exists a unique $\mathscr{L}$-ideal that covers L. Let $\Gamma = \Gamma(G)$ be the set of all pairs of $\mathscr{L}$-ideals (G^γ, G_γ) such that G_γ is regular and G^γ covers G_γ. We shall frequently identify Γ with the set of pairs (G^γ, G_γ). In particular, define $\alpha < \beta$ in Γ if $G^\alpha \subseteqq G_\beta$. Then $(\Gamma, \leqq)$ is a root system. If $g \in G^\gamma \backslash G_\gamma$, then we say that γ is a *value* of g. If $0 < g$ has exactly one value, then g is called *special* and in this case its unique value will be denoted by $v(g)$. If $g \in G$ has exactly one value then g is comparable with zero and so either g or $-g$ is special. If $a, b, \in G$ are special, then $a \wedge b = 0$ if and only if $v(a) \parallel v(b)$. If L is an $\mathscr{L}$-ideal of G such that G/L is an o-group and $0 < g \in G\backslash L$ implies that $g > L$, then G is called a *lex-extension* of L. It follows that each coset $L \neq L + x$ consists entirely of positive elements or entirely of negative elements. If a and b are positive elements of an $\mathscr{L}$-ring G, then $a \ll b$ will mean that $na < b$ for all integers $n > 0$. If $a \ll b, c > 0$, and $bc \neq 0$, then $nac < bc$ for all n and so $ac \ll bc$.

3.1. In [4] it is shown that for $0 < g \in G$, the following are equivalent:

(1) g is special;

(2) $G(g) = \{z \in G \mid |z| \leqq ng$ for some integer $n > 0\}$ has exactly one maximal $\mathscr{L}$-ideal;

(3) $G(g)$ is a lex-extension of a proper $\mathscr{L}$-ideal L.

Consequently, if a is special and L is the unique maximal $\mathscr{L}$-ideal of $G(a)$, then $G(a)/L$ is an archimedian o-group and $G(a)$ is a lex-extension of L.

LEMMA 3.2. *If G is an abelian $\mathscr{L}$-group and $0 < g \in G$, then $Tg = \{z \in G \mid 0 \leqq z \ll g\}$ is a convex semigroup that contains 0 but not g and so $[Tg] = \{y - z \mid y, z \in Tg\}$ is an $\mathscr{L}$-ideal of G and $[Tg]^+ = Tg$.*

Proof. By Theorem 11 on page 81 in [8] it suffices to show that Tg is a semigroup. But this is well known for o-groups, and since G is a subdirect sum of o-groups, it follows that Tg is a semigroup.

COROLLARY. *$[Tg]$ is the largest (proper) $\mathscr{L}$-ideal of $G(g)$ if and only if g is special.*

Proof. If $[Tg]$ is the largest $\mathscr{L}$-ideal of $G(g)$, then g is special by 3.1 (2). Conversely, suppose that g is special and let L be the largest

$\mathscr{L}$-ideal of $G(g)$. Since $g \notin [Tg]$ it follows that $[Tg] \subseteqq L$ and since $nL^+ \subseteqq L^+ < g$ for all positive integers $n, L \subseteqq [Tg] = L$.

LEMMA 3.3. *Suppose that a and b are special elements in an $\mathscr{L}$-ring G with an identity and that a^{-1} and b^{-1} exist.*

(i) *If $a^{-1} \in G^+$, then a^{-1} is special.*

(ii) *If $a^{-1}, b^{-1} \in G^+$, then $Tab = TaG(b)^+ = G(a)^+Tb$ and $[Tab]$ is the largest $\mathscr{L}$-ideal in $G(ab)$. Thus ab is special.*

Proof. (i) Let L be a proper $\mathscr{L}$-ideal of $G(a^{-1})$ and consider $0 < q \in L$. Since $q < na^{-1}$ for some $n > 0$, we have $qa^2 < na$ and so $qa^2 \in G(a)$. If $qa^2 \notin Ta$, then since $G(a)/[Ta]$ is an archimedean o-group, $[Ta] + nqa^2 > [Ta] + a$ for some $n > 0$. Then since $G(a)$ is a lex-extension of $[Ta]$, $nqa^2 > a$ and so $nq > a^{-1}$. But then $L \supseteqq G(a^{-1})$, a contradiction. Thus $qa^2 \in Ta$ and so $qa^2 \ll a$, and hence $q \ll a^{-1}$. Therefore $L^+ \subseteqq Ta^{-1}$ and hence by the above corollary a^{-1} is special.

(ii) If $x \in Ta$ and $y \in G(b)^+$, then $knx < a$ and $y < kb$ for some $k > 0$ and all $n > 0$. Thus $knxy \leqq ay \leqq kab$ and hence $nxy \leqq ab$ for all n, and since $ab \neq 0, nxy < ab$ for all n. Thus $TaG(b)^+ \subseteqq Tab$. If $z \in Tab$, then $z \ll ab$ and $zb^{-1} \ll a$. Then $z = (zb^{-1})b \in TaG(b)^+$. Therefore $Tab = TaG(b)^+$ and similarly $Tab = G(a)^+Tb$.

Now suppose that L is a proper $\mathscr{L}$-ideal of $G(ab)$ and $0 < q \in L$. Since $q < nab$ for some n, $qb^{-1} < na$ shows that $qb^{-1} \in G(a)$. If $qb^{-1} \notin Ta$, then as above $mqb^{-1} > a$ *for some* $m > 0$ and so $L \supseteqq G(ab)$, a contradiction. Thus $qb^{-1} \in Ta$ and hence $q = (qb^{-1})b \in TaG(b)^+ = Tab$. Hence $L^+ \subseteqq Tab$ and ab is special.

Conditions (4) and (5) in the next theorem show that special elements behave like elements from an f-ring. A commutative $\mathscr{L}$-field is totally ordered if and only if the positive cone is closed under division (see [8], p. 139). This is one reason for putting requirements on the special elements, rather than on all the positive elements.

THEOREM I. *For a lattice ordered division ring G with an identity the following are equivalent.*

(1) *The special elements form a multiplicative group or the null set.*

(2) *If a is special, then $a^{-1} > 0$.*

(3) *If a is special, then a^{-1} is special.*

(4) *$a(c \vee 0) = ac \vee 0$ for special elements a and all $c \in G$.*

(5) *$a(y \vee z) = ay \vee az$ for special elments a and all $y, z \in G$. Six additional conditions each equivalent to* (4) *and* (5) *are obtained by writing $(c \vee 0)a = ca \vee 0$, $(y \vee z)a = ya \vee za$, and by replacing "$\vee$" with "$\wedge$".*

Proof. The implications (1) $\to$ (3) $\to$ (2) and (5) $\to$ (4) are trivial and (2) $\to$ (1) follows from Lemma 3.3.

(2) $\to$ (4). Since a and a^{-1} are both positive, the left multiplications by a and a^{-1} are inverse order preserving mappings and hence are lattice automorphisms.

(4) $\to$ (2). If a is special, then $a(1 \vee 0) = a \vee 0 = a$ and so $1 \vee 0 = 1$. Thus $a(a^{-1} \vee 0) = 1 \vee 0 = 1$ and hence $a^{-1} = a^{-1} \vee 0 > 0$.

(4) $\to$ (5).

$$\begin{aligned} a(y \vee z) &= a[((y - z) \vee 0) + z] \\ &= ((ay - az) \vee 0) + az = ay \vee az \ . \end{aligned}$$

The equation

$$\begin{aligned} a(y \wedge z) &= a(-(-y \vee -z)) = -(a(-y \vee -z)) \\ &= -(-ay \vee -az) = ay \wedge az \end{aligned}$$

shows that "$\vee$" may be replaced by "$\wedge$" throughout. Finally, each of the above arguments applies equally well to $(c \vee 0)a$, $(y \vee z)a$, $(c \wedge 0)a$, and $(y \wedge z)a$.

Suppose that each element in the lattice ordered division ring G has at most a finite number of values and that the special elements in G form a multiplicative group S. Then each $\gamma \in \Gamma$ is the value of a special element (see [4], p. 118) and the map v of $s \in S$ onto its value $v(s)$ is an o-homomorphism of S onto Γ. In particular, Γ is a partially ordered group and of course a root system.

PROPOSITION 3.4. *If G is a finite valued $\mathscr{L}$-field, i.e., each element has only a finite number of values, and if the special elements of G form a group and the associated value group Γ of G is torsion free, then the order of G can be extended to a total order.*

Proof. Extend the partial order of Γ to a total order. An element $0 \neq g \in G$ has a unique representation $g = g_1 + \cdots + g_n$ where each g_i or $-g_i$ is special and $|g_i| \wedge |g_j| = 0$ if $i \neq j$ (see [4]). One of the $v(g_i)$ will be the largest in the total ordering of Γ, say $\gamma = v(g_j)$. Define g to be positive if $G_\gamma + g > G_\gamma$. Clearly this is a total order of the set G that extends the given lattice order and a straightforward computation shows that G is an o-field.

An element $0 < b$ of an $\mathscr{L}$-group G is *basic* if $\{g \in G \mid 0 \leqq g \leqq b\}$ is totally ordered. A *basis* for G is a maximal pairwise disjoint subset of G which, in addition, consists of basic elements. G has a finite basis if there exists a basis consiting of n elements or equivalently if G contains n disjoint elements but not $n + 1$ such elements. For a structure theorem for a group with a finite basis see ([8], p. 86).

If G is a lattice ordered division ring with a finite basis, if the special elements form a group and if $\Gamma(G)$ is torsion free, then there exists an extension of the lattice order of G to a total order of G. The proof of this fact is the same as the proof of the last proposition.

4. An embedding theorem for o-fields. In this section it is shown that an arbitrary totally ordered field F can be embedded in the o-field $V(\Gamma(F), R)$. Only the statement of this embedding theorem and not the method of proof will be used in subsequent sections. The proof assumes some familiarity with the valuation theory of fields.

Let F be an o-field and F^* be the multiplcative group of all strictly positive elements of F. Then F^* is the set of all special elements, and the mapping v of $f \in F^*$ upon its value $v(f)$ in $\Gamma = \Gamma(F)$ is an o-homomorphism. Thus Γ may be regarded as an additive o-group with identity θ, and v is the natural order valuation of F (see [1] or [11]). Note that $1 \in F^\theta \backslash F_\theta$, F^θ is the *valuation ring* of F, and F^θ / F_θ is the residue class field. Also, F^θ / F_θ is an archimedean o-field and hence essentially a subfield of the real numbers. As before, $V = V(\Gamma, R)$ is the o-field of formal power series with exponents from Γ and with real coefficients. For $\gamma \in \Gamma$, let x^γ be the element in V such that

$$x^\gamma(\alpha) = \begin{cases} 1 \text{ if } \alpha = \gamma \\ 0 \text{ otherwise} . \end{cases}$$

Note that $x^\theta = 1$. Although $V(\Gamma, R)$ in general contains several o-isomorphic copies of the reals, it contains Rx^θ as a distinguished copy, and $V(\Gamma, R)$ is an o-algebra over the reals under component-wise multiplication by Rx^θ.

Let E be a not necessarily ordered division ring with a valuation $w: E \backslash \{0\} \to \Gamma(E)$ in the sense of [16] except with the order of $\Gamma(E)$ reversed. Thus in case E is ordered w would be an order preserving map. If $E \subset D$ where D is another valuated division ring whose valuation extends w, then D is called an *immediate extension* of (E, w) provided the value group of E, that is $w(E)$, is also the value group of D, and if the residue class fields of E and D are isomorphic. By Zorn's lemma, every (E, w) has a maximal immediate extension.

THEOREM II. (i) *If F is an o-field with value group Γ, then there exists a value and order preserving isomorphism π of F into the o-field $V(\Gamma, R)$.* (ii) *Moreover, if $\Delta \subset \Gamma$ is a rationally independent basis for the divisible hull of Γ, and for each $\delta \in \Delta$, $0 < x_\delta \in F$ is arbitrary with value δ, then π can be chosen so that $x_\delta \pi = x^\delta$.* (iii) *Now assume in addition that $\bar{R} \subset F$ is any o-isomorphic copy* *of the reals and* $r \to \bar{r}$ *is the unique o-isomorphism of* R *onto* $\bar{R}$. *Then in addition to satisfying* (ii), π *can be so chosen that* $\bar{r}\pi = rx^\theta$.

Proof. We only outline a proof in the sense that [13] and [16] are quoted for all the difficult steps (also see [1], p. 328). By [13], any totally ordered field F can be embedded in a totally ordered field E so that the order induced on F from $F \subset E$ is the orginal order of F, both E and F have the same value group $\Gamma = \Gamma(F) = \Gamma(E)$, and E contains an isomorphic copy of the reals, i.e., $R \cong \bar{R} \subset E$. Since $v(1) = \theta$, necessarily, $\bar{R}\backslash\{0\} \subseteqq E^\theta\backslash E_\theta$ and also $E^\theta/E_\theta \cong R$. The reader should recall that the real field R has no nontrivial automorphisms, since R admits exactly one total order. Let $r \to \bar{r}$ denote the o-isomorphism of R onto $\bar{R}$. The field E with the natural order valuation $v: E/\{0\} \to \Gamma$ has a maximal immediate extension $E \subseteqq M$. Denote the valuation on M also by v. We define $d \in M$ with value γ to be positive if $M_\theta + df^{-1}$ is positive in M^θ/M_θ for some $0 < f \in F$ with value γ. This is the unique extension of the order of E to M. Let $M^* = \{d \mid 0 < d \in M\}$.

It will be shown next that the subgroup

$$M^* \cap (M^\theta\backslash M_\theta) = \{0 < d \in M \mid v(d) = \theta\}$$

is divisible. If $0 < d \in M$ with $v(d) = \theta$, define $\bar{c} \in \bar{R}$ by $\bar{c} = \inf\{\bar{r} \mid d < r1\}$. Then $v(d - \bar{c}1) < \theta$ and $d = \bar{c}(1 + \lambda)$, $\lambda = (1/\bar{c})d - 1$ with $v(\lambda) < \theta$. If $m > 1$ is any integer, then in order to show that $d^{1/m} \in M^* \cap (M^\theta\backslash M_\theta)$ take $\bar{c} = 1$ and define $p_n \in M^* \cap (M^\theta\backslash M_\theta)$ by taking terms up to λ^n from the formal power series expansion of $(1 + \lambda)^{1/m}$. Then $\{p_n \mid n = 1, 2, \cdots\}$ defines a so called pseudo convergent sequence (see [16], p. 39]). If this sequence has a pseudo limit ([16], p. 47), then that limit is $d^{1/m}$. However, by ([16], p. 51, Th. 8), M contains a pseudo limit for each of its pseudo convergent sequences. Thus $d^{1/m} \in M^* \cap (M^\theta\backslash M_\theta)$ and hence M^* splits, $M^* = T \times M^* \cap (M^\theta\backslash M_\theta)$, where T is some complement of $M^* \cap (M^\theta\backslash M_\theta)$. For $t \in T$ define $t\,\pi = x^{v(t)}$ and for $\bar{r} \in \bar{R}$ define $\bar{r}\pi = rx^\theta$. Then this determines a value and order preserving isomorphism π of the subfield K of M that is generated by $\bar{R} \cup T$ into V. Moreover, M and V are maximal immediate extensions of K and $K\pi$ respectively. By ([16], p. 222, Th. 4), π can be extended to a value preserving isomorphism of M onto V so that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & M \arrow[d, "\pi", "\cong"'] \\ K\pi \arrow[r] & V(\Gamma, R)\,. \end{tikzcd}$$

It is asserted that $\pi: M \to V$ preserves order. Since each element of M^θ is congruent modulo M_θ to an element of the form $\bar{r}$ with $r \in R$, and since $\bar{r}\pi = rx^\theta$, it follows that π induces an order preserving isomorphism $\pi^\theta\colon M^\theta/M_\theta \to V^\theta/V_\theta = R$. But $d \in M$ is positive by definition, provided for any $0 < k \in K$ with $v(k) = v(d)$, we have $M_\theta < M_\theta + dk^{-1}$. However, since $k\pi > 0$, and since $(M_\theta + dk^{-1})\pi^\theta = (V_\theta + d\pi)(V_\theta + k^{-1}\pi)$, it necessarily follows that $d\pi > 0$.

The set $\{x_\delta \mid \delta \in \Delta\}$ described in the theorem generates a subgroup of M^* whose intersection with $M^* \cap (M^\theta \backslash M_\theta)$ is zero and so we may pick $T \supset \{x_\delta \mid \delta \in \Delta\}$. Finally, in performing the embedding any subfield $\bar{R} \subset M$ isomorphic to R could have been used.

REMARK. Hahn's theorem for an abelian o-group G states that G can be o-embedded in $V(\Gamma, R)$. (See [8], p. 60). There are now several short elementary proofs of this result in the literature. It would be a considerable achievement to also have such a direct proof of the above theorem.

5. An embedding theorem for a class of $\mathscr{L}$-fields. The embedding theorem for an o-field is actually a special case of the more general embedding theorem for $\mathscr{L}$-fields which is developed in this section.

Suppose that H is a subgroup of finite index in a torsion free abelian group Γ. Then Γ/H is a direct sum of cyclic groups.

$$\Gamma/H = [H + s(1)] \oplus \cdots \oplus [H + s(k)]$$

where the order of $[H + s(i)] = d(i)$, $d(1) \geqq \cdots \geqq d(k)$ and $d(i+1) \mid d(i)$.

LEMMA 5.1. *The subgroup of Γ generated by the $s(i)$ is a direct sum $[s(1)] \oplus \cdots \oplus [s(k)]$. In particular, $d(1)s(1), \cdots, d(k)s(k)$ are rationally independent elements of H.*

Proof. Suppose that $\sum m(i)s(i) = 0$, where the integers $m(i)$ are not all zero. Since Γ is torsion free, the g.c.d. of the $m(i)$ can be factored out and so we may assume that the $m(i)$ have g.c.d. 1. But since the linear combination must become trivial modulo H, $d(i) \mid m(i)$ and hence $d(k) \mid m(i)$ for all i, a contradiction.

THEOREM III. *Suppose that G is an $\mathscr{L}$-field with a finite basis and that the special elements of G form a group. Then the set of values Γ of G is a po-group and a root system. If Γ is torsion free then there exists a value preserving $\mathscr{L}$-isomorphism of G into the $\mathscr{L}$-field $V(\Gamma, R)$.*

Proof. It follows from § 3 that Γ is a *po*-group and a root system. Then by Lemma 2.1 there exists a totally ordered subgroup H of Γ

such that the index $|\Gamma : H| = n$ of H in Γ is finite and H^+ is the positive cone for Γ. Thus $\Gamma = \bigcup\{H + \gamma_k \mid k = 1, \cdots, n\}$ is a disjoint union of totally ordered cosets, where $\gamma_1 \in \Gamma$ is chosen as $\gamma_1 = \theta$. Just as in the proof of Proposition 3.4 each $g \in G$ is uniquely of the form $g = g_1 + \cdots g_n$, where if $g_i \neq 0$, then either g_i or $-g_i$ is special, where $|g_i| \wedge |g_j| = 0$ if $i \neq j$, and where g_i "lives" on $H + \gamma_i$, that is $g_i \in G_\gamma$ for all $\gamma \in \Gamma \backslash (H + \gamma_i)$. Let F be the set of all elements that "live" on H, that is

$$F = \{g \in G \mid g \in G_\gamma \text{ for all } \gamma \in \Gamma \backslash H\} .$$

Then F is a totally ordered subfield of G. For clearly, F is a totally ordered convex subring of G, and if g is special, then by hypothesis g^{-1} is also special. Thus g^{-1} lives on $H + \gamma_i$ for some i. If $i \neq 1$, then $g\, g^{-1} = 1$ lives on $H + \gamma_i$ which is impossible. Therefore $g^{-1} \in F$ and thus F is a field. Now assume that Γ is torsion free; then by Proposition 2.2 $V(\Gamma, R)$ is an $\mathscr{L}$-field. As before, for each $\gamma \in \Gamma$ define $x^\gamma \in V$ by

$$x^\gamma(\alpha) = \begin{cases} 1 \text{ if } \alpha = \gamma \\ 0 \text{ otherwise} . \end{cases}$$

In particular $x^\theta = 1$. As previously

$$\Gamma/H = [H + s(1)] \oplus \cdots \oplus [H + s(k)]$$

with orders $d(1) \geqq \cdots \geqq d(k)$ so that $d(i+1) \mid d(i)$. The reader should note that $n = d(1) \cdots d(k)$ and that $d(k)^k \mid n$. For each $i = 1, \cdots, k$ pick $0 < z_i \in G$ that lives on $H + \gamma_i$ and has value γ_i. In particular, each z_i is special. By Lemma 5.1, the $d(1)s(1), \cdots, d(k)s(k)$ are rationally independent elements of H and hence by Theorem II there exists a value and order preserving isomorphism π of the o-field F into $V(\Gamma, R)$ such that

(i) the support of $f\pi$ is contained in H for each $f \in F$, and

(ii) $z_i^{d(i)}\pi = x^{d(i)s(i)}$.

We shall extend π to an isomorphism of G into V. Consider

$$g(1)(H + s(1)) + \cdots + g(k)(H + s(k)) \in \Gamma/H$$

where $g(i)$ are integers in $0 \leqq g(i) < d(i)$ and let $g \in G$ live on this coset. Then

$$g = \bar{g} z_1^{g(1)} z_2^{g(2)} \cdots z_k^{g(k)}$$

where $\bar{g} \in F$. Since g lives on one of the n distinct cosets of Γ/H, g is special and conversely every special element is of the above form. Define

$$g\pi = \bar{g}\pi x^{g(1)s(1)} x^{g(2)s(2)} \cdots x^{g(k)} s^{(k)} .$$

Thus we have extended π to a one to one mapping of all special elements S of G. (Note that the map $S \to F, g \to \bar{g}$ is *not* a homomorphism of multiplicative groups unless $\Gamma = H$, while $g \to g(i)$ is a homomorphism of S into the integers modulo $d(i)$.) If $h \in S$ also lives on this same coset as g, then so does $h + g$ and $g(i) = h(i) = (h + g)(i)$ for all i, thus

$$h + g = \bar{h} z_1^{h(1)} + \cdots z_k^{h(k)} + \bar{g} z_1^{g(1)} \cdots z_k^{g(k)} = (\bar{h} + \bar{g}) z_1^{g(1)} \cdots z_k^{g(k)} .$$

Therefore $(h + g)^- = \bar{h} + \bar{g}$ and so $(h + g)\pi = h\pi + g\pi$. Next it will be shown that $\pi: S \to F$ is a homomorphism of multiplicative groups. Take $g, h \in S$ and write

$$g(i) + h(i) = n(i)d(i) + r(i), 0 \leqq r(i) < d(i), i = 1, \cdots, k .$$

Then since

$$\begin{aligned} hg &= \overline{hg} z_1^{g(1)+h(1)} \cdots z_k^{g(k)+h(k)} \\ &= \overline{hg} z_1^{n(1)d(1)} \cdots z_k^{n(k)d(k)} z_1^{r(1)} \cdots z_k^{r(k)} , \end{aligned}$$

it follows that

$$(hg)^- = \overline{hg} z_1^{n(1)d(1)} \cdots z_k^{n(k)d(k)};\ (hg)(i) = r(i), i = 1, \cdots k .$$

Thus

$$(hg)\pi = (\overline{hg})^- z_1^{r(1)} \cdots z_k^{r(k)} = (h\pi)(g\pi) .$$

Now each $a \in G$ has the above mentioned unique representation $a = a_1 + \cdots + a_n$ where a_i lives on $H + \gamma_i$; define $a\pi = a_1\pi + \cdots + a_n\pi$. Clearly, π is a map of G into V that preserves addition and values. If $b \in G$ with $b = b_1 + \cdots + b_n$ and $ab = c_1 + \cdots + c_n$ where b_i, c_i live on $H + \gamma_i$, it remains to show that $c_1\pi + \cdots + c_n\pi = \sum (a_i\pi)(b_j\pi)$. Each c_t is of the form $c_t = \sum' a_i b_j$ where $\sum'$ denotes the sum over those distinct pairs (i, j) for which $\gamma_i\gamma_j \in H + \gamma_t$. It suffices to show that $c_t\pi = \sum' (a_i\pi)(b_j\pi)$. However, first, since a_i, $b_j \in -S \cup S$ we have $(a_i b_j)\pi = (a_i\pi)(b_j\pi)$; and, secondly, since π preserves addition, $\sum' (a_i\pi)(b_j\pi) = (\sum' a_i b_j)\pi$. Thus it follows that $(ab)\pi = (a\pi)(b\pi)$. Therefore π is a homomorphism of the field G into the field $V(\Gamma, R)$ that is clearly not zero and so it must be an isomorphism. If $a = a_1 + \cdots + a_n$ where the a_i live on $H + \gamma_i$, then $a \vee 0$ is just the sum of the positive a_i. Therefore $(a \vee 0)\pi = a\pi \vee 0$ and π is a value preserving $\mathscr{L}$-isomorphism of G into V. This completes the proof of the theorem.

An $\mathscr{L}$-field F is an a-extension of an $\mathscr{L}$-field G, if for each

$0 < f \in F$, there exists an element $0 < g \in G$ such that $f < mg$ and $g < nf$ for some positive integers m and n, and G is *a-closed* if it does not admit such an extension.

The next corollary shows that the field V, into which G was embedded in the last theorem, has an intrinsic characterization.

COROLLARY. *Under the same hypotheses as in the previous theorem, V is the unique a-closed a-extension of $G\pi$.*

Proof. V is a-closed as an $\mathscr{L}$-group and, clearly, it is an a-extension of $G\pi$. In order to prove the uniqueness of V, let $G \subset D$ be any other a-extension of G. Since D satisfies all the hypotheses of Theorem III, for $g \in D\backslash G$, $g\pi$ can be defined exactly as in the proof of Theorem III to yield and $\mathscr{L}$-embedding of D into V that extends π. Furthermore, $D\pi \subseteqq V$ is an a-extension. Finally, if D is a-closed, then so is also $D\pi$ and hence $D\pi = V$. Thus π extends to an $\mathscr{L}$-isomorphism of D onto V leaving G elementwise fixed.

REMARK. Under the hypotheses of Theorem III we can extend the order of G to a total order (Proposition 3.4) and hence by Theorem II there is an o-isomorphism of the o-field G into the $\mathscr{L}$-field $V(\Gamma, R)$. It would be nice to be able to prove that this isomorphism is also an $\mathscr{L}$-isomorphism, but this we have not been able to do.

6. **Examples and questions.** The first example shows that Γ need not be torsion free even if G is an $\mathscr{L}$-field with a finite basis in which the special elements form a multiplicative group. Similar examples exist in which G is actually a real algebra.

6.1. Take an algebraic extension $G = Q[w]$ of the rationals Q, where $w \in R$, $w^n = 2$, i.e., $w = 2^{1/n}$ for some $n \geqq 2$. For

$$y = c_0 + c_1 w + \cdots + c_{n-1}w^{n-1} \in Q[w]$$

with $c_i \in Q$ define $y \geqq 0$ if and only if all $c_i \geqq 0$. Note that this order differs from the natural order of $Q[w]$ as a subset of R. Then in the context of the notation of § 5, the multiplicative group of special elements S is generated by $S = [\{cw \mid 0 < c \in Q\}]$, $H = \{\theta\}$; Γ is the cyclic group of order n and hence not torsion free.

6.2. Take $n = 2$ above in 6.1 but redefine $y = c_0 + c_1 w > 0$ if and only if $c_1 \leqq 0$ and $c_0 \geqq 0$.

6.3. Let Γ be a cancellative multiplicative semigroup with identity that contains an element k in the center such that $k^m \neq k^n$ for all distinct positive integers m and n. For $a, b \in \Gamma$, define $a \geqq b$ if $a = k^n b$ for some integer $n \geqq 0$ where $k^0 = 1$. Then a straightforward computation shows that Γ is a strictly *po*-semigroup and a root system; in fact, Γ is the join of disjoint totally ordered sets each of which is countable.

6.4. In the multiplicative abelian semigroup Γ generated by a, b, k with $k^0 = 1$, define $a^i b^j k^n > a^p b^q k^m$ provided one of the following four cases holds.

Case 1. $i > p$.

Case 2. $i = p = 0, j = q$, but $n > m$.

Case 3. $i = p > 0$ and $j > q$.

Case 4. $i = p > 0, j = q$, but $n > m$.

Note that the subsemigroup $\{a^i b^j k^n \mid i \geqq 1\}$ is lexiographically ordered. Aside from being a strictly *po*-semigroup and a root system, Γ has two noteworthy features. It is not the union of disjoint chains such that the elements from distinct chains are incomparable, and it has no convex semigroup ideals.

In conclusion we list some questions we could not answer.

(a) Can the partial order of each $\mathscr{L}$-field be extended to a total order?

(b) If F is an $\mathscr{L}$-field in which each square is positive, then is F an o-field?

(c) Does each $\mathscr{L}$-field contain a unique maximal totally ordered subfield?

(d) When can a lattice order of a commutative integral domain be extended to a lattice order of its field of fractions?

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SUMMABILITY OF FOURIER SERIES BY TRIANGULAR MATRIX TRANSFORMATIONS

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Hille and Tamarkin have proved a result for the Nörlund summability of the Fourier series of $f(t)$ at $t = x$, under the hypothesis (i) $\varphi(t) = \{f(x+t) + f(x-t) - 2f(x)\}/2 = o(1), t \to 0$, which includes as a special case the corresponding result for the Cesàro summability. However, under the lighter condition (ii) $\int_0^t \varphi(u)du = o(t), t \to 0$, Astrachan has proved a theorem for the Nörlund summability which does not cover the corresponding Cesàro case. The object of the present paper is to prove theorems for the Nörlund summability and another triangular matrix method of summability which are subtler than Astrachan's theorem in the sense that they include as a special case the corresponding result for the Cesàro summability.

1. Definitions and notations. Let $\sum_{n=0}^{\infty} v_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. We shall consider sequence-to-sequence transformation of the type

$$u_n = \sum_{k=0}^{\infty} d_{nk} s_k \tag{1.1}$$

in which the elements of the matrix $D = ((d_{nk}))$ are real or complex constants and $d_{nk} = 0$ for $k > n$. The sequence $\{u_n\}$ is said to be the sequence of D-means of $\{s_n\}$. If $\lim_{n\to\infty} u_n$ exists and is equal to u then we say that the series $\sum_{n=0}^{\infty} v_n$ or the sequence $\{s_n\}$ is summable D to the sum u.

Let $\{p_n\}$ be a sequence of constants, real or complex and let us write $P_n = p_0 + p_1 + \cdots + p_n \neq 0, P_{-1} = p_{-1} = 0$. Then the matrix D defines a Nörlund matrix (N, p_n) [7], if

$$d_{nk} = p_{n-k}/P_n\,, \qquad (n \geqq k \geqq 0)\,. \tag{1.2}$$

The conditions for the regularity of the (N, p_n) mean are

$$\lim_{n\to\infty} p_n/P_n = 0 \quad \text{and} \quad \sum_{k=0}^{n} |p_k| = O(|P_n|)\,, \qquad n \to \infty\,. \tag{1.3}$$

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha > -1) \tag{1.4}$$

the (N, p_n) mean reduces to the familiar (C, α) mean.

The product of the matrix $(C, 1)$ with the matrix (N, p_n) defines the matrix $(C, 1)\cdot(N, p_n)$. Thus D defines the matrix $(C, 1)\cdot(N, p_n)$ if

$$d_{nk} = \frac{1}{n+1}\sum_{\nu=k}^{n} p_{\nu-k}/P_\nu \ , \qquad (0 \leqq k \leqq n) \ . \tag{1.5}$$

Similarly, one defines the $(N, p_n)\cdot(C, 1)$ matrix as a product of the (N, p_n) matrix with the $(C, 1)$ matrix. In Astrachan's notations [1] the $(N, p_n)\cdot(C, 1)$ summability is denoted by $(N, p_n)\cdot C_1$.

Let $f(t)$ be a periodic function, with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without any loss of generality that the constant term in the Fourier series of $f(t)$ is zero, so that $\int_{-\pi}^{\pi} f(t)dt = 0$ and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \ . \tag{1.6}$$

We write throughout:

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \ ;$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du, \ \alpha > 0; \ \Phi_0(t) = \varphi(t) \ ;$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\Phi_\alpha(t)/t^\alpha; \ \alpha \geqq 0 \ ;$$

$$R_n = np_n/P_n; \ S_n = \sum_{\nu=0}^{n} P_\nu(\nu+1)^{-1}/P_n \ ;$$

$$\Delta\mu_n, \text{ or more precisely } \Delta_n\mu_n = \mu_n - \mu_{n+1} \ ;$$

$$\tau = [1/t]; \ P_{[\lambda]} = P(\lambda); \ p_{[\lambda]} = p(\lambda) \ ;$$

where $[\lambda]$ denotes the greatest integer not greater than λ.

K, denotes a positive constant not necessarily the same at each occurrence.

2. Introduction.

Concerning the Cesàro summability of Fourier series Bosanquet [2] has proved the following.

THEOREM A. *If $\varphi_\alpha(t) = o(1)$ as $t \to 0$, then the Fourier series of $f(t)$, at $t = x$, is summable $(C, \alpha + \delta)$ for every $\delta > 0$ and $\alpha \geqq 0$.*

Theorem A is known to be the best possible in the sense that it breaks down if $\delta = 0$.

For the Nörlund summability of Fourier series we have the following result due to Hille and Tamarkin [5].

THEOREM B. *A regular (N, p_n) method is Fourier effective, if the sequence $\{p_n\}$ satisfies the hypotheses:*

$$R_n = O(1)\ , \tag{2.1}$$

$$\sum_{k=1}^{n} k\,|\Delta p_{k-1}| = O(|P_n|)\ , \tag{2.2}$$

$$\sum_{k=1}^{n} |P_k|/k = O(|P_n|)\ , \tag{2.3}$$

as $n \to \infty$.

Theorem B implies *inter alia* that if $\varphi(t) = o(1)$ as $t \to 0$, and $\{p_n\}$ satisfies the hypotheses (2.1)–(2.3), then the Fourier series of $f(t)$ is summable by a regular (N, p_n) method.

Replacing the hypothesis: $\varphi(t) = o(1)$ as $t \to 0$ of Theorem B by the lighter hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$, Astrachan [1] proved the following.

THEOREM C. *A regular* (N, p_n) *method is* K_α *effective* $(0 < \alpha \leqq 1)$, *if the sequence* $\{p_n\}$ *satisfies the hypotheses* (2.1), (2.2) *and*

$$\sum_{k=1}^{n} k(n-k)\,|\Delta^2 p_{k-2}| = O(|P_n|)\ , \tag{2.4}$$

$$\sum_{k=1}^{n} |P_k|/k^2 = O(|P_n|/n)\ , \tag{2.5}$$

as $n \to \infty$.

Hille and Tamarkin have also pointed out in [5] that the sequence $\{p_n\}$ defined by (1.4) satisfies the hypotheses of Theorem B for $1 > \alpha > 0$ and therefore, (C, α) summability for such a α is Fourier effective. Thus Bosanquet's Theorem A when $\alpha = 0$ is an immediate consequence of Theorem B. It is therefore natural to expect that the hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$, may lead to (N, p_n) summability of the Fourier series of $f(t)$ and that such a result may include Theorem A when $\alpha = 1$, as a special case. However, Astrachan's Theorem C in this direction only implies the summability (C, δ) for $\delta \geqq 2$, whereas one needs the summability $(C, \delta), \delta > 1$, in order to cover Bosanquet's Theorem A when $\alpha = 1$. Thus there is a gap of approximately 1 between the orders of (C) summability implied by Theorem C and the corresponding case of Theorem A. This emerges from the following reasoning.

The result of Lemma 8.1 in Astrachan [1], which is required for the proof of his Theorem C states that

$$\sum_{k=0}^{n} (n-k)\,|\Delta^2 p_{k-2}| = O(|P_n|/n)\ , \tag{2.6}$$

as $n \to \infty$. Since the left hand side of (2.6) is greater than Kn we observe that $Kn^2 \leqq |P_n|$. It may be pointed out that for Astrachan's proof of Lemma 8.1 one has to assume $p_0 = 0$.

The object of our Theorem 1 is to show that it is indeed, possible to obtain a result for the (N, p_n) summability of Fourier series which has also the scope of covering Bosanquet's Theorem A for $\alpha = 1$.

Astrachan [1, Th. II] has also obtained the following result for the $(N, p_n)\cdot(C, 1)$ summability of the Fourier series.

THEOREM D. *The $(N, p_n)\cdot(C, 1)$ method is K_α effective* $(0 < \alpha \leqq 1)$ *provided the sequence $\{p_n\}$ satisfies the hypotheses* (2.1)–(2.3) *and the regularity condition* (1.3).

Due to possible oversight, Astrachan has not shown that the regularity conditions follow from his statement of Theorem D. Further, his proof of Theorem D contains a deficiency, which has been pointed out and supplied by the present author in [4].

Silverman has shown in [8, Th. 1] that a necessary and sufficient condition for a (N, p_n) matrix to be permutable with the $(C, 1)$ matrix is that it be a Cesàro matrix. This implies that

$$(C, 1)\cdot(N, p_n) \neq (N, p_n)\cdot(C, 1)$$

except when $\{p_n\}$ is defined by (1.4). In view of this Astrachan's technique of obtaining his Theorem D from Theorem B fails in the case of the $(C, 1)\cdot(N, p_n)$ summability and one has to give a direct proof to conclude the $(C, 1)\cdot(N, p_n)$ summability of Fourier series of $f(t)$ under the hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$. More precisely, we observe that since the $(C, 1)$ mean is a very special case of the (N, p_n) mean viz. the case in which $p_n = 1$, the convenience of expressing the $(C, 1)$ mean of the Fourier series of $f(t)$, essentially as a difference of the Fejér's and Dirichlet's kernels of $\varphi_1(t)$ [1, p. 546], disappears totally in the case of the (N, p_n) mean.

Thus for the $(C, 1)\cdot(N, p_n)$ summability of Fourier series, we obtain Theorem 2 which also covers Theorem A when $\alpha = 1$.

3. We prove the following results.

THEOREM 1. *If $\varphi_1(t) = o(1)$ as $t \to 0$ and $\{p_n\}$ is nonnegative, monotonic nondecreasing sequence such that $p_n \to \infty$ as $n \to \infty$, $\{p_{n+1} - p_n\}$ is nonincreasing, $R_n = O(1)$ and* (2.5) *holds, then the Fourier series of $f(t)$, at $t = x$, is summable (N, p_n).*

THEOREM 2. *If $\varphi_1(t) = o(1)$ as $t \to 0$ and $\{p_n\}$ is a nonnegative, monotonic nonincreasing sequence such that $S_n = O(1)$, then the Fourier series of $f(t)$, at $t = x$, is summable $(C, 1)\cdot(N, p_n)$.*

REMARKS. It is easy to see that if $\{p_n\}$ is nonnegative and nondecreasing then $(n+1)p_n \geqq P_n$ and therefore $S_n = O(1)$. Further, in this case

$$\sum_{k=1}^{n} k\,|\Delta p_{k-1}| = -\sum_{k=1}^{n-1}\sum_{\mu=1}^{k}(p_\mu - p_{\mu-1}) + n\sum_{\mu=1}^{n}(p_\mu - p_{\mu-1}) = O(P_n)\ ,$$

if $R_n = O(1)$. Thus the sequence $\{p_n\}$ used in Theorem 1 also satisfies the hypotheses of Theorem B.

As demonstrated by the present author in [3] if $\{p_n\}$ is a nonnegative sequence then the hypotheses: $R_n = O(1)$ and $S_n = O(1)$ imply that

$$P_k \sum_{n=k+1}^{\infty} \frac{1}{(n+1)P_{n-1}} = O(1)\ , \qquad (k = 1, 2, 3, \cdots)\ ,$$

from which it is immediate that $P_n \to \infty$ as $n \to \infty$. It may be observed that with a slight modification in author's analysis in [3] it is possible to even drop the condition $R_n = O(1)$ to get the same conclusion.

4. We require the following lemmas for the proof of our results.

LEMMA 1. *If $\{q_n\}$ is nonnegative and nonincreasing, then for $0 \leqq a \leqq b \leqq \infty$ and $0 \leqq t \leqq \pi$,*

$$\left|\sum_{k=a}^{b} q_k \exp ikt\right| \leqq KQ_\tau\ ,$$

where $\tau = [1/t]$ and $Q_m = q_0 + q_1 + \cdots + q_m$.

This lemma may be proved by following the technique of proof of Lemma 5.11 in McFadden [6].

LEMMA 2. *If $\{p_n\}$ is a nonnegative and monotonic nondecreasing sequence such that $\{p_{n+1} - p_n\}$ is nonincreasing and $R_n = O(1)$, then as $n \to \infty$*

$$\sum_{k=0}^{n} p_k(n-k)\exp(ikt) = O(nP_\tau) + O(t^{-2}p_n)$$

uniformly in $0 < t \leqq \pi$.

Proof. We write by Abel's transformation

$$\begin{aligned}
&\sum_{k=0}^{n} p_k(n-k)\exp(ikt)\\
&= \sum_{k=0}^{n-1} \Delta_k\{p_k(n-k)\}\sum_{\nu=0}^{k}\exp(i\nu t)\\
&= (1-\exp it)^{-1}\left[\sum_{k=0}^{n-1}\Delta_k\{p_k(n-k)\} - \sum_{k=0}^{n-1}\Delta_k\{p_k(n-k)\}\exp i(k+1)t\right]
\end{aligned}$$

$$= (1 - \exp it)^{-1}\Big[np_0 - \sum_{k=0}^{n-1}(n-k)\Delta p_k \exp i(k+1) - \sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big]$$

$$= (1 - \exp it)^{-1}\Big[np_0 - \sum_{k=0}^{n-1}\sum_{\nu=0}^{k}\Delta p_\nu \exp i(\nu+1)t - \sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big].$$

Thus

$$\Big|\sum_{k=0}^{n} p_k(n-k)\exp ikt\Big|$$

$$\leqq |1 - \exp it|^{-1}\Big[np_0 + \sum_{k=0}^{n-1}\Big|\sum_{\nu=0}^{k}\Delta p_\nu \exp i(\nu+1)t\Big| + \Big|\sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big|\Big]$$

$$\leqq Kt^{-1}\Big[np_0 + K\sum_{k=0}^{n-1}\sum_{\nu=0}^{\tau}(p_{\nu+1} - p_\nu) + p_n \max_{1\leqq\nu\leqq n}\Big|\sum_{k=1}^{\nu}\exp ikt\Big|\Big]$$

(by Lemma 1 and Abel's Lemma, since $\{p_{\nu+1} - p_\nu\}$ is nonnegative, nonincreasing and $\{p_n\}$ is nondecreasing)

$$\leqq Kt^{-1}[np_0 + Knp_{\tau+1} + Kp_n t^{-1}]$$

$$\leqq Knt^{-1}p_{\tau+1} + Kt^{-2}p_n$$

$$\leqq KnP_\tau + Kt^{-2}p_n\ ,$$

since $\{p_n\}$ is nondecreasing and $R_n = O(1)$ which also implies $P_{n+1}/P_n = O(1)$.

This completes the proof of Lemma 2.

LEMMA 3. *If $\{p_n\}$ is nonnegative and nonincreasing, then as $n\to\infty$*

$$\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}(\nu-k)p_k \exp i(\nu-k)t = O(t^{-2}) + O\Big(t^{-1}P_t\sum_{\nu=\tau}^{n}\frac{1}{P_\nu}\Big) + O\Big(\frac{nt^{-1}P_\tau}{P_{n+1}}\Big),$$

uniformly in $0 < t \leqq \pi$.

Proof. Applying Abel's transformation we get

$$\sum_{\nu=k}^{n}\frac{\nu-k}{P_\nu}\exp i(\nu-k)t$$

$$= \sum_{\nu=k}^{n}\Delta_\nu\Big(\frac{\nu-k}{P_\nu}\Big)\sum_{\mu=k}^{\nu}\exp i(\mu-k)t + \frac{n-k+1}{P_{n+1}}\sum_{\mu=k}^{n}\exp i(\mu-k)t$$

$$= (1 - \exp it)^{-1}\Big[\sum_{\nu=k}^{n}\Delta_\nu\Big(\frac{\nu-k}{P_\nu}\Big)\{1 - \exp i(\nu-k+1)t\}$$

$$+ \frac{n-k+1}{P_{n+1}}\{1 - \exp i(n-k+1)t\}\Big]$$

$$= (1 - \exp it)^{-1}\Big[-\sum_{\nu=k}^{n}\frac{p_{\nu+1}}{P_\nu P_{\nu+1}}(\nu-k)\exp i(\nu-k+1)t$$

$$+ \sum_{\nu=k}^{n}\frac{1}{P_{\nu+1}}\exp i(\nu-k+1)t - \frac{n-k+1}{P_{n+1}}\exp i(n-k+1)t\Big].$$

Changing the order of summation of the inner sums, thus we have

$$
\begin{aligned}
\Sigma &\equiv \left|\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}(\nu-k)p_k\exp i(\nu-k)t\right| \\
&= \left|\sum_{k=0}^{n}p_k\sum_{\nu=k}^{n}\frac{\nu-k}{P_\nu}\exp i(\nu-k)t\right| \\
&\leqq Kt^{-1}\Bigg[\left|\sum_{k=0}^{n}p_k\sum_{\nu=k}^{n}\frac{p_{\nu+1}}{P_\nu P_{\nu+1}}(\nu-k)\exp i(\nu-k+1)t\right| \\
&\quad + \left|\sum_{k=0}^{n}p_k\sum_{\nu=k}^{n}\frac{1}{P_{\nu+1}}\exp i(\nu-k+1)t\right| \\
&\quad + \frac{1}{P_{n+1}}\left|\sum_{k=0}^{n}p_k(n-k+1)\exp i(n-k+1)t\right|\Bigg] \\
&= \sum_1 + \sum_2 + \sum_3 ,
\end{aligned}
$$

say.

Again by a change of order of summation we have

$$
\begin{aligned}
\sum_1 &\leqq Kt^{-1}\sum_{\nu=0}^{n}\frac{p_{\nu+1}}{P_\nu P_{\nu+1}}\left|\sum_{k=0}^{\nu}p_k(\nu-k)\exp i(\nu-k+1)t\right| \\
&\leqq Kt^{-1}\sum_{\nu=0}^{\tau-1}\frac{R_{\nu+1}}{P_\nu}\sum_{k=0}^{\nu}p_k + Kt^{-1}\sum_{\nu=\tau}^{n}\frac{R_{\nu+1}}{P_\nu}\max_{0\leqq\rho\leqq\nu}\left|\sum_{k=0}^{\rho}p_k\exp i(\nu-k+1)t\right|
\end{aligned}
$$

(by Abel's Lemma. If $\tau = 0$ the first part is taken as 0.)

$$
\leqq Kt^{-2} + Kt^{-1}P_\tau\sum_{\nu=\tau}^{n}\frac{1}{P_\nu},
$$

by virtue of Lemma 1 and the fact that $(n+1)p_n \leqq P_n$.

Similarly,

$$
\begin{aligned}
\sum_2 &\leqq Kt^{-1}\sum_{\nu=0}^{n}\frac{1}{P_{\nu+1}}\left|\sum_{k=0}^{\nu}p_k\exp i(\nu-k+1)t\right| \\
&\leqq Kt^{-1}\sum_{\nu=0}^{\tau-1}\frac{1}{P_{\nu+1}}\sum_{k=0}^{\nu}p_k + Kt^{-1}\sum_{\nu=\tau}^{n}\frac{1}{P_{\nu+1}}P_\tau \\
&\leqq Kt^{-1} + Kt^{-1}P_\tau\sum_{\nu=\tau}^{n}\frac{1}{P_\nu},
\end{aligned}
$$

by Lemma 1.

Finally, by Lemma 1 and Abel's Lemma we have

$$
\sum_3 \leqq Kt^{-1}\frac{n}{P_{n+1}}P_\tau .
$$

This completes the proof of Lemma 3.

5. *Proof of Theorem* 1. For the Fourier series of $f(t)$, at $t = x$ we have

$$s_k(x) - f(x) = \frac{1}{\pi}\int_0^\pi \varphi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt\ .$$

Therefore, if t_n denotes the (N, p_n) mean of $\{s_k(x)\}$ then

$$t_n - f(x) = \frac{1}{\pi P_n}\int_0^\pi \varphi(t)\left\{\sum_{k=0}^n p_{n-k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right\}dt\ .$$

Integrating by parts, we get

$$\begin{aligned} t_n - f(x) &= \frac{\Phi_1(\pi)}{\pi P_n}\sum_{k=0}^n p_{n-k}(-1)^k \\ &\quad - \frac{1}{\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{k=0}^n p_{n-k}k\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &\quad - \frac{1}{2\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{k=0}^n p_{n-k}\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &\quad + \frac{1}{2\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\tan(t/2)}\left\{\sum_{k=0}^n p_{n-k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right\}dt \\ &= L_1 + L_2 + L_3 + L_4\ , \end{aligned}$$

say.

Thus, in order to prove the theorem it is sufficient to show that as $n \to \infty$,

(5.1) $$L_j = o(1)\ ; \qquad (j = 1, 2, 3 \text{ and } 4)\ .$$

Since $\Phi_1(t)\cot t/2 = o(1)$ as $t \to 0$, it follows from Theorem B that $L_4 = o(1)$ as $n \to \infty$, when one appeals to the remarks contained in § 3 of the present paper.

We write

$$\frac{1}{P_n}\left|\sum_{k=0}^n p_{n-k}(-1)^k\right| \leqq K\frac{p_n}{P_n} = o(1)\ ,$$

as $n \to \infty$, since $\{p_n\}$ is nonnegative and nondecreasing and $R_n = O(1)$. Thus, we have $L_1 = o(1)$ as $n \to \infty$.

Also, $L_3 = o(1)$ as $n \to \infty$, by virtue of Riemann-Lebesgue Theorem and the regularity of the (N, p_n) mean which is implied by the hypotheses: $\{p_n\}$ is nonnegative and $R_n = O(1)$.

Finally, to show that $L_2 = o(1)$ as $n \to \infty$, we observe that

$$\frac{\Phi_1(t)}{\sin t/2} = o(1)$$

as $t \to 0$ and that the kernel occuring in L_2 is the real part of the complex valued function

$$\frac{1}{\pi P_n}\left\{\exp - i\left(n+\frac{1}{2}\right)t\right\}\sum_{k=0}^n p_k(n-k)\exp ikt = M_n(t)\ ,$$

say.

Therefore, in order to prove that $L_2 = o(1)$ as $n \to \infty$, it is enough to show that as $n \to \infty$

$$I \equiv \int_0^\pi g(t) M_n(t) dt = o(1) , \tag{5.2}$$

where $g(t) = o(1)$ as $t \to 0$.

We write, for a fixed δ such that $0 < \delta \leqq \pi$,

$$I = \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\delta} + \int_\delta^\pi\right) g(t) M_n(t) dt = I_1 + I_2 + I_3 , \tag{5.3}$$

say.

Since

$$M_n(t) = O\left(\frac{1}{P_n} \sum_{k=0}^{n} p_k(n-k)\right) = O(n) ,$$

we have, as $n \to \infty$

$$I_1 = O\left(n \int_0^{n^{-1}} |g(t)| dt\right) = o(1) . \tag{5.4}$$

For the interval $o < \delta \leqq t \leqq \pi$, we have from Lemma 2

$$M_n(t) = O\left(\frac{n}{P_n}\right) + O\left(\frac{p_n}{P_n}\right) = O\left(\frac{1}{p_n}\right) + o(1) = o(1) ,$$

as $n \to \infty$, by the hypotheses: $R_n = O(1)$ and that $p_n \to \infty$ as $n \to \infty$. Therefore, as $n \to \infty$,

$$I_3 = o(1) . \tag{5.5}$$

Since $g(t) = o(1)$ as $t \to 0$, to demonstrate the truth of $I_2 = o(1)$ as $n \to \infty$ we prove that

$$I_2^* \equiv \int_{n^{-1}}^{\delta} |M_n(t)| dt \leqq K .$$

By Lemma 2, we have

$$\begin{aligned} I_2^* &\leqq K \frac{n}{P_n} \int_{n^{-1}}^{\delta} P(1/t) dt + K \frac{p_n}{P_n} \int_{n^{-1}}^{\delta} t^{-2} dt \\ &= K \frac{n}{P_n} \int_{\delta^{-1}}^{n} \frac{P(s)}{s^2} ds + K R_n , \\ &\leqq K , \end{aligned}$$

by virtue of the hypotheses: $R_n = O(1)$ and (2.5). Thus, as $n \to \infty$.

$$I_2 = o(1) . \tag{5.6}$$

Combining (5.3)–(5.6), we get (5.2) and therefore $L_2 = o(1)$ as $n \to \infty$. This completes the proof of Theorem 1.

6. *Proof of Theorem* 2. If t_n^1 denotes the $(C, 1)\cdot(N, p_n)$ mean of the sequence $\{s_k(x)\}$ then

$$\begin{aligned} t_n^1 - f(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\sum_{\nu=k}^{n}\frac{p_{\nu-k}}{P_\nu}s_k(x) - f(x) \\ &= \frac{1}{n+1}\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}s_k(x) - f(x) \\ &= \frac{1}{(n+1)\pi}\int_0^\pi \varphi(t)\left\{\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right\}dt\,. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} t_n^1 - f(x) = &\ \frac{\Phi_1(\pi)}{\pi(n+1)}\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}(-1)^k \\ &- \frac{1}{\pi(n+1)}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}k\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &- \frac{1}{2\pi(n+1)}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &+ \frac{1}{2\pi(n+1)}\int_0^\pi \frac{\Phi_1(t)}{\tan(t/2)}\left\{\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}p_{\nu-k}\frac{\sin[k+(1/2)]t}{\sin(t/2)}\right\}dt \\ = &\ C_1 + C_2 + C_3 + C_4\,, \end{aligned}$$

say.

Thus, in order to prove the theorem it is sufficient to show that as $n \to \infty$

$$C_j = o(1)\,; \qquad (j = 1, 2, 3 \text{ and } 4)\,. \tag{6.1}$$

Since $\{p_n\}$ is nonnegative and nonincreasing, we have by Abel's Lemma

$$\frac{1}{P_\nu}\left|\sum_{k=0}^{\nu}p_{\nu-k}(-1)^k\right| \leqq K\frac{p_0}{P_\nu} = o(1)\,,$$

as $\nu \to \infty$, by virtue of the fact that $P_n \to \infty$ as $n \to \infty$. By virtue of the regularity of the $(C, 1)$ mean we now get $C_1 = o(1)$ as $n \to \infty$.

Further, since $[\Phi_1(t)/\sin(t/2)]\cos t/2 = o(1)$ as $t \to 0$ and the $(C, 1)$ mean is regular, Theorem B implies that $C_4 = o(1)$ as $n \to \infty$, when one observes that the sequence $\{p_n\}$ used in our Theorem 2 satisfies all the hypotheses of Theorem B.

That $C_3 = o(1)$ as $n \to \infty$, follows from the Riemann-Lebesgue Theorem and the fact that the $(C, 1)$ and the (N, p_n) mean are both regular.

Finally, we observe that $[\Phi_1(t)/\sin(t/2)] = o(1)$ as $t \to 0$ and therefore, in order to prove that $C_2 = o(1)$ as $n \to \infty$, it is sufficient to show that as $n \to \infty$

$$E \equiv \int_0^\pi g(t) J_n(t) dt = o(1) , \tag{6.2}$$

where $g(t) = o(1)$ as $t \to 0$ and

$$J_n(t) = \frac{\exp(it/2)}{\pi(n+1)} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k) p_k \exp i(\nu - k)t .$$

Let us write for a fixed δ such that $0 < \delta \leqq \pi$,

$$E = \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\delta} + \int_\delta^\pi\right) g(t) J_n(t) dt = E_1 + E_2 + E_3 , \tag{6.3}$$

say. Since

$$|J_n(t)| < \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k) p_k \leqq Kn ,$$

we have as $n \to \infty$

$$E_1 = O\left(n \int_0^{n^{-1}} (g(t)) dt\right) = o(1) . \tag{6.4}$$

For the interval $0 < \delta \leqq t \leqq \pi$, we have by Lemma 3

$$J_n(t) = o(1) + O\left(\frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu}\right) + O\left(\frac{1}{P_{n+1}}\right) = o(1)$$

as $n \to \infty$, since $P_n \to \infty$ as $n \to \infty$ and $(C, 1)$ mean is regular. Thus, as $n \to \infty$,

$$E_3 = o(1) . \tag{6.5}$$

Since $g(t) = o(1)$ as $t \to 0$, to prove that $E_2 = o(1)$ as $n \to \infty$, it is enough to demonstrate that

$$E_2^* = \int_{n^{-1}}^{\delta} |J_n(t)| \, dt \leqq K .$$

By Lemma 3 we get

$$\begin{aligned} E_2^* &\leqq \frac{K}{n+1} \int_{n^{-1}}^{\delta} \frac{dt}{t^2} + \frac{K}{n+1} \int_{n^{-1}}^{\delta} \frac{P(1/t)}{t} \left\{ \sum_{\nu=[1/t]}^{n} \frac{1}{P_\nu} \right\} dt \\ &\quad + \frac{K}{P_n} \int_{n^{-1}}^{\delta} \frac{P(1/t)}{t} dt \\ &\leqq K + \frac{K}{n+1} \int_{\delta^{-1}}^{n} \frac{P(s)}{s} \left\{ \sum_{\nu=[s]}^{n} \frac{1}{P_\nu} \right\} ds + K \frac{1}{P_n} \int_{\delta^{-1}}^{n} \frac{P(s)}{s} ds \\ &\leqq K + \frac{K}{n+1} \int_{\delta^{-1}}^{n} \frac{P(s)}{s} \left\{ \sum_{\nu=[s]}^{n} \frac{1}{P_\nu} \right\} ds , \end{aligned}$$

since $S_n = O(1)$. That $E_2^* \leqq K$, now follows from the fact that

$$\frac{1}{n+1}\sum_{k=1}^{n}\frac{P_k}{k}\sum_{\nu=k}^{n}\frac{1}{P_\nu} = \frac{1}{n+1}\sum_{\nu=1}^{n}\frac{1}{P_\nu}\sum_{k=1}^{\nu}\frac{P_k}{k}$$
$$= \frac{1}{n+1}\sum_{\nu=1}^{n} S_\nu \leqq K\,,$$

since $S_n = O(1)$. Therefore, as $n \to \infty$

$$E_2 = o(1)\,. \tag{6.6}$$

Combining (6.3)-(6.6), we get (6.2) and therefore, $C_2 = o(1)$ as $n \to \infty$.

This completes the proof of Theorem 2.

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LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS PRESERVING A FIXED RANK

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In this paper T is a linear transformation from a tensor product $X \otimes Y$ into $U \otimes V$, where X, Y, U, V are vector spaces over an infinite field F. The main result gives a characterization of surjective transformations T for which there is a positive integer k ($k < \dim U$, $k < \dim V$) such that whenever $z \in X \otimes Y$ has rank k then also $Tz \in U \otimes V$ has rank k. It is shown that $T = A \otimes B$ or $T = S \circ (C \otimes D)$ where A, B, C, D are appropriate linear isomorphisms and S is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

Let F be an infinite field and X, Y, U, V vector spaces over F. We denote by T a linear transformation of the tensor product $X \otimes Y$ into $U \otimes V$. The rank of a tensor $z \in X \otimes Y$ is denoted by $\rho(z)$. By definition $\rho(o) = 0$. The subspace of X spaned by the vectors $x_1, \cdots, x_n \in X$ will be denoted by $< x_1, \cdots, x_n >$.

LEMMA 1. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. Then $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$ for all z.*

Proof. If this is not true then for some $z \in X \otimes Y, z \neq 0$, we have $\rho(z) < k$ and $\rho(Tz) > k$. There exists $t \in X \otimes Y$ such that $\rho(t) + \rho(z) = k$ and moreover $\rho(z + \lambda t) = k$ for all $\lambda \neq 0, \lambda \in F$. Let

$$Tz \; \sum_{i=1}^{m} u_i \otimes v_i \;, \qquad m = \rho(Tz) \;.$$

Since $u_i \in U$ are linearly independent and also $v_i \in V$ we can consider them as contained in a basis of U and V, respectively. The matrix of coordinates of Tz has the form

$$\left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0 \end{array}\right)$$

where I_m is the identity $m \times m$ matrix. Let

$$\left(\begin{array}{c|c} A_m & B \\ \hline C & D \end{array}\right)$$

be the matrix of coordinates of Tt. Then the minor $|I_m + \lambda A_m|$ of the matrix of $T(z + \lambda t)$ has the form

$$1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots .$$

Since F is infinite we can choose $\lambda \neq 0$ so that $|I_m + \lambda A_m| \neq 0$. For this value of λ we have

$$\rho(z + \lambda t) = k , \qquad \rho(T(z + \lambda t)) \geqq m > k$$

which contradicts our assumption. This proves the lemma.

LEMMA 2. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply $\rho(Tz) \leqq k$. If T is surjective and $k < \dim U$, $k < \dim V$ then $\rho(z) \geqq \rho(Tz)$ for all z.*

Proof. Assume that for some z we have $\rho(z) < \rho(Tz)$. Clearly, we can assume in addition that $\rho(z) = 1$. Therefore $k > 1$. By assumption $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$. Let $s \leqq k$ be the maximal integer such that there exists $z \in X \otimes Y$ satisfying $\rho(z) < s$ and $\rho(Tz) = s$. Let

$$Tz = \sum_{i=1}^{s} u_i \otimes v_i .$$

We can choose $u_{s+1} \in U$, $v_{s+1} \in V$ such that $u_{s+1} \notin < u_1, \cdots, u_s >$ and $v_{s+1} \notin < v_1, \cdots, v_s >$. Since $u_i \in U$ are linearly independent and $v_i \in V$ also linearly independent we can assume that these vectors are contained in a basis of U and V, respectively. Since T is surjective there exists $t \in X \otimes Y$ such that $\rho(t) = 1$ and the $(s + 1, s + 1)$-coordinate $a_{s+1,\, s+1}$ of Tt is nonzero. The minor of order $s + 1$ in the upper left corner of the matrix of $T(z + \lambda t)$ has the form

$$a_{s+1,s+1}\lambda + \alpha_2\lambda^2 + \cdots .$$

Since $a_{s+1,s+1} \neq 0$ we can choose $\lambda \neq 0$ so that the minor is nonzero. For this value of λ we have

$$\rho(z + \lambda t) \leqq \rho(z) + 1 \leqq s \leqq k ,$$
$$\rho(T(z + \lambda t)) \geqq s + 1 .$$

If $s = k$ this contradicts our assumption. If $s < k$ this contradicts the maximality of s. Hence, Lemma 2 is proved.

LEMMA 3. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If T is surjective and $k < \dim U$, $k < \dim V$ then $\rho(z) = \rho(Tz)$ for each $z \in X \otimes Y$ satisfying $\rho(z) \leqq k$.*

Proof. The assertion is trivial if $\rho(z) = 0$ or k. Let $0 < \rho(z) < k$. Choose $t \in X \otimes Y$ such that

$$\rho(z + t) = \rho(z) + \rho(t) = k .$$

Using this and Lemmas 1 and 2 we deduce

$$\begin{aligned}
&\rho(T(z + t)) = \rho(Tz + Tt) = k , \\
&\rho(Tz) + \rho(Tt) \geqq k , \\
&\rho(Tz) + \rho(t) \geqq k , \\
&\rho(Tz) \geqq \rho(z) .
\end{aligned}$$

Since by Lemma 2, $\rho(Tz) \leqq \rho(z)$ we are ready.

The following Theorem is an immediate consequence of Lemma 3 and Theorem 3.4 of [3]:

THEOREM 1. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If T is surjective and $k <$ dim U, $k <$ dim V then*

$$T = A \otimes B , \tag{1}$$

or

$$T = S \circ (C \otimes D) , \tag{2}$$

where

$$\begin{aligned}
A : X \rightarrow U , \qquad B : Y \rightarrow V , \\
C : X \rightarrow V , \qquad D : Y \rightarrow U ,
\end{aligned}$$

are bijective linear transformations and S is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

This theorem gives a partial answer to a conjecture of Marcus and Moyls [2].

From Lemma 2 and Theorem 3.4 of [3] we get the following variant:

THEOREM 2. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply that $\rho(Tz) \leqq k$. If T is bijective and $k <$ dim U, $k <$ dim V then* (1) *or* (2) *holds.*

When $X = Y = U = V$, dim $X = n$, $k = n - 1$ we get a result of Dieudonné [1].

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EXTENSIONS OF A FOURIER MULTIPLIER THEOREM OF PALEY

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Let A be the class of continuous power series on the unit circle T, that is those continuous functions f whose Fourier coefficients $\hat{f}(n)$ are 0 for negative indices n. It is known that the most that can be said about the size of the coefficients of such f is that they are square summable. For instance Paley proved the following: Suppose that $\sum_0^\infty |w(n)|^2 = \infty$. Then there is an f in A with $\sum_0^\infty |\hat{f}(n)w(n)| = \infty$. In other words the l^2 sequences are the only multipliers which map A into the class of absolutely convergent power series.

The main result of this paper is that Paley's theorem can be generalized as follows: Let G be a compact Abelian group with a partially ordered dual group Γ. Denote by A the class of continuous functions f on G whose Fourier coefficients $\hat{f}(\gamma)$ vanish off the non-negative cone S of Γ. Let E be a totally ordered subset of S and w be a function defined on E which is not square summable. Then $\sum_E |\hat{f}(\gamma)w(\gamma)| = \infty$ for some f in A.

The class A when Γ is in fact a totally ordered group is a frequently considered generalization of the algebra of continuous power series. In this situation S itself is totally ordered so that $\sum_S |w(\gamma)|^2 < \infty$, whenever $\sum |\hat{f}(\gamma)w(\gamma)| < \infty$ for all f in A. This was obtained for $G = T^n$ by Helson [4] and in general by Rudin [8, p. 222]. Their proofs differed from Paley's although his method can be made to work in the situations they considered.

Now the power series discussed in the first paragraph are the restrictions to the circle of those functions which are continuous on the closure of the unit disc and analytic in its interior. From this point of view it would be natural, when $G = T^2$, to let A be the class of restrictions, to the distinguished boundary of the unit bidisc, of functions which are continuous on the closure and analytic in the interior of the bidisc. These are precisely the continuous functions on T^2 whose Fourier coefficients $\hat{f}(m, n)$ vanish off the first quadrant S of Z^2. The full analogue of Paley's theorem would be that every sequence w with the Paley multiplier property, $\sum |w(N)\hat{f}(N)| < \infty$ for all f in A, is square summable.

It is not known whether this strong version of theorem holds. The Helson-Rudin proofs for the case when S is a half space depend on a property of the analytic projection L taking trigonometric polynomials $\sum_\Gamma \hat{f}(\gamma)\gamma(x)$ into $\sum_S \hat{f}(\gamma)\gamma(x)$. Specifically, $\|Lf\|_p \leqq K_p \|f\|_1$

for $p < 1$. The corresponding projection when S is the first quadrant does not have this property [12, Th. 4] and [13, p. 208].

Except for this, however, the above mentioned proofs work in the double power series case. A simple counterexample to the full analogue of Paley's theorem would provide a simple proof that the double analytic projection is not bounded from L^1 to L^p for any $p<1$. As Helson observed, by a theorem of Bohr [2, p. 468, Th. 5], there are Paley multipliers on power series in infinitely many variables which do not even tend to 0; so the infinite dimensional version of Paley's theorem is false. This paper is the result of an attempt to settle the question for two or more variables.

What our main theorem says about Paley multipliers w on double power series is that $\sum_{k=1}^{\infty} |w(N_k)|^2 < \infty$ for any sequence $\{N_k\}_{k=1}^{\infty}$, of pairs of nonnegative integers, which is increasing in the strong sense that the N_k are distinct and the sequences of first and second components are nondecreasing. It follows easily that all such Paley multipliers w tend to 0 but perhaps not fast enough to make $\sum_S |w(N)|^2 < \infty$. So it is still not known if the only Paley multipliers on double power series are square summable. The proof of the main theorem does not involve properties of the analytic projection, however, and this suggests that Paley's theorem may not be as closely related to the boundedness of the projection as the previous proofs suggest.

As we shall see in § 3, Paley multipliers can be thought of as coefficients in a semi-lacunary series on a somewhat larger group than G. The proof of the main theorem takes advantage of this fact and the method can be applied to lacunary Fourier series in other situations. In order to present the idea in a simple setting, we begin in § 2 with such an application to semi-lacunary trigonometric series. In § 3 we use the same general approach to prove the main theorem. Section 4 contains a discussion of Paley multipliers on power series in several variables; a number of special results not depending on the main theorem are obtained. In § 5, we investigate Bohr sets, that is those subsets of S whose characteristic functions are Paley multipliers. It turns out that all such sets are finite unions of sets in each of which no two elements are related under the partial ordering of Γ. Finally, in § 6, we return to the subject of Fourier series whose restrictions to S are lacunary and obtain some information about such series from our main theorem.

Notation and terminology have been taken from [8], which is a good source for the facts which we shall assume in what follows.

2. We begin with an illustration of our method in a simple setting.

THEOREM 1. *Let* $E = \{m_n\}_{n=1}^{\infty}$ *be a set of positive integers with*

$m_{n+1} > 2\, m_n$ *for all* n. *Suppose that* f *is a function in* $L^1(T)$ *with* $\hat{f}(m) = 0$ *for all nonnegative* m *which do not belong to* E. *Then* $\sum_E |\hat{f}(m)|^2 \leqq 4\, \|f\|_1^2$.

Proof. We can assume that $\|f\|_1 = 1$. Factor $f = g\bar{h}$ where g and h are in $L^2(T)$ with $\|g\|_2 = \|h\|_2 = 1$.

Then $\hat{f}(m) = 1/2\pi \int_{-\pi}^{\pi} g(\theta)\bar{h}(\theta) \exp(-im\theta)d\theta$

$$\text{i.e.,}\quad \hat{f}(m) = \langle g, \chi^m n\rangle \tag{1}$$

where $\chi(\theta) = \exp(i\theta)$ and $\langle .,. \rangle$ is the usual inner product in the Hilbert space $L^2(T)$.

By assumption the inner product in (1) is 0 for most nonnegative m. The theorem is a consequence of the following result about such inner products.

LEMMA 2. *Let* H *be a Hilbert space and* $M_1 \subset M_2 \subset \cdots \subset M_N$ *be closed subspaces of* H. *Let* $A_1, A_2, \cdots, A_N$ *be unitary linear operators on* H *with* $A_1M_1 \subset A_2M_2 \subset \cdots \subset A_NM_N$. *Suppose that* g *and* h *are elements of* H *satisfying*:

(i) $A_nh \in A_{n+1}M_{n+1}$ *for* $n = 1, 2, \cdots, N-1$

(ii) g *is orthogonal to the subspaces* $A_{n+1}M_n$ *for* $n = 1, 2, \cdots, N-1$.

Then $\sum_1^N |\langle g, A_nh\rangle|^2 \leqq 4\, \|g\|^2 \cdot \|h\|^2$.

To prove the theorem let H be $L^2(T)$ and take M_n to be the closed subspace of $L^2(T)$ generated by $\{\chi^m h \mid -m_n \leqq m < 0\}$. Clearly

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots .$$

Define A_n by $A_nk = \chi^{m_n}k$ for all k in H.

The subspaces A_nM_n are the closed linear spans in $L^2(T)$ of $\{\chi^m h \mid 0 \leqq m < m_n\}$. So, $A_1M_1 \subset A_2M_2 \subset \cdots \subset A_nM_n \subset A_{n+1}M_{n+1} \subset \cdots$. Also as $m_n < m_{n+1}$, $A_nh \in A_{n+1}M_{n+1}$ for all n.

Finally $A_{n+1}M_n$ is the closed subspace generated by

$$\{\chi^m h \mid m_{n+1} - m_n \leqq m < m_{n+1}\}\, .$$

Now $m_{n+1} - m_n > 2\, m_n - m_n = m_n$ so that $m_{n+1} - m_n \leqq m < m_{n+1}$ implies $m_n < m < m_{n+1}$. For such m, $\langle g, \chi^m h\rangle = \hat{f}(m) = 0$ by assumption. Therefore $\langle g, k\rangle = 0$ for every generator k of $A_{n+1}M_n$ and hence for all k in $A_{n+1}M_n$ and (ii) holds.

The lemma applies for any fixed N to yield

$$\sum_{n=1}^{N} |\hat{f}(m_n)|^2 = \sum_{1}^{N} |\langle g, A_nh\rangle|^2 \leqq 4\, .$$

Therefore $\sum_{n=1}^{\infty} |\hat{f}(m_n)|^2 \leqq 4$.

Proof of Lemma 2. Again normalize by assuming $||g|| = ||h|| = 1$. By M_0 we shall mean the subspace of H consisting of 0 alone.

For $n = 0, 1, \cdots, N$ let k_n be the orthogonal projection of h onto M_n. Then the sequence $k_1 - k_0, k_2 - k_1, \cdots, k_N - k_{N-1}$ is orthogonal and

$$\sum_1^N ||k_n - k_{n-1}||^2 = ||k_N - k_0||^2 \leqq 1 . \tag{2}$$

Now for each $n, A_n(h - k_n)$ is orthogonal to A_nM_n. But for $m < n$, $A_mh \in A_{m+1}M_{m+1} \subset A_nM_n$, and $A_mk_m \in A_mM_m \subset A_nM_n$. So for distinct m and n, $A_n(h - k_n)$ and $A_m(h - k_m)$, are orthogonal with norm at most 1.

Write $\langle g, A_nh\rangle = \langle g, A_n(h-k_n)\rangle + \langle g, A_n(k_n - k_{n-1})\rangle + \langle g, A_nk_{n-1}\rangle = a_n + b_n + c_n$ say.

By (ii) $c_n = 0$ for all n.

By Bessel's inequality,

$$\sum_1^N |a_n|^2 \leqq ||g||^2 = 1 .$$

Finally $|b_n| \leqq ||g||\cdot||A_n(k_n - k_{n-1})|| = ||k_n - k_{n-1}||$ so that by (2) $\sum_1^N |b_n|^2 \leqq 1$.

The triangle inequality for l^2 yields $[\sum_1^N |\langle g, A_nh\rangle|^2]^{1/2} \leqq 2$.

Results like Theorem 1 are well known for lacunary series, i.e., series with $\hat{f}(m) = 0$ for *all* m off E [14, p. 205, Remark (a)]. The fact that the same is true for semi-lacunary series is implicit in an argument of Rudin, [9, § 5.7], and seems to be well known among Fourier analysts. So the novelty of Theorem 1 lies in the method of proof rather than the conclusion. On the other hand, the most general situation in which our method works seems different from the one in which the usual technique works; we shall compare them in § 6.

For the moment, let us remark that a simple modification of the above handles the case when, for some λ strictly between 1 and 2, $m_{n+1} > \lambda m_n$ for all n. It turns out that if f is as in Theorem 1 then $\sum_E |\hat{f}(m)|^2 \leqq (\sqrt{k} + 1)^2 ||f||_1^2$, where k is an integer chosen so that $\lambda^k \geqq \lambda/(\lambda - 1)$.

3. In what follows, G will be a compact Abelian group and Γ will be the dual group of G, with the group operations written additively. S will denote a semigroup in Γ which contains 0. We let A be the algebra of continuous functions f on G for which $\hat{f}(\gamma)$ is 0 off S. For definiteness, the reader may find it convenient to imagine that $G = T^2$, $\Gamma = Z^2$, and that S is the first quadrant in Z^2.

Let M be the class of Paley multipliers on A, that is those sequences $\{w(\gamma)\}_{\gamma \in S}$ with $\sum_S |\hat{f}(\gamma)w(\gamma)|$ finite for all f in A. In other words for each w in M, the mapping $f \to \{\hat{f}(\gamma)w(\gamma)\}_{\gamma \in S}$ sends A into $l^1(S)$. In fact, by the closed graph theorem, this is a bounded linear operator. So M is a normed linear space with the operator norm:

$$\| w \|_M = \sup \sum_S |\hat{f}(\gamma)w(\gamma)| \qquad (f \text{ in } A \text{ and } \| f \|_\infty = 1) .$$

Observe that if v is a sequence with $|v(\gamma)| \leqq |w(\gamma)|$ for all γ in S then $\| v \|_M \leqq \| w \|_M$. In particular this is true if v is a truncation of w which agrees with w on part of S and is 0 elsewhere.

For any sequence $\varepsilon(\gamma) = \pm 1$ we have that $|\sum_S \varepsilon(\gamma)f(\gamma)w(\gamma)| \leqq \| w \|_M \cdot \| f \|_\infty$. Therefore the mapping $f \to \sum_S \varepsilon(\gamma)w(\gamma)\hat{f}(\gamma)$ is a bounded linear functional on A of norm no greater than $\| w \|_M$. By the Hahn-Banach theorem it has a norm preserving extension to all of the continuous functions on G. This means that there is a bounded regular Borel measure μ on G with $\| \mu \| \leqq \| w \|_M$ and

$$\sum_S \varepsilon(\gamma)w(\gamma)\hat{f}(\gamma) = \int_G f(x)d\mu(-x)$$

for all f in A. Taking $f = \gamma$ for any γ in S we obtain:

$$(1) \qquad \varepsilon(\gamma)w(\gamma) = \int_G \gamma(x)d\mu(-x) = \int_G \gamma(-x)d\mu(x) = \hat{\mu}(\gamma) .$$

The property that for every choice of signs $\varepsilon(\gamma)$ there are measures μ satisfying (1) characterizes M and was used by Helson and Rudin in their proofs of Paley's theorem ([4] and [8, p. 222]).

Now S induces a partial ordering of Γ under the rule: $\gamma_1 \leqq \gamma_2$ if and only if $\gamma_2 - \gamma_1 \in S$. The order relation is transitive and invariant under addition but it may happen that $\gamma_1 \leqq \gamma_2 \leqq \gamma_1$ without $\gamma_1 = \gamma_2$.

We can now state and prove our main theorem.

THEOREM 3. *Let $w \in M$ and $E \subset S$ be totally ordered under the order induced by S. Then*

$$\sum_E |w(\gamma)|^2 \leqq 4 \| w \|_M^2 .$$

Proof. It is enough to prove the theorem for $\| w \|_M = 1$ and E finite. Let $\gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_N$ be the elements of E. Denote by v the truncation of w to E: $v(\gamma) = w(\gamma)$ if $\gamma \in E$ and $v(\gamma) = 0$ otherwise. As observed above $\| v \|_M \leqq 1$.

Let $\varepsilon(\gamma)$ be any sequence of ± 1 on S. There is a bounded regular Borel measure μ on G with $\| \mu \| \leqq 1$ and $\hat{\mu}(\gamma) = \varepsilon(\gamma)v(\gamma)$ for all

γ in S.

Fix $K > 1$. As E is finite there is a trigonometric polynomial P on G with $\|P\|_1 \leqq K^2$ and $\hat{P}(\gamma) = 1$ on E [8, Th. 2.6.8]. Then $f \equiv P*\mu$ is a trigonometric polynomial with the following properties:
(2) $\|f\|_1 \leqq K^2, \hat{f}(\gamma) = \varepsilon(\gamma)w(\gamma)$ on E and 0 elsewhere in S, and $\hat{f}(\gamma) = 0$ off the support of $\hat{P}$.

It is a theorem of Littlewood [5] that, if for each choice of signs there is an f satisfying (2) and with $\hat{f} = 0$ off E, then $\sum_E |w(\gamma)|^2 \leqq BK^4$ where B is a fixed constant. Our problem is to reach the same conclusion assuming only that $\hat{f}(\gamma) = 0$ on the rest of S.

In order to make use of the random signs $\varepsilon(\gamma)$ in the above, we introduce the Rademacher functions. Let Q be the Cartesian product of N copies of Z_2, the additive cyclic group of order 2. Denote the elements of Q by $t = (t_1, \cdots, t_n)$ with each $t_j = 0$ or 1. Define the n'th Rademacher function r_n by

$$r_n(t) = \begin{cases} -1 & \text{if } t_{n+1} = 1 \\ 1 & \text{if } t_{n+1} = 0 \end{cases} \qquad n = 0, 1, \cdots, N-1 .$$

By (2), for each t in Q we can find a trigonometric polynomial $f(t, x)$ on G so that $\int_G |f(t, x)|\, dx \leqq K^2$, $[f(t, \cdot)]^\frown(\gamma) = 0$ off the support of $\hat{P}$ and

$$(3) \qquad [f(t, \cdot)]^\frown(\gamma) = \begin{cases} r_{n-1}(t)w(\gamma_n) & \text{if } \gamma = \gamma_n \in E \\ 0 \text{ for all other } \gamma \text{ in } S \end{cases} .$$

Letting dt be the Haar measure on Q which assigns mass 2^{-N} to each point, we get $\int_Q \int_G |f(t, x)|\, dx dt \leqq K^2$. That is, if we assume for the moment that f is a measurable function on $Q \times G$, then $f \in L^1(Q \times G)$ with norm no greater than K^2.

In fact, f is a trigonometric polynomial on $Q \times G$, that is, a finite linear combination of continuous characters on $Q \times G$, but to be sure of this we must look at the set of such characters, i.e., the dual group of $Q \times G$.

To begin with, the complete set of characters on Q is the set of Walsh functions $\psi(m)(t)$, $m = 0, 1, \cdots, 2^N - 1$ [3, pp. 376–377], which are defined as follows. We write

$$m = 2^{n_1} + \cdots + 2^{n_k}, 0 \leqq n_1 < n_2 < \cdots < n_k$$

and let $\psi(m)(t) = r_{n_1}(t) \cdot r_{n_2}(t) \cdots r_{n_k}(t)$, with the convention that $\psi(0)(t) \equiv 1$. R, the dual group of Q is the set of all such functions, under multiplication.

The dual group of $Q \times G$ is $R \times \Gamma$ [8, Th. 2.2.2] so that the products $\psi(m)(t)\gamma(x)$ form the complete set of continuous characters on $Q \times G$.

For fixed $t, f(t, x)$ is a trigonometric polynomial on G whose coefficients are 0 off the support of $\hat{P}$. As Q and R are finite, all functions on Q are trigonometric polynomials and in particular $[f(t, \cdot)]^\wedge(\gamma)$ is a trigonometric polynomial on Q for each γ. So, the finite sum $f(t, x) = \sum_{\gamma \in \operatorname{supp} \hat{P}} \gamma(x)[f(t, \cdot)]^\wedge(\gamma)$ is a trigonometric polynomial on $Q \times G$.

Write $f(t, x) = \sum_{m=0}^{2N-1} \sum_{\gamma \in \Gamma} a(m, \gamma)\psi(m)(t)\gamma(x)$. Clearly $a(m, \gamma) = 0$ unless $\gamma \in \operatorname{supp} \hat{P}$ and in view of (3) we have

$$a(m, \gamma) = \begin{Bmatrix} w(\gamma_n) \text{ if } \gamma = \gamma_n \text{ and } m = 2^{n-1} \\ 0 \text{ otherwise for } \gamma \text{ in } S \end{Bmatrix} . \tag{4}$$

We have no information about $a(m, \gamma)$ when γ is in supp $\hat{P}$ but not in S but this will not matter.

The proof now proceeds much as in Theorem 1 with the products $\psi(2^{n-1})\gamma_n$ playing the role of the thin set of characters $\{\chi^{m_n}\}_{n=1}^{\infty}$.

Factor $f(t, x) = g(t, x)\bar{h}(t, x)$ where $g, h \in L^2(Q \times G)$ and $\| g \|_2 = \| h \|_2 = \| f \|_1^{1/2} \leqq K$.

For all m, γ

$$\begin{aligned} a(m, \gamma) &= \int_Q \int_G g(t, x)\bar{h}(t, x)\bar{\gamma}(x)\bar{\psi}(m)(t)dxdt , \\ \text{i.e., } a(m, \gamma) &= \langle g, \psi(m)\gamma h \rangle . \end{aligned} \tag{5}$$

We wish to apply Lemma 2 with $A_n k \equiv \psi(2^{n-1})\gamma_n k$ for all k in $L^2(Q \times G)$. Assume for a moment that subspaces M_n can be chosen so that the hypotheses of the lemma hold. Then, in view of (4) and (5)

$$\sum_1^N | w(\gamma_n) |^2 = \sum_1^N | a(2^{n-1}, \gamma_n) |^2 = \sum_1^N | \langle g, A_n h \rangle |^2 \leqq 4K^2$$

which is what we want, as K is any constant larger than 1.

It remains to chose the M_n so that the assumptions of the lemma are satisfied. This is the only part of the proof where the total ordering of E is used.

Let M_n be the closed linear span in $L^2(Q \times G)$ of $\{\psi(m)\gamma h \mid (m, \gamma) \neq (0, 0), 0 \leqq m < 2^n, -\gamma_n \leqq \gamma \leqq 0\}$. As $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ it is clear that $M_1 \subset M_2 \subset \cdots \subset M_n$.

The set $\{\psi(m) \mid 0 \leqq m < 2^n\}$ is the subgroup of the Walsh functions generated by $r_0, r_1, \cdots, r_{n-1}$. Therefore, $A_n M_n$ is the closed subspace generated by $\{\psi(m)\gamma h \mid (m, \gamma) \neq (2^{n-1}, \gamma_n), 0 \leqq m < 2^n, 0 \leqq \gamma \leqq \gamma_n\}$. Certainly $A_1 M_1 \subset A_2 M_2 \subset \cdots \subset A_N M_N$. Moreover

$$A_n h = \psi(2^{n-1})\gamma_n h \in A_{n+1} M_{n+1}$$

for $n<N$ and (i) holds. Again this depends on the fact that $\gamma_n \leqq \gamma_{n+1}$.

Finally if $m < 2^n$, r_n is not one of the factors of $\psi(m)$. So

$$\psi(m)\psi(2^n) = \psi(m)r_n = \psi(m + 2^n) .$$

Therefore, $A_{n+1}M_n$ is the closed linear span of

$$\{\psi(m)\gamma h \mid (m, \gamma) \neq (2^n, \gamma_{n+1}),\, 2^n \leqq m < 2^{n+1},\, \gamma_{n+1} - \gamma_n \leqq \gamma \leqq \gamma_{n+1}\} .$$

Now when m and γ satisfy these restrictions, then $\gamma \in S$ as $\gamma_{n+1} - \gamma_n \geqq 0$, and by (5) and (4), $\langle g, \psi(m)\gamma h\rangle = a(m, \gamma) = 0$. That is $\langle g, k\rangle = 0$ for every generator k of $A_{n+1}M_n$ and hence for every k in $A_{n+1}M_n$. (ii) holds and the proof of the theorem is complete.

In fact the argument actually works under somewhat weaker assumptions than those above.

DEFINITION. A set $I \subset \Gamma$ is called convex if $\{\gamma \mid \gamma_1 \leqq \gamma \leqq \gamma_2\} \subset I$ whenever γ_1 and γ_2 are in I.

THEOREM 4. *Let $I \subset \Gamma$ be convex and w be a sequence defined on I. Define*

$$\| w \|_I = \sup \sum_I | \hat{f}(\gamma) w(\gamma) | \qquad (f \in C_I(G),\, \| f \|_\infty = 1) ,$$

where $C_I(G) = \{f \in C(G) \mid \hat{f}(\gamma) = 0$ off $I\}$. Let $E \subset I$ be totally ordered. Then $\sum_E | w(\gamma) |^2 \leqq 4 \| w \|_I^2$.

Outline of proof. The method of Theorem 3 works in this situation. The main change is that statements which held for all γ in S in the proof of Theorem 3 now hold for all γ in I. Lemma 2 applies with M_n taken to be the closed linear span in $L^2(Q \times G)$ of $\{\psi(m)\gamma h \mid (m, \gamma) \neq (0, 0),\, 0 \leqq m < 2^n$, and $\gamma_1 - \gamma_n \leqq \gamma \leqq 0\}$. We omit the details.

In the special case $G = T$, $S = \{n \mid n \geqq 0\}$, $I = \{n \mid n_1 \leqq n \leqq n_2\}$, I itself is totally ordered and we conclude that $\sum_I | w(n) |^2 \leqq 4 \| w \|_I^2$. With the constant 4 replaced by a much larger one, this was obtained by Steckin [11, Lemma 2] as a consequence of Paley's theorem.

The definition of $\| w \|_I$ makes sense for any set I and does not depend at all on S. Furthermore a set may be convex with respect to several orderings of Γ. For instance, let $G = T^2$, $\Gamma = Z^2$, I = first quadrant in Z^2. We can take S to be any quadrant and in each case I is convex with respect to the order induced by S. So, S plays an indirect role in Theorem 4 which may be restated as follows:

THEOREM 4′. *Let I be a subset of Γ. Define $\| w \|_I$ as before. Then $\sum_E | w(\gamma) |^2 \leqq 4 \| w \|_I^2$ for any set $E \subset I$ which is totally ordered* *under some ordering of Γ with respect to which I is convex.*

4. We now treat the case of power series in a finite number of variables. That is, $G = T^n$, $\Gamma = Z^n$, and $S = \{N = (N_1, N_2, \cdots, N_n) \mid N_j \geqq 0 \text{ for all } j\}$.

The main theorem tells us that, when w is a Paley multiplier, 4 $\| w \|_M^2$ is a uniform bound on $\sum_E | w(N) |^2$ for all totally ordered subsets E of S. Unfortunately, such sets E are essentially one dimensional. For instance, the N_1 axis $\{N \text{ in } S \mid N_j = 0 \text{ for } j > 1\}$ is a maximal totally ordered subset of S.

It is possible, however, to give bounds on $\sum_D | w(N) |^2$ for some sets D which are not as thin as totally ordered sets. For simplicity we prove the following theorem only for the case $n = 2$.

THEOREM 5. *Fix an integer $L > 0$ and let D be the set*

$$\{N \mid L \leqq N_1 \leqq 3L \quad \textit{and} \quad N_2 \geqq 0\} .$$

Then for every w in M, $\sum_D | w(N) |^2 \leqq 36 \| w \|_M^2$.

Proof. Since truncation does not increase norms in M, assume that $w = 0$ off D.

Let $K_n(\theta)$ be the Fejer kernel $\sum_{m=-n}^{n} [1 - | m |/(n+1)] \exp(im\theta)$. Put $K(\theta) = \exp(i2L\theta)[2K_{2L-1}(\theta) - K_{L-1}(\theta)]$. Then $(1/2\pi)\int_{-\pi}^{\pi} | K(\theta) | d\theta \leqq 3$, $\hat{K}(m) = 1$ if $L \leqq m \leqq 3L$, and $\hat{K}(m) = 0$ if $m < 0$. Define a measure ν on T^2 by $\int_{T^2} f(\theta, \varphi) d\nu(-\theta, -\varphi) = (1/2\pi)\int_{-\pi}^{\pi} f(\theta, 0) K(-\theta) d\theta$. Then $\| \nu \| \leqq 3$, $\hat{\nu} = 1$ on D, and $\hat{\nu}(N) = 0$ if $N_1 < 0$.

Now let S_1 be the half space $\{N \mid N_2 > 0, \text{ or } N_2 = 0 \text{ and } N_1 \geqq 0\}$. w can be thought of as a Paley multiplier on $C_{S_1}(T^2)$, the continuous functions whose coefficients vanish off S_1. For, suppose that f is such a function. Let $g = f * \nu$. Then $g \in A$ and $\| g \|_\infty \leqq 3 \| f \|_\infty$. Extend w to S_1 by setting it equal to 0 on the rest of S_1. Now,

$$\sum_{S_1} | \hat{f}(N) w(N) | = \sum_{D} | \hat{g}(N) w(N) | \leqq \| w \|_M \| g \|_\infty \leqq 3 \| w \|_M \| f \| .$$

So, as a multiplier on C_{S_1}, w has norm at most 3 $\| w \|_M$. It follows from the main theorem that

$$\sum_{D} | w(N) |^2 = \sum_{S_1} | w(N) |^2 \leqq 36 \| w \|_M^2 .$$

The same kind of conclusion can be obtained in dimension n for sets D of the form:

$$\{N \mid N_n \geqq 0 \text{ and } L_j \leqq N_j \leqq 3L_j \text{ for } j = 1, 2, \cdots, n-1\}$$

where $L_1, L_2, \cdots, L_{n-1}$ are any fixed nonnegative integers. Of course the last coordinate need not be the one that is free in the above.

As we shall see in the next section, the main theorem implies that, in the context of power series in a finite number of variables, $w(N) \to 0$ as $N \to \infty$ whenever $w \in M$. This fact can also be derived from the following lemma of Helson.

LEMMA (*Translation Lemma*). *Let G be a compact Abelian group. Suppose that μ is a finite regular Borel measure on G and $\{\gamma_n\}_1^\infty$ is a sequence of distinct elements of Γ. Define measures λ_n by $d\lambda_n(x) = \gamma_n(x)d\mu(x)$. Let $\lambda_n \to \sigma$ in the weak star topology. Then σ is singular with respect to the Haar measure of G* [8, Lemma 3.5.1].

THEOREM 6. *With $G = T^n$ and S as above, $w(N) \to 0$ as $N \to \infty$ for every Paley multiplier w.*

Proof. Suppose that the theorem is false. Let w have the property that $|w(N)| \geqq 1$ on an infinite set B of N's.

First assume that B contains an infinite sequence $\{N^{(k)}\}_1^\infty$ such that for each k and all j, $N_j^{(k)} \geqq k$. Then in fact the sequence can be chosen to be lacunary in the sense that for each k and j, $N_j^{(k+1)} > 2N_j^{(k)}$. Let v be equal to 1 on this sequence and 0 elsewhere. As v is dominated by w, it is a Paley multiplier.

Therefore there is finite regular Borel measure μ on T^n so that $\hat{\mu}(N^{(k)}) = 1$ for all k and $\hat{\mu}(N) = 0$ for all other N in S. Consider the measures λ_k defined by $d\lambda_k(x) = \exp(-iN^{(k)}\cdot x)d\mu(x)$. For all k, $\|\lambda_k\| = \|\mu\|$, so that a subsequence of the λ_k's converges in the weak star sense to a measure σ.

$\hat{\lambda}_k(0) = 1$ for all k, so that $\hat{\sigma}(0) = 1$. For any $N \neq 0$, $\hat{\lambda}_k(N) = 0$ for all large k. Hence $\hat{\sigma}(N) = 0$ for $N \neq 0$. This means that $d\sigma$ is dx, the Haar measure on T^n. But by the translation lemma, $d\sigma$ is singular with respect to dx, a contradiction.

The preceding three paragraphs prove the theorem for the case $n = 1$ as then any infinite B would contain such a sequence $\{N^{(k)}\}$.

For $n > 1$, we conclude that B contains no such sequence. It follows that there is an integer k for which the cone $\{N \mid N_j > k$ for all $j\}$ does not intersect B. In other words B is contained in the union of the $(k+1)\cdot n$ hyperplanes $\{N$ in $S \mid N_j = h\}$ where j runs from 1 to n and h from 0 to k. The intersection of B with one of these hyperplanes, for particular choices of j and h, is infinite. Let S_1 be the positive cone in Z^{n-1}. Define a sequence v on S_1 by

$$v(N_1, N_2, \cdots, N_{n-1}) = w(N_1, N_2, \cdots, N_{j-1}, h, N_{j+1}, \cdots, N_{n-1}) .$$

It is not hard to see the that v is a Paley multiplier on $C_{S_1}(T^{n-1})$

with $|v(N)| \geqq 1$ for infinitely many N in S_1. The theorem follows by induction on the dimension n.

The idea for the application of the translation lemma in the above came from Rider's treatment of the infinitite dimensional case [7, § 3]. In fact Rudin made a similar application in [9, Th. 4] to obtain the result in the one variable case.

So, for the case of power series in two or more variables Theorems 3, 5, and 6 provide a variety of restrictions which must be satisfied by any Paley multiplier. We now give an example for the case of two variables of a sequence which satisfies these restrictions but is not square summable. It resembles one given by Bohr in infinitely many variables [2, p. 468, Th. 5] and arose from a suggestion of Professor Walter Rudin.

For $m \geqq 0, n \geqq 0$, let $w(m, n) = 1/(m + n + 1)$.

Observe that any totally ordered set E intersects the line $l_k = \{(m, n) \mid m + n = k\}$ in at most one point. For any such E,

$$\sum_E |w(N)|^2 = \sum_{k=0}^{\infty} \sum_{E \cap l_k} |w(N)|^2 \leqq \sum_0^{\infty} 1/(k+1)^2 = \pi^2/6 .$$

Hence w satisfies the conclusion of Theorem 3.

Next let D be any set of the type considered in Theorem 5. For $k < L$, $l_k \cap D$ is empty and for any k, $l_k \cap D$ has at most $2L + 1$ elements. Therefore,

$$\begin{aligned}\sum_D |w(N)|^2 &= \sum_{k=L}^{\infty} \sum_{l_k \cap D} |w(N)|^2 \leqq \sum_L^{\infty} (2L+1)(k+1)^{-2} \\ &\leqq \frac{2L+1}{L} \leqq 3 ,\end{aligned}$$

and the conclusion of Theorem 5 holds for w.

Finally, it is clear that $w(N) \to 0$ as $N \to \infty$.

On the other hand,

$$\sum_S |w(N)|^2 = \sum_{k=0}^{\infty} \sum_{l_k \cap S} |w(N)|^2 = \sum_{k=0}^{\infty} (k+1)(k+1)^{-2} = \infty .$$

It is not known whether w is a Paley multiplier sequence. This example shows, however, that in the context of power series in two as more variables, our results do not imply that $M = l^2(S)$. The question is therefore still open for the case of n variables, $1 < n < \infty$.

5. We modify a definition of Rider [7, p. 558].

DEFINITION. Let G, S, and A be as in § 3. A subset B of S will be called a Bohr set if there is a constant K so that $\sum_B |\hat{f}(\gamma)| \leqq K\|f\|_\infty$ for all f in A.

In other words, B is a Bohr set if the sequence which is 1 on B and 0 elsewhere in S is a Paley multiplier.

The reason for the name Bohr set is the following theorem of Bohr [2, p. 468, Th. 5]. Let G be the complete direct sum T^ω of countably many circles and Γ be the direct sum Z^∞, [8, § 8.7.9]. Let $S = \{N \in Z^\infty \mid N_j \geqq 0 \text{ for all } j\}$ and let A be the space of continuous functions on T^ω with coefficients supported by S. Let $B = \{N \in Z^\infty \mid N_j = \delta_{ij} \text{ for some } i\}$. Then $\sum_B |\hat{f}(N)| \leqq \|f\|_\infty$ for all f in A. Other examples of Bohr sets and an account of the connection with Dirichlet series appear in [7].

We use Theorem 3 to obtain necessary arithmetic conditions on Bohr sets.

THEOREM 7. *Let B be a Bohr set and K the constant of the definition. Then every totally ordered subset of B has at most $4K^2$ elements.*

Proof. By assumption the multiplier w which is 1 on B and 0 elsewhere has norm at most K. If $E \subset B$ is totally ordered, $\sum_E 1 = \sum_E |w(\gamma)|^2 \leqq 4K^2$.

Observe that the theorem certainly holds for Bohr's example B. Totally ordered subsets of B have one element as no two elements of B are related under the order induced by S.

DEFINITION. A subset B or Γ will be called unrelated if no two elements of B are related under the order induced by S.

LEMMA 8. *A subset B of Γ contains no totally ordered set with more than K elements if and only if B is the union of at most K unrelated sets.*

Proof. It is obvious that such a union contains no totally ordered set with more than K elements.

Conversely, suppose that the totally ordered subsets of B have at most K elements. Let E be a totally ordered subset of B, maximal with respect to containment. We shall find a set B_1 consisting of exactly one minimal element from each such E. As E is finite the set F of minimal elements of E is nonempty. By the maximality of E, F is a maximal equivalence class in B: i.e., for any γ in F, $F = \{\gamma' \text{ in } B \mid \gamma \leqq \gamma' \leqq \gamma\}$. Thus if E' is another maximal totally ordered subset of B and F' is the set of minimal elements of E' then either $F = F'$ or F and F' are disjoint. The axiom of choice yields a set B_1 consisting of one element from each such F, that is one minimal element from each such E. By the maximality of the E's, B_1 is unrelated. Moreover as every totally ordered subset of B is contained in such an $E, B \sim B_1$ contains no totally ordered set with more than $K - 1$ elements. The lemma follows by induction on K.

THEOREM 9. *Every Bohr set is the union of at most $4K^2$ unrelated sets, where K is the constant in the definition of Bohr set.*

Proof. Combine 7 and 8.

It can be shown that every unrelated subset of the positive cone S of Z^n is finite. This means that, for the case of power series in n variables, Bohr sets are finite. This statement is equivalent to Theorem 6, as it is easy to see in any case that there is an infinite Bohr set in S if and only if there is a Paley multiplier which does not tend to 0. In fact we can use Theorem 7 in place of the Translation Lemma in the proof of Theorem 6. Simply observe that the lacunary sequence discussed in the second paragraph of the proof of Theorem 6 is increasing with respect to the order induced by S and can have at most $4K^2$ elements, contrary to the assumption that it is infinite. Therefore there is no such sequence and the last paragraph of the proof of Theorem 6 applies.

We now turn to the case of power series in infinitely many variables; i.e., G is the complete direct sum T^ω, Γ is the direct sum Z^∞ and $S = \{N \mid N_j \geqq 0 \text{ for all } j\}$. Bohr's example shows that there are infinite Bohr sets in this case.

In [7, p. 560] Rider gives sufficient arithmetic conditions for a set to be a Bohr set: Let $B \subset S$ satisfy:

(c) the elements of B are linearly independent over the integers.

(d) whenever $N \in S$ and $N = \sum_1^k \beta_i N^{(i)}$ where the β_i are integers, $\sum_1^k \beta_i = 1$, and the $N^{(i)} \in B$ for all i, then $N \in B$.

Then B is a Bohr set.

It is easy to see that these conditions force any such B to be unrelated. For if $N^{(1)} < N^{(2)}$ are in B, then by (d)

$$M^{(k)} \equiv N^{(1)} + k(N^{(2)} - N^{(1)})$$

is in B for all $k \geqq 0$. But $M^{(2)} + N^{(1)} - 2N^{(2)} = 0$ contrary to (c).

On the other hand, an unrelated set need not be a Bohr set. For instance let $B_k = \{N \text{ in } S \mid N_j = 0 \text{ unless } j = 2k - 1 \text{ or } 2k, \text{ and } \sum_1^\infty N_j = k\}$. Let $B = \bigcup_1^\infty B_k$. It is easy to see that B is unrelated. Let w be the sequence which is 1 on B and 0 elsewhere. Apply

Theorem 4′ with $I = S$, using the order induced by $\{N \text{ in } Z^\infty \mid N_j \geqq 0$ for $j \neq 2k$ and $N_{2k} \leqq 0\}$, to obtain: $k + 1 = \sum_{B_k} |w(N)|^2 \leqq 4 \|w\|_M^2$. Therefore w is not a Paley multiplier and B is not a Bohr set. We can modify this example so that it becomes an unrelated Sidon set which is not a Bohr set.

Nevertheless any set consisting of exactly one element from each B_k satisfies (c) and (d) and is therefore a Bohr set. It is not clear whether every infinite unrelated set must contain an infinite Bohr set.

It is shown in [2] that there is a connection between Dirichlet series and power series in infinitely many variables. Theorem 9 can be restated as follows:

THEOREM. *Suppose that B is a set of positive integers so that there is a constant K, with $\sum_B |c(n)| \leqq K$ whenever there is a Dirichlet series $f(s + it) = \sum_1^\infty c(n)n^{-s-it}$ with $|f(s + it)| \leqq 1$ for all $s > 0$. Then B is the union of at most $4K^2$ sets in each of which no element divides any other.*

6. Conclusions similar to Theorem 1 can be obtained under weaker assumptions. Once again G, Γ, S, and A are as in § 3.

DEFINITION. A set $B \subset \Gamma$ is called a Sidon set if there is a constant K so that $\sum_B |\hat{f}(\gamma)| \leqq K \|f\|_\infty$ for every trigonometric polynomial f for which $\hat{f}$ is 0 off B, [8, § 5.7].

THEOREM 10. *Let B be a Sidon set and I be a convex subset of Γ. Suppose that f is in $L^1(G)$ and $\hat{f}(\gamma) = 0$ whenever γ is in I but not in B. Let E be a totally ordered subset of $B \cap I$. Then $\sum_E |\hat{f}(\gamma)|^2 \leqq 4K^2 \|f\|_1^2$. The constant K is the one appearing in the definition of Sidon set and does not depend on I or E.*

Proof. Let g be a trigonometric polynomial with $\hat{g} = 0$ off I. Put $h = f * g$. Then $\hat{h} = 0$ off $B \cap I$ and in particular off B. By the definition of Sidon set,

$$(1) \qquad \sum_I |\hat{g}(\gamma)\hat{f}(\gamma)| = \sum_B |\hat{h}(\gamma)| \leqq K \|h\|_\infty \leqq K \|f\|_1 \cdot \|g\|_\infty .$$

Since the trigonometric polynomials with coefficients supported on I are dense in $C_I(G)$, (1) holds for all g in $C_I(G)$. Putting $w(\gamma) = \hat{f}(\gamma)$ we have that $\|w\|_I \leqq K \|f\|_1$. By Theorem 4, $\sum_E |w(\gamma)|^2 \leqq 4K^2 \|f\|_1^2$.

Observe that in the above $\hat{f}$ is arbitrary off I.

COROLLARY. *Let $E = \{m_n\}_{n=1}^\infty$ be any Hadamard set of positive integers (i.e., there is a $\lambda > 1$ so that $m_{n+1} \geqq \lambda m_n$ for all n). Suppose* *that* $f \in L^1(T)$ *and* $\hat{f}(m) = 0$ *for all* $m \geqq 0$ *which are not in* E. *Then* $\sum_E |\hat{f}(m)|^2 \leqq K \| f \|_1^2$ *for some constant* K *depending on* E.

Proof. It is well known that every Hadamard set is a Sidon set. Theorem 10 applies with $I = S = \{m \mid m \geqq 0\}$ and $B = E$.

As a Sidon set need not be a Hadamard set Theorem 10 generalizes Theorem 1.

If S is a half space, it is not necessary to know that E is a Sidon set to obtain the conclusion of Theorem 10.

DEFINITION. A set $E \subset \Gamma$ is said to be of type $\Lambda(s)$, $s > 0$, if for some $r < s$, there is a constant B_{rs} so that $\| f \|_s \leqq B_{rs} \| f \|_r$ for every trigonometric polynomial f whose coefficients are 0 off E.

In [9, Th. 1.4], Rudin shows that if there is such a constant B_{rs} for one $r < s$, then there are such constants $B_{r's}$ for all $r' < s$.

The following argument was shown to us by F. Forelli. It resembles the one used by Rudin in proving Paley's theorem for half spaces [8, p. 222], and is the technique mentioned at the end of § 2.

THEOREM 11. *Suppose that* S *is a half-space, that is, that* Γ *is totally ordered. Let* $E \subset S$ *be a* Λ (2) *set. Then there is a constant* K *so that* $\sum_E |\hat{f}(\gamma)|^2 \leqq K \| f \|_1^2$ *for every* f *with* $\hat{f} = 0$ *on* $S \sim E$.

Proof. First suppose that f is a trigonometric polynomial. Let $g(x) = \sum_S \hat{f}(\gamma)\gamma(x)$ be the analytic projection of f. There is a constant K_1 so that $\| g \|_{1/2} \leqq K_1 \| f \|_1$ [8, Th. 8·7·6]. The coefficients of g vanish off E so that $\| g \|_2 \leqq B_{(1/2)2} \| g \|_{1/2} \leqq K_2 \| f \|_1$, say. Then

$$\sum_E |\hat{f}(\gamma)|^2 = \| g \|_2^2 \leqq (K_2)^2 \| f \|_1^2 ,$$

the desired result with $K = (K_2)^2$.

We obtain the same conclusion for arbitrary f by convoluting f with a sequence of trigonometric polynomials which form an approximate identity.

Every Sidon set is of type Λ (2) [8, § 5·7·7]. So when S is a half space and $I = S$, Theorem 10 is a special case of Theorem 11. When S is smaller than a half space, however, the proof of 11 breaks down for the same reason as Rudin's proof of Paley's theorem: The analytic projection may not be a bounded operator from L^1 to L^r for $r<1$.

One reason for considering theorems like these is that by an argument due to Banach [1, Th. a], they are equivalent to theorems about interpolating l^2 sequences by Fourier coefficients of continuous functions. We demonstrate this idea by applying it to Theorem 10.

THEOREM 12. *Let B be a Sidon set and I a convex subset of Γ. Let E be a totally ordered subset of $B \cap I$ and suppose that a sequence v is defined on E so that $\sum_E |v(\gamma)|^2 < \infty$. Then there is a function f in $C_I(G)$ with $\hat{f}(\gamma) = v(\gamma)$ for all γ in E. Moreover f can be chosen with $\|f\|_\infty \leqq 2K' \|v\|_2$ for any fixed K' larger than the constant K associated with B.*

Proof. Let D be the closed subspace of elements f of C_I with $\hat{f} \equiv 0$ on E. Consider the bounded linear operator $L: C_I/D \to l^2(E)$ defined by $L([f]) = \{\hat{f}(\gamma)\}_{\gamma \in E}$. We must show that L is onto and that $\|L^{-1}\| \leqq 2K$. The range of L is dense in $l^2(E)$ and L is one so that L is onto if and only if L^* is [8, p. 259, C 11].

Now $(C_I/D)^* = D^\perp$ the annihilator of D in $(C_I)^*$. Also $(C_I)^* = M(G)/(C_I)^\perp$ where $M(G)$ is the space of bounded regular Borel measures on G and $(C_I)^\perp$ is the set of such measures μ for which $\int f(x)d\mu(-x) = 0$ for all f in C_I. Since the trigonometric polynomials in C_I are dense, $(C_I)^\perp = \{\mu \text{ in } M(G) \mid \hat{\mu} = 0 \text{ on } I\}$. Then $D^\perp = \{\mu + (C_I)^\perp \mid \hat{\mu} = 0 \text{ off } E\}$.

To any $l^2(E)$ sequence w associate the $L^2(G)$ function $g(x) = \sum_{\gamma \in E} w(\gamma)\gamma(x)$. $L^*(w)$ is the coset $g + (C_I)^\perp$ in $D^\perp$.

Pick μ in $(C_I)^\perp$ and a finite subset F of E. Let P be a trigonometric polynomial with $\hat{P} = 1$ on F. Then the function $h \equiv (g + \mu)*P$ is a trigonometric polynomial. On I, $\hat{\mu}(\gamma) = 0$ so that $\hat{h}(\gamma) = \hat{g}(\gamma)\hat{P}(\gamma)$. In particular $\hat{h}(\gamma) = 0$ on $I \sim B$. By Theorem 10,

$$\sum_F |w(\gamma)|^2 = \sum_F |\hat{h}(\gamma)|^2 \leqq 4K^2 \|h\|_1^2 .$$

But $\|h_1\| \leqq \|g + \mu\| \cdot \|P\|_1$ and $\|P\|_1$ can be taken arbitrarily close to 1. Therefore $\sum_F |w(\gamma)|^2 \leqq 4K^2 \|g + \mu\|^2$ for all finite subsets F of E. Hence $\|g + \mu\| \geqq (1/2K)(\sum_E |w(\gamma)|^2)^{1/2}$ for all μ in $(C_I)^\perp$. That is,

$$\begin{aligned} \|L^*(w)\| &= \inf \|g + \mu\| \qquad (\mu \in (C_I)^\perp) \\ &\geqq (1/2K) \|w\|_2 . \end{aligned}$$

This means that L^* is onto [8, p. 259, C11] and $\|(L^*)^{-1}\| \leqq 2K$. Therefore L is onto.

Moreover $\|L^{-1}\| = \|(L^{-1})^*\| = \|(L^*)^{-1}\| \leqq 2K$.

A similar interpolation theorem can be derived from Theorem 11.

For the circle group, for instance, it is well known that if B is a Sidon set of integers and v is a $l^2(B)$ sequence then there is a continuous f with $\|f\|_\infty \leqq 2K\|v\|_2$ and $\hat{f}(n) = v(n)$ on B [10, Th. 5.1]. Also if $I = \{n \mid n_1 \leqq n \leqq n_2\}$ and v is 0 off $I \cap B$ then the trigonometric polynomial $g(\theta) = \sum_{I \cap B} v(n) \exp(in\,\theta)$ has the right coefficients but, as B is a Sidon set, $\|g\|_\infty \geqq (1/K) \sum_{I \cap B} |v(n)|$, which may be much larger than $2K\|v\|_2$. So the interpolating continuous function, in order to have small norm, may need some nonzero coefficients off B. Theorem 12 says that such an f can still be taken as a trigonometric polynomial with coefficients supported by the smallest interval I containing the support of v.

This paper is based on my Ph. D. dissertation at the University of Wisconsin. Many of the ideas arose in conversations with various faculty members there. I would especially like to thank Prof. Frank Forelli for suggesting the problem and supervising my research.

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A SUBCOLLECTION OF ALGEBRAS IN A COLLECTION OF BANACH SPACES

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Let $D(p, r)$ with $1 \leqq p < \infty$ and $-\infty < r < +\infty$ denote the Banach space consisting of certain analytic functions $f(z)$ defined in the unit disk. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a member of $D(p, r)$ if and only if

$$\sum_{n=0}^{\infty} (n+1)^r \,|\, a_n \,|^p < \infty .$$

We define the norm of f in $D(p, r)$ by

$$||\, f\,||_{p,r} = \left(\sum_{n=0}^{\infty} (n+1)^r \,|\, a_n \,|^p\right)1/p .$$

By the product of two functions f and g in $D(p, r)$ we shall mean their product as functions, i.e., $[f.\, g](z) = f(z)g(z)$. The purpose of this paper is to discover which of the spaces $D(p, r)$ are algebras.

THEOREM 1. *If $D(p, r)$ is an algebra, then there exists a real $c > 0$ with $||\, fg\,|| \leqq c\,||\, f\,||\,||\, g\,||$ for every $f, g \in D(p, r)$.*

Proof. Let h be a fixed element of $D(p, r)$. It suffices to show the map $f \rightarrow hf$ is a bounded linear transformation from $D(p, r)$ to itself. The proof is based on the closed graph theorem [2, p. 306]. Suppose h is a multiplier from $D(p_1, r_1)$ to $D(p_2, r_2)$ and suppose

(i) $f_n \rightarrow f$ in $D(p_1, r_1)$ and

(ii) $hf_n \rightarrow g$ in $D(p_2, r_2)$.

Then $f_n(z) \rightarrow f(z)$ for each z in the unit disk and so $h(z)f_n(z) \rightarrow h(z)f(z)$. On the other hand by (ii), $h(z)f_n(z) \rightarrow g(z)$ for each z in the unit disk. Hence $g = hf$, and so by the closed graph theorem multiplication by h is a continuous linear transformation. It follows from this [2, p. 183] that $D(p, r)$ is equivalent to a Banach algebra, and from this the theorem follows immediately.

COROLLARY 1. *If $D(p, r)$ is an algebra and $c > 0$ as above, then* $|\, f(z)\,| \leqq c\,||\, f\,||\ \forall f \in D(p, r)$ and $|\, z\,| < 1$.

Proof. For each f in $D(p, r)$ let T_f denote the multiplication operator from $D(p, r)$ to itself determined by f, i.e., $T_f(g) = fg$. Then for z_0 satisfying $|\, z_0\,| < 1$ the map $T_f \rightarrow f(z_0)$ is a multiplicative linear functional on the Banach algebra of multiplication operators

$$T_f,\, f \in D(p, r)$$

with the usual norm. Hence

$$|f(z_0)| \leqq \|T_f\| = \sup_{\|g\|=1} \|fg\| \leqq c\|f\|, g \in D(p, r) .$$

THEOREM 2. *If $p = 1$, then $D(p, r)$ is no algebra for $r < 0$. And if $1 < p < \infty$, then $D(p, r)$ is no algebra for $r \leqq p - 1$.*

Proof. The function $f(z) = \sum_{n=0}^{\infty} [1/(n+1)]z^n$ is an unbounded function on $|z| < 1$ but lies in $D(1, r)$ if $r \leqq 0$. And similarly the function $f(z) = \sum_{n=0}^{\infty} 1/[(n+1)\log(n+1)]z^n$ is an unbounded function on $|z| < 1$ in $D(p, r)$ if $p > 1$ and $r \leqq p - 1$. Therefore by Corollary 1 the spaces are not algebras.

THEOREM 3. *If $p = 1$, then $D(p, r)$ is an algebra for $r \geqq 0$, and if $1 < p < \infty$ then $D(p, r)$ is an algebra for $r > p - 1$.*

Proof. (i) Suppose first $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ lie in $D(1, r)$ with $r \geqq 0$. We will show $fg \in D(1, r)$

$$\begin{aligned}
\|fg\| &= \sum_{n=0}^{\infty} (n+1)^r \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \leqq \sum_{n=0}^{\infty} (n+1)^r \sum_{k=0}^{n} |a_k|\,|b_{n-k}| \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (n+1)^r |a_k|\,|b_{n-k}| \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j+k+1)^r |a_k|\,|b_j| \quad \text{where} \quad j = n-k \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j+k+1)^r/[(k+1)^r(j+1)^r](k+1)^r |a_k| (j+1)^r |b_j| \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} [(j+k+1)/(jk+j+k+1)]^r (k+1)^r |a_k| (j+1)^r |b_j| \\
&\leqq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k+1)^r |a_k| (j+1)^r |b_j| \\
&= \|f\|\,\|g\| .
\end{aligned}$$

(ii) Now suppose $r > p - 1$, and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be two elements of $D(p, r)$. We will show there is a constant K such that $\|fg\| \leqq K\|f\|\,\|g\|$. Define q by the equation $1/p + 1/q = 1$.

$$\|fg\|^p = \sum_{n=0}^{\infty} (n+1)^r \left| \sum_{k=0}^{n} a_k b_{n-k} \right|^p \leqq \sum_{n=0}^{\infty} (n+1)^r$$
$$\left\{ \sum_{k=0}^{n} 1/[(k+1)^{r/p}(n-k+1)^{r/p}](k+1)^{r/p} |a_k| (n-k+1)^{r/p} |b_{n-k}| \right\}^p .$$

Applying Holder's inequality we get

$$\begin{aligned}
&\| fg \|^p \\
&\leqq \sum_{n=0}^{\infty} (n+1)^r \left\{ \left(\sum_{k=0}^{n} [1/\{(k+1)^{r/p}(n-k+1)^{r/p}\}]^q \right)^{1/q} \right. \\
&\qquad\qquad \left. \left(\sum_{k=0}^{n} (k+1)^r \mid a_k \mid^p (n-k+1)^r \mid b_{n-k} \mid^p \right)^{1/p} \right\}^p \\
&= \sum_{n=0}^{\infty} [C_n] \sum_{k=0}^{n} (k+1)^r \mid a_k \mid^p (n-k+1)^r \mid b_{n-k} \mid^p \\
&\leqq \sup_n [C_n] \| f \|^p \| g \|^p
\end{aligned}$$

where

$$C_n = (n+1)^r \left(\sum_{k=0}^{n} \{1/[k+1)^{r/p}(n-k+1)^{r/p}]\}^q \right)^{p/q} .$$

We complete the proof of the theorem by showing

$$\sup_n [C_n] < \infty .$$

$$\begin{aligned}
C_n &= (n+1)^r \left(\sum_{k=0}^{n} \{1/[(k+1)^{r/p}(n-k+1)^{r/p}]\}^q \right)^{p/q} \\
&= (n+1)^r \left(\sum_{k=0}^{n} 1/(n+2)^{rq/p} \{1/(k+1) + 1/(n-k+1)\}^{rq/p} \right)^{p/q} \\
&= [(n+1)/(n+2)]^r \left[\sum_{k=0}^{n} \{1/(k+1) + 1/(n-k+1)\}^{rq/p} \right]^{p/q} \\
&\leqq \left[\sum_{k=0}^{n} \{2/(k+1)\}^{rq/p} \right]^{p/q} \\
&\leqq 2^r \left[\sum_{k=0}^{\infty} 1/(k+1)^{rq/p} \right]^{p/q}
\end{aligned}$$

since

$$rq/p = r/(p-1) > 1 .$$

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TWISTED COHOMOLOGY AND ENUMERATION OF VECTOR BUNDLES

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In the present paper we give a technique for completely enumerating real 4-plane bundles over a 4-dimensional space, real 5-plane bundles over a 5-dimensional space, and real 6-plane bundles over a 6-dimensional space. We give a complete table of real and complex vector bundles over real projective space P_k, for $k \leqq 5$. Some interesting results are:

(0.1.1.) Over P_5, there are four oriented 4-plane bundles which could be the normal bundle to an immersion of P^5 in R^9, i.e., have stable class $2h + 2$, where h is the canonical line bundle. Of these, two have a unique complex structure.

(0.1.2.) Over P_5 there is an oriented 4-plane bundle which we call C, which has stable class $6h - 2$, which has two distinct complex structures. D, the conjugate of C, i.e., reversed orientation, has no complex structure.

(0.1.3) Over P_5, there are no 4-plane bundles of stable class $5h - 1$ or $7h - 3$.

0.2. In reading the tables (4.5.2) and (4.6), remember that if ξ: $P_k \rightarrow BO(n)$ or $\xi: P_k \rightarrow BU(n)$ is a locally oriented (i.e., oriented over base-point) real or complex vector bundle, and if

$$a \in H^k(P_k; \pi_k(BO(n), \xi))$$

(local coefficients if ξ unoriented) or $a \in H^k(P_k; \pi_k(BU(n))$, then $\xi + a$ is a vector bundle obtained by cutting out a disk in the top cell of P_k and joining a sphere with some vector bundle on it.

0.3. Since some of the homotopy groups of $BO(n)$ are acted upon nontrivially by $Z_2 \cong \pi_1(BO(n))$ for n even, we study cohomology with local coefficients in § 3.

1.2. From here on, we assume that all spaces are connected C. W.-complexes with base-point, all maps are b.p.p. (base-point-preserving) and that all homotopies are b.p.p.

For any space Y, we choose a Postnikov system for Y, that is: for each integer $n \geqq 0$, a space $(Y)_n$ and a map P_n: $Y \rightarrow (Y)_n$ which induces an isomorphism in homotopy through dimension n, where all homotopy groups of $(Y)_n$ are zero above n; for each $n \geqq 1$ a fibration $p_n: (Y)_n \rightarrow (Y)_{n-1}$ such that $p_n P_n = P_{n-1}$. The fiber of each p_n is then an Eilenberg-MacLane space of type $(\pi_n(Y), n)$. If X is a space of finite dimension m, then $[X; Y]$, the set of homotopy classes of maps from X to Y, is in one-to-one correspondence with $[X; (Y)_m]$.

DEFINITION (1.2.1). For any integer $n \geqq 1$, let $G_n(Y)$ be the sheaf over $(Y)_1$ whose stalk over every y is defined to be $\pi_n(p^{-1}y)$, which is isomorphic to $\pi_n(Y)$ (where $p = p_2 \cdots p_n \colon (Y)_n \to (Y)_1$) if $n \geqq 2$; $\pi_1((Y)_1, y)$ if $n = 1$. If X is any space and $f\colon X \to (Y)_1$ is a map, let $\pi_n(Y, f)$ be the sheaf $f^{-1}G_n(Y)$ over X. This sheaf depends only on the homotopy class of f. If $g\colon X \to (Y)_m$ is a map for any integer $m \geqq 1$, or if $h\colon X \to Y$ is a map, let $\pi_n(Y, g)$ denote $\pi_n(Y, p_2 \cdots p_m g)$ and let $\pi_n(Y, n)$ denote $\pi_n(Y, P_1 h)$.

DEFINITION (1.2.2). If f and g are maps from X to $(Y)_n$ for any $n \geqq 2$, which agree on A, and if $F\colon X \times I \to (Y)_{n-1}$ is a homotopy of $p_n f$ with $p_n g$ which holds A fixed, let $\delta^n(f, g; F) \in H^n(X, A; \pi_n(Y, f))$ be the obstruction to lifting F to a homotopy of f with g which holds A fixed.

REMARK (1.2.3). If $g\colon X \to (Y)_n$ is another map which agrees with f on A, and if G is a homotopy of $p_n g$ with $p_n h$ which holds A fixed, then $\delta^n(f, g; F) + \delta^n(g, h; G) = \delta^n(f, h; F + G)$, where, for each $(x, t) \in X \times I$,

$$(F + G)(x, t) = \begin{cases} F(x, 2t) & \text{if} \quad 0 \leqq t \leqq \frac{1}{2} \\ G(x, 2t - 1) & \text{if} \quad \frac{1}{2} \leqq t \leqq 1 \,. \end{cases}$$

DEFINITION (1.2.4). Let X be a space, let $A \subset X$ be any subcomplex (possible empty), let $f\colon X \to (Y)_n$ be a map for some integer $n \geqq 2$, and let a be an element of $H^n(X, A; \pi_n(Y, f))$. We define $f + a$ to be that map from X to $(Y)_n$, unique up to fiber homotopy with A held fixed, such that $p_n(f + a) = p_n f$ and $\delta^n(f, f + a) = a$, where C is the constant homotopy.

REMARK (1.2.5). If b is any other element of $H^n(X, A; \pi_n(Y, f))$, then $f + (a + b) = (f + a) + b$.

REMARK (1.2.6). If $g\colon (X', A') \to (X, A)$ is a map, where $(X'A')$ is any other C. W. pair, then $(f + a)g = gf + g^*a$.

MAIN THEOREM (1.2.7). *For any* $a \in H^n(X, A; \pi_n(Y, f))$, $f + a$ *is homotophic to* f, $\operatorname{rel} A$, *if and only if* $\delta^n(f, f; F) = a$ *for some homotopy* F *of* $p_n f$ *with itself which holds* A *fixed.*

Proof. Let C be the constant homotopy of $p_n f$ with itself. On the one hand, if F is any homotopy of $p_n f$ with itself which holds

A fixed, let $a = \delta^n(f, f; F)$. Then $\delta^n(f + a, f; F) = \delta^n(f + a, f; C) + \delta^n(f, f; F) = - a + a = 0$. Thus F may be lifted to a homotopy of $f + a$ with f. On the other hand, if G is a homotopy of $f + a$ with f, then $\delta^n(f, f; p_nG) = \delta^n(f, f + a; C) + \delta^n(f + a, f; p_nG) = a + 0 = a$.

DEFINITION (1.2.8). Let L_f be the subgroup of $H^n(X, A; \pi_n(Y, f))$ consisting of all a such that $f + a$ is homotopic to $f \operatorname{rel} A$. Then the set of all homotopy ($\operatorname{rel} A$) classes of liftings of p_nf to $(Y)_n$ which agree with f on A is in a one-to-one correspondence with the quotient group $H^n(X, A; \pi_n(Y, f))/L_f$; each coset $a + L_f$ corresponds to $f + a$. If $g: X \to Y$ is a map such that $p_ng = f$, let $L_g^n = L_f$. If $h: X \to (Y)_m$ is a map such that $p_{n+1} \cdots p_mh = f$, for $m \geqq n$, let $L_h^n = L_f$.

REMARK (1.2.9). If $a \in H^n(X, A; \pi_n(Y, f))$, then $L_{f+a} = L_f$.

Proof. Let F be any homotopy of $p_nf = p_n(f + a)$ with itself, and let C be the constant homotopy. Then $\delta^n(f + a, f + a; F) = \delta^n(f + a, f; C) + \delta^n(f, f; F) + \delta^n(f, f + a; C) = - a + \delta^n(f, f; F) + a = \delta^n(f, f; F)$.

1.3. In order to calculate L_f in specific cases, such as X a projective space, A = base-point, and $Y = BO(m)$ for some m, we use a spectral sequence which has the following properties:

(1.3.1) ${}^fE_2^{p,q} = E_2^{p,q} = H^p(X, A; \pi_q(Y, f))$ if $2 \leqq q \leqq n, 1 \leqq p \leqq q + 1$.

(1.3.2) $E_2^{p,q} = 0$ for all other values of p and q.

(1.3.3) $d_r: E_r^{p,q} \to E_r^{p+r,q+r-1}$ for all $r \geqq 2$.

(1.3.4) $E_\infty^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, which, by (1.2.7) and (1.2.8) can be put into one-to-one correspondence with the set of $\operatorname{rel} A$ homotopy classes of maps $X \to (Y)_n$ whose projection to $(Y)_{n-1}$ is $\operatorname{rel} A$ homotopic to p_nf.

Basically, what is happening is as follows (where, for any space Z and any map $g: A \to Z$, the set of $\operatorname{rel} A$ homotopy classes of maps $X \to Z$ which agree with g on A is denoted "$[X; Z: g]$"); consider the function:

$$[X; (Y)_n: f \mid A] \xrightarrow{(p_n)_\#} [X; (Y)_{n-1}: p_nf \mid A] .$$

Now $(p_n)_\#$ is just a function of sets, but $(p_n)_\#^{-1}(p_nf)$ is an Abelian group with 0 the homotopy class of f itself. This group, $E_\infty^{n,n}$ of our spectral sequence, depends on the choice of f.

We define our spectral sequence via an exact couple:

$$\begin{tikzcd} D_2^{*,*} \arrow[rr, "i_2"] & & D_2^{*,*} \arrow[dl, "j_2"] \\ & E_2^{*,*} \arrow[ul, "k_2"] & \end{tikzcd}$$

where $E_2^{p,q}$ is as defined in (1.3.1) and (1.3.2), where i_2, j_2, and k_2 have bi-degrees $(-1, -1), (2, 1)$, and $(0, 0)$ respectively; and where (for all $t \leqq n$, M_t = space of maps from X to $(Y)_t$ which agree with $p_t^n f$ on A, compact-open topology):

(1.3.5) $D_2^{p,q} = \pi_{q-p}(M_q, p_q^n f)$ if $0 \leqq q \leqq n$, and $p \leqq q$.

(1.3.6) $D_2^{p,q} = 0$ if $q < p$ or $q < 0$.

(1.3.7) $D_2^{p,q} = D_2^{p-1,q-1}$ if $q > n$.

Note that $D_2^{p,q}$ is only a group if $q = p + 1$ and only a set if $q = p$. This will not affect our computation, however.

We proceed to define the homomorphisms i_2, j_2 and k_2.

(1.3.8) If $q > n$, let i_2 be the identity. If $q \leqq n$, let $i_2 = (p_q)_\#$.

(1.3.9) If $p \leqq q$ and $0 \leqq q < n$, any $x \in D_2^{p,q}$ represents a map g: $X \times I^{q-p} \to (Y)_q$, where $g(x, v) = p_q^n f(x)$ for all $(x, v) \in X \times \partial I^{q-p} \cup A \times I^{q-p}$. Let $j_2(x) = (s^{q-p})^{-1}\gamma^{q+2}(g)$, where $s^{q-p}\colon H^{p+2}(X, A; \pi_{q+1}(Y, f)) \to H^{q+2}(X \times I^{q-p}, X \times \partial I^{q-p} \cup A \times I^{q-p}; \pi_{q+1}(Y, g))$ is the $(q-p)$-fold suspension and $\gamma^{q+2}(g)$ is the obstruction to finding a lifting $h\colon X \times I^{q-p} \to (Y)_{q+1}$ of g such that $h(x, v) = p_{q+1}^n f(x)$ for all $(x, v) \in X \times \partial I^{q+p} \cup A \times I^{q-p}$. (If $p > q$ or $q < 0$ or $q \geqq n$, $j_2\colon D_2^{p,q} \to E_2^{p+2,q+1}$ is obviously the zero map, since $E_2^{p+2,q+1} = 0$.) This obstruction is zero if and only if g can be lifted; it follows immediately that:

(1.3.10) The sequence $D_2^{p+1,q+1} \xrightarrow{i_2} D_2^{p,q} \xrightarrow{j_2} E^{p+2,q+1}$ is exact.

Furthermore, since every homotopy, rel A, of $p_n f$ with itself represents a loop in M_{n-1}:

(1.3.11) L_f is the image of $j_2\colon D_2^{n-2,n-1} \to E_2^{n,n}$. For any $2 \leqq q \leqq n$, $1 \leqq p \leqq q$, and any $a \in E_2^{p,q}$, let

$$b = s^{q-p}a \in H^q(X \times I^{q-p}, X \times \partial I^{q-p} \cup A \times I^{q-p}; \pi_q(Y, C)) ,$$

where $C(x, v) = p_q^n f(x)$ for every $(x, v) \in X \times I^{q-p}$. Let $k_2(a) \in D_2^{p,q}$ be that element represented by the map $C + b$ (cf. 1.2.2). It follows from (1.2.3) that k_2 is a homomorphism if $p < q$; if $p = q$ then $D_2^{p,q}$ is only a set anyway. (For other values of p and q, $k_2 = 0$.) Since $p_q(C+b) = p_q C$, and C represents $0 \in D_2^{p,q}$:

(1.3.12) $\operatorname{Im} k_2 \subset \operatorname{Ker} i_2$.

If, on the other hand, a map g: $X \times I^{q-p} \to (Y)_q$ such that $g = C$ on $X \times \partial I^{q-p} \cup A \times I^{q-p}$ is a representative of a given $a \in \operatorname{Ker} i_2$, then $p_q g$ is homotopic, rel $X \times \partial I^{q-p} \cup A \times I$, to $p_q C$ via a homotopy F, then $a = k_2((s^{q-p})^{-1}\delta^q(C, g; F))$. Thus:

(1.3.13) $\operatorname{Ker} i_2 \subset \operatorname{Im} k_2$.

Somewhat more difficult to show is:

(1.3.14) $\operatorname{Ker} k_2 = \operatorname{Im} j_2$ if $p \leqq q$.

Proof. Let $2 \leqq q \leqq n, 1 \leqq p \leqq q$. Let $g(x, v) = p_q^n f(x) \in (Y)_q$ for all $(x, v) \in X \times I^{q-p}$; g represents $0 \in D_2^{p,q}$. Let $b \in E_2^{p,q}$. Then $b \in \operatorname{Ker} k_2$

if and only if $s^{q-p}b \in L_g$ (cf. 1.2.7). If $b = j_2 a$, then a represents F, a homotopy, $\operatorname{rel} X \times \partial I^{q-p} \cup A \times I^{q-p}$ of $p_q q$ with itself, and $s^{q-p}b = \delta^q(g, g; F) \in L_g$. If, on the other hand, $s^{q-p}b \in L_g$, then $s^{q-p}b = \delta^q(g, g; F)$ for some homotopy F, $\operatorname{rel} X \times \partial I^{q-p} \cup A \times I^{q-p}$, of $p_q g$ with itself; let $a = [F] \in D^{p-2,q-1}$, and $j_2 a = b$.

1.4. Since only finitely many of the E_2 terms are nonzero, we obtain E_∞ after a finite number of steps. We also have, by straightforward algebra, an exact sequence

$$0 \longrightarrow E_\infty \xrightarrow{k_\infty} D_\infty \xrightarrow{i_\infty} D_\infty \longrightarrow 0 \ .$$

Consider now the commutative diagram with exact columns:

$$\begin{array}{ccccl}
 & & D_2^{n-2,n-1} & = \pi_1(M_{n-1}, p_n f) & \qquad [F] \\
 & & \downarrow{\scriptstyle j_2} & & \qquad \downarrow \\
E_\infty^{n,n} & \xleftarrow{\text{epi}} & E_2^{n,n} & = H^n(X, A; \pi_n(Y, f)) & \delta^n(f, f; F) \\
{\scriptstyle\text{mono}}\downarrow{\scriptstyle k_\infty} & & \downarrow{\scriptstyle k_2} & & \qquad x \\
D_\infty^{n,n} & = & D_2^{n,n} & = [X; (Y)_n : f \mid A] & \qquad \downarrow \\
{\scriptstyle\text{epi}}\downarrow{\scriptstyle i_\infty} & & \downarrow{\scriptstyle i_2} & & \quad f + x \ . \\
D_\infty^{n-1,n-1} & \xrightarrow[\text{mono}]{} & D_2^{n-1,n-1} & = [X; (Y)_{n-1} : p_n f \mid A] &
\end{array}$$

A typical element of $D_2^{n-2,n-1}$ is a $\operatorname{rel} X \times \partial I \cup A \times I$ homotopy class of homotopies of $p_n f$ with itself; if F is such a homotopy, $j_2[F] = \delta^n(f, f; F)$, by (1.3.9). If $x \in H^n(X, A; \pi_n(Y, f))$, $k_2 x = f + x$, by (1.3.11). Thus $\operatorname{Im} j_2 = L_f$, and $E_\infty^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, the set of $\operatorname{rel} A$ homotopy classes of liftings of $p_n f$.

1.5. If $g\colon (X', A') \to (X, A)$ is a map, g induces a map of spectral sequences.

(1.5.1) $g^*\colon {}^f E_r^{p,q} \to {}^{fg}E_r^{p,q}$ for all p, q, r. If $h\colon Y \to Z$ is a map, where Z is any other space, h determines a map $h_m\colon (Y)_m \to (Z)_m$ for each $m \geqq 0$ [1]. Then $h_\#\colon \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ induces a sheaf homomorphism from $G_n(Y)$ to $(h_1)^{-1}G_n(Z)$ which in turn induces a homomorphism.

(1.5.2) $h_*\colon H^*(X, A; \pi_m(Y, f)) \to H^*(X, A; \pi_m(Z, hf))$ for all $m \geqq 0$ and a map of spectral sequences

(1.5.3) $h_*\colon {}^f E_r^{p,q} \to {}^{hf}E_r^{p,q}$ for all p, q, r.

2. Nonbase-point-preserving homotopies.

2.1. Using the techniques of § 1, we can compute all b.p.p.

homotopy classes of maps from a finite-dimensional space X to a space Y. What if we want to know, instead, all free homotopy classes of maps?

2.2. Let $f: X \to Y$ be any b.p.p. map, and let $a \in \pi_1(Y, y_0)$. By the homotopy extension property, we can find a free homotopy F: $X \times I \to Y$ of f such that $F \mid \{x_0\} \times I$ represents a. Let $f^a(x) = F(x, 1)$ for any $x \in X$; f^a is unique up to b.p.p. homotopy, and $f^{ab}(f^a)^b$ for any other $b \in \pi_1(Y, y_0)$.

THEOREM (2.2.1). *If f and g are any* b.p.p. *maps from X to Y, then f is freely homotopic to g if and only if f^a is* b.p.p. *homotopic to g for some $a \in \pi_1(Y, y_0)$.*

Proof. If f^a is b.p.p. homotopic to g, then f is obviously freely homotopic to g since f is freely homotopic to f^a. If, on the other hand, $F: X \times I \to Y$ is a free homotopy of f with g, let a be that element of $\pi_1(Y, y_0)$ represented by the loop $F \mid \{x_0\} \times I$. Then $f^a = g$ (up to b.p.p. homotopy).

THEOREM (2.2.2). *If $n \geqq 2, f: X \to (Y)_n$ is a map,*

$$a \in H^n(X, x_0; \pi_n(Y, f)) ,$$

and $b \in \pi_1(Y, y_0)$, then $(f + a)^b = f^b + 1^b_*(a)$, where 1^b_* is the homomorphism induced by the map 1^b (cf. 1.5.2), *where 1 is the identity map on $(Y)_n$.*

Proof. The theorem follows from naturality of obstruction theory.

3. Sheaves of local coefficients.

3.1. The homotopy groups of $BO(n)$ are sometimes acted on nontrivially by π_1. We must therefore study twisted sheaves.

DEFINITION (3.1.1). A twisted group is an ordered pair (G, T), G an Abelian group, $T: G \to G$ an automorphism of order 2. If X is a space, a (G, T)-sheaf over X is a fiber bundle over X with fiber G and structural group Z_2, action determined by T. Let $G^T[u]$ be the (G, T)-sheaf over P_∞ obtained by identifying (x, g) with (Tx, Tg) for all $(x, g) \in S^\infty \times G$, where $T: S^\infty \to S^\infty$ is the antipodal map.

DEFINITION (3.1.2). If $a \in H^1(X, x_0; Z_2)$ and $f: (X, x_0) \to (P_\infty, *)$ is a map where $f^*u = a$ ($u =$ fundamental class of P_∞), let $G^T[a] = f^{-1}G^T[u]$. We call a the twisting class of $G^T[a]$.

PROPOSITION (3.1.3). *$G^T[u]$ is universal in the sense of Steenrod [6], that is, if G is a (G, T)-sheaf over a space $X, G \cong G^T[a]$ for some unique $a \in H^1(X, x_0; Z_2)$.*

Proof. $P_\infty = BZ_2$.

REMARK (3.1.4). If $F: X \times I \to P_\infty$ is a *free* homotopy of f with itself, where $f^*u = a$, then F induces an automorphism of $G^T[a]$; 1 or T depending on whether $F | \{x_0\} \times I$ is a trivial loop in P_∞ or not.

3.2. If X is a space, $B \subset A \subset X$ are closed, and S is a sheaf over X, we have a long exact sequence:

$$\cdots \longrightarrow H^n(X, A; S) \longrightarrow H^n(X, B; S) \longrightarrow H^n(A, B; S) \xrightarrow{\delta} H^{n+1}(X, A; S) \longrightarrow \cdots$$

PROPOSITION (3.2.1). *If S is a sheaf over a space X, and $A \subset X$ is closed, we may find an isomorphism*

$$s: H^*(X, A; S) \longrightarrow H^*(X \times I, X \times \partial I \cup A \times I; S \times I),$$

called the suspension, of degree 1, *where $S \times I = p^{-1}S$; $p: X \times I \to X$ being the projection.*

Proof. Let S' be that subsheaf of S such that $S' | A = 0$ and $S' | (X - A) = S | (X - A)$. According to Bredon [1],

$$H^*(X, A; S) = H^*(X; S')$$

and

$$H^*(X \times I, X \times \partial I \cup A \times I; S \times I) = H^*(X \times I, X \times \partial I; S' \times I).$$

Now $H^*(X \times I, X \times \{t\}; S') = 0$ for any $t \in I$ [1], and by the long exact sequence of $(X \times I, X \times \partial I, X \times \{1\})$ and excision we have an isomorphism $H^*(X \times \{0\}; S' \times I) \xrightarrow{\cong} H^*(X \times I, X \times \partial I; S' \times I)$ of degree 1; the left group is isomorphic to $H^*(X; S')$.

3.3. Let X be a space, $A \subset X$ closed. If $\alpha: S \to S'$ is a homomorphism of sheaves over X, we get a homomorphism $\alpha_*: H^*(X, A; S) \to H^*(X, A; S')$. If S and S' are sheaves over X and

$$E: 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

is an extension of S' by S, then E determines a long exact sequence

$$\cdots \longrightarrow H^n(X, A; S) \xrightarrow{i_*} H^n(X, A; S'') \xrightarrow{p_*} H^n(X, A; S')$$
$$\xrightarrow{\delta^E} H^{n+1}(X, A; S) \longrightarrow \cdots$$

where δ^E is called the Bockstein of E.

PROPOSITION (3.3.1). *If S and S' are sheaves over X and if*

$$E\colon 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

and

$$F\colon 0 \longrightarrow S \xrightarrow{j} U \xrightarrow{q} S' \longrightarrow 0$$

are elements of $\mathrm{Ext}\,(S', S)$, *then* $\delta^{E+F} = \delta^E + \delta^F$.

Proof. We use the Baer sum construction to find

$$E + F\colon 0 \longrightarrow S \longrightarrow V \longrightarrow S' \longrightarrow 0;$$

our result follows from the commutative diagram, where each row is exact:

$$\begin{tikzcd}
0 \arrow[r] & S \times S \arrow[r] & S'' \times U \arrow[r] & S' \times S' \arrow[r] & 0 \\
0 \arrow[r] & S \times S \arrow[r] \arrow[u, "1"] \arrow[d, "\alpha"'] & W \arrow[r] \arrow[u] \arrow[d] & S' \arrow[r] \arrow[u, "\Delta"] \arrow[d, "1"] & 0 \\
0 \arrow[r] & S \arrow[r] & V \arrow[r] & S' \arrow[r] & 0 .
\end{tikzcd}$$

3.4. As Abelian groups $\mathrm{Ext}\,(Z_2, Z_2) \cong Z_2$; the nonzero extension is Z_4. Fix a space X; we study Ext of sheaves over X.

PROPOSITION 3.4.1. *As sheaves over X,*

$$\mathrm{Ext}\,(Z_2, Z_2) \cong Z_2 + H^1(X, x_0; Z_2) .$$

For any $a \in H^1(X, x_0; Z_2)$, $(0, a)$ corresponds to the extension

$$E_a^0\colon 0 \longrightarrow Z_2 \xrightarrow{i_1} (Z_2 + Z_2)^T[a] \xrightarrow{p_2} Z_2 \longrightarrow 0 ,$$

where $T(x, y) = (x + y, y)$, $i_1(x) = (x, 0)$, and $p_2(x, y) = y$; $(1, a)$ corresponds to

$$E_a^1\colon 0 \longrightarrow Z_2 \xrightarrow{m} Z_4^T[a] \xrightarrow{e} Z_2 \longrightarrow 0 ,$$

where $T(x) = -x$ for all $x \in Z_4$, $m(1) = 2$, and $e(1) = 1$.

Proof. Routine computation shows that $E_a^x + E_b^y = E_{a+b}^{x+y}$ for any $x, y \in Z_2$ and $a, b \in H^1(X, x_0; Z_2)$. On the other hand, suppose that

$$E\colon 0 \longrightarrow Z_2 \xrightarrow{i} G \xrightarrow{p} Z_2 \longrightarrow 0$$

is some extension. Then the stalk of G at x_0 is Z_4, in which case $G = Z_4^T[a]$ for some $a \in H^1(X, x_0; Z_2)$, or it is $Z_2 + Z_2$. In that case, we have an exact sequence of stalks at x_0:

$$0 \longrightarrow Z_2 \xrightarrow{i_1} Z_2 + Z_2 \xrightarrow{p_2} Z_2 \longrightarrow 0 .$$

Since G is locally isomorphic to $Z_2 + Z_2$, it is a fiber bundle with fiber $Z_2 + Z_2$ and structural group $\mathrm{Aut}\,(Z_2 + Z_2)$. But the only nontrivial automorphism which commutes with $i_1\colon Z_2 \to Z_2 + Z_2$ and $p_2\colon Z_2 + Z_2 \to Z_2$ is T given above. So the structural group of G may be reduced to Z_2; $G = (Z_2 + Z_2)^T[a]$ for some $a \in H^1(X, x_0; Z_2)$. This gives us the isomorphism.

We have the following commutative diagram with both rows exact, for any $a \in H^1(X, x_0; Z_2)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^T[a] & \xrightarrow[2]{} & Z^T[a] & \xrightarrow[\Pi]{} & Z_2 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \Pi} & & \big\downarrow{\scriptstyle \Pi} & & {\scriptstyle 1}\big\downarrow & & \\
0 & \longrightarrow & Z_2 & \xrightarrow[m]{} & Z_4^T[a] & \xrightarrow[e]{} & Z_2 & \longrightarrow & 0 .
\end{array}$$

DEFINITION (3.4.2). Let $\beta^T[a]$ (or simply β^T, when a is understood) denote the Bockstein of the top row of the above diagram, and let $(S_q^1)^T[a]$ (or $(S_q^1)^T$) denote the Bockstein of the bottom row.

REMARK (3.4.3). $\Pi_* \beta^T = (S_q^1)^T$.

PROPOSITION (3.4.4). *For any* $n \geqq 0$ *and any* $x \in H^n(X, A\colon Z_2)$, $(S_q^1)^T x = S_q^1 x + x \cup a$.

Proof. Samelson [5].

PROPOSITION (3.4.5). *For any* $n \geqq 0$ *and any* $x \in H^n(X, A; Z_2)$ $\delta(x) = x \cup a$, *where* δ *is the Bockstein of* $E_a^0\colon 0 \to Z_2 \to (Z_2 + Z_2)^T[a] \to Z_2 \to 0$.

Proof. The result follows immediately from (3.3.1), (3.4.1), and (3.4.4).

3.5. Let $T(n, m) = (m - n, m)$ for any $(n, m) \in Z + Z$. If S and S' are sheaves over a space X, and if $\mu\colon S \otimes S' \to S''$ is a sheaf homomorphism, then we have a cup product defined from

$$H^*(X, A; S) \otimes H^*(X, B; S')$$

to $H^*(X, A \cup B; S'')$ for any closed $A \subset X$ and $B \subset X$. We have thus cup products generated by the following relations:

$$\begin{aligned} Z^T[a] \otimes Z^T[b] &= Z^T[a+b],\ Z_2 \otimes (Z_2 + Z_2)^T[a] \\ &= (Z_2 + Z_2)^T[a],\ Z \otimes (Z + Z)^T[a] \\ &= (Z + Z)^T[a],\ Z^T[a] \otimes (Z + Z)^T[a] = (Z + Z)^T[a] \end{aligned}$$

(where $\quad n \otimes (p, q) = (np, 2np - nq)),\ Z_4^T[a] \otimes Z_4^T[b] = Z_4^T[a+b]$,

and many others.

Let (X, A) be a C. W.-pair. Let $a \in H^1(X, x_0; Z_2)$ and

$$\alpha = \beta^T[a](1) \in H^1(X; Z^T[a]) .$$

We have the following commutative diagram; where

$$i_1 x = (x, 0),\ T(x, y) = (y - x, y),\ j_1 x = (x, 2x),$$

and $q_2(x, y) = y - 2x$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^T[a] & \xrightarrow{i_1} & (Z+Z)^T[a] & \xrightarrow{p_2} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} & & \\ 0 & \longrightarrow & Z_2 & \xrightarrow{i_1} & (Z_2+Z_2)^T[a] & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \Pi} & & \uparrow{\scriptstyle \Pi} & & \uparrow{\scriptstyle \Pi} & & \\ 0 & \longrightarrow & Z & \xrightarrow{j_1} & (Z+Z)^T[a] & \xrightarrow{p_2} & Z^T[a] & \longrightarrow & 0 . \end{array}$$

PROPOSITION (3.5.1). *The Bockstein homomorphisms δ_1 and δ_2 are both cup products with α.*

Proof. By (3.4.3) and (3.4.4) we may compute that

$$H^1(P_\infty; Z^T[u]) \cong Z_2$$

and is generated by $\bar{u} = \beta^T(1)$.

Let $x \in H^n(X, A; Z)$. If $n = 0$, then the universal example is $X = P_\infty, A = \varnothing, x = 1$. Then $\alpha = \bar{u}$. Now $H^0(P_\infty; Z^T) = 0$, so $(j_1)_*$: $H^0(P_\infty; Z) \leftarrow H^0(P_\infty; (Z+Z)^T)$ is an isomorphism, and $p_2 j_1 = 2$. Thus $1 \notin \operatorname{Im}(p_2)_*$, so $\delta_1(1) = \bar{u}$. If $n \geqq 1$, the universal example is $X = K(Z, n) \times P_\infty, A = * \times P_\infty, x = v_n \times 1$. Then $\alpha = p^* \bar{u}$, where $p: X \to P_\infty$ is projection onto the second factor. Now routine computations using (3.4.3) and (3.4.4) show that $H^{n+1}(X, A; Z^T) \cong Z_2$ and is generated by $(v_n \times 1) \cup p^* \bar{u}$, which is mapped onto $\Pi_* v_n \times u$ under $\Pi_*: H^*(; Z^T) \to H^*(; Z_2)$. The result follows from (3.4.5).

Let $x \in H^n(X, A; Z^T)$. If $n = 0, x = 0$. If $n = 1$, the universal example is $X = K(Z^T, n), A = P_\infty$, and $x = v_n^T$, where $K(Z^T, n)$ is obtained as follows:[1] Let $K(Z, n)$ be a topogical group, let $T(g, y) = (g^{-1}, Ty)$ for all $g \in K(Z,n)$ and $y \in S^\infty$. Let

[1] Personal communication from C. T. C. Wall.

$$K(Z^T, n) = K(Z, n) \times S^\infty/T .$$

We have inclusion and projection

$$P_\infty \xrightarrow{i} K(Z^T, n) \xrightarrow{p} P_\infty$$

where $i[y] = [*, y]$ and $p[g, y] = [y]$; P_∞ may thus be considered to be a subset of $K(Z^T, n)$, and its cohomology group is a direct summand[1]. Then $v_n^T \in H^n(K(Z^T, n), P_\infty; Z^T[u])$ is the fundamental class.

$$H^n(X, A; Z_2) \cong Z_2$$

is generated by $\Pi_* v_n^T$; $H^{n+1}(X, A; Z_2) \cong Z_2$ generated by $\Pi_* v_n^T \cup u$. Thus, by (3.4.3) and (3.4.4), $H^{n+1}(X, A; Z) \cong Z_2$ generated by $v_n^T \cup \bar{u}$, and the result follows from (3.4.5).

(3.5.2). We summarize the results of (3.4.5) and (3.5.1) in the following commutative diagram with all rows exact:

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X, A; Z^T) \arrow[r, "(i_1)_*"] \arrow[d, "\Pi_*"] & H^n(X, A; (Z+Z)^T) \arrow[r, "(p_2)_*"] \arrow[d, "\Pi_*"] & H^n(X, A; Z) \arrow[r, "\cup\alpha"] \arrow[d, "\Pi_*"] & H^{n+1}(X, A; Z^T) \arrow[r] \arrow[d, "\Pi_*"] & \cdots \\
\cdots \arrow[r] & H^n(X, A; Z_2) \arrow[r, "(i_1)_*"] & H^n(X, A; (Z_2+Z_2)^T) \arrow[r, "(p_2)^*"] & H^n(X, A; Z_2) \arrow[r, "\cup\alpha"] & H^{n+1}(X, A; Z_2) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^n(X, A; Z) \arrow[r, "(j_1)_*"] \arrow[u, "\Pi_*"] & H^n(X, A; (Z+Z)^T) \arrow[r, "(q_2)^*"] \arrow[u, "\Pi_*"] & H^n(X, A; Z^T) \arrow[r, "\cup\alpha"] \arrow[u, "\Pi_*"] & H^{n+1}(X, A; Z) \arrow[r] \arrow[u, "\Pi_*"] & \cdots
\end{tikzcd}$$

3.6. Applying the results of 3.4 and 3.5, we compute the cohomology of real projective space P_k, for $k \geqq 1$:

(3.6.1) $$H^n(P_k; Z_2) \cong \begin{cases} Z_2, & \text{generated by } u^n, \text{ if } n \leqq k \\ 0 & \text{if } n > k . \end{cases}$$

(3.6.2) $$H^n(P_k; Z) \cong \begin{cases} Z_2, & \text{generated by } \bar{u}^n, \text{ if } n \text{ even, } 0 < n \leqq k \\ Z, & \text{generated by } 1, \text{ if } n = 0 \\ 0, & \text{if } n \text{ odd, } 0 < n < k \\ Z, & \text{generated by } t(P_k), \text{ the top class, if } n = k \text{ odd} \\ 0 & \text{if } n > k . \end{cases}$$

(3.6.3) $$H^n(P_k; Z^T[u]) \cong \begin{cases} Z_2, & \text{generated by } \bar{u}^n, \text{ if } n \text{ odd, } 0 < n \leqq k \\ 0, & \text{if } n \text{ even, } 0 < n < k \\ Z, & \text{generated by } t(P_k), \text{ the top class, if } n = k \text{ even} \\ 0, & \text{if } n > k . \end{cases}$$

$$(3.6.4)\qquad H^nP_k, *; Z^T[u]) \cong \begin{cases} 0, & \text{if } n = 0 \\ Z, & \text{generated by } \bar{u}, \text{ if } n = 1 . \\ H^n(P_k; Z^T[u]) & \text{if } n > 1 \end{cases}$$

$$(3.6.5)\qquad H^n(P_k; Z_2 + Z_2) \cong H^n(P_k; Z_2) \oplus H^n(P_k; Z_2) .$$

$$(3.6.6)\qquad H^n(P_k; Z + Z) \cong H^n(P_k; Z) \oplus H^n(P_k; Z) .$$

$$(3.6.7)\qquad H^n(P_k; (Z + Z)^T[u]) \cong \begin{cases} Z, & \text{generated by } (j_1)_*1, \\ & \text{if } n = 0 \\ 0, & \text{if } 0 < n < k \\ Z, & \text{generated by } \tfrac{1}{2}(i_1)_*t(P_k) = \\ & (q_2)_*^{-1}t(P_k) \text{ if } n = k \text{ is even} \\ Z, & \text{generated by } \tfrac{1}{2}(j_1)_*t(P_k) = \\ & (p_2)_*^{-1}t(P_k) \text{ if } n = k \text{ is odd} \\ 0, & \text{if } n > k \end{cases}$$

$$(3.6.8)\qquad H^n(P_k; (Z_2 + Z_2)^T[u]) \cong \begin{cases} Z_2, & \text{generated by } (i_1)_*1 \\ & \text{if } n = 0 \\ 0, & \text{if } 0 < n < k \\ Z_2, & \text{generated by } (p_2)_*^{-1}u^k \\ & (= \Pi_*\tfrac{1}{2}(i_1)_*t(P_k)) \text{ if } k \\ & \text{even, } = \Pi_*\tfrac{1}{2}(j_1)_*t(P_k) \text{ if } k \\ & \text{odd) if } n = k \\ 0, & \text{if } n > k . \end{cases}$$

4. Evaluation of the differentials.

4.1. We need two remarks.

(4.1.1) If Y_1 and Y_2 are spaces, and $h\colon Y_1 \to Y_2$ is a map, h induces a map $(Y_1)_{n-1} \to (Y_2)_{n-1}$ and a sheaf homomorphism $\tilde{h}\colon \pi_n(Y_1, 1) \to \pi_n(Y_2, h)$. If k_1^{n+1} and k_2^{n+1} are the n^{th} k-invariants of Y_1 and Y_2 respectively, $\tilde{h}_* k_1^{n+1} = h^* k_1^{n+2} \in H^{n+1}((Y_1)_{n-1}; \pi_n(Y_2, h))$.

(4.1.2) Let X and Y be spaces, $2 \leqq m < n$ integers such that $\pi_k(Y) = 0$ for all $m < k < n$, and $f\colon X \to (Y)_n$ a map. If the k-invariant k^{n+1} of Y is based on the relation $\theta(1, k^{m+1}) = 0$, where θ is a map cohomology operation and $1\colon (Y)_{m-1} \to (Y)_{m-1}$ is the identity map, then; for any

$$x \in H^{m-1}(X; \pi_m(Y, f)),\ d_r(x) = s^{-2}\theta(p^n_{m-1}fP, s^2x),\ r = n - m + 1 ,$$

where $P\colon X \times S^2 \to X$ is projection,

$$s^2\colon H^*(X, x_0) \to H^{*+2}(X \times S^2, X \times * \cup x_0 \times S^2)$$

is suspension and $p^n_{m-1} = p_m \cdots p_n \colon (Y)_n \to (Y)_{m-1}$.

Proof. Let $(S^1, *)$ be a circle, which we think of as the unit interval with end-points identified. Let $C\colon X \times S^1 \to (Y)_m$ be the constant homotopy of $p^n_m f$ with itself. Now $p_m(C + sx) = p_m C$, where $C + sx$ is as defined in (1.2.2) and $d_r(x) = \delta^n(f, f; C + sx)$ by (1.3). Finally, $s\delta^n(f, f; C + sx) = (C + sx)^* k^{n+1} = s^{-1}\theta(p^n_{m-1} fP, s^2 x)$.

4.2. Kervaire [3, p. 162] gives us the following table of homotopy groups:

	$BO(1)$	$BO(2)$	$BO(3)$	$BO(4)$	$BO(5)$	$BO(6)$	$BO(n)$ for $7 \leqq n \leqq \infty$
π_1	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2
π_2	0	Z	Z_2	Z_2	Z_2	Z_2	Z_2
π_3	0	0	0	0	0	0	0
π_4	0	0	Z	$Z + Z$	Z	Z	Z
π_5	0	0	Z_2	$Z_2 + Z_2$	Z_2	0	0
π_6	0	0	Z_2	$Z_2 + Z_2$	Z_2	Z	0 .

Now $\pi_1(BO(n)) = Z_2$ acts on $\pi_k(BO(n))$ for all $n \geqq 1, k \geqq 1$; this action is trivial if $\pi_k(BO(n))$ is stable, that is, $k < n$; because BO is simple. For n even, Z_2 acts nontrivially on $\pi_n(BO(n))$, because the first relative k-invariant of $BO(n) \to BO$ is

$$k^{n+1} = \beta^T[w_1]w_n \in H^{n+1}(BO; Z^T[w_1]) \ .$$

(Because $\Pi_* k^{n+1}$, the reduction mod 2, must be w_{n+1}). Z_2 acts trivially on $\pi_4(BO(3))$ because if acts trivially on $\pi_4(BO)$ and the map $Z \cong \pi_4(BO(3)) \to \pi_4(BO) \cong Z$ is just multiplication by 2. Since Z_2 can only act trivially on Z_2, we need only now examine the action on $\pi_4(BO(4))$ for $k = 4, 5, 6$.

PROPOSITION (4.2.1). *We may choose generators x and y of $\pi_4(BO(4))$ such that $T(x) = -x$, $T(y) = x + y$, and the maps*

$$i^3_4\colon \pi_4(BO(3)) \longrightarrow \pi_4(BO(4)) \quad \textit{and} \quad i^4_4\colon \pi_4(BO(4)) \longrightarrow \pi_4(BO(5))$$

have the properties $i^3_4(1) = x + 2y$, $i^4_4(x) = 0$ and $i^4_4(y) = 1$.

Proof. We know that i^4_4 is onto. Choose x to be a generator of $\operatorname{Ker} i^4_4$, and pick a such that $i^4_4 a = 1$. Now $2a - i^3_4(1) \in \operatorname{Ker} i^4_4$, since $i^4_4 i^3_4 = 2$. So $2a - i^3_4(1)$ is a multiple of x. It can't be an even multiple, because then $i^3_4(1)$ would be divisible by 2, and $i^3_4\pi_4(BO(3))$ is a direct summand of $\pi_4(BO(4))$. So for some k, $2a - i^3_4(1) = (2k-1)x$. Let $y = a - kx$; then $i^3_4(1) = x + 2y$, $i^4_4(x) = 0$, and $i^4_4(y) = 1$. Now $T(x) \in \operatorname{Ker} i^4_4$, so $T(x)$ must be $-x$. $T(x + 2y) = x + 2y$ so $T(y) = \frac{1}{2}(x + 2y - Tx) = x + y$. We are done.

We represent $\pi_4(BO(4))$ as ordered pairs of integers, where (p, q) represents $px + qy$.

PROPOSITION (4.2.2). *$\pi_5(BO(4))$ and $\pi_6(BO(4))$ may be represented as ordered pairs of elements of Z_2, such that $i_5^3(x) = i_6^3(x) = (x, 0)$, $i_5^4(x, y) = i_6^4(x, y) = y$, and $T(x, y) = (x + y, y)$ for all $x, y \in Z_2$.*

Proof. $\pi_5(BO(n))$ and $\pi_6(BO(n))$ are the images, under η and η^2 respectively, of $\pi_4(BO(n))$, for $n = 3$, 4, or 5. Apply (4.2.1).

REMARK (4.2.3). There are two possible choices of x in (4.2.1) we retroactively make that choice such that the image of $\pi_5(BU(2)) \cong Z_2$, under the classifying map of the reallification $BU(2) \to BO(4)$, is generated by $(0, 1) \in \pi_5(BO(4))$.

4.3. We need to describe k-invariants for $BO(n)$.

(4.3.1) For all n, k^3 of $BO(n)$ is zero, since the projection

$$P_1: BO(n) \longrightarrow (BO(n))_1 = K(Z_2, 1) = BO(1)$$

has a lifting, namely, the map induced by the inclusion of $O(1)$ in $O(n)$. Also $k^4 = 0$, since $\pi_3(BO(n)) = 0$.

(4.3.2) For $BO(3), k^5 = \pm \beta_4 \mathfrak{P} w_2$, where β_4 is the Bockstein of $Z \to Z \to Z_4$ and $\mathfrak{P}: H^2(; Z_2) \to H^4(; Z_4)$ is the Pontrjagin square [2], and k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.3) For $BO(5)$, $k^5 = 2\beta_4 \mathfrak{P} w_2 = \beta w_2^2$ (see [4]), and $k^6 = w_6$, based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.4) Using (4.3.2), (4.3.3), we get that for $BO(4)$, $k^5 = \iota \beta_4 \mathfrak{P} w_2$, where $\iota: H^*(; Z) \to H^*(; (Z + Z)^T)$ is $(j_1)_*$ as described in (3.5.2), and k^5 is of order 4 and generates $H^5((BO(4))_4; (Z + Z)^T[w_1])$. Also, k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5$, where

$$S_q^2: H^*(; (Z_2 + Z_2)^T[a]) \longrightarrow H^{*+2}(; (Z_2 + Z_2)^T[a])$$

is that unique operation which is ordinary S_q^2 on each factor when $a = 0$, and $w_2 \cup$ is as described in (3.5).

(4.3.5) For $BO(6)$, $k^5 = 2\beta_4 \mathfrak{P} w_2 = \beta w_2^2$, and $k^7 = \beta^T[w_1] w_6$, based on the relation $\beta^T(S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5) = 0$.

4.4. Using (4.1.1) and (4.1.2) we can now evaluate some differentials $d_r = d_r^f$ for a map $f: X \to (Y)_k$.

(4.4.1) If $Y = BO(1)$ or $BO(2)$, $d_r = 0$.

(4.4.2) If $Y = BO(3)$ and $k < 4, d_r = 0$. If $k = 4, d_2 = 0$: by (4.1.2), $d_3(x) = \beta(x^3 + x \cup f^* w_2) \in H^4(X; Z)$ for all $x \in H^1(X; Z_2)$. This was also known to Dold and Whitney [2]. If

$$k = 5, d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; Z_2) ,$$

for all $x \in H^3(X; Z)$ by (4.1.2); $d_3 = 0$, and d_4 requires special computation.

(4.4.3) If $Y = BO(4)$ and $k < 4, d_r = 0$. If $k = 4, d_2 = 0$; and by (4.1.2),

$$d_3(x) = \iota\beta(x^3 + x \cup f^* w_2) \in H^4(X; (Z + Z)^T[f^* w_1])$$

for all $x \in H^1(X; Z_2)$; if

$$k = 5, d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; (Z_2 + Z_2)^T[f^* w_1])$$

for all $x \in H^3(X; (Z + Z)^T[f^* w_1])$ by (4.1.2), $d_3 = 0$, and d_4 must be computed specially.

(4.4.4) If $Y = BO(5)$ and $k < 5, d_r = 0$. If

$$k = 5, d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; Z_2)$$

for all $x \in H^3(X; Z), d_3 = 0$, and

$$\begin{aligned} d_4(x) &= x^5 + f^* w_1 \cup x^4 + f^* w_2 \cup x^3 + f^* w_3 \cup x^2 \\ &\quad + f^* w_4 \cup x + \operatorname{Im} d_2 \in E_4^{5,5} = H^5(X; Z_2)/\operatorname{Im} d_2 \end{aligned}$$

for all $x \in H^1(X; Z_2)$.

Proof. We have a map $S: \Sigma K(Z, 1)—BSO$, such that $S^* w_{i+1} = su^i$ for all $i \geqq 1$, where u is the fundamental class. Now $(BO(5))_4 = (BO)_4$ has the same homotopy as BO up through dimension 7, so we identify $H^k((BO(5))_4$ with $H^k(BO)$ for $0 \leqq k \leqq 7$. Let $h: \Sigma K(Z_2, 1)—(BO(5))_4$ be given by the commutative diagram:

$$\begin{array}{ccc} \Sigma K(Z_2, 1) & \xrightarrow{h} & (BO(5))_4 = (BO)_4 \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle P_4} \\ BSO & \longrightarrow & BO\,. \end{array}$$

$(BO(5))_4$ has an H-space structure $\mu: (BO(5))_4 \times (BO(5))_4 \to (BO(5))_4$ and $\mu^* w_6 = \sum_{i=0}^{6} w_i \times w_{6-i}$. Let QX be the space obtained from $X \times S^1$ by collapsing $x_0 \times S^1$; let $J: QX \to \Sigma X$ be the map which collapses $X \times *$, and let $p_1: QX \to X$ be projection onto the first factor. For any $x \in (H^* X)$, let $qx = p_1^* x$ and let $Qx = J^* sx$, both in $H^*(QX)$. We showed in [4, 5.1] that $qa \cup qb = q(a \cup b), qa \cup Qb = Q(a \cup b)$, and $Qa \cup Qb = 0$ for all $a, b \in H^*(X)$. Let $C: X \to K(Z_2, 1)$ be a classifying map for a given $x \in H^1(X; Z_2)$, and let $F: QX \to (BO(5))_4$ be a map, which represents a homotopy of $p_5 f$ with itself, defined by composing the following maps:

$$QX \xrightarrow{\Delta} QX \times QX \xrightarrow{J \times p_1} \Sigma X \times X \xrightarrow{\Sigma C \times p_5 f} \Sigma K(Z_2, 1) \times (BO(5))_4$$
$$\xrightarrow{h \times 1} (BO(5))_4 \times (BO(5))_4 \xrightarrow{\mu} (BO(5))_4 .$$

By (1.3), $d_4(x)$ contains $\delta^5(f, f; F)$. Now routine computation shows that $f^*w_6 = Q(x^5 + x^4 f^* w_1 + x^3 f^* w_2 + x^2 f^* w_3 + x f^* w_4)$, and the result follows from [4, 5.2].

(4.4.5) If $Y = BO(6)$ and $k < 6$, $d_r = 0$. If $k = 6$, $d_2 = 0$ and $d_3(x) = \beta^T(S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x) \in H^6(X; Z^T[f^* w_1])$ for all $x \in H^3(X; Z)$; $d_4 = 0$ and

$$d_5(x) = \beta^T(x^5 + x^4 f^* w_1 + x^3 f^* w_2 + x^2 f^* w_3 + x f^* w_4)$$
$$+ \operatorname{Im} d_2 \in E_5^{6,6} = H^6(X; Z^T[f^* w_1])/\operatorname{Im} d_3$$

for all $x \in H^1(X; Z_2)$.

Proof. same as (4.4.4).

4.5. We are now ready to classify real vector bundles over P_k, for $k \leqq 5$.

DEFINITION (4.5.1). A locally oriented real n-dimensional vector bundle over a space X shall be a b.p.p. homotopy class of maps from X to $BO(n)$. If $f: X \to BO(n)$ represents a locally oriented v.b. ξ, let $\sim \xi$, or ξ conjugate, be that locally oriented v.b. given by a map $g: X \to BO(n)$ which is connected to f via a free homotopy which sends the base-point of X around a nontrivial loop of $BO(n)$. Obviously $\sim \xi \cong \xi$, and conjugate classes of locally oriented vector bundles correspond to equivalence classes of vector bundles.

TABLE (4.5.2). For $k \geqq 1$, let $h: P_k \to BO(1)$ be the canonical line bundle. Let "$\oplus$" denote Whitney sum. We give a complete list of all locally oriented real n-dimensional vector bundles over P_k, each n and k; all bundles are self-conjugate unless otherwise specified.

Let G denote $(q_1)_*^{-1} t(P_4) = \frac{1}{2}(i_1)_* t(P_4)$ which generates

$$H^4(P_4; (Z + Z)^T[u]) .$$

Also $(p_2^*)^{-1} u^5$ generates $H^5(P_5; (Z_2 + Z_2)^T[u])$. Locally oriented real n-dimensional vector bundles over P_k, for $n - 1 \leqq k \leqq 5$:

Over P_1		Over P_2		
1	2	1	2	3
h	$h \oplus 1$	h	$T_p = (h \oplus 1) + pt(P_2)$, for all $p \in Z$; stable class $h + 1$ if p even, $3h - 1$ if p odd; $\sim T_p = T_{-p}$.	$h \oplus 2$
			$2h = 2 + \bar{u}^2$	$2h \oplus 1 = 3 + u^2$
				$3h = (h \oplus 2) + u^2$

Over P_3				Over P_4				
1 h	2 $h\oplus 1$ $2h$	3 $h\oplus 2$ $2h\oplus 1$ $3h$	4 $h\oplus 3$ $2h\oplus 2$ $3h\oplus 1$	1 h	2 $h\oplus 1$ $2h$	$3=3+\bar{u}^4$ $h\oplus 2$ $(h\oplus 2)+\bar{u}^4$ $2h\oplus 1$ $(2h\oplus 1)+\bar{u}^4$ $3h=3h+\bar{u}^4$	$4=4+(\bar{u}^4, 0)$ $2h\oplus 2$ $2h\oplus 2+(\bar{u}^4, 0)$ $4h=4+(0,\bar{u}^4)=4h+(\bar{u}^4,0)$ $2h\oplus 2+(0, \bar{u}^4)$; stable class $6h-2$ $2h\oplus 2+(\bar{u}^4, \bar{u}^4)=\sim(2h\oplus 2+(0, \bar{u}^4))$ $E_p=h\oplus 3+pG$ for all $p\in Z$; stable class $h+3$ if p even, $5h-1$ if p odd; $\sim E_p=E_{-p}$ $F_p=3h\oplus 1+pG$ for all $p\in Z$; stable class $3h+1$ if p even, $7h-3$ if p odd; $\sim F_p=F_{-p}$	5 $h\oplus 4$ $2h\oplus 3$ $3h\oplus 2$ $4h\oplus 1$ $5h$ $((2h\oplus 2)+(0, \bar{u}^4))\oplus 1$; stably $6h-1$ $F_1\oplus 1$; stable class $7h-2$

Over P_5					
1 h	2 $h\oplus 1$ $2h$	3 $3+u^5$ $h\oplus 2$ $h\oplus 2+u^5$ $A=A+u^5$; $A\vert P_4=h\oplus 2+\bar{u}^4$ $2h\oplus 1$ $2h\oplus 1+u^5$ $B=B+u^5$; $B\vert P_4=2h\oplus 1+\bar{u}^4$ $3h$ $3h+u^5$	4 $4+(u^5, 0)$ $4+(0, u^5)$ $4+(u^5, u^5)=\sim(4+(0, u^5))$ $h\oplus 3$ $h\oplus 3+(p_2^*)^{-1}u_5$ $2h\oplus 2$ $2h\oplus 2+(u^5, 0)$ $2h\oplus 2+(0, u^5)$ $2h\oplus 2+(u^5, u^5)=\sim(2h\oplus 2+(0,u^5))$ $B\oplus 1=B\oplus 1+(u^5, 0)$ $B\oplus 1+(0, u^5)=B\oplus 1+(u^5, u^5)$ $3h\oplus 1$ $3h\oplus 1+(p_2^*)^{-1}u_5$ $4h$ $4h+(u^5, 0)$ $4h+(0, u^5)$ $4h+(u^5, u^5)=\sim(4h+(0, u^5))$ $C=C+(0,u^5); C\vert P_4=2h\oplus 2+(0,\bar{u}^4)$ $D=D+(0, u^5)=\sim C$ $C+(u^5, 0)=C+(u^5, u^5)$ $D+(u^5, 0)=\sim(C+(u^5, 0))=D+(u^5, u^5)$	$5=5+u^5$ $h\oplus 4$ $h\oplus 4+u^5$ $2h\oplus 3$ $2h\oplus 3+u^5$ $3h\oplus 2$ $3h\oplus 2+u^5$ $4h\oplus 1$ $4h\oplus 1+u^5$ $5h=5h+u^5$ $C\oplus 1=C\oplus 1+u^5$ $C\oplus h=C\oplus h+u^5$	6 $h\oplus 5$ $2h\oplus 4$ $3h\oplus 3$ $4h\oplus 2$ $5h\oplus 1$ $6h$ $C\oplus h\oplus 1$

4.6. Similarly, we can classify all complex vector bundles over P_k, for $k \leqq 5$. We give a table of homotopy groups:

	$BU(1)$	$BU(2)$	$BU(n)$ for $3 \leqq n \leqq \infty$
π_1	0	0	0
π_2	Z	Z	Z
π_3	0	0	0
π_4	Z	Z	Z
π_5	0	Z_2	0

The only nonzero k-invariant in this range is k^6 of $BU(2)$, which is $\Pi_*(c_1c_2) + S_q^2\Pi_*c_2$, where $c_i \in H^{2i}(BU(2); Z)$ are the Chern classes. We thus have:

REMARK (4.6.1). For any space X, all complex line bundles over X correspond to $H^2(X; Z)$.

REMARK (4.6.2). For any space X of dimension $\leqq 5$, all complex n-bundles, for $n \geqq 3$, over X correspond to $KU(X)$, satisfying the exact sequence $0 \to H^4(X; Z) \to KU(X) \to H^2(X; Z) \to 0$.

REMARK (4.6.3). If $f\colon X \to (BU(2))_5$ is a map, then

$$d_2(x) = \Pi_*(c_1x) + S_q^2\Pi_*x \in H^5(X; Z_2)$$

for all $x \in H^3(X; Z)$; $d_3 = 0$; $d_4(x) = \Pi_*(f^*c_2 \cup x) + \operatorname{Im} d_2$ for all

$$x \in H^1(X; Z)\ .$$

Proof. Let $S\colon S^2 = \Sigma K(Z, 1) \to BU$ be the generator of $\pi_2(BU)$; then $S^*c_1 = \sigma$, the fundamental class of S^2, and $S^*c_2 = 0$. The result follows just as in (4.4.4).

TABLE (4.6.4). We summarize complex n-bundles over P_k, $2n - 1 \leqq k \leqq 5$. The reallification is given in square brackets.

Over P_2				Over P_3			
1	[2]	2	[4]	1	[4]	2	[4]
H	$[2h]$	$H \oplus 1 = 2 + u^2$	$[2h \oplus 2]$	H	$[2h]$	$H \oplus 1$	$[2h \oplus 2]$

Over P_4					
1	[2]	2	[4]	3	[6]
H	$[2h]$	$H \oplus 1$	$[2h \oplus 2]$	$H \oplus 2$	$[2h \oplus 4]$
		$2H = 2 + \bar{u}^4$	$[4h]$	$2H \oplus 1 = 3 + \bar{u}^4$	$[4h \oplus 2]$
		$H \oplus 1 + \bar{u}^4$	$[2h \oplus 2 + (\bar{u}^4, 0)]$	$3H = H \oplus 2 + \bar{u}^4$	$[6h]$
		Stable class $3H - 1$			

Over P_5			
1	$[2]$	2	$[4]$
H	$[2h]$	$2 + u^5$	$[4 + (0, u^5)]$
		$H \oplus 1$	$[2h \oplus 2]$
		$H \oplus 1 \oplus u^5$	$[2h \oplus 2 + (0, u^5)]$
		$2H$	$[4h]$
		$2H + u^5$	$[4h + (0, u^5)]$
		C	$[C]$
		$C + u^5$	$[C]$

4.7. We give a few representative examples of evaluating those difficult differentials. Is $f: P_5 \to (BO_4)_5$ is a map representing a 4-plane bundle ξ, then $d_4^f(u)$ is defined if and only if

$$d_2^f(u) = (j_1)_*\beta(u^3 + uf^*w_2) = 0 \in H^4(P_5; (Z + Z)^T[f^*w_1]) .$$

If $d_2(u) = 0$, then $d_4^f(u) = 0$ if and only if there is a map $F: QP_5 \to (BO_4)_5$ which represents a homotopy of f with itself, such that $F^*w_2 = qf^*w_2 + Qu$, where QX is as given in [4; 5].

EXAMPLE (4.7.1). If $\xi = 4$ or $4h$, then $f^*w_2 = 0$, so $d_2(u) = (\bar{u}^4, 0)$ and $d_4(u)$ is not defined. Thus $4, 4 + (u^5, 0), 4 + (0, u^5)$, and $4 + (u^5, u^5)$ are all distinct oriented vector bundles.

EXAMPLE (4.7.2). If $\xi = 2h \oplus 2$, then $f^*w_2 = u^2$, so $d_2(u) = 0$.

Let η_1 be that line bundle over QP_5 such that $w_1(\eta_1) = qu$; now 2-plane bundles over a space X with $w_1 = x$ are classified by $H^2(X; Z^T[x])$; let η_2 be that 2-plane bundle over QP_5 with $w_1(\eta_2) = qu$ classified by $Q\bar{u}$. Then $w_2(\eta_2) = Qu$. Let $c: QP_5 \to BO(4)$ be the classifying map of $\eta_1 \oplus \eta_2 \oplus 1$; $c^*w_2 = qu^2 + Qu$ and $(\eta_1 \oplus \eta_2 \oplus 1) \mid P_5 = 2h \oplus 2$. Thus F, the projection of c onto $(BO(4))_5$, and $d_4^f(u) = 0$.

EXAMPLE (4.7.3). If $\xi = C$, then $f^*w_2 = u^2$, so $d_2^f(u) = 0$, and $d_4^f(u)$ is defined. Now $p_5C = p_5(2h \oplus 2) + (0, \bar{u}^4)$,

$$\begin{tikzcd}
QP_5 \arrow[r, "F"] \arrow[dr, "C"'] & (BO(4))_5 \arrow[d, "p_5"] \\
P_5 \arrow[u, phantom, "\cup"] \arrow[ur, "2h\oplus 2"] \arrow[r, shift left, "p_5C"] \arrow[r, shift right, "p_5(2h\oplus 2)"'] & (BO(4))_4
\end{tikzcd}$$

and so $d_4(u) = 0$ if and only if we can lift the map

$$p_5F + q(0, \bar{u}^4): Qp_5 \longrightarrow ((BO(4))_4$$

to $(BO(4))_5$, where F is the map given in (4.7.2). Now the k-invariant k^6 is based on the relation $S_q^2\Pi_*k^5 + w_2 \cup \Pi_*k^5 = 0$, and $(p_5F)^*k^6 = 0$, so $(p_5F + a)^*k^6 = S_q^2\Pi_*a + (p_5F)^*w_2 \cup \Pi_*a$ which, when $a = q(0, \bar{u}^4)$, equals $S_q^2q(0, u^4) + (qu^2 + Qu) \cup q(0, u^4) = Q(0, u^5)$. So, by [4; 5.2], $d_4(u) = (0, u^5)$. Thus $C + (0, u^5) = C$, but $C + (u^5, 0)$ is different. We also have that there are two complex structures on C, because since C is the reallification of the complex bundle C, $C = C + (0, u^5)$ is the reallification of $C + u^5$.

4.8. We would like to know how vector bundles behave under tensor products. If L is any line bundle over any space, $L \otimes L = 1$. Furthermore:

REMARK (4.8.1). If η_1 and η_2 are locally oriented real n-plane bundles over a space X, which agree on X^{k-1}, and if ξ is a locally oriented real m-plane bundle over X, then $i_*\delta^k(\eta_1, \eta_2) = \delta^k(\eta_1 \oplus \xi, \eta_1 \oplus \xi)$ and $j_*\delta^k(\eta_1, \eta_2) = d^k(\eta_1 \otimes \xi, \eta_2 \otimes \xi)$, where $i: BO(n) \rightarrow BO(n + m)$ and $j: BO(n) \subset BO(nm)$ are the maps induced by the inclusion of $O(n)$ in $O(n + m)$ and $O(nm)$. Similarly for complex vector bundles.

REMARK (4.8.2). If ξ is an oriented real vector bundle which has a complex structure, and if η is any other locally oriented real vector bundle, then $\xi \otimes \eta$ also has a complex structure.

Proof. Let $C(\eta)$ be the complexification of η, and let ξ' be a complex bundle whose reallification is ξ. Then we can see routinely that the reallification of $\xi' \otimes C(\eta)$ is $\xi \otimes \eta$.

With the above information, we can almost completely determine the action of "$\oplus$" and "$\otimes$" on all locally oriented real vector bundles over P_k, $k \leqq 5$. For example,

$$A \otimes h = B, C \otimes h = C, 4 \otimes h = 4h, (4 + (0, u^5)) \otimes h = 4h + (0, u^5),$$
$$T_p \otimes h = T_p, E_p \otimes h = F_p, (4h + (u^5, u^5)) \oplus 1 = 4h \oplus 1 + u^5 .$$

The only unsolved questions are whether $A \oplus h = B \oplus 1$; it is also possible that $A \oplus h = B \oplus 1 + (0, u^5)$; and whether $B \oplus 2$ equals $2h \oplus 3$ or $2h \oplus 3 + u^5$.

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A RADICAL COINCIDING WITH THE LOWER RADICAL IN ASSOCIATIVE AND ALTERNATIVE RINGS

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In a recent paper by the second author a construction was given which was shown to coincide with the lower radical in all associative rings. In the present paper this construction is considered in various classes of not necessarily associative rings. It is shown that while the construction still defines a radical, it will in general properly contain the lower radical. More precisely, it is shown that the radical constructed coincides with the lower radical if the semisimple class of the lower radical is hereditary (or, equivalently, if the radical of a ring always contains the radicals of all its ideals).

From this condition it follows that the construction coincides with the lower radical in all associative and alternative rings, but an example is given which shows that this is not true in general. We conclude by showing that an apparently quite different construction due to J. F. Watters [5] yields exactly the same class of rings.

We will assume that all rings considered in this paper are from some universal class $\mathscr{U}$ of not necessarily associative rings. We will use the following construction, which is equivalent to that of [4]. Let $\mathscr{A}$ be an arbitrary class of rings and $\mathscr{A}_0$ its homomorphic closure. Then define $\mathscr{A}_n = \{R \in \mathscr{U} \mid R$ has a nonzero ideal $I \in \mathscr{A}_{n-1}\}$, and $A_\omega = \bigcup_n \mathscr{A}_n$. Then define $\mathscr{Y}(\mathscr{A}) = \{R \in \mathscr{U} \mid R/I \in \mathscr{A}_\omega$ for all ideals I of $R\}$. It is clear from this definition that we have

LEMMA 1. $\mathscr{A} \subseteqq \mathscr{A}_0 \subseteqq \mathscr{Y}(\mathscr{A})$.

LEMMA 2. *$\mathscr{A} \subseteqq \mathscr{B}$ implies $\mathscr{Y}(\mathscr{A}) \subseteqq \mathscr{Y}(\mathscr{B})$.*

It is also easy to check that the proof of [4, Th. 1] makes no use of associativity. Thus we may state

THEOREM 1. *$\mathscr{Y}(\mathscr{A})$ is a radical class.*

We will replace [4, Th. 2] by the following generalization:

THEOREM 2. *If $\mathscr{P}$ is a radical sub-class of $\mathscr{U}$, then $\mathscr{P} = \mathscr{Y}(\mathscr{P})$ if either of the following two equivalent conditions is satisfied:*
(i) *The semisimple class $\mathscr{S}\mathscr{P}$ of $\mathscr{P}$ is hereditary,*
(ii) *Writing $\mathscr{P}(R)$ for the $\mathscr{P}$-radical of R, then $\mathscr{P}(I) \subseteqq \mathscr{P}(R)$*

for every ideal I of every $R \in \mathscr{U}$.

Proof. The equivalence of (i) and (ii) follows from [1, Lemma 2, p. 595]. Thus assume that $\mathscr{S}\mathscr{P}$ is hereditary. By Lemma 1 we have $\mathscr{P} \subseteqq \mathscr{Y}(\mathscr{P})$ and suppose there could exist $R \in \mathscr{Y}(\mathscr{P}), R \notin \mathscr{P}$. Then R has a nonzero homomorphic image in $\mathscr{S}\mathscr{P}$, so (without loss of generality) assume $R \in \mathscr{Y}(\mathscr{P}) \cap \mathscr{S}\mathscr{P}$. Thus $R \in \mathscr{P}_n$ for some n, and since $\mathscr{P} \cap \mathscr{S}\mathscr{P} = 0$ it is clear that $n \geqq 1$. Let m be the smallest integer such that there exists a nonzero $R \in \mathscr{P}_m \cap \mathscr{S}\mathscr{P}$. Then R has a nonzero ideal $I \in \mathscr{P}_{m-1}$. Since $\mathscr{S}\mathscr{P}$ is hereditary $I \in \mathscr{S}\mathscr{P}$ contrary to the minimality of m. Thus $\mathscr{P} = \mathscr{Y}(\mathscr{P})$.

COROLLARY 1. *If $\mathscr{P}$ is a radical class then in any associative or alternative ring the $\mathscr{P}$-radical and the $\mathscr{Y}(\mathscr{P})$-radical coincide.*

Proof. This is clear since the intersection of $\mathscr{P}$ with any universal associative or alternative class is again a radical class and semisimple classes are always hereditary in associative [2, Corollary 2, p. 125] or alternative classes [1, Corollary 2, p. 602].

Note that a sufficient condition for property (ii) is that $\mathscr{P}(I)$ shall be an ideal of R. This is already known to be true in associative rings [2, Th. 47, p. 124] or alternative rings [1, Th. 2, p. 600]. From this last remark it also follows that the proof of [4, Th. 2] could have been applied equally well to alternative rings.

THEOREM 3. *Let $\mathscr{L}(\mathscr{A})$ be the lower radical for an arbitrary class $\mathscr{A}$. Then $\mathscr{L}(\mathscr{A}) = \mathscr{Y}(\mathscr{A})$ if $\mathscr{S}\mathscr{L}(\mathscr{A})$ is hereditary.*

Proof. Suppose $\mathscr{S}\mathscr{L}(\mathscr{A})$ is hereditary. From Lemma 1 and the minimality of $\mathscr{L}(\mathscr{A})$ among radical classes containing $\mathscr{A}$ [2, Lemma 5, p. 13] it follows that $\mathscr{L}(\mathscr{A}) \subseteqq \mathscr{Y}(\mathscr{A})$. But by Lemma 2, $\mathscr{A} \subseteqq \mathscr{L}(\mathscr{A})$ implies $\mathscr{Y}(\mathscr{A}) \subseteqq \mathscr{Y}(\mathscr{L}(\mathscr{A}))$. Then if $\mathscr{S}\mathscr{L}(\mathscr{A})$ is hereditary if follows from Theorem 2 that $\mathscr{Y}(\mathscr{L}(\mathscr{A})) = \mathscr{L}(\mathscr{A})$ and so $\mathscr{L}(\mathscr{A}) = \mathscr{Y}(\mathscr{A})$.

We can thus conclude that the $\mathscr{Y}(\mathscr{A})$-radical coincides with the lower radical in any associative or alternative ring.

Note. The class $\mathscr{I}$ of all idempotent rings is a radical class whose semisimple class is nonhereditary [3, Th. 2, p. 1116]. It is also true that $\mathscr{Y}(\mathscr{I}) = \mathscr{I}$ for if $R \notin \mathscr{I}$ then since all subrings of R/R^2 are zero rings, R/R^2 has no accessible subrings in $\mathscr{I}$. Thus $R \notin \mathscr{Y}(\mathscr{I})$ and so $\mathscr{Y}(\mathscr{I}) = \mathscr{I}$. This example shows that the conditions of Theorems 2 and 3 are not necessary.

Also remark that there are classes $\mathscr{A}$ for which $\mathscr{Y}(\mathscr{A})$ is not the lower radical. One example is the class $\mathscr{B} = \mathscr{L}(\mathscr{Z})$ where $\mathscr{Z}$

is the class of all zero rings. Let R be the ring constructed [see 3] over Z_2 in symbols u, v, w satisfying relations $u^2 = w^2 = 0, uv = vu = uw = u$, and $wu = vw = wv = v^2 = v$. The only ideal of R is $H = \{0, u, v, u + v\}$ for which $H^2 = H$. Now $\mathscr{S}$ is a hereditary class so by a result of A.E. Hoffman [see 6, Theorem 1,] we have $\mathscr{S} \cap \mathscr{L}(\mathscr{X}) = 0$. Now the lower radical of a hereditary class is hereditary. Thus $\mathscr{L}(\mathscr{X})$ is hereditary and so $R \notin \mathscr{L}(\mathscr{X})$. On the other hand, $R/H \in \mathscr{X}$ and R has the accessible subring $J = \{0, u\} \in \mathscr{X}$. Hence $R \in \mathscr{Y}(\mathscr{X})$.

It should also be noted that while $\mathscr{Y}(\mathscr{A})$ need not equal $\mathscr{L}(\mathscr{A})$, it is nevertheless true for all classes $\mathscr{A}$ that $\mathscr{Y}(\mathscr{L}(\mathscr{A})) = \mathscr{Y}(\mathscr{A})$. This is an easy consequence of the fact that $\mathscr{Y}(\mathscr{Y}(\mathscr{A})) = \mathscr{Y}(\mathscr{A})$.

In a paper [5] which is soon to appear the following construction is given: Let $\mathscr{M}$ be an arbitrary homomorphically closed subclass of some universal class $\mathscr{U}$. For $R \in \mathscr{U}$ define $M_{\sigma 0} = 0$, and for an arbitrary ordinal α $M_{\sigma\alpha} = \bigcup_{\beta<\alpha} M_{\sigma\beta}$, if α is a limit ordinal, or $M_{\sigma\alpha}/M_{\sigma\beta}$ is the ideal of $R/M_{\sigma\beta}$ generated by all accessible $\mathscr{M}$-subrings of $R/M_{\sigma\beta}$, whenever $\alpha = \beta + 1$. If γ is the ordinal for which $M_{\sigma\gamma} = M_{\sigma\gamma+1}$, write $M_\sigma(R) = M_{\sigma\gamma}$ and let $\mathscr{M}'_\sigma = \{R \in \mathscr{U} \mid M_\sigma(R) = R\}$.

THEOREM 4. $\mathscr{M}'_\sigma = \mathscr{Y}(\mathscr{M})$.

Proof. Let R be a ring for which $M_\sigma(R) = R$, and let $I \neq R$ be an ideal of R. Then $0 = M_{\sigma 0} \subseteqq I$, and there must exist some ordinal α such that $M_{\sigma\alpha} \subseteqq I$ but $M_{\sigma\alpha+1} \nsubseteqq I$. Write $A = M_{\sigma\alpha}, B = M_{\sigma\alpha+1}$. Since I is an ideal of R, it follows from the definition that B/A contains an accessible $\mathscr{M}$-subring W/A of R/A such that $W \nsubseteqq I$. Then the natural homomorphism $R/A \rightarrow R/I$ gives $W/A \rightarrow W'$ with W' accessible. Thus since $\mathscr{M}$ is homomorphically closed, we have a nonzero $W' \in \mathscr{M}$. It follows that $R/I \in \mathscr{M}_n$ for some n and since I was arbitrary, $R \in \mathscr{Y}(\mathscr{M})$. Thus $\mathscr{M}_\sigma' \subseteqq \mathscr{Y}(\mathscr{M})$.

The converse is clear, for suppose $R \in \mathscr{Y}(\mathscr{M})$ and $I = M_\sigma(R) = M_{\sigma\gamma}(R)$. If $I \neq R$, it follows that R/I has an accessible $\mathscr{M}$-subring, contradicting $M_{\sigma\gamma}(R) = M_{\sigma\gamma+1}(R)$. Thus $M_\sigma(R) = R$ whence $R \in \mathscr{M}_\sigma'$ and so $\mathscr{M}_\sigma' = \mathscr{Y}(\mathscr{M})$.

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CHARACTERIZATION OF CERTAIN INVARIANT SUBSPACES OF H^p AND L^p SPACES DERIVED FROM LOGMODULAR ALGEBRAS

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Let $A = A(X)$ be a logmodular algebra and m a representing measure on X associated with a nontrivial Gleason part. For $1 \leqq p \leqq \infty$, let $H^p(dm)$ denote the closure of A in $L^p(dm)$ (w^* closure for $p = \infty$). A closed subspace M of $H^p(dm)$ or $L^p(dm)$ is called *invariant* if $f \in M$ and $g \in A$ imply that $fg \in M$. The main result of this paper is a characterization of the invariant subspaces which satisfy a weaker hypothesis than that required in the usual form of the generalized Beurling theorem, as given by Hoffman or Srinivasan.

For $1 \leqq p \leqq \infty$, let I^p be the subspace of functions in $H^p(dm)$ vanishing on the Gleason part of m and let $A_m = \left\{f \in A\colon \int f dm = 0\right\}$.

THEOREM. *Let M be a closed invariant subspace of $L^2(dm)$ such that the linear span of $A_m M$ is dense in M but the subspace $R = \{f \in M\colon f \perp I^\infty M\}$ is nontrivial and has the same support set E as M. Then M has the form $\chi_E \cdot F \cdot (\bar{I}^2)^\perp$ for some unimodular function F.*

A modified form of the result holds for $1 \leqq p \leqq \infty$. This theorem is applied to give a complete characterization of the invariant subspaces of $L^p(dm)$ when A is the standard algebra on the torus associated with a lexicographic ordering of the dual group and m is normalized Haar measure.

1. Invariant subspaces. In 1949 Beurling [1], using function analytic methods, showed that all the closed invariant subspaces of H^2 of the circle have the form $M = FH^2$, where $|F| \equiv 1$ a.e. In 1958 Helson and Lowdenslager [3] and [4] extended the result to some but not all subspaces of the H^2 space of the torus, using Hilbert space methods. In the past 10 years the latter arguments have been extended by Hoffman [5, Th. 5.5, p. 293], Srinivasan [8], [9], and others to prove the following generalized Beurling theorem. If m is a representing measure for a logmodular algebra A and if M is an invariant subspace of $L^2(dm)$ which is *simply* invariant, i.e., if

(1) the linear span of $A_m M$ is not dense in M,

then $M = FH^2$ for $|F| \equiv 1$. In the general case (even the torus case) not all invariant subspaces satisfy this hypothesis. Our purpose is to extend the characterization by weakening hypothesis (1).

We assume throughout the paper that $A = A(X)$ is a logmodular algebra [5] of continuous complex-valued functions on a compact Hausdorff space X and that m is the unique representing measure on X for a complex homomorphism of A, i.e., $\int fgdm = \int fdm \int gdm$ for all $f, g \in A$. Furthermore we assume that this complex homomorphism lies in a Gleason part $P(m)$ containing more than one element. A function $f \in H^\infty(dm)$ is called *inner* if $|f| \equiv 1$. For each $f \in H^2(dm)$ we write $\hat{f}(\varphi) = \int fd\varphi$ for φ in $P(m)$, where φ also denotes the representing measure for the homomorphism φ.

In [10] Wermer showed (for A a Dirichlet algebra) that there exists an inner function Z such that $\hat{Z}$ maps $P(m)$ onto $\{\lambda: |\lambda| < 1\}$ and such that the equation

(2) $G(\hat{Z}(\varphi)) = \hat{f}(\varphi)$

associates with each f in $H^2(dm)$ an analytic function $G(\lambda) = \sum_{n=0}^\infty a_n\lambda^n$ for $|\lambda| < 1$ where $a_n = \int \bar{Z}^n fdm$. (See [5] for the extension to logmodular algebras.) Denote by F the boundary value function of G (i.e., the function in $L^2(d\theta)$ whose Fourier coefficients are a_n, where $d\theta$ is normalized Lebesgue measure on $\{|\lambda| = 1\}$).

Elementary arguments (including the Riesz-Fischer theorem) establish that the mapping $\Phi(f) = F$ can be extended to a bounded linear transformation of $L^2(dm)$ onto $L^2(d\theta)$, using the fact that $L^2(dm) = H^2(dm) \oplus \bar{H}^2_m(dm)$ [5, Th. 5.4, p. 293].

Denote by $\mathscr{Z}^p$ the closure (in $L^p(dm)$) of the polynomials in Z; denote by $\mathscr{L}^p$ the closure (in $L^p(dm)$) of the polynomials in Z and $\bar{Z}$. (For $p = \infty$, the closure is taken in the w^* topology.) Thus $\mathscr{L}^2 = \overline{\mathscr{Z}}^2 \oplus \mathscr{Z}^2_m$ and Φ, restricted to $\mathscr{L}^2$, is an isometric isomorphism onto $L^2(d\theta)$, induced by the correspondence $Z \to e^{i\theta}$.

Actually Φ can be extended to a continuous transformation of $L^1(dm)$ onto $L^1(d\theta)$ induced by formula (2) and for $1 \leqq p \leqq \infty$ carrying $\mathscr{L}^p$ isometrically onto $L^p(d\theta)$. (This map also carries $H^p(dm)$ onto $H^p(d\theta)$.) This follows from the following result of Lumer [6, Th. 3, p. 285] (and our Lemma 5 below): The correspondence $Z \to e^{i\theta}$ induces an isometric isomorphism of $\mathscr{L}^p$ onto $L^p(d\theta)$ for each p, $1 \leqq P \leqq \infty$, which carries $\mathscr{L}^p$ onto $H^p(d\theta)$. See also Merrill [7, Proof of Th. 1]. For f and $g \in L^2(dm)$, $\Phi(fg) = \Phi(f)\Phi(g)$ (see the proof of Lemma 10 in Wermer [10]). We call Φ the natural homomorphism of $L^1(dm)$ onto $L^1(d\theta)$.

Define $I^p = \left\{f \in H^p(dm): \int \bar{Z}^n fdm = 0, n = 0, 1, 2, \cdots\right\}$ for $1 \leqq p \leqq \infty$, so that $H^2(dm) = \mathscr{Z}^2 \oplus I^2$. Using (2) it is not hard to check that $I^p = \{f \in H^p(dm): \hat{f}(\varphi) = 0, \varphi \in P(m)\}$. For any subset $S \subseteqq L^2(dm)$, denote by $[S]$ the closed linear span of S.

DEFINITION. Let M be a closed invariant subspace of $L^p(dm)$. M

is called *simply invariant* if A_mM is not dense in M (w^* dense for $p = \infty$) and doubly invariant if $\bar{A}M \subseteqq M$. We call M *sesqui-invariant* if $\bar{Z}M \subseteqq M$ but M is not invariant under $\bar{A}$.

There exist closed invariant subspaces of $L^2(dm)$ which are sesqui-invariant, i.e., neither simply nor doubly invariant. For example, let $M = I^2$. If I^2 satisfied (1) so that it had the form FH^2, F inner, then F would be in I^2, so that $\bar{Z}F$ would be in I^2 by Lemma 1 below. But if $I^2 = FH^2$, then $\bar{Z} \in H^2$, which is not the case.

Our main purpose in § 2 is to relax hypothesis (1) and to obtain a characterization of certain invariant subspaces of $L^2(dm)$ not covered by the Beurling theorem, in terms of the support set of M, a unimodular function, and I^2. At the end we extend the result to $1 \leqq p \leqq \infty$. Examples in which I^2 is nontrivial are given in § 3 together with applications of the main theorem. First we give three lemmas of a preliminary nature which collect elementary and known facts.

LEMMA 1. *If $f \in I^2$, then $\bar{Z}^n f \in I^2$.*

Proof. Clearly it suffices to show that $\bar{Z}f \in H^2$, for then $\bar{Z}f \perp \mathscr{Z}^2$ and hence $\bar{Z}f \in I^2$. Let $h \in H^2_m(dm)$ and write

$$a_n = \int \bar{Z}^n f dm,\ b_n = \int \bar{Z}^n h dm\ .$$

Then $\int \bar{Z}fhdm = a_0b_1 + a_1b_0 = 0$ so $\bar{Z}f \in H^2$.

LEMMA 2. *Let $M \subseteqq L^2(dm)$ be a closed subspace. Then the following are equivalent*

(i) $AM \subseteqq M$

(ii) $H^\infty M \subseteqq M$

(iii) $H^\infty_m M = ZM = [A_mM]$.

Proof. That (i) implies (ii) follows from the w^* density of A in $H^\infty(dm)$. To see that (ii) implies (iii) observe that by definition of Z, $H^2_m = ZH^2$ and hence $H^1_m = ZH^1$, by taking closure in L^1. By considering conjugate spaces and applying Corollary to Theorem 6.1 in Hoffman [5, p. 298], we have $H^\infty_m = ZH^\infty$. Using (ii), $H^\infty_m M = ZH^\infty M \subseteqq ZM \subseteqq H^\infty_m M$. In any case $H^\infty_m M = [A_mM]$ by the w^* density of A_m in H^∞_m. This establishes (iii).

To show that (iii) implies (i), it suffices to show (iii) implies (ii). We have seen that $H^\infty_m = ZH^\infty$ or $\bar{Z}H^\infty_m = H^\infty$. Using (iii) this yields $H^\infty M = \bar{Z}H^\infty_m M \subseteqq \bar{Z}ZM = M$.

LEMMA 3. *Let $M \subseteqq L^2(dm)$ be a closed invariant subspace. Then* *the following are equivalent.*

(a) $M = FH^2$ *for some unimodular function* F.

(b) $M \ominus [A_m M] \neq \{0\}$.

(c) $M \ominus ZM \neq \{0\}$.

(d) M *is not invariant under* $\bar{Z}$.

Proof. The equivalence of (a) and (b) is the generalized Beurling theorem. Items (b) and (c) are equivalent by Lemma 2. If (a) holds then so does (d). For if M were invariant under $\bar{Z}$ then since $F \in M$, $\bar{Z}F \in M = FH^2$, so that $\bar{Z} \in H^2$ which is not the case. On the other hand, if (d) holds, ZM is a proper closed subspace of M, i.e., (c) holds.

DEFINITION. If $f \in L^1(dm)$, we define the *support set* of f (denoted by E_f) as the complement of a set of maximal measure on which f is null. If M is a closed subspace of $L^1(dm)$, the support set of M (denoted by E_M) is defined as the complement of a set of maximal measure on which all $f \in M$ are null. Clearly E_f and E_M are defined only up to sets of measure zero.

2. The invariant subspace theorem.

THEOREM 1. *Let A be a logmodular algebra and m a fixed representing measure such that the part $P(m)$ contains more than one element. Let M be a closed sesqui-invariant subspace of $L^2(dm)$ and let E be the support set of M. Let $R = M \ominus [I^\infty M]$ and $L = M^\perp \ominus [\bar{I}^\infty M^\perp]$ where $M^\perp = \{f \in \chi_E L^2(dm): f \perp M\}$. Then*

(3) *L is nontrivial and the support set of L is E if and only if $\chi_E \in \mathscr{L}^2$ and M has the form $M = \chi_E \cdot F \cdot I^2$ for some unimodular function F, and*

(4) *R is nontrivial and the support set of R is E if and only if $\chi_E \in \mathscr{L}^2$ and M has the form $M = \chi_E \cdot F \cdot (\bar{I}^2)^\perp = \chi_E \cdot F \cdot (\mathscr{L}^2 \oplus I^2)$ for some unimodular function F.*

We need several lemmas, the key fact being Lemma 8.

LEMMA 4. *Let Z be the Wermer embedding function. If θ is Lebesgue measure on T, then $\theta\{Z(x): x \in X\} = 1$ and $m(Z^{-1}(E)) = 0$ if and only if $\theta(E) = 0$, for each measurable subset E of T. Moreover, if F in $L^1(d\theta)$ corresponds to $f \in \mathscr{L}^1$ under the natural homomorphism Φ, then $f(x) = F(Z(x))$ a.e.*

Proof. Suppose that $\theta(Z(X)) < 1$. Then there exists a closed set $K \subseteqq T \backslash Z(X)$ such that $\theta(K) > 0$. The functions $f_n(t) = 1/(1 + n\rho(t, K))$, where ρ denotes distance, are continuous for each n and converge to

$\chi_K(t)$ pointwise everywhere and in $L^2(d\theta)$. Let g_n and g denote the images in $\mathscr{L}^2$ of f_n and χ_K, respectively, under the natural correspondence. Hence $g_n \to g$ in $L^2(dm)$ and by passing to a subsequence we may assume that $g_n(x) \to g(x)$ a.e. (dm). Since the f_n may be approximated by trigonometric polynomials, $g_n(x) = f_n(Z(x))$ a.e. (dm), and the latter sequence converges to zero a.e. (dm) by the definition of the f_n. Hence $g(x) = 0$ a.e. (dm). But this contradicts the fact that g corresponds to a nonzero function. Thus $\theta(Z(X)) = 1$.

This also proves that if $\theta(E) > 0$, then $m(Z^{-1}(E)) > 0$. Now suppose that $\theta(E) = 0$, i.e., that $\chi_S(t) = 1$ a.e. $(d\theta)$, where $S = T\backslash E$. Choose closed sets $K_1 \subseteqq K_2 \subseteqq, \cdots, \subseteqq S$, such that $\theta(K_n) \to \theta(S)$. Using the argument of the previous paragraph, we can show that the characteristic function of K_n corresponds to that of $Z^{-1}(K_n)$. Thus the characteristic function of $Z^{-1}(K_n)$ converges in $L^2(dm)$ to the function 1. But the characteristic function of $Z^{-1}(K_n)$ also converges to that of $Z^{-1}(\cup K_n)$. Thus the latter function is 1 a.e. Thus $m(Z^{-1}(S)) = 1$ so that $m(Z^{-1}(E)) = 0$.

To obtain the last assertion of the lemma, let $F \in L^1(d\theta)$ and f the corresponding function in the isomorphic image of $L^1(d\theta)$ in $L^1(d\mathrm{m})$. Choose a sequence F_n of polynomials in $e^{i\theta}$ and $e^{-i\theta}$ which converge to F in $L^1(d\theta)$ and a.e. Let f_n correspond to F_n so that $f_n \to f$ in $L^1(dm)$ and can be replaced by a subsequence which converges a.e.

Since F_n are polynomials, $f_n(x) = F_n(Z(x))$ a.e. (dm). Since $F_n(t) \to F(t)$ a.e. $(d\theta)$, the first part of the lemma implies that $F_n(Z(x)) \to F(Z(x))$ a.e. (dm). Thus $f(x) = F(Z(x))$ a.e.

LEMMA 5. *If* $1 \leqq p \leqq \infty$, *then*

$$H^p(dm) = \mathscr{X}^p \oplus I^p$$

where $\oplus$ *denotes algebraic direct sum. Denote by* N^p *the closure of* $\bar{I}^p \oplus I^p$ *in* $L^p(dm)$ *(norm closure for* $1 \leqq p < \infty$; w^* *closure for* $p = \infty$). *Then*

$$L^p(dm) = \mathscr{L}^p \oplus N^p .$$

Proof. First assume $1 < p \leqq \infty$. If $f \in H^p(dm)$, then f defines a bounded linear functional on $L^q(dm)$ which (via Lumer's isometry) induces a bounded linear functional on $L^q(d\theta)$, which in turn is represented by some $F \in L^p(d\theta)$. It is easy to show that

$$\int Z^n f dm = \int e^{in\theta} F d\theta$$

for all integers n. Hence $F \in H^p(d\theta)$, and by Lumer's isometry there exists $g \in \mathscr{X}^p$ with

$$\int Z^n f dm = \int Z^n g dm$$

so that $f - g \in I^p$. Hence $H^p(dm) = \mathscr{Z}^p \oplus I^p, 1 < p \leqq \infty$.

Now let $p = 1$ and $f \in H^1(dm)$. Since the lemma holds for $p = 2$ and H^1 is the closure of $\mathscr{Z}^2 \oplus I^2$, there exists $g_n \in \mathscr{Z}^2$ and $h_n \in I^2$ such that the functions $f_n = g_n + h_n$ converge in L^1 to f. We will have shown that $H^1(dm) = \mathscr{Z}^1 \oplus I^1$ if we can establish that $\{g_n\}$ forms a Cauchy sequence. For this it suffices to show that whenever $f = g + h$ for $g \in \mathscr{Z}^2$ and $h \in I^2$, then $\| g \|_1 \leqq \| f \|_1$.

Applying Lumer's isometry for $p = 1$ for the second equality and for $p = \infty$ for the fourth, we have

$$\| g \|_1 = \int_X | g | dm = \int_T | \Phi(g) | d\theta = \sup_{\| q \|_\infty \leqq 1} \left| \int_T \Phi(g)\Phi(q) d\theta \right|$$
$$= \sup_{\| q \|_\infty \leqq 1} \left| \int_X gq dm \right| = \sup_{\| q \|_\infty \leqq 1} \left| \int_X fq dm \right| \leqq \| f \|_1 ,$$

where q ranges over $\mathscr{L}^\infty$. Thus $H^p(dm) = \mathscr{Z}^p \oplus I^p, 1 \leqq p \leqq \infty$.

For the second part of the lemma, denote

$$M^p = \left\{ f \in L^p(dm) \colon \int Z^n f dm = 0 \text{ all integers } n \right\} .$$

It can be shown that $L^p(dm) = \mathscr{L}^p \oplus M^p$ by the same arguments we used for the H^p case. We can complete the proof of the lemma by showing that $M^p = N^p, 1 \leqq p \leqq \infty$.

Clearly $N^p \subseteqq M^p$. Let $f \in M^p$. Since $\bar{H}^p_m(dm) \oplus H^p(dm)$ is dense in $L^p(dm)$ [5, Th. 6.7, p. 305] and $H^p(dm) = \mathscr{Z}^p \oplus I^p$ by the first part of the lemma, we can choose $g_n \in \mathscr{L}^p$ and $h_n \in N^p$ such that

$$\int k(g_n + h_n) dm \longrightarrow \int k f dm$$

for all $k \in L^q(dm)$. Write $k = k_1 + k_2$ where $k_1 \in \mathscr{L}^q$ and $k_2 \in M^q$. Thus

$$\int k_1 g_n dm = \int k_1(g_n + h_n) dm \longrightarrow \int k_1 f dm = 0 .$$

Also $\int k_2 g_n dm = 0$. Thus $\int k g_n dm \to 0$. Since the subspace N^p is norm closed for $1 \leqq p \leqq \infty$, it is also weakly closed, so $f \in N^p$. If $p = \infty$, clearly $f \in N^\infty$.

LEMMA 6. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$, and let $R = M \ominus [I^\infty M]$. If $f \in R$ and E_f is the support set of f, write $\tilde{f}$ for the characteristic function of E_f. Then $\tilde{f} \perp I^2$.*

Proof. Observe that for any $f, g \in R$ the function $f\bar{g}$ is orthogonal to both I^∞ and $\bar{I}^\infty$. For if $h \in I^\infty$, $gh \in I^\infty M$ so that $f \perp gh$, i.e., $f\bar{g} \perp h$. Similarly $f\bar{g} \perp \bar{I}^\infty$. In particular $|f|^2 = f\bar{f} \perp I^\infty$ and $\bar{I}^\infty$. It follows easily from Lemma 5 that $|f|^2$ lies in $\mathscr{L}^1$. If F is the function in $L^1(d\theta)$ corresponding to $|f|^2$, we have $|f(x)|^2 = F(Z(x))$ by Lemma 4. In particular $f(x) = 0$ if and only if $F(Z(x)) = 0$ so that $\tilde{f} = \tilde{F} \circ Z$. Since $\tilde{F} \in L^2(d\theta)$, it follows that $\tilde{f} \in \mathscr{L}^2$, i.e., $\tilde{f} \perp I^2$.

LEMMA 7. *Suppose that M is a closed sesqui-invariant subspace of $L^2(dm)$ and let $R = M \ominus [I^\infty M]$. Then there exists $f \in R$ with $E_f = E_R$.*

Proof. If $f, g \in R$, note that there exists $h \in R$ with $E_h = E_f \cup E_g$. For let $F = E_g \backslash E_f$. Since $\chi_F \in \mathscr{L}^2$ by Lemma 6, $\chi_F g \in R$. Then $f + \chi_F g \in R$ and has support set $E_f \cup E_g$. Now let $\alpha = \sup\{m(E_f): f \in R\}$. Choose $f_n \in R$ with $m(E_{f_n}) \to \alpha$ and $E_{f_1} \subseteqq E_{f_2} \subseteqq \cdots$. Alter the functions f_n by the technique above so that their supports are disjoint. Then $f_0 = \sum_{n=1}^\infty 2^{-n} f_n \in R$ and has support G with $m(G) = \alpha$. If $m(E_R) > \alpha$, then there would exist a set of positive measure in $E_R \backslash G$ and a function $g \in R$ such that g would not vanish on that set. But then $E_{f_0} \cup E_g$ is the support set for some function in R, although $m(E_{f_0} \cup E_g) > \alpha$. This contradiction shows that $E_{f_0} = E_R$.

LEMMA 8. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$, $R = M \ominus [I^\infty M]$, and let E be the support set of R. Then there exists a unimodular function $F \in L^2(dm)$ such that $\chi_E F \in R$. If $m(E) = 1$, then $F \in R$.*

Proof. By Lemma 7, there exists $f \in R$ with $E_f = E$. Define

$$F(x) = \begin{cases} f(x)/|f(x)|, & x \in E \\ 1, & x \notin E. \end{cases}$$

Then $|F(x)| = 1$ a.e., and $f = F|f|$.

As in the proof of Lemma 6, since $f \in R$, there exists a function $F \in L^1(d\theta)$ such that $|f(x)|^2 = F(Z(x))$ a.e. Thus $F \geqq 0$ a.e. and $\sqrt{F} \in L^2(d\theta)$. Let h be the function in the isomorphic image of $L^2(d\theta)$ corresponding to $\sqrt{F}$. By Lemma 4, $\sqrt{F}(Z(x)) = h(x)$ a.e., i.e., $|f| = h \in \mathscr{L}^2$. It follows that $f = F|f| \in F\mathscr{L}^2$. Clearly $[Z^n f] \subseteqq F\mathscr{L}^2$ for all integers n. Writing $N = [Z^n f]$, we have $\bar{F}N = [Z^n \bar{F} f]$. But $Z^n \bar{F} f = Z^n(|f|/f)f = Z^n|f|$ on E, and is zero off E. Therefore $Z^n \bar{F} f \in \mathscr{L}^2$, so that $\bar{F}N \subseteqq \mathscr{L}^2$. However, $\bar{F}N$ is invariant under Z and $\bar{Z}$, so that its isomorphic image in $L^2(d\theta)$ is doubly invariant and must have the form $QL^2(d\theta)$ where $Q = Q^2 \in L^2(d\theta)$. Thus $\bar{F}N = q\mathscr{L}^2$ where q is the corresponding idempotent in $\mathscr{L}^2$. It is clear from the definition of N that $q = \chi_E$. Hence $N = F\chi_E \mathscr{L}^2$, so that $F\chi_E \in N \subseteqq R$.

REMARK. If M is a closed sesqui-invariant subspace of $L^2(dm)$, then $M^\perp$ (as defined earlier) is a closed subspace of $L^2(dm)$ invariant under $\bar{H}^\infty(dm)$ and Z. Let $L = M^\perp \ominus [\bar{I}^\infty M^\perp]$. Then dual forms of Lemma 6, 7, and 8 hold with L in place of R.

Proof of Theorem 1. First we assume that $M = \chi_E F I^2$ for some unimodular function F and that $\chi_E \in \mathscr{L}^2$ and show that $\chi_E F \in L$, so that $E_L = E$. To this end let $h \in I^2$. Then

$$\int \chi_E \bar{F} \chi_E F h dm = \int \chi_E h dm = 0$$

by assumption, so that $\chi_E F \in M^\perp$. To see that $\chi_E F \perp \bar{I}^\infty M^\perp$, let $h \in I^\infty$ and $k \in M^\perp$. It suffices to show that $\chi_E F \perp \bar{h}k$, i.e., that $\chi_E F h \perp k$. But this follows since $k \perp M$. A dual argument shows that $M = \chi_E F(\bar{I}^2)^\perp$ and $\chi_E \in \mathscr{L}^2$ imply that $\chi_E F \in R$ so that $E_R = E$.

Conversely, let us suppose that $E_L = E$. By Lemma 8, there exists a unimodular function $F \in L^2(dm)$ such that $\chi_E F \in L$. It follows that

$$FH^2(dm) \supseteqq M \supseteqq \chi_E F I^2 \,. \tag{5}$$

To prove the first inclusion in (5) it suffices to show that $M^\perp \supseteqq F\bar{H}^2_m$ where this time $M^\perp$ denotes the orthogonal complement in *all* of $L^2(dm)$. Thus let $h \in A_m$, so that $hM \subseteqq M$ and $\chi_E F \perp hM$. Since the functions in M vanish off E by assumption it follows that $F \perp hM$, i.e., $F\bar{h} \perp M$, so that $F\bar{H}^2_m \subseteqq M^\perp$ as required.

To obtain the second inclusion, let $g \in I^\infty$ and suppose that $f \perp M$ in $\chi_E L^2(dm)$. It follows easily from Lemma 5 that I^∞ is dense in I^2. Thus it suffices to show that $\chi_E F g \perp f$, i.e., that $\chi_E F \perp \bar{g} f$. But this follows since $\chi_E F \perp \bar{I}^\infty M^\perp$ by construction.

Multiplying (5) by $\bar{F}$ we have

$$H^2(dm) \supseteqq \bar{F} M \supseteqq \chi_E I^2 \,. \tag{6}$$

We use the invariance of M under $\bar{Z}$ to show that $\bar{F}M = \chi_E I^2$. For let $f \in \bar{F}M$ and write $f = f_1 + f_2$ where $f_1 \in \mathscr{Z}^2$, $f_2 \in I^2$. By Lemma 6, $\chi_E \in \mathscr{L}^2$ so that

$$f = \chi_E f = \chi_E f_1 + \chi_E f_2$$

is the unique orthogonal decomposition of f into $\mathscr{L}^2$ and I^2. However, since f and $\chi_E f_2$ are both in H^2 (Lemma 1), it follows that $\chi_E f_1 \in H^2$. Therefore $\chi_E f_1 \in \mathscr{Z}^2$. But $\chi_E f_1$ vanishes on the complement of E so that either (i) $m(E) = 1$, or (ii) $\chi_E f_1 \equiv 0$.

If case (i) holds, $H^2 \supseteqq \bar{F}M \supseteqq I^2$ so that either $\bar{F}M = I^2$ or there exists $f \in \bar{F}M$ with $\int \bar{Z}^n f dm \neq 0$ for some nonnegative integer n. By considering the least integer for which such an f exists, it is not hard to see that $\bar{F}M$ would not be invariant under $\bar{Z}$. Thus $M = FI^2$.

If case (ii) holds, $f = \chi_E f_2 \in I^2$ and $\chi_E f = f \in \chi_E I^2$. Thus $\bar{F}M \subseteqq \chi_E I^2$. Together with (6) this implies that $\bar{F}M = \chi_E I^2$. So that $M = \chi_E \cdot F \cdot I^2$.

We turn now to case (4) in which R is nontrivial and the support of $(R) = E$. Let $N = M^\perp = \{f \in L^2(dm) \colon E_f \subseteqq E \text{ and } f \perp M\}$. Then N is the complex conjugate of a sesqui-invariant subspace and

$$N^\perp \ominus [I^\infty N^\perp] = M \ominus [I^\infty M] = R \, .$$

We apply (a trivial modification of) the first part of the theorem to N. For this we need to know that $E_N = E$. If $G = E \backslash E_N$ is not the null set, then $\chi_G \cdot L^2(dm) \subseteqq M$ which is not possible. Thus $E_N = E$ and $N = \chi_E \cdot F \cdot \bar{I}^2$ for some unimodular function F. Hence

$$M = N^\perp = \chi_E F \cdot (\bar{I}^2)^\perp = \chi_E \cdot F \cdot (\mathscr{L}^2 \oplus I^2) \, .$$

We now extend the main result to a more general class of subspaces of $L^2(dm)$.

THEOREM 2. *Let M be a closed sesqui-invariant subspace of $L^2(dm)$. Let $M_1 = \{f \in M \colon f \cdot L^\infty(dm) \subseteqq M\}$ and $M_2 = M \ominus M_1$, and $R_2 = M_2 \ominus [I^\infty M_2]$. Assume that E_2, the support set of M_2 is the same as the support set of R_2. Then*

$$M = \chi_{E_1} \cdot L^2(dm) \oplus \chi_{E_2} \cdot F \cdot I^2$$

where F is unimodular, E_1 is the support set of M_1 and $\chi_{E_2} \perp I^2$.

Proof. Since M_1 is a closed doubly invariant subspace of $L^2(dm)$, there exists a measurable set $E_1 \subseteqq X$ such that $M_1 = \chi_{E_1} \cdot L^2(dm)$ (see Helson [2, Th. 2, p. 7]). It is easy to check that

$$M_2 = \{f \in M \colon f \equiv 0 \quad \text{on} \quad E_1\} \, .$$

Since M is sesqui-invariant, $M_2 \neq \{0\}$, and is itself sesqui-invariant. By Theorem 1, $M_2 = \chi_{E_2} \cdot F \cdot I^2$ for some $\chi_{E_2} \perp I^2$ and F unimodular.

The final theorem of this section characterizes the invariant subspaces of $L^p(dm)$ for $1 \leqq p \leqq \infty$.

THEOREM 3. *Fix p in the range $1 \leqq p \leqq \infty$. Let M be a closed sesqui-invariant subspace of $L^p(dm)$ and let E be the support set of M. Let $R = \{f \in M \cap L^q \colon f \perp I^\infty M\}$ and $L = \{f \in M^\perp \cap L^p \colon f \perp \bar{I}^\infty M^\perp\}$ where q is the conjugate index to p and $M^\perp = \{f \in \chi_E \cdot L^q(dm) \colon f \perp M\}$.*

Then

(i) $M = \chi_E \cdot F(\mathscr{L}^p + I^p)$ *where* $\chi_E \in \mathscr{L}^2$ *and* F *is a unimodular function if and only if* E *is the support set for* R.

(ii) $M = \chi_E \cdot F \cdot I^p$ *where* $\chi_E \in \mathscr{L}^2$ *and* F *is a unimodular function if and only if* E *is the support set for* L.

Proof. It is easy to show that if M has form (i) or (ii) then E is the support set of R or L, respectively. Let us prove the converse. First we prove the theorem for $p = 1$. Suppose that E is the support set of R. Let $N = M \cap L^2(dm)$; N is a closed sesqui-invariant subspace of $L^2(dm)$. Let $R^* = \{f \in N: f \perp I^\infty N\}$. Since $R \subset R^*$, we get E is the support set of R^* which in turn is the support set for N. Applying the L^2 invariant subspace theorem to N, we get $N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$. Since $N \subseteqq M$, we get $\chi_E \cdot F(\mathscr{L}^1 + I^1) \subseteqq M$. For $f \in M$, define $k = |f|^{1/2}$ for $|f| \geqq 1$ and 1 for $|f| < 1$. Take $h \in H^2(dm)$ outer such that $|h| = k$. It is easy to see that $1/h \in H^\infty(dm)$ and therefore $f/h \in M$. Since $f/h \in L^2(dm)$ also, we get $f/h \in N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$ and therefore $f \in \chi_E \cdot F(\mathscr{L}^1 + I^1)$. Thus we get $M = \chi_E \cdot F \cdot (\mathscr{L}^1 + I^1)$. When E is the support set for L, we get $M = \chi_E \cdot F \cdot I^1$ by applying an argument similar to the above.

Now let us prove the theorem for $p = \infty$. Suppose that E is the support set for R. Let $N = [M]$ (where [ ] denotes closure in $L^2(dm)$. Let $R^* = \{f \in N: f \perp I^\infty N\}$. It is clear that E is the support set for N which in turn is the support set for R^*. By the L^2 invariant subspace theorem we get $N = \chi_E \cdot F(\mathscr{L}^2 + I^2)$. Since $M \subseteqq N \cap L^\infty(dm)$, we get $M \subseteqq \chi_E \cdot F(\mathscr{L}^\infty + I^\infty)$. By applying the L^1 invariant subspace theorem to $\overline{M^\perp}$, we get $M^\perp = \chi_E \cdot \bar{G} \cdot \overline{I^1}$, $|G| \equiv 1$. It is easy to see that $\chi_E \cdot \overline{GI^1} \perp \chi_E F(\mathscr{L}^\infty + I^\infty)$ and therefore $M = \chi_E \cdot F(\mathscr{L}^\infty + I^\infty)$. When E is the support set for L, we get $M = \chi_E \cdot F \cdot I^\infty$ by applying an argument similar to the above. The proof for $1 < p < 2$ is similar to the one for $p = 1$ and that for $2 < p < \infty$ is similar to the one for $p = \infty$. Thus the theorem is true for $1 \leqq p \leqq \infty$.

3. **Applications.** We give an example of a logmodular algebra and a representing measure m for which I^2 is nontrivial and show that the above theorems, together with known results, completely characterize the invariant subspaces of $L^2(dm)$.

EXAMPLE 1. Let $T = \{\lambda \in C: |\lambda| = 1\}$ and let $A = A(T^2)$ be the logmodular algebra of continuous functions on T^2 which are uniform limits of polynomials in $e^{in\theta}e^{im\varphi}$ where

$$(n, m) \in S = \{(n, m): n > 0\} \cup \{(0, m): m \geqq 0\} .$$

The maximal ideal space of A can be identified with

$$(\{\theta: |\theta| \leqq 1\} \times T) \cup (\{0\} \times \{\varphi: |\varphi| \leqq 1\}) ,$$

with normalized Haar measure m identified with $\theta = \varphi = 0$. The part of m is $\{0\} \times \{\varphi: |\varphi| < 1\}$. The Wermer embedding function is given by $Z = e^{i\varphi}$, $\mathscr{Z}^2$ is the L^2 closure of the polynomials in $e^{im\varphi}$, $m = 0, 1, \cdots$, and I^2 is the L^2 closure of the polynomials in $e^{in\theta}e^{im\varphi}$ for $n \geqq 1$.

Let now M be a closed invariant subspace of $L^2(dm)$. Observe that M is doubly invariant if and only if $e^{i\theta}M = M$. In this case $M = \chi_E \cdot L^2(dm)$, for some measurable set $E \subseteqq T^2$.

If $M \ominus e^{i\theta}M \neq \{0\}$ and $M = e^{i\varphi}M$ we show that $R \neq \{0\}$ and that $E_R = E_2$ (see Theorem 2). To see that $R = M \ominus e^{i\theta}M$, let $g \in M$, $g \perp e^{i\theta}M$. Since M is sesqui-invariant $g \perp e^{-im\varphi}e^{i\theta}M$, for $m = 1, 2, \cdots$. Hence $g \perp [I^\infty M]$.

Define $M_1 = \{f \in M: e^{-in\theta}f \in M, n = 1, 2, \cdots\}$ and $M_2 = M \ominus M_1$. Then $M_1 = \chi_{E_1} \cdot L^2(dm)$ for some measurable E_1. We show that Theorem 2 applies to M_2. Let K be the complement of E_R in T^2.

Since $\chi_K \in \mathscr{L}^2$, we get $\chi_K M_2 \subseteqq M_2$. Also $\chi_K \cdot M_2 \perp R$ so $\chi_K M_2 \subset e^{i\theta}M_2$ and therefore $\chi_K M_2 = \chi_K(e^{i\theta}M_2)$. But M_2 cannot contain a doubly invariant subspace, so $E_R = E_2$. Theorem 2 applies and

$$M_2 = \chi_{E_2} \cdot F'(\bar{I}^2)^\perp$$

for some unimodular function F'. Writing $F = e^{-i\theta}F'$, we have $M_2 = \chi_{E_2} \cdot F \cdot I^2$. Note that the proofs of Lemmas 4, 6, and 7 are much simpler for the torus case than for the general case.

If $M \ominus e^{i\varphi}M \neq \{0\}$, then $M = FH^2$ by the generalized Beurling theorem.

Suppose that we now replace $T \times T$ with $B \times T$, where B is the Bohr compactification of the real line and consider $A = A(B \times T)$. Again Haar measure is associated with a nontrivial part. Denote by $\chi_\tau(x)$ the characters on B, where $\tau \in R$. I^2 is generated by the characters $\chi_\tau(x)e^{im\varphi}$ for $\tau > 0$. Clearly (3) holds for $M = \chi_\tau I^2$ and (4) holds for $M = \chi_\tau(I^2 \oplus \mathscr{L}^2)$, for any fixed τ. However one can use the example in Helson and Lowdenslager [4] to construct a sesqui-invariant subspace of $H^2(dm)$ for which both L and R are trivial.

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MULTI-VALUED CONTRACTION MAPPINGS

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Some fixed point theorems for multi-valued contraction mappings are proved, as well as a theorem on the behaviour of fixed points as the mappings vary.

In § 1 of this paper the notion of a multi-valued Lipschitz mapping is defined and, in § 2, some elementary results and examples are given. In § 3 the two fixed point theorems for multi-valued contraction mappings are proved. The first, a generalization of the contraction mapping principle of Banach, states that a multi-valued contraction mapping of a complete metric space X into the nonempty closed and bounded subsets of X has a fixed point. The second, a generalization of a result of Edelstein, is a fixed point theorem for compact set-valued local contractions. A counterexample to a theorem about (ε, λ)-uniformly locally expansive (single-valued) mappings is given and several fixed point theorems concerning such mappings are proved.

In § 4 the convergence of a sequence of fixed points of a convergent sequence of multi-valued contraction mappings is investigated. The results obtained extend theorems on the stability of fixed points of single-valued mappings [19].

The classical contraction mapping principle of Banach states that if (X, d) is a complete metric space and $f: X \rightarrow X$ is a contraction mapping (i.e., $d(f(x), f(y)) \leqq \alpha d(x, y)$ for all $x, y \in X$, where $0 \leqq \alpha < 1$), then f has a unique fixed point. Edelstein generalized this result to mappings satisfying a less restrictive Lipschitz inequality such as local contractions [4] and contractive mappings [5]. Knill [13] and others have considered contraction mappings in the more general setting of uniform spaces.

Much work has been done on fixed points of multi-valued functions. In 1941, Kakutani [10] extended Brouwer's fixed point theorem for the n-cell to upper semi-continuous compact, nonempty, convex set-valued mappings of the n-cell. In 1946 Eilenberg and Montgomery [7] generalized Kakutani's result to acyclic absolute neighborhood retracts and upper semicontinuous mappings F such that $F(x)$ is nonempty, compact, and acyclic for each x. In 1953, Strother [22] showed that every continuous multi-valued mapping of the unit interval of I into the nonempty compact subsets of I has a fixed point but that the analogous result for the 2-cell is false. In [22] Strother also proved some fixed point theorems for multi-valued mappings with restrictions on the manner in which the images of points are embedded under a homeomorphism of the space onto a retract of a Tychonoff cube. Plunkett [20], Ward [23], and others have shown that the spaces which have the fixed point property for continuous compact set-valued mappings constitute a fairly small subclass of those which have the fixed point property for continuous single-valued mappings.

In this paper, we combine the ideas of set-valued mapping and Lipschitz mapping and prove some fixed point theorems about multi-valued contraction mappings. These theorems place no severe restrictions on the images of points and, in general, all that is required of the space is that it be complete metric. Some results in this paper were presented to the American Mathematical Society on November 18, 1967; an abstract of that talk may be found in [18]. A slightly different version of Theorem 5 below was announced later in [15].

1. Basic definitions and conventions. If (X, d) is a metric space, then

(a) $\underline{CB(X)} = \{C \mid C \text{ is a nonempty closed and bounded subset of } X\}$,

(b) $2^X = \{C \mid C \text{ is a nonempty compact subset of } X\}$,

(c) $\underline{N(\varepsilon, C)} = \{x \in X \mid d(x, c) < \varepsilon \text{ for some } c \in C\}$ if $\varepsilon > 0$ and $C \in CB(X)$, and

(d) $\underline{H(A, B)} = \inf\{\varepsilon \mid A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}$ if $A, B \in CB(X)$.

The function H is a metric for $CB(X)$ called the *Hausdorff metric*. We note that the metric H actually depends on the metric for X and that two equivalent metrics for X may not generate equivalent Hausdorff metrics for $CB(X)$ (see [11, p. 131]). We shall not notate this dependency except where confusion may arise. *It will be understood, unless otherwise stated, that the symbol H stands for the Hausdorff metric obtained from a fixed preassigned metric.*

Let (X, d_1) and (Y, d_2) be metric spaces. A function $F: X \to CB(Y)$ is said to be a *multi-valued Lipschitz mapping* (abbreviated m.v.l.m.) of X into Y if and only if $H(F(x), F(z)) \leqq \alpha d_1(x, z)$ for all $x, z \in X$, where $\alpha \geqq 0$ is a fixed real number. The constant α is called a *Lipschitz constant for F*. If F has a Lipschitz constant $\alpha < 1$, then F is called a *multi-valued contraction mapping* (abbreviated m.v.c.m.). A m.v.l.m. is continuous.

A point x is said to be a *fixed point* of a single-valued mapping f (multi-valued mapping F) provided $f(x) = x (x \in F(x))$. Since the mapping $i: X \to CB(X)$, given by $i(x) = \{x\}$ for each $x \in X$, is an isometry, the fixed point theorems in this paper for multi-valued mappings are generalizations of their single-valued analogues.

2. Preliminary results. In this section we present some elementary results which will be used in later sections and introduce some notation and terminology. The proofs of many of the theorems are straightforward. From a remark in [23, p. 161] if $F: X \to 2^Y$ is a m.v.l.m. and $K \in 2^X$, then $\bigcup \{F(x) \mid x \in K\} \in 2^Y$.

LEMMA 1. *Let $F: X \to 2^Y$ be a* m.v.l.m. *with Lipschitz constant α. If $A, B \in 2^X$, then $H(\bigcup \{F(a) \mid a \in A\}, \bigcup \{F(b) \mid b \in B\}) \leqq \alpha H(A, B)$.*

THEOREM 1. *Let $F: X \to 2^Y$ be a* m.v.l.m. *with Lipschitz constant α and let $G: Y \to 2^Z$ be a* m.v.l.m. *with Lipschitz constant β. If $G \circ F: X \to 2^Z$ is defined by $(G \circ F)(x) = \bigcup \{G(y) \mid y \in F(x)\}$ for all $x \in X$, then $G \circ F$ is a* m.v.l.m. *with Lipschitz constant $\alpha \cdot \beta$.*

THEOREM 2. *Let $F: X \to 2^Y$ be a* m.v.l.m. *with Lipschitz constant α and let $\hat{F}: 2^X \to 2^Y$ be given by $\hat{F}(A) = \bigcup \{F(a) \mid a \in A\}$ for all $A \in 2^X$. Then $\hat{F}$ is a Lipschitz mapping with Lipschitz constant α.*

Let (X, d) be a complete metric space and let $F: X \to 2^X$ be a multi-valued contraction mapping. By Theorem 2 $\hat{F}$ is a contraction mapping and therefore, since $(2^X, H)$ is complete [2, p. 59], has a unique fixed point $A \in 2^X$. In the next section (see Theorem 5) we prove that such an F has fixed points. The existence of the fixed point A of $\hat{F}$ does not seem to imply the existence of a fixed point of F and in fact, as the next example illustrates, there seems to be little relation between the set S of fixed points of F and the fixed point A of $\hat{F}$ (except the containment of S in A; see the last part of the proof of Theorem 9).

EXAMPLE 1. Let $I = [0, 1]$ denote the unit interval of real numbers (with the usual metric) and let $f: I \to I$ be given by

$$f(x) = \begin{cases} \dfrac{1}{2} \cdot x + \dfrac{1}{2}, \ 0 \leqq x \leqq \dfrac{1}{2} \\ -\dfrac{1}{2} \cdot x + 1, \ \dfrac{1}{2} \leqq x \leqq 1 \end{cases} . \qquad \text{Define } F: I \to 2^I \text{ by}$$

$F(x) = \{0\} \cup \{f(x)\}$ for each $x \in I$. It is easy to verify that (a) F is a multi-valued contraction mapping, (b) the set of fixed points of F is $\{0, 2/3\}$, and (c) the fixed point of $\hat{F}$ is

$$\left\{\frac{2}{3}, 0, f(0), f(f(0)), f(f(f(0))), \cdots\right\} .$$

THEOREM 3. *Let $F: X \to CB(Y)$ be a* m.v.l.m. *with Lipschitz constant α and let $G: X \to CB(Y)$ be a* m.v.l.m. *with Lipschitz constant β.* If *$F \cup G: X \to CB(Y)$ is given by $(F \cup G)(x) = F(x) \cup G(x)$ for all $x \in X$, then $F \cup G$ is a* m.v.l.m. *with Lipschitz constant* $\max\{\alpha, \beta\}$.

The following example shows it is not in general true that the intersection of two multi-valued contraction mappings is continuous (we define the intersection of two multi-valued mappings only when the image sets have a nonempty intersection at each point).

EXAMPLE 2. Let $I^2 = \{(x, y) \mid 0 \leqq x \leqq 1 \text{ and } 0 \leqq y \leqq 1\}$, let $F: I^2 \rightarrow CB(I^2)$ be defined by $F(x, y)$ is the line segment in I^2 from the point $\{(1/2)\cdot x, 0\}$ to the point $\{(1/2)\cdot x, 1\}$ for each $(x, y) \in I^2$, and let $G: I^2 \rightarrow CB(I^2)$ be defined by $G(x, y)$ is the line segment in I^2 from the point $\{(1/2)\cdot x, 0\}$ to the point $\{(1/3)\cdot x, 1\}$ for each $(x, y) \in I^2$. It is easy to see that F and G are each multi-valued contraction mappings and that $F \cap G$, which is given by

$$(F \cap G)(x, y) = \begin{cases} \left\{\left(\frac{1}{2}\cdot x, 0\right)\right\}, & x \neq 0 \\ \{(x, y) \in I^2 \mid x = 0\}, & x = 0 \end{cases} \quad \text{for}$$

all $(x, y) \in I^2$, is not continuous.

Let X be a closed convex subset of a Banach space. If $A \in CB(X)$, then let $\overline{\text{co}}\,(A)$ denote the intersection of all closed convex sets containing A. We may think of $\overline{\text{co}}$ as a function from $CB(X)$ into $CB(X)$.

LEMMA 2. *Let X be a closed convex subset of a Banach space (with norm $\| \ \|$). Then $\overline{\text{co}}: CB(X) \rightarrow CB(X)$ is nonexpansive, i.e., if $A, B \in CB(X)$, then $H(\overline{\text{co}}(A), \overline{\text{co}}(B)) \leqq H(A, B)$.*

Proof. Let $A, B \in CB(X)$ and let $\varepsilon > 0$. Choose $p \in \overline{\text{co}}(A)$. Then there exist $a_1, a_2, \cdots, a_n \in A$ and $t_1, t_2, \cdots, t_n \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$ and $\| p - \sum_{i=1}^n t_i a_i \| < \varepsilon/2$. For each $i = 1, 2, \cdots, n$ there is a point $b_i \in B$ such that $\| a_i - b_i \| < H(A, B) + \varepsilon/2$. Let $q = \sum_{i=1}^n t_i \cdot b_i$. Then $q \in \overline{\text{co}}\,(B)$ and $\| p - q \| \leqq \| p - \sum_{i=1}^n t_i \cdot a_i \| + \| \sum_{i=1}^n t_i \cdot a_i - \sum_{i=1}^n t_i \cdot b_i \| < \varepsilon/2 + \sum_{i=1}^n t_i \| a_i - b_i \| < H(A, B) + \varepsilon$. This proves that

$$\overline{\text{co}}\,(A) \subset N(H(A, B) + \varepsilon, \overline{\text{co}}(B)) .$$

Similarly it can be shown that $\overline{\text{co}}\,(B) \subset N(H(A, B) + \varepsilon, \overline{\text{co}}\,(A))$. Since ε was arbitrary, the result follows.

The proof of the next theorem is immediate from Lemma 2.

THEOREM 4. *Let X be a closed convex subset of a Banach space and let $F: X \rightarrow CB(X)$ be a* m.v.l.m. *with Lipschitz constant α. If $\overline{\text{co}}\,F: X \rightarrow CB(X)$ is given by $(\overline{\text{co}}\,F)(x) = \overline{\text{co}}\,(F(x))$ for all $x \in X$, then $\overline{\text{co}}\,F$ is a* m.v.l.m. *with Lipschitz constant α.*

REMARK. Theorem 3 gives a technique for constructing a multi-valued Lipschitz mapping from a finite number of single-valued Lipschitz mappings by "unioning their graphs at each point". Theorem 3 can be generalized to an arbitrary family $\{F_\lambda\}_{\lambda \in \Lambda}$ of multi-valued Lipschitz mappings if it is assumed that (1) $\bigcup\{F_\lambda(x) \mid \lambda \in \Lambda\}$ is a closed and bounded subset of X for each $x \in X$ and (2) there is a real number μ such that $\alpha_\lambda \leqq \mu$ for all $\lambda \in \Lambda$ where α_λ is a Lipschitz constant for F_λ.

REMARK. Note that if, in Theorem 4, F is compact set-valued, then so is $\overline{\text{co}}\, F$. This is an immediate consequence of a result of Mazur's [3, pp. 416–417].

REMARK. Requiring a multi-valued mapping to be Lipschitz is placing a very strong continuity condition on the mapping. The literature on continuous selections suggests that, for a multi-valued mapping F to have a continuous selection, conditions on the individual sets $F(x)$ are just as important (if not more important) as restrictions on the continuity of F [17]. We substantiate this by pointing out that a multi-valued contraction mapping need not have a continuous selection, as may be seen by defining F on the unit circle in the complex plane by $F(z)$ is the two square roots of z.

3. Fixed point theorems. The first theorem of this section is proved by an iteration procedure similar to that used in proving the contraction mapping principle of Banach [14, pp. 40–42].

THEOREM 5. *Let (X, d) be a complete metric space. If $F\colon X \to CB(X)$ is a* m.v.c.m., *then F has a fixed point.*

Proof. Let $\alpha < 1$ be a Lipschitz constant for F, (we may assume $\alpha > 0$) and let $p_0 \in X$. Choose $p_1 \in F(p_0)$. Since $F(p_0), F(p_1) \in CB(X)$ and $p_1 \in F(p_0)$, there is a point $p_2 \in F(p_1)$ such that

$$d(p_1, p_2) \leqq H(F(p_0), F(p_1)) + \alpha$$

(see the remark which follows this proof). Now, since

$$F(p_1), F(p_2) \in CB(X) \quad \text{and} \quad p_2 \in F(p_1)\ ,$$

there is a point $p_3 \in F(p_2)$ such that $d(p_2, p_3) \leqq H(F(p_1), F(p_2)) + \alpha^2$. Continuing in this fashion we produce a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X such that $p_{i+1} \in F(p_i)$ and $d(p_i, p_{i+1}) \leqq H(F(p_{i-1}), F(p_i)) + \alpha^i$ for all $i \geqq 1$. We note that

$$\begin{aligned} d(p_i, p_{i+1}) &\leqq H(F(p_{i-1}), F(p_i)) + \alpha^i \leqq \alpha d(p_{i-1}, p_i) + \alpha^i \\ &\leqq \alpha[H(F(p_{i-2}), F(p_{i-1})) + \alpha^{i-1}] + \alpha^i \\ &\leqq \alpha^2 d(p_{i-2}, p_{i-1}) + 2\alpha^i \leqq \cdots \leqq \alpha^i d(p_0, p_1) + i \cdot \alpha^i \end{aligned}$$

for all $i \geqq 1$. Hence

$$\begin{aligned} d(p_i, p_{i+j}) &\leqq d(p_i, p_{i+1}) + d(p_{i+1}, p_{i+2}) + \cdots + d(p_{i+j-1}, p_{i+j}) \\ &\leqq \alpha^i d(p_0, p_1) + i\cdot\alpha^i + \alpha^{i+1}d(p_0, p_1) + (i+1)\cdot\alpha^{i+1} + \cdots \\ &\quad + \alpha^{i+j-1}d(p_0, p_1) + (i+j-1)\cdot\alpha^{i+j-1} \\ &= \left(\sum_{n=i}^{i+j-1} \alpha^n\right) d(p_0, p_1) + \sum_{n=i}^{i+j-1} n\alpha^n \end{aligned}$$

for all $i, j \geqq 1$.

It follows that the sequence $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since (X, d) is complete, the sequence $\{p_i\}_{i=1}^{\infty}$ converges to some point $x_0 \in X$. Therefore, the sequence $\{F(p_i)\}_{i=1}^{\infty}$ converges to $F(x_0)$ and, since $p_i \in F(p_{i-1})$ for all i, it follows that $x_0 \in F(x_0)$. This completes the proof of the theorem.

REMARK. Let $A, B \in CB(X)$ and let $a \in A$. If $\eta > 0$, then it is a simple consequence of the definition of $H(A, B)$ that there exists $b \in B$ such that $d(a, b) \leqq H(A, B) + \eta$ (in the proof of the previous theorem the Lipschitz constant α and subsequently α^i play the role of such an η). However, there may not be a point $b \in B$ such that $d(a, b) \leqq H(A, B)$ (if B is compact, then such a point b does exist). For example, let l_2 denote the Hilbert space of all square summable sequences of real numbers; let $a = (-1, -1/2, \cdots, -1/n, \cdots)$ and; for each $n = 1, 2, \cdots$, let e_n be the vector in l_2 with zeros in all its coordinates except the n^{th} coordinate which is equal to one. Let $A = \{a, e_1, e_2. \cdots, e_n, \cdots\}$ and let $B = \{e_1, e_2, \cdots, e_n, \cdots\}$. Since $\|a - e_n\| = (\|a\|^2 + 1 + 2/n)^{\frac{1}{2}}$ for each $n = 1, 2, \cdots$, $H(A, B) = (\|a\|^2 + 1)^{\frac{1}{2}}$ and there is no e_n in B such that $\|a - e_n\| \leqq H(A, B)$.

In [4] Edelstein proved that if X is a complete ε-chainable metric space and $f: X \to X$ is an (ε, λ)-uniformly locally contractive mapping, then there is an $x \in X$ such that $f(x) = x$. We generalize this result to multi-valued functions in Theorem 6, but first we give some definitions.

A metric space (X, d) is said to be *ε-chainable* (where $\varepsilon > 0$ is fixed) if and only if given $a, b \in X$ there is an ε-chain from a to b (that is, a finite set of points $x_0, x_1, \cdots, x_n \in X$ such that $x_0 = a$, $x_n = b$, and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \cdots, n$). A function $F: X \to CB(X)$ is said to be an *(ε, λ)-uniformly locally contractive multi-valued mapping* (where $\varepsilon > 0$, $0 \leqq \lambda < 1$) provided that, if $x, y \in X$ and $d(x, y) < \varepsilon$, then $H(F(x), F(y)) \leqq \lambda d(x, y)$. This definition is modeled after Edelstein's definition for single-valued mappings in [4]. Formally this definition, in the case of single-valued mappins, is less restrictive than Definition 2.2 in [4], but Edelstein uses only the properties of this type of uniform condition in the proof of his Theorem 5.2 [4].

The proof of Theorem 6 is substantially different from the proof of Theorem 5.2 of [4]. The basic idea was inspired by Remark 2.34 of [6, p. 691].

THEOREM 6. *Let (X, d) be a complete ε-chainable metric space. If $F: X \to 2^X$ is an (ε, λ)-uniformly locally contractive multi-valued mapping, then F has a fixed point.*

Proof. If $(x, y) \in X \times X$, then let $d_\varepsilon(x, y) = \inf \{\sum_{i=1}^n d(x_{i-1}, x_i) \mid x_0 = x, x_1, \cdots, x_n = y$ is an ε-chain from x to $y\}$. It is easy to verify that d_ε is a metric for X satisfying (1) $d(x, y) \leqq d_\varepsilon(x, y)$ for all $x, y \in X$ and (2) $d(x, y) = d_\varepsilon(x, y)$ for all $x, y \in X$ such that $d(x, y) < \varepsilon$. From (1) and (2) and the completeness of (X, d) it follows that (X, d_ε) is complete. Let H_ε be the Hausdorff metric for 2^X obtained from d_ε. Note that if $A, B \in 2^X$ and $H(A, B) < \varepsilon$, then $H_\varepsilon(A, B) = H(A, B)$. We now show that $F: X \to 2^X$ is a m.v.c.m. with respect to d_ε and H_ε. Let $x, y \in X$ and let $x_0 = x, x_1, \cdots, x_n = y$ be an ε-chain from x to y. Since $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \cdots, n$, $H(F(x_{i-1}), F(x_i)) \leqq \lambda d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \cdots, n$. Therefore,

$$\begin{aligned} H_\varepsilon(F(x), F(y)) &\leqq \sum_{i=1}^n H_\varepsilon(F(x_{i-1}), F(x_i)) \\ &= \sum_{i=1}^n H(F(x_{i-1}), F(x_i)) \leqq \lambda \sum_{i=1}^n d(x_{i-1}, x_i) , \end{aligned}$$

i.e., $H_\varepsilon(F(x), F(y)) \leqq \lambda \sum_{i=1}^n d(x_{i-1}, x_i)$. Since $x_0 = x, x_1, \cdots, x_n = y$ was an arbitrary ε-chain from x to y, it follows that $H_\varepsilon(F(x), F(y)) \leqq \lambda d_\varepsilon(x, y)$. This proves that F is a m.v.c.m. with respect to d_ε and H_ε. By Theorem 5, F has a fixed point. This completes the proof of Theorem 6.

In [4] Edelstein defines a single-valued mapping f to be (ε, λ)-uniformly locally expansive (where $\varepsilon > 0$ and $\lambda > 1$) provided that, if x is in the domain of f, then, for any distinct points p and q in the domain of f such that $d(p, x) < \varepsilon$ and $d(q, x) < \varepsilon$, $d(f(p), f(q)) > \lambda d(p, q)$. Corollary 6.1 of [4] states "If f is a one-to-one (ε, λ)-uniformly locally expansive mapping of a metric space Y onto an ε-chainable complete metric space $X \supset Y$ then there exists a unique ξ such that $f(\xi) = \xi$". The proof offered for this corollary is that f^{-1} is (ε, β)-uniformly locally contractive for some $\beta < 1$. In the following example we show that this is not necessarily the case and, in fact, that Corollary 6.1 as stated is false.

EXAMPLE 3. Let $S = \{1, 2, 3, 4, 5, 6\}$ with absolute value distance. Define $f: S \to S$ by

$$f(x) = \begin{cases} 2, x = 1 \\ 4, x = 2 \\ 6, x = 3 \\ 1, x = 4 \\ 3, x = 5 \\ 5, x = 6 \end{cases}.$$

It is easy to verify that f satisfies all the hypotheses of Corollary 6.1 of [4] where $\varepsilon = 1\frac{1}{4}$ and $\lambda = 1\frac{1}{4}$ (note also that S is $1\frac{1}{4}$-chainable). However, f has no fixed point.

Next we prove two fixed point theorems for single-valued (not necessarily one-to-one) uniformly locally expansive mappings. Conditions are placed on the inverse of a uniformly locally expansive mapping which reflect the degree of chainability of the space or the degree of local expansiveness of the mapping. Example 3 is the motivating factor for such conditions (note that the mapping of Example 3 has a uniformly continuous inverse).

We shall use a slightly weaker definition of uniform local expansiveness than Edelstein's definition given above. Specifically, a single-valued mapping f is said to be (ε, λ)-*uniformly locally expansive* (where $\varepsilon > 0$ and $\lambda > 1$) provided that, if x and y are in the domain of f and $d(x, y) < \varepsilon$, then $d(f(x), f(y) \geqq \lambda d(x, y)$.

We need several more definitions before stating the next theorem. A metric space is *well-chained* if and only if it is ε-chainable for each $\varepsilon > 0$ (for compact spaces well-chained is equivalent to connected but $\{(x, \tan(x)) \mid 0 \leqq x < \pi/2\} \cup \{(\pi/2, y) \mid y \geqq 0\}$ is a well-chained complete space which is not connected). A function g from a space X to a space Y is said to be ε-*continuous* (for fixed $\varepsilon > 0$) if and only if each point x of X admits a neighborhood U_x such that the diameter of $g(U_x)$ is less than ε (in [12], where ε-continuity was apparently first defined, the requirement was that the diameter of $g(U_x)$ be less than or equal to ε). A function $F: X \rightarrow CB(X)$ is said to be an ε-*nonexpansive multi-valued* mapping (where $\varepsilon > 0$ is fixed) if and only if $H(F(x), F(y)) \leqq d(x, y)$ for all $x, y \in X$ such that $d(x, y) < \varepsilon$ (this definition is modeled after Definition 1.1 of [6] for single-valued functions).

THEOREM 7. *Let* (X, d) *be a complete* ε-*chainable* (*well-chained*) *metric space, let* A *be a nonempty subset of* X, *and let* $f: A \rightarrow X$ *be an* (ε, λ)-*uniformly locally expansive mapping of* A *onto* X. *If* $f^{-1}(x) \in 2^A$ *for each* $x \in X$ *and* $f^{-1}: X \rightarrow 2^A$ *is* ε-*nonexpansive* (*uniformly* ε-*continuous*), *then* f *has a fixed point.*

Proof. We first prove the theorem for the case when X is ε-chainable and f^{-1} is ε-nonexpansive. We shall show that $f^{-1}: X \to 2^A$ is $(\varepsilon, 1/\lambda)$-uniformly locally contractive. Let $x, y \in X$ such that $0 < d(x, y) < \varepsilon$ and choose $\eta > 0$. Let $p \in f^{-1}(x)$. Since f^{-1} is ε-nonexpansive, $H(f^{-1}(x), f^{-1}(y)) \leqq d(x, y) < \varepsilon$. Hence, there exists a point $q \in f^{-1}(y)$ such that $d(p, q) < \varepsilon$. Therefore, $d(f(p), f(q)) \geqq \lambda d(p, q)$, i.e., $d(p, q) < [1/\lambda + \eta] d(x, y)$. This proves that

$$f^{-1}(x) \subset N\left(\left[\frac{1}{\lambda} + \eta\right] d(x, y), f^{-1}(x)\right) .$$

Similarly, it can be shown that $f^{-1}(y) \subset N([1/\lambda + n] d(x, y), f^{-1}(x))$. Since η was arbitrary, it now follows that f^{-1} is $(\varepsilon, 1/\lambda)$-uniformly locally contractive. Since X is ε-chainable we may now apply Theorem 6 to conclude that there is a point $x_0 \in X$ such that $x_0 \in f^{-1}(x_0)$. Clearly, $f(x_0) = x_0$. We now prove the theorem for the case where x is well-chained and f^{-1} is uniformly ε-continuous. Since f^{-1} is uniformly ε-continuous, there exists a $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $H(f^{-1}(x_1), f^{-1}(x_2)) < \varepsilon$. Using a procedure similar to that employed above, it follows that f^{-1} is $(\delta, 1/\lambda)$-uniformly locally contractive. Since X is well-chained, X is δ-chainable and we may now use Theorem 6 to obtain, as above, a fixed point for f. This proves Theorem 7.

A metric space (X, d) is said to be *convex* (in the sense of Menger) provided that, if $x, y \in X$, $x \neq y$, then there exists a point $z \in X$, $z \neq x$ and $z \neq y$, such that $d(x, y) = d(x, z) + d(z, y)$. If (X, d) is a complete convex metric space and $F: X \to CB(X)$ is (ε, λ)-uniformly locally contractive, then F is actually a multi-valued contraction mapping. The proof is the same as the proof of the corresponding statement for single-valued mappings in [4]. Using this fact we may now prove the following:

THEOREM 8. *Let (X, d) be a complete convex metric space, let A be a nonempty subset of X, and let $f: A \to X$ be an (ε, λ)-uniformly locally expansive mapping of A onto X. If $f^{-1}(x) \in CB(X)$ for each $x \in X$ and $f^{-1}: X \to CB(X)$ is uniformly ε-continuous, then f has a fixed point.*

Proof. Proceeding as in the second part of the proof of Theorem 7 we can show that f^{-1} is $(\delta, 1/\lambda)$-uniformly locally contractive for some $\delta > 0$. From the comments immediately preceding this theorem it follows that f^{-1} is actually a multi-valued contraction mapping. Hence, by Theorem 5, there is an $x_0 \in X$ such that $x_0 \in f^{-1}(x_0)$. Clearly $f(x_0) = x_0$ and the proof of Theorem 8 is completed.

REMARK. The author does not know if Theorem 6 remains true when stated for mappings into $CB(X)$. The proof of Theorem 5.2 of [4] does not seem to generalize for mappings into $CB(X)$ and the proof of Theorem 6 is not valid for mappings into $CB(X)$ because d_ε may not be bounded even though d is. If Theorem 6 were valid when stated for mappings into CB(X), then Theorem 7 would be valid in the more general setting and Theorem 8 would be superfluous. (Cf. § 5).

4. Sequences of multi-valued contraction mappings and fixed points. Suppose (X, d) is a complete metric space, F_i: $X \rightarrow CB(X)$ is a multi-valued contraction mapping with a fixed point x_i for each $i = 1, 2, \cdots$, and F_0: $X \rightarrow CB(X)$ is a multi-valued contraction mapping. In this section we investigate the following question: If the sequence $\{F_i\}_{i=1}^\infty$ converges (in some sense) to F_0, does some subsequence $\{x_{i_j}\}_{j=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ converge to a fixed point of F_0?

Without further assumptions on the images of points it is easy to see that the answer to the above question is no; simply let $F_i(x)$ be the set of real numbers (with a bounded metric) for all $i = 0, 1, 2, \cdots$ and for all real numbers x and let $x_i = i$ for each $i = 1, 2, \cdots$. For this reason we shall assume from now on (except in Lemma 3) that $F_i(x)$ is compact for all i and for all x.

In this section we shall prove the following:

THEOREM 9. *Let (X, d) be a complete metric space, let F_i: $X \rightarrow 2^X$ be a* m.v.c.m. *with fixed point x_i for each $i = 1, 2, \cdots$, and let F_0: $X \rightarrow 2^X$ be a* m.v.c.m. *If any one of the following holds*:

(1) *each of the mappings $F_1, F_2, \cdots$ has the same Lipschitz constant $\alpha < 1$ and the sequence $\{F_i\}_{i=1}^\infty$ converges pointwise to F_0;*

(2) *the sequence $\{F_i\}_{i=1}^\infty$ converges uniformly to F_0;*

or

(3) *the space (X, d) is locally compact and the sequence $\{F_i\}_{i=1}^\infty$ converges pointwise to F_0;*

then there is a subsequence $\{x_{i_j}\}_{j=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ such that $\{x_{i_j}\}_{j=1}^\infty$ converges to a fixed point of F_0.

Before giving a proof of this theorem we need several preliminary results. A proof of Proposition 1 below may be found in [1, pp. 6–7], Proposition 2 is a special case of Theorem 1 of [19], and Proposition 3 is Theorem 2 of [19]. In each of these propositions f_i is a single-valued contraction mapping of a metric space (X, d) into itself with fixed point a_i for each $i = 0, 1, 2, \cdots$.

PROPOSITION 1. *If all the mappings $f_1, f_2, \cdots$ have the same Lipschitz constant $\alpha < 1$ and if the sequence $\{f_i\}_{i=1}^\infty$ converges pointwise to f_0, then the sequence $\{a_i\}_{i=1}^\infty$ converges to a_0.*

PROPOSITION 2. *If the sequence* $\{f_i\}_{i=1}^{\infty}$ *converges uniformly to* f_0, *then the sequence* $\{a_i\}_{i=1}^{\infty}$ *converges to* a_0.

PROPOSITION 3. *If the space* (X, d) *is locally compact and the sequence* $\{f_i\}_{i=1}^{\infty}$ *converges pointwise to* f_0, *then the sequence* $\{a_i\}_{i=1}^{\infty}$ *converges to* a_0.

The following lemma is a generalization of the lemma in [19].

LEMMA 3. *Let* (X, d) *be a metric space, let* $F_i\colon X \to CB(X)$ *be a* m.v.c.m. *with fixed point* x_i *for each* $i = 1, 2, \cdots$, *and let* $F_0\colon X \to CB(X)$ *be a* m.v.c.m. *If the sequence* $\{F_i\}_{i=1}^{\infty}$ *converges pointwise to* F_0 *and if* $\{x_{i_j}\}_{j=1}^{\infty}$ *is a convergent subsequence of* $\{x_i\}_{i=1}^{\infty}$, *then* $\{x_{i_j}\}_{j=1}^{\infty}$ *converges to a fixed point of* F_0.

Proof. Let $x_0 = \lim_{j\to\infty} x_{i_j}$ and let $\varepsilon > 0$. Choose an integer M such that $H(F_{i_j}(x_0), F_0(x_0)) < \varepsilon/2$ and $d(x_{i_j}, x_0) < \varepsilon/2$ for all $j \geqq M$. Then, if $j \geqq M$,

$$\begin{aligned} H(F_{i_j}(x_{i_j}), F_0(x_0)) &\leqq H(F_{i_j}(x_{i_j}), F_{i_j}(x_0)) + H(F_{i_j}(x_0), F_0(x_0)) \\ &< d(x_{i_j}, x_0) + H(F_{i_j}(x_0), F_0(x_0)) < \varepsilon\ . \end{aligned}$$

This proves that $\lim_{j\to\infty} F_{i_j}(x_{i_j}) = F_0(x_0)$. Therefore, since $x_{i_j} \in F_{i_j}(x_{i_j})$ for each $j = 1, 2, \cdots$, it follows that $x_0 \in F_0(x_0)$. This proves the lemma.

Proof of Theorem 9. For each $i = 0, 1, 2, \cdots$, let $\hat{F}_i\colon 2^X \to 2^X$ be defined in terms of F_i as in Theorem 2. Then, by Theorem 2, $\hat{F}_i$ is a contraction mapping and therefore has a unique fixed point $A_i \in 2^X$ for each $i = 0, 1, 2, \cdots$. If the sequence $\{F_i\}_{i=1}^{\infty}$ converges pointwise to F_0 as assumed in 1 and 3, then $\{F_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of X to F_0 [21, p. 156]; and hence, the sequence $\{\hat{F}_i\}_{i=1}^{\infty}$ converges pointwise on 2^X to $\hat{F}_0$. A direct argument shows that if the sequence $\{F_i\}_{i=1}^{\infty}$ converges uniformly to F_0 as assumed in 2, then the sequence $\{\hat{F}_i\}_{i=1}^{\infty}$ converges uniformly on 2^X to $\hat{F}_0$. In any case we may use Proposition 1 in connection with 1, Proposition 2 in connection with 2, and Proposition 3 in connection with 3 to conclude that the sequence $\{A_i\}_{i=1}^{\infty}$ converges to A_0. Hence, $K = \bigcup \{A_i \mid i = 0, 1, 2, \cdots\}$ is a compact subset of X. Note that, by the iteration procedure of Banach [14, pp. 40–42], the sequence $\{\hat{F}_i^n(x_i)\}_{n=1}^{\infty}$ converges to A_i (where $\hat{F}_i^n(x_i) = \hat{F}(\hat{F}(\cdots(\hat{F}(x_i))\cdots))$, n times); and therefore, since $x_i \in \hat{F}_i^n(x_i)$ for all $n = 1, 2, \cdots$, it follows that $x_i \in A_i$ for each $i = 1, 2, \cdots$. Thus we have that $\{x_i\}_{i=1}^{\infty}$ is a sequence in the compact set K. Hence, $\{x_i\}_{i=1}^{\infty}$ has a convergent subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ which, by Lemma 3, converges to a fixed point of F_0. This completes the proof of Theorem 9.

We now make several remarks concerning Theorem 9.

REMARK. If F_0 has only one fixed point x_0, then (with the hypotheses of Theorem 9) the sequence $\{x_i\}_{i=1}^\infty$ itself converges to x_0. To see this suppose $\{x_i\}_{i=1}^\infty$ does not converge to x_0. Then there is a subsequence $\{x_{i_k}\}_{k=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ such that no subsequence of $\{x_{i_k}\}_{k=1}^\infty$ converges to x_0. Applying Theorem 9 in the context of the two sequences $\{F_{i_k}\}_{k=1}^\infty$ and $\{x_{i_k}\}_{k=1}^\infty$, we see that there is a subsequence of $\{x_{i_k}\}_{k=1}^\infty$ which converges to a fixed point of F_0. This establishes a contradiction. (This remark shows that Theorem 9 is an extension of Propositions 1, 2, and 3 stated above).

REMARK. To see that local compactness is a necessary hypothesis in Proposition 3 and, therefore, in part (3) of Theorem 9, the reader is referred to Example 1 of [19].

REMARK. Let (X, d) be a compact metric space. In this setting Theorem 9 is a direct consequence of Lemm 3. Let $Mf(X) = \{G: X \to 2^X \mid G$ is continuous and G has fixed points$\}$ and, if G_1 and G_2 are in $Mf(X)$, let $\rho(G_1, G_2) = \sup\{H(G_1(z), G_2(x)) \mid x \in X\}$. Define $\varphi: Mf(X) \to 2^X$ by $\varphi(G) = \{x \in X \mid x \in G(x)\}$ for each $G \in Mf(X)$. Using a modification of Lemma 3 together with the fact that convergence in $(Mf(X), \rho)$ is uniform convergence, it can be shown that φ is upper semi-continuous (this is a generalization of a result of Wehausen [24] which also appears in [8]). It follows from a result in [9] that φ is continuous on a dense subspace of $Mf(X)$. However, φ may be discontinuous even at some constant functions. In the next example we construct a sequence $\{G_n\}_{n=1}^\infty$ of multi-valued contraction mappings defined on the unit interval $[0, 1]$ which converges uniformly to the mapping given by $G(x) = [0, 1]$ for all $x \in [0, 1]$ but for which the sequence $\{\varphi(G_n)\}_{n=1}^\infty$ does not converge to $\varphi(G) = [0, 1]$. It is interesting to compare this phenomenon with results in [8] and [16].

EXAMPLE 4. Let $I = [0, 1]$ denote the unit interval of real numbers (with the usual metric). For each $n = 1, 2, \cdots$, let $G_n: I \to 2^I$ be given by

$$G_n(x) = \left\{y \mid 0 \leqq y \leqq \frac{n}{n+1} \cdot x\right\} \cup \left\{y \,\middle|\, \frac{n}{n+1} \cdot x + \frac{1}{n+1} \leqq y \leqq 1\right\}$$

for all $x \in I$. Using Theorem 3 it is easy to see that G_n is a multi-valued contraction mapping for each $n = 1, 2, \cdots$. Clearly, the sequence $\{G_n\}_{n=1}^\infty$ converges uniformly to the mapping $G: I \to 2^I$ defined by $G(x) = I$ for each $x \in I$. Since $\varphi(G_n) = \{0, 1\}$ for all $n = 1, 2, \cdots$ (see the preceding remark), it follows that $\{\varphi(G_n)\}_{n=1}^\infty$ does not converge to $\varphi(G) = I$.

5. *Added in proof.* In a forthcoming paper with Professor Covitz on multi-valued contraction mappings in generalized metric spaces the author has extended Theorems 5 and 6 of this paper to mappings into $CL(X) = \{C \mid C$ is a nonempty closed subset of $X\}$ with the generalized Hausdorff distance. These results give an affirmative answer to problems posed in this remark and show that even boundedness of point images is not necessary. In addition, it was discovered by the author that a generalized version of the iteration procedure of Edelstein [4] can be carried out to give a proof of Theorem 6 above even for mappings into the more general space $CL(X)$.

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SEMI-GROUPS OF SCALAR TYPE OPERATORS IN BANACH SPACES

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This paper deals with the spectral representation theorems of semi-groups of scalar type operators in Banach spaces. These results generalize the corresponding ones on semi-groups of hermitian, normal and unitary operators in Hilbert spaces. In the beginning sections we study some interesting properties of a $W^*(||\cdot||)$-algebra-which generalizes the notion of an abelian von Neumann algebra to Banach spaces-and unbounded spectral operators arising out of $E(\cdot)$-unbounded measurable functions where $E(\cdot)$ is a resolution of the identity. These results are applied later to prove the spectral representation theorems on semi-groups of scalar type operators. The last theorem of this paper gives an extension of Stone's theorem on strongly continuous one parameter group of unitary operators to arbitrary Banach spaces.

This paper mainly deals with the spectral representation theorems of semi-groups of scalar type operators in Banach spaces, generalizing those of semi-groups of hermitian, normal and unitary operators in Hilbert spaces. Since all the classical proofs of these theorems vitally depend on the inner-product structure of the Hilbert space they cannot be adapted to Banach spaces. However, Phillips has obtained in [15] these spectral representation theorems on Hilbert spaces by making use of the theory of abelian W^* (von Neumann) algebras. Here we adapt his method of proof by suitably generalizing the notion of an abelian W^* algebra to Banach spaces.

In [3] Bade has developed the theory of operator algebras W on Banach spaces, which are generated in the weak operator topology by a σ-complete Boolean algebra of projections. Such an algebra W has its maximal ideal space extremally disconnected, just as in the case of an abelian W^* algebra. However, W is not a B^*-algebra. To this end we exploit the work on hermitian operators in Banach spaces by Berkson [4, 5, 6], Lumer [12, 13] and Vidav [18] and we define an algebra called a $W^*(||\cdot||)$-algebra in § 2 of this paper, which is a B^*-algebra generated in the weak operator topology by a σ-complete Boolean algebra of projections. The involution * of this algebra is also strongly continuous (Theorem 1 of § 2) as its counterpart in the abelian W^*algebra. Thus $W^*(||\cdot||)$-algebras have all the essential properties of abelian W^* algebras, though the double commutant theorem fails for such algebras. (See Dieudonné [7]).

Further in § 2 we introduce an ordering relation among hermitian operators on a Banach space and prove that in a $W^*(||\cdot||)$-algebra a bounded monotonic net of operators converges strongly. This result is given here not only for its own interest, but also for its application later in Lemma 6 of § 4.

The spectral operators arising from unbounded measurable functions are studied in § 3 and a generalization of Lemma 6 of Dunford [8] is obtained here in Theorem 4. This is a basic result, which is used in the theory of semi-groups of scalar type operators to show that the infinitesimal generator of the semi-group is a spectral operator of scalar type. Theorem 5 of this section, which states that the residual spectrum of an unbounded spectral operator of scalar type is empty, generalizes the corresponding result for maximal normal operators in Hilbert spaces.

In §§ 4, 5 and 6 we study the semi-groups of scalar type operators making use of the tools developed in § 2 and § 3. We generalize Theorems 22.3.1, 22.3.2, 22.4.1, 22.4.2 and 22.4.3 of Hille and Phillips [11] to Banach spaces. Because of the lack of the inner-product in our case the relations (13) and (23) are to be obtained here in a way completely different from [11]. Also Theorem 8 of § 4 has to be weaker than the corresponding Theorem 22.3.2 of [11] as we have to apply here the generalized Lebesgue bounded convergence theorem which is available only for a sequence of functions.

As the $W^*(||\cdot||)$-algebra W does not in general satisfy the double commutant theorem, we have to assume explicitly in Theorem 9 of § 5, that the resolvent operators $R(\lambda, A)$ of the infinitesimal generator A of the semi-group belong to W. However, this explicit assumption is not needed in the particular case when the operators of the semi-group are all unitary. (See Theorem 10 of § 6).

1. Preliminaries. The terminology and notation in this paper are as follows. By a Banach space we mean a complex Banach space. X always denotes a Banach space. For definitions of boundedness, σ-completeness and completeness of a Boolean algebra (abbreviated as B.A. hereafter) of projections in X, one may refer to Bade [3]. For a B.A. $\mathfrak{B}$ of projections in X, $\bar{\mathfrak{B}}^s$ denotes the strong closure of $\mathfrak{B}$. The results on spectral operators that are used in the sequel, can be found in Dunford [8] and Bade [1, 2, 3].

DEFINITION 1. An element h of a Banach algebra A with the norm $||\cdot||$ will be called hermitian in the norm if for r real,

$$||e + irh|| = 1 + o(r)$$

as $r \to 0$, where e is the identity of A.

If T is an operator on a Banach space X then T is called hermitian in the equivalent norm $\|\cdot\|$ of X in the sense of Vidav if T, as an element of the algebra $\mathfrak{B}(X)$, is hermitian in the induced operator norm $\|\cdot\|$ of $\mathfrak{B}(X)$.

DEFINITION 2. An operator T on a Banach space X is said to be hermitian in the equivalent norm $\|\cdot\|$ of X in the sense of Lumer, if $[Tx, x]$ is real for all x in X with $\|x\| = 1$ where $[\ ,\ ]$ is a semi-inner-product (see Lumer [12]) on X consistent with the norm $\|\cdot\|$.

In [12], Lumer has shown that T is hermitian in $\|\cdot\|$ in the sense of Lumer if and only if T is hermitian in the sense of Vidav. Hence we write that T is hermitian in the norm $\|\cdot\|$ to mean hermiticity in either sense.

(1.1) VIDAV'S THEOREM. *Let A be a Banach algebra with the identity and with the norm $\|\cdot\|$. Let H be the set of elements of A which are hermitian in $\|\cdot\|$. If $A = H + iH$ and if for every $h \in H$, h^2 can be expressed in the form $h^2 = u + iv$ with u, v in H and $uv = vu$, then*

(i) *for each x in A, the decomposition $x = u + iv$ u, v in H, is unique;*

(ii) *the map $*$ which assigns to each element $x = u + iv$ (where u, v are in H) the element $x^* = u - iv$ is an involution on A; (we call x^* the Vidav adjoint of x);*

(iii) *$\|\cdot\|_0$, defined by $\|x\|_0 = \|x^*x\|^{1/2}$ is a Banach algebra norm on A equivalent to the given norm and moreover $\|h\|_0 = \|h\|$ for every $h \in H$;*

(iv) *the algebra A with the involution $*$ and the norm $\|\cdot\|_0$, is a B^*-algebra.*

DEFINITION 3. If a Banach algebra A with the norm $\|\cdot\|$ satisfies the hypothesis of Vidav's theorem (1.1) and is equipped with the involution $*$ defined in (ii) of (1.1), then we call A, following Berkson [5], a V^*-algebra in the norm $\|\cdot\|$.

The Vidav's theorem (1.1) has been sharpened recently by Berkson in [5] to the following form.

(1.2) A is a V^*-algebra in the norm $\|\cdot\|$ if and only if A is a B^*-algebra in the norm $\|\cdot\|$.

As pointed out to the author by Bade and Phillips, the above result of Berkson has the following important consequence for the theory of spectral operators.

THEOREM. (1.3) *Let $\mathfrak{B}$ be a bounded B.A. of projections on a Banach space X and let A be the uniformly closed Banach algebra generated by $\mathfrak{B}$. Then there exists an equivalent norm $|||\cdot|||$ on X (i.e., a norm $|||\cdot|||$ which is equivalent to the given norm of X) such that the norm of each operator in A, computed relative to $|||\cdot|||$, is equal to its spectral norm. Thus the Gelfand map of A is an isometric isomorphism onto $C(\mathfrak{m})$, $\mathfrak{m}$ the space of maximal ideals of A.*

Proof. Since $\mathfrak{B}$ is bounded, all the members of $\mathfrak{B}$ are hermitian in some equivalent norm $|||\cdot|||$ of X by remarks in § 3 of Lumer [13]. Hence all the members of the closure of the real linear span of $\mathfrak{B}$ are hermitian in $|||\cdot|||$. Now arguing as in the proof of Theorem 3.1 of Berkson [4], it can be shown that each T in A may be written as $T = R + iJ$, with R, J in A and hermitian in $|||\cdot|||$. Thus A is a V^*-algebra in the operator norm $|||\cdot|||$ induced by the Banach space norm $|||\cdot|||$ of X and hence by Berkson's result (1.2), the theorem follows.

For results on semi-groups of operators one may refer to Hille and Phillips [11].

2. **$W^*(||\cdot||)$-algebras.** This section deals with the theory of algebras of operators W on a Banach space X which are generated in the weak operator topology by a σ-complete Boolean algebra $\mathfrak{B}$ of projections. Such algebras are the natural generalization of abelian von Neumann algebras. The theory of such algebras is developed quite fully in [2], [3] and [9]. It is shown there that

(i) W is the algebra generated in the uniform operator topology by $\bar{\mathfrak{B}}^s$;

(ii) $\bar{\mathfrak{B}}^s$ is a complete B.A. of projections whose Stone representation space $\mathfrak{m}$ is the maximal ideal space of W;

(iii) The Gelfand map $A \to A(.)$ is an isomorphism of W onto $C(\mathfrak{m})$ and W and $C(\mathfrak{m})$ are topologically equivalent under this map;

(iv) Every operator S in W is scalar type of class X^*; $S = \int \lambda E(d\lambda)$, whose spectral projections $E(\sigma)$ belong to $\bar{\mathfrak{B}}^s \subseteq W$. Further, the space $\mathfrak{m}$ is extremally disconnected.

Here we make the additional assumption on W that the norm $||\cdot||$ on X is such that the operator norm on W is isometric to the supremum norm in $C(\mathfrak{m})$. Such an algebra is called here a $W^*(||\cdot||)$-algebra.

DEFINITION 4. By a $W^*(||\cdot||)$-algebra W on a Banach space X, we mean a pair, consisting of a commutative subalgebra W of $\mathfrak{B}(X)$ generated by a σ-complete B.A. of projections in X in the weak operator topology and some equivalent norm $||\cdot||$ on X such that every element S in W has the representation of the form $S = R + iJ$ where R and J satisfy the following conditions (V):

$$(V)\begin{cases} (\text{i}) & RJ = JR \text{ with } R \text{ and } J \text{ in } W; \\ (\text{ii}) & R^m J^n (m, n = 0, 1, 2 \cdots) \text{ are hermitian in the norm } ||\cdot|| . \end{cases}$$

We make the following observations in regard to a $W^*(||\cdot||)$-algebra.

REMARK 1. A $W^*(||\cdot||)$-algebra W on a Banach space X is precisely an abelian subalgebra of $\mathfrak{B}(X)$, which is a V^*-algebra in the operator norm $||\cdot||$, induced by the Banach space norm $||\cdot||$ on X together with the property that it is generated weakly by a σ-complete B.A. of projections in X.

REMARK 2. A $W^*(||\cdot||)$-algebra W is a commutative B^*-algebra in the operator norm $||\cdot||$ induced by the Banach space norm $||\cdot||$ of X and hence the Gelfand map is an isometric isomorphism of W onto the space $C(\mathfrak{m})$ of complex valued continuous functions, where $\mathfrak{m}$ is the maximal ideal space of W.

For, the above remark follows from the fact that a σ-complete B.A. of projections is bounded and from Theorem (1.3) of § 1.

REMARK 3. The Banach algebra W generated weakly by a σ-complete B.A. $\mathfrak{B}$ of projections on a Banach space X is a $W^*(||\cdot||)$-algebra, under a suitable equivalent norm $||\cdot||$ on X. If X is weakly complete, the hypothesis that $\mathfrak{B}$ is σ-complete may be replaced by the hypothesis that $\mathfrak{B}$ is bounded.

For, the σ-completeness of $\mathfrak{B}$ implies that $\overline{\mathfrak{B}}^s$ is complete and bounded. Hence the weakly closed algebra generated by $\mathfrak{B}$ coincides with the uniformly closed algebra generated by $\overline{\mathfrak{B}}^s$. Now the remark follows by appealing to Theorem (1.3) of § 1.

REMARK 4. An operator S on a Banach space X is scalar type if and only if it belongs to a $W^*(||\cdot||)$-algebra on X.

REMARK 5. If an operator S belongs to a $W^*(||\cdot||)$-algebra W, then S is scalar type and all its spectral projections are in W. Further every projection in W is hermitian in $||\cdot||$.

Now we shall show that the *-operation in a $W^*(||\cdot||)$-algebra is strongly continuous. Though this result is noted in [9] on page 544, the proof that we are giving here is based on the notions of hermiticity and semi-inner-product. In addition, this proof is more direct. To this end, we prove the following lemma.

LEMMA 1. *If E is a nonzero projection operator on a Banach space X then there is an equivalent norm $\|\cdot\|$ on X in which E is hermitian and the norm of E computed with respect to $\|\cdot\|$ is unity, i.e.,* $\|E\| = 1$.

Proof. The B.A. $\mathfrak{B}$ of projections, consisting of $0, I, E$ and $I - E$ is bounded and hence there is an equivalent norm $\|\cdot\|$ on X, in which the members of $\mathfrak{B}$ are hermitian. By Theorem 1.3 of § 1, the Banach algeba $\mathscr{E}$ generated by $\mathfrak{B}$ is a B^*-algebra in the operator norm induced by the Banach space norm $\|\cdot\|$ of X. Hence $\|E\| = \sup_{m \in \mathfrak{m}} E(m) = 1$ as $E \neq 0$.

THEOREM 1. *If W is a $W^*(\|\cdot\|)$-algebra on a Banach space X, then the involution * defined in W which makes it a V^*-algebra (see Definition 3 of § 1) is continuous in the strong operator topology.*

Proof. Since the maximal ideal space $\mathfrak{m}$ of W is extremally disconnected, every operator T in W admits a spectral representation of the form

$$T = \int_{\mathfrak{m}} T(m)E(dm) \tag{1}$$

where $T(m)$ is the Gelfand function associated with T and $E(.)$ is a strongly countably additive spectral measure, having its range in W. Further, for Borel sets σ of the maximal ideal space $\mathfrak{m}$, the projections $E(\sigma)$ are hermitian in $\|\cdot\|$ by Remark 5 and hence $\|E(\sigma)\| \leqq 1$ by Lemma 1.

Let T_α in W converge strongly to T. Since W is strongly closed, T belongs to W. Hence there exist operators R_α, J_α, R and J in W such that $T = R + iJ$, $T_\alpha = R_\alpha + iJ_\alpha$ and R_α, J_α and R, J satisfy conditions (V) of Definition 4. By following an argument similar to that of Bade (p. 408, [2]), which is available here in view of the spectral representation (1) of any operator T in W and the fact that $\|E(\sigma)\| \leqq 1$ for Borel sets σ of the maximal ideal space $\mathfrak{m}$, we can show that $\lim_\alpha R_\alpha$ and $\lim_\alpha J_\alpha$ exist in W in the strong operator topology. Let $\lim_\alpha R_\alpha x = R_1 x$ and $\lim_\alpha J_\alpha x = J_1 x$ for x in X.

Now we shall show that $R_1 = R$ and $J_1 = J$. If $[\ ,\ ]$ is a semi-inner-product on X consistent with the norm $\|\cdot\|$, then,

$$\text{(2)} \qquad \text{and} \quad \begin{matrix} [R_\alpha x, x] \to [R_1 x, x] \\ [J_\alpha x, x] \to [J_1 x, x] \end{matrix}$$

for $\|x\| = 1$. Since R_α, J_α are hermitian in $\|\cdot\|$, $[R_\alpha x, x]$ and $[J_\alpha x, x]$ are real for each α when $\|x\| = 1$ and hence $[R_1 x, x]$ and $[J_1 x, x]$ are real for $\|x\| = 1$ by (2). Thus R_1 and J_1 are hermitian in $\|\cdot\|$. Similary $R_1^m J_1^n (m, n = 0, 1, 2, \cdots)$ are hermitian in $\|\cdot\|$. Hence R_1 and J_1 satisfy conditions (V) of Definition 4. Clearly $Tx = R_1 x + iJ_1 x$ for $x \in X$. But $T = R + iJ$ by assumption, with R, J satisfying conditions (V). Hence from the uniqueness of the representation of T in W, it follows that $R = R_1$ and $J = J_1$.

Now $\lim_\alpha T_\alpha^* x = \lim_\alpha (R_\alpha - iJ_\alpha)x = (R_1 - iJ_1)x = (R - iJx) = T^*x$ for x in X. This establishes the strong continuity of the involution $*$.

In the rest of this section we generalize the notion of positivity of operators on a Hilbert space to operators on a Banach space. We recall that in a Hilbert space H, an operator T is called positive if $(Tx, x) \geqq 0$ for x in H, where $(,)$ is the inner-product of H. Also it is known there that T is positive if and only if $\sigma(T)$ is nonnegative.

DEFINITION 5. An operator T on a Banach space X is called positive in the equivalent norm $\|\cdot\|$ on X (which we denote by $T \geqq 0$ in $\|\cdot\|$) if $[Tx, x] \geqq 0$ for x in X, with $\|x\| = 1$, where $[,]$ is a semi-inner-product consistent with the norm $\|\cdot\|$ on X; i.e., if the numerical range $W(T)$ with respect to the semi-inner-product $[,]$ is nonnegative.

The above definition calls for several comments. Since there may be an infinite number of semi-inner-products consistent with a given norm, the definition looks ambiguous at first sight. But the ambiguity disappears in the light of Theorem 14 of Lumer [12], according to which the numerical range has the same convex hull relative to any two semi-inner-products inducing the same norm. It may be noted that this definition also coincides with the classical one in a Hilbert space.

LEMMA 2. *If E is a projection operator on a Banach space X and is hermitian in the equivalent norm $\|\cdot\|$ on X then E is positive in $\|\cdot\|$.*

Proof. The cases in which $E = 0$ or $E = I$ are trivial. Hence suppose $E \neq 0, I$. Then by Lemma 1 $\|E\| = 1$ and $\|I - E\| = 1$, since E and $I - E$ are hermitian in $\|\cdot\|$. Now for x in X with $\|x\| = 1$ and a semi-inner- product $[,]$ consistent with the norm $\|\cdot\|$, we have

$$\begin{aligned}[Ex, x] &= [\{I - (I - E)\}x, x] \\ &= 1 - [(I - E)x, x] \\ &\geqq 0\end{aligned}$$

as $[(I - E)x, x]$ is real and $|[(I - E)x, x]| \leqq \| I - E \| = 1$. Hence the lemma.

The above lemma on projections enables us to link the positivity of an operator in a $W^*(\|\cdot\|)$-algebra, with the nonnegativeness of its spectrum.

THEOREM 2. *If T is an operator belonging to a $W^*(\|\cdot\|)$-algebra W then the following are equivalent.*

(i) *$\sigma(T)$ is nonnegative.*

(ii) *The Gelfand function $T(m)$ in $C(\mathfrak{m})$ is nonnegative where $\mathfrak{m}$ is the maximal ideal space of W.*

(iii) *T is positive in $\|\cdot\|$.*

Proof. The equivalence of (i) and (ii) is clear from the results that the spectrum of T in W, viz. $\sigma_W(T)$, is the range of $T(m)$ and that $\sigma_W(T) = \sigma(T)$ (see Corollary 3.7.6 of Rickart [16]).

To prove the theorem, therefore it suffices to show that (i) and (iii) are equivalent. Let (i) hold. Then, as T belongs to the $W^*(\|\cdot\|)$-algebra W it is scalar type and its spectral projections are in W. Further they are hermitian in $\|\cdot\|$ by Remark 5. Hence, if $E(.)$ is the resolution of the identity of T, then $E(\sigma)$ are hermitian in $\|\cdot\|$ for Borel sets σ of the complex plane, so that $E(\sigma)$ are positive in $\|\cdot\|$ by Lemma 2. Now let $[\ ,\ ]$ be a semi-inner-product on X consistent with the norm $\|\cdot\|$. Then for x in X, with $\| x \| = 1$, $[E(.)x, x]$ is a positive measure and hence

$$\begin{aligned}[Tx, x] &= \left[\int_{\sigma(T)} \lambda E(d\lambda)x, x\right] \\ &= \int_{\sigma(T)} \lambda [E(d\lambda)x, x] \\ &\geqq 0\end{aligned}$$

as $\sigma(T)$ is nonnegative. Hence T is positive in $\|\cdot\|$; i.e., (iii) holds.

Conversely, let (iii) hold. Then as T is in W, T is scalar type. Hence by Theorem 5, §4, of Foguel [10], $\sigma(T) = \pi(T)$ where $\pi(T)$ is the approximate point spectrum of T. Since T is bounded, by Theorem 4 of Lumer [12], we have $\pi(T) \subseteqq \overline{W(T)}$ where $W(T)$ is the numerical range of T with respect to some semi-inner-product consistent with the norm $\|\cdot\|$. But by hypothesis $W(T)$ is nonnegative and hence $\overline{W(T)}$ is nonnegative. Hence $\sigma(T) = \pi(T)$ is nonnegative. Therefore (i) holds.

This completes the proof of the theorem.

DEFINITION 6. For two operators T, T' on a Banach space X we say T is greater than T' in the equivalent norm $||\cdot||$ on X (briefly $T \geqq T'$ in $||\cdot||$) if (i) T, T' are hermitian in $||\cdot||$ and (ii) $T - T'$ is positive in $||\cdot||$.

DEFINITION 7. A net $\{T_\alpha\}$ is said to be monotonic increasing (decreasing) in the equivalent norm $||\cdot||$ on X if $T_\alpha \geqq T_\beta (T_\beta \geqq T_\alpha)$ in $||\cdot||$ whenever $\alpha \geqq \beta$. In symbols we write this as $\{T_\alpha\}$ *m.i.* (*m.d.*) in $||\cdot||$.

We recall that in a Hilbert space, if $\{T_\alpha\}$ is a bounded monotonic net of commuting hermitian operators, then $\{T_\alpha\}$ converges strongly to a Hermitian operator. We generalize this result to Banach spaces below.

LEMMA 3. *Let $T_\alpha \downarrow 0$ be a monotonic decreasing net in a $W^*(||\cdot||)$-algebra W. Then $T_\alpha \to 0$ in the strong operator topology.*

Proof.[1] By Theorem 2, for elements S_1 and S_2 in W, $S_1 \geqq S_2$ in $||\cdot||$ if and only if $S_1(m) \geqq S_2(m)$ in $C(\mathfrak{m})$. Since $\mathfrak{m}$ is stonean, the hermitian elements of $C(\mathfrak{m})$ and hence of W form a conditionally complete lattice under this partial ordering.

For $\alpha \geqq \alpha_0$, $T_\alpha \leqq T_{\alpha_0}$ in $||\cdot||$. Hence $T_\alpha(m) \leqq T_{\alpha_0}(m)$ so that $|| T_\alpha|| = \sup | T_\alpha(m) | = \sup T_\alpha(m) \leqq \sup T_{\alpha_0}(m) = || T_{\alpha_0} ||$. Hence

$$|| T_\alpha || \leqq || T_{\alpha_0} || \tag{3}$$

for $\alpha \geqq \alpha_0$.

Let $x \in X$ and $\varepsilon > 0$. If $e_\alpha = \{m: T_\alpha(m) < \varepsilon\}$ then $\bigvee_\alpha E(e_\alpha) = I$ so that $E(e_\alpha) \to I$ strongly where $E(.)$ is a countably additive spectral measure on the Borel sets of $\mathfrak{m}$ with respect to which T_α have the spectral representation

$$T_\alpha = \int_{\mathfrak{m}} T_\alpha(m) E(dm) .$$

Thus for $x \in X$,

$$|| T_\alpha x || \leqq || T_\alpha E(e_\alpha) x || + || T_\alpha || \, || (I - E(e_\alpha)) x || < \varepsilon || x || + \varepsilon$$

for $\alpha \geqq \alpha_0$ in view of (3) and Lemma 1. Thus $T_\alpha \to 0$ strongly.

THEOREM 3. *Let $\{T_\alpha\}$ be a net in a $W^*(||\cdot||)$-algebra W of operators on a Banach space X such that*

(i) *$\{T_\alpha\}$ is monotonic in $||\cdot||$; and*

(ii) *for some R in W, $T_\alpha \leqq R$ in $||\cdot||$ if $\{T_\alpha\}$ is* m.i. *in $||\cdot||$ and*

[1] The present short proof is due to the referee to whom the author is thankful.

$T_\alpha \geqq R$ *in* $\|\cdot\|$ *if* $\{T_\alpha\}$ *is* m.d. *in* $\|\cdot\|$; *i.e., the net* $\{T_\alpha\}$ *is bounded. Then* $\lim_\alpha T_\alpha x$ *exists for each* x *in* X. *Further* $\lim_\alpha T_\alpha x = \bigvee_\alpha T_\alpha x$ $(\bigwedge_\alpha T_\alpha x)$ *if* $\{T_\alpha\}$ *is* m.i. (m.d.) *in* $\|\cdot\|$.

Proof. Without loss of generality we may assume that $\{T_\alpha\}$ is m.i. in $\|\cdot\|$. Since $T_\alpha \leqq R$ in $\|\cdot\|$ and since the hermitian elements in W form a conditionally complete lattice, it follows that $T_\alpha \uparrow \bigvee_\alpha T_\alpha$ in W. Hence $(\bigvee_\alpha T_\alpha - T_\alpha) \downarrow 0$. Then it follows from Lemma 3 that $\lim_\alpha T_\alpha x = \bigvee_\alpha T_\alpha x$ for x in X. Hence the theorem.

3. Unbounded spectral operators of scalar type. In this section we obtain some interesting results on unbounded spectral operators of scalar type, which will be needed in the sequel. For definitions and results on such operators which are used here, the reader may refer to Bade [1].

Let $\mathfrak{m}$ be a set and Σ be a σ-algebra of subsets of $\mathfrak{m}$. Let $E(.)$ be an X^*-countably additive spectral measure on Σ. Suppose f is a complex valued $E(.)$-essentially unbounded Σ-measurable function on $\mathfrak{m}$. Then we define a linear transformation $f(E)$ as below.

DEFINITION 8. Let $e_n = \{m: m \in \mathfrak{m}, |f(m)| \leqq n\}$. Then define f_n as follows.

$$f_n(m) = f(m),\ m \in e_n$$
$$= 0,\ m \notin e_n\ .$$

We define

$$D(f) = \left\{x: x \in X \text{ and } \lim_{n\to\infty} \int_{e_n} f(m)E(dm)x \text{ exists}\right\}$$

and

$$f(E)x = \lim_{n\to\infty} \int_{e_n} f(m)E(dm)x,\ x \in D(f)\ .$$

It is easy to check that $D(f)$ is a dense linear manifold in X and $f(E)$ is a linear transformation over $D(f)$ with its range in X.

LEMMA 4. *Let* $A(f)$ *be the set of members in* Σ *on which* f *is bounded. Then we have*:

(i) $A(f)$ *is closed under finite unions and contains any subset of its members, if the subset belongs to* Σ;

(ii) *If* $e \in A(f)$, *then* $E(e)X \subseteqq D(f)$ *and* $f(E)$ *is bounded in* $E(e)X$, *where* $D(f)$ *and* $f(E)$ *are as in Definition* 8;

(iii) $E(e)f(E)E(e) = f(E)E(e),\ e \in A(f)$;

(iv) $A(f)$ *contains an increasing sequence* $\{\delta_n\}$ *such that*

$$E(\textstyle\bigcup_{n=1}^\infty \delta_n) = I\ .$$

Proof. The statement (i) is obvious. The statement (iv) is clear, if we take for $\{\delta_n\}$ the sequence $\{e_n\}$ in Definition 8.

Since f is bounded on $e \in A(f)$ and since $\lim_{n\to\infty} E(e_n)x = x$ for $x \in X$, we have

$$\begin{aligned} f(E)E(e)x &= \lim_{n\to\infty}\int_{e_n} f(m)E(dm)E(e)x \\ &= \lim_{n\to\infty}\int_{e} f(m)E(dm)E(e_n)x \\ &= \int_e f(m)E(dm)x \,. \end{aligned} \tag{4}$$

Now from (4) the assertions (ii) and (iii) follow.

LEMMA 5. *Let $\{\delta_n\}$ be any other increasing sequence from $A(f)$ for which $E(\bigcup_{n=1}^{\infty}\delta_n) = I$. Then:*

(i) $\lim_{n\to\infty} f(E)E(e_n)x$ *exists if and only if x is in $D(f)$;*

(ii) *If $\lim_{n\to\infty} f(E)E(e_n)x$ exists, then*

$$\lim_{n\to\infty} f(E)E(e_n)x = \lim_{n\to\infty} f(E)E(\delta_n)x$$

and conversely;

(iii) *If in the definition of $f(E)$ the sequence $\{e_n\}$ is replaced by any other sequence $\{\delta_n\}$ in $A(f)$ such that $\delta_n \supseteqq \delta_{n-1}$ and $E(\bigcup_{n=1}^{\infty}\delta_n) = I$, we obtain the same linear transformation $f(E)$.*

Proof. Let x be in X. Then by relation (4) in the above, we have

$$f(E)E(e_n)x = \int_{e_n} f(m)E(dm)x$$

from which the assertion (i) of the lemma follows.

The statement (ii) of the lemma can be proved by following an argument similar to that of Lemma 2.1 of Bade [1].

Finally, to prove (iii) let x be in $D(f)$. Then by relation (4) and statement (ii) of the lemma we have

$$\begin{aligned} f(E)x &= \lim_{n\to\infty}\int_{e_n} f(m)E(dm)x \\ &= \lim_{n\to\infty} f(E)E(e_n)x \\ &= \lim_{n\to\infty} f(E)E(\delta_n)x \\ &= \lim_{n\to\infty}\Big(\lim_{k\to\infty}\int_{e_k} f(m)E(dm)\Big)E(\delta_n)x \\ &= \lim_{n\to\infty}\lim_{k\to\infty}\int_{e_k\cap\delta_n} f(m)E(dm)x \\ &= \lim_{n\to\infty}\lim_{k\to\infty} E(e_k)\int_{\delta_n} f(m)E(dm)x \end{aligned}$$

so that

$$f(E)x = \lim_{n\to\infty}\int_{\delta_n} f(m)E(dm)x\ . \tag{5}$$

Now the assertion (iii) is clearly a consequence of equation (5) and statements (i) and (ii) of the lemma. Hence the lemma.

As a consequence of the above lemmas, we prove the following main theorem of this section. We also remark that this theorem is a generalization of Lemma 6 of Dunford [8], to the unbounded case.

THEOREM 4. *Let f be a complex valued $E(\cdot)$-essentially unbounded Σ-measurable function on the set $\mathfrak{m}$, where $E(\cdot)$ is an X^*-countably additive spectral measure on Σ, a σ-algebra of subsets of $\mathfrak{m}$. Then:*

(i) *The set $D(f) = \left\{x\colon \lim_{n\to\infty}\int_{e_n} f(m)E(dm)x \text{ exists}\right\}$ is a dense linear manifold of X where*

$$e_n = \{m\colon m \in \mathfrak{m},\ |f(m)| \leqq n\}\ ;$$

(ii) *The operator $f(E)$ defined by*

$$f(E)x = \lim_{n\to\infty}\int_{e_n} f(m)E(dm)x$$

is an unbounded spectral operator of scalar type with domain $D(f)$ whose resolution of the identity is given by $E_f(\cdot)$ where

$$E_f(\sigma) = E(f^{-1}(\sigma))$$

for Borel sets σ of the complex plane;

(iii) *Any other increasing sequence $\{\delta_n\}$ in $A(f)$* (see Lemma 4 for definition of $A(f)$) *such that $E(\bigcup_{n=1}^{\infty}\delta_n) = I$ would also define the same linear transformation $f(E)$.*

Proof. The proof of (i) is trivial. The assertion (iii) follows from Lemma 5.

To prove (ii), let $E_f(\cdot) = E(f^{-1}(\cdot))$ on the family $\mathfrak{B}$ of Borel sets of the complex plane. $E_f(\cdot)$ is also an X^*-countably additive spectral measure. Defining f_n as in Definition 8, we have

$$\int_{e_n} f(m)E(dm) = \int_{\mathfrak{m}} f_n(m)E(dm) = \int_{\overline{f_n(\mathfrak{m})}} \lambda E_{f_n}(d\lambda) \tag{6}$$

by Lemma 6 of Dunford [8], where $E_{f_n}(\cdot) = E(f_n^{-1}(\cdot))$.

Now by Lemma 1, § 3, of Foguel [10], the operator $\int_{\overline{f_n(\mathfrak{m})}} \lambda E_{f_n}(d\lambda)$ belongs to the uniformly closed algebra generated by $E_{f_n}(\alpha)$ for Borel sets α of the complex plane such that $0 \notin \bar{\alpha}$. Let α be such a Borel set of $\overline{f_n(\mathfrak{m})}$. Since $\alpha \not\ni 0$

$$f_n^{-1}(\alpha) = \{m; f(m) \in \alpha \text{ and } m \in e_n\} \tag{7}$$

as $f_n = f$ on e_n. But, as $\overline{f_n(\mathfrak{m})} \subseteqq \{\lambda: |\lambda| \leqq n\}$, $\alpha \subseteqq \overline{f_n(\mathfrak{m})} \subseteqq \{\lambda: |\lambda| \leqq n\}$. Hence $f^{-1}(\alpha) \subseteqq e_n$. Thus

$$f^{-1}(\alpha) = \{m: f(m) \in \alpha \text{ and } m \in e_n\} . \tag{8}$$

Hence from (7) and (8) it follows that $f_n^{-1}(\alpha) = f^{-1}(\alpha)$ so that $E_{f_n}(\alpha) = E_f(\alpha)$. Hence by the above lemma of Foguel [10] we have

$$\begin{aligned}\int_{e_n} f(m)E(dm) &= \int_{\overline{f_n(\mathfrak{m})}} \lambda E_{f_n}(d\lambda)\\ &= \int_{\overline{f(e_n)}} \lambda E_{f_n}(d\lambda)\\ &= \int_{\overline{f(e_n)}} \lambda E_f(d\lambda)\end{aligned}$$

as $f = f_n$ on e_n and $f_n(m) = 0$ for $m \notin e_n$. Thus,

$$\begin{aligned} f(E)x &= \lim_{n\to\infty} \int_{e_n} f(m)E(dm)x\\ &= \lim_{n\to\infty} \int_{\overline{f(e_n)}} \lambda E_f(d\lambda)x .\end{aligned} \tag{9}$$

Clearly the left hand side and the right hand side of (9) exist if and only if x is in $D(f)$. Hence the right hand side of (9) defines the same operator $f(E)$.

Now as $\overline{f(e_n)} \subseteqq \{\lambda: |\lambda| \leqq n\}$, the sequence $\{\overline{f(e_n)}\}$ is a sequence of bounded closed sets in the complex plane. Clearly it is an increasing sequence. Also it is easy to check that $E_f(\bigcup_{n=1}^{\infty} \overline{f(e_n)}) = I$. Hence by Theorem 3.3. of Bade [1] and by the definition of scalar type operators (see p. 379, Bade [1]) it follows that the operator $f(E)$ defined by

$$f(E)x = \lim_{n\to\infty} \int_{\overline{f(e_n)}} \lambda E_f(d\lambda)x,\ x \in D(f)$$

is an unbounded spectral operator of scalar type with the resolution of the identity $E_f(\cdot)$.

This completes the proof of the theorem.

In Theorem 3.3 of Bade [1] the operator $f(S)$ (see p. 379 of Bade [1] for definition) is proved to be spectral and nothing has been said whether $f(S)$ is scalar type. But the above theorem asserts that $f(S)$ is scalar type and hence we state this result separately below.

COROLLARY. *If f is an $E(\cdot)$-essentially unbounded Borel measu-*

rable function over the complex plane where $E(\cdot)$ is an X^*-countably additive spectral measure over the Borel sets of the complex plane, then the operator $f(S)$ is an unbounded spectral operator of scalar type.

In [10] Foguel has proved that the residual spectrum of a (bounded) spectral operator of finite type is empty. In the following theorem we generalize this result to unbounded spectral operators of scalar type.

THEOREM 5. *Let S be an unbounded spectral operator of scalar type on a Banach space X, with its resolution of the identity $E(\cdot)$. Then a point l in the complex plane belongs to* (i) *the point spectrum $\sigma_P(S)$ if and only if $E(l) \neq 0$ and* (ii) *the continuous spectrum $\sigma_c(S)$ if and only if $l \in \sigma(S)$ and $E(l) = 0$. Consequently, the residual spectrum of S is empty.*

Proof.[2] Suppose $E(l) = 0$. Then the function $f(\lambda) = (\lambda - l)^{-1}$ is analytic and single valued in the complement of the single point closed set l for which $E(l) = 0$. Hence $f(\lambda) \in \mathfrak{R}$ (see p. 387 of Bade [1] for definition of $\mathfrak{R}$). Now taking

$$e_n = \left\{\lambda : |\lambda| \leqq n, \, dist. \; (\lambda, l) \geqq \frac{1}{n}\right\}$$

$\{e_n\}$ is an increasing sequence of bounded closed sets for which

$$E(\bigcup_{n=1}^{\infty} e_n) = I .$$

Therefore defining

$$f(S)x = \lim_{n\to\infty} \int_{e_n} f(\lambda)E(d\lambda)x \tag{10}$$

on the set $D(f(S))$ of x for which the limit in (10) exists, we see that $f(S)$ is a closed operator with its domain $D(f(S))$ dense in X.

Now for x in $D(f(S))$ we have

$$\begin{aligned}
&\int_{e_n} (\lambda - l)E(d\lambda)f(S)x \\
&= \int_{e_n} (\lambda - l)E(d\lambda) \lim_{k\to\infty} \int_{e_k} f(\lambda)E(d\lambda)x \\
&= \lim_{k\to\infty} \int_{e_n} (\lambda - l)E(d\lambda)\left[\int_{e_n} f(\lambda)E(d\lambda)x + \int_{e_k - e_n} f(\lambda)E(d\lambda)x\right] \\
&= \int_{e_n} (\lambda - l)f(\lambda)E(d\lambda)x = E(e_n)x \, .
\end{aligned}$$

[2] The method we adopt here is an extension of that on p. 325 of Stone [**17**] for maximal normal operators on Hilbert spaces.

Hence for x in $D(f(S))$ we have

$$(S - lI)f(S)x = \lim_n E(e_n)x = x \,. \tag{11}$$

Similarly we have for x in $D(S) = D(S - lI)$, $f(S)(S - lI)x = x$. Thus $f(S)$ is the inverse $(S - lI)^{-1}$. From (11) and that $D(f(S))$ is dense in X it follows that the range of $S - lI$ is dense in X. Hence l is not in the residual spectrum of S. Also as $(S - lI)^{-1}$ exists, l is not in the point spectrum $\sigma_P(S)$. Thus if l belongs to $\sigma(S)$ and $E(l) = 0$ then $l \in \sigma_c(S)$. This proves the direct part of assertion (ii). The converse part of (ii) clearly follows from the result (i), which we shall now prove.

Suppose $l \in \sigma_P(S)$. If $E(l) = 0$ then from the above, we must have l in the continuous spectrum $\sigma_c(S)$ which is a contradiction as

$$\sigma_P(S) \cap \sigma_c(S) = \varphi \,.$$

Hence $E(l) \neq 0$.

Conversely, suppose $E(l) \neq 0$. Clearly $l \in \sigma(S)$. Since $E(l) \neq 0$ there is a vector $x_0 \in X$ such that $E(l)x_0 \neq 0$. Then

$$SE(l)x_0 = \lim_{n\to\infty} \int_{\sigma_n} \lambda E(d\lambda)E(l)x_0$$

where $\sigma_n = \{\lambda : |\lambda| \leqq n\}$. Therefore

$$\begin{aligned} SE(l)x_0 &= \lim_{n\to\infty} \int_{\sigma_n \cap \{l\}} \lambda E(d\lambda)x_0 \\ &= \int_{\{l\}} \lambda E(d\lambda)x_0 = lE(l)x_0 \end{aligned}$$

and since $E(l)x_0 \neq 0$, l is in $\sigma_P(S)$.

This completes the proof of the theorem.

The following theorem on $W^*(\|\cdot\|)$-algebras which plays a key role in the spectral representation of semi-groups of scalar type operators, is a consequence of the preceding results.

THEOREM 6. *Let W be a $W^*(\|\cdot\|)$-algebra. If $\mathfrak{m}$ is the maximal ideal space of W, then there exists an X^*-countably additive spectral measure $P(\cdot)$ on the Borel sets Σ of $\mathfrak{m}$ such that each bounded Borel measurable function f on $\mathfrak{m}$ corresponds to a unique operator F in W, the correspondence being given by*

$$F = \int_{\mathfrak{m}} f(m)d_mP(E)^3$$

[3] $\int_{\mathfrak{m}} f(m)P(dm)$ is denoted by $\int_{\mathfrak{m}} f(m)d_mP(E)$ hereafter following the notation of Hille and Phillips [**11**].

the above integral existing in the uniform operator topology.

Also if f is a $P(\cdot)$-essentially unbounded Borel measurable function on $\mathfrak{m}$ and if $\{e_n\}$ is an increasing sequence of Borel sets of $\mathfrak{m}$ on which f is bounded and if $P(\bigcup_{n=1}^{\infty} e_n) = I$ then the set $D(f)$ of all x of X for which

$$f(P)x = \lim_{n\to\infty}\int_{e_n} f(m)d_m P(E)x \tag{12}$$

exists, is dense in X and the transformation $f(P)$ defined by (12) *is an unbounded spectral operator of scalar type with domain $D(f)$. Consequently, the residual spectrum of $f(P)$ is empty.*

REMARK 6. If the Borel measurable function f in the above theorem is real valued, an equivalent representation for $f(P)$ in (12) is obtained as follows.

If $E_\lambda = \{m: f(m) \leqq \lambda\}$, then denote the projection $P(E_\lambda)$ by $P(\lambda)$. The family of projections $[P(\lambda): -\infty < \lambda < \infty]$ which generates the resolution of the identity $P_f(\cdot)$ of $f(P)$ (see Theorem 4) has the following properties (ψ).

$$(\psi)\begin{cases}(\mathrm{i}) & \bigwedge P(\lambda) = 0,\ \bigvee P(\lambda) = I\,;\\ (\mathrm{ii}) & P(\lambda)P(\mu) = P(\lambda) \text{ for } \lambda \leqq \mu\,;\\ (\mathrm{iii}) & P(\lambda) = \bigwedge_{\mu>\lambda} P(\mu)\,.\end{cases}$$

Also the property (iii) is equivalent to

(iii)′ $P(\lambda)x = \lim_{\mu\to\lambda+} P(\mu)x$, $x \in X$ in view of Lemma 2.3 of Bade [3].

Further, by Theorem 4 the equation (12) can be written as

$$f(P)x = \lim_{n\to\infty}\int_{\sigma_n} \lambda dP_f(\lambda)x,\ x \in D(f)$$

where $P_f(\cdot) = P(f^{-1}(\cdot))$ and $\{\sigma_n\}$ is any increasing sequence of bounded Borel sets of the real line such that $P_f(\bigcup_{n=1}^{\infty} \sigma_n) = I$.

Since the resolution of the identity $P_f(\cdot)$ of $f(P)$ is generated by the family $[P(\lambda): -\infty < \lambda < \infty]$, (12) is written as

$$f(P)x = \int_{-\infty}^{\infty} \lambda dP(\lambda)x,\ x \in D(f)\,. \tag{12′}$$

4. Semi-groups of real scalar type operators. In this section we obtain the spectral representation of a strongly measurable (and hence strongly continuous) semi-groups of real scalar type operators. We also obtain an ergodic theorem for such semi-groups of operators. Henceforth we closely follow the notations of Hille and Phillips [11] and often omit details of analogous proofs.

DEFINITION 9. Let $\mathfrak{S} \equiv [T(\xi): \xi > 0]$ be a semi-group of operators on X. Then by X_0 we denote the set $\bigcup_{\xi>0} T(\xi)X$. (see p. 307 of [11]).

THEOREM 7. *Let $\mathfrak{S} \equiv [T(\xi): \xi > 0]$ be a strongly measurable semi-group of real scalar type operators of class X^* on X and let the members of $\mathfrak{S}$ belong to a $W^*(||\cdot||)$-algebra W. Then $\mathfrak{S}$ is of finite type ω_0 (say) and $|| T(\xi) || = \exp(\omega_0\xi)$.* Also,

$$\lim_{\xi\to 0^+} T(\xi)x = Jx, x \in X \tag{13}$$

where J is a projection with its range $\mathfrak{R}(J) = \bar{X}_0$ and

$$T(\xi)J = JT(\xi) = T(\xi)$$

for all $\xi > 0$.

Finally, $T(\xi)$ has a holomorphic extension $T(\tau)$ having either the whole plane or the right half-plane as its maximal domain of analytic existence and there exists a unique representation of $T(\tau)$ of the form

$$T(\tau)x = \int_{-\infty}^{\omega_0} e^{\lambda\tau} dP(\lambda)x, x \in X .$$

Here $[P(\lambda)]$ generates the resolution of the identity relative to $\bar{X}_0$ for the infinitesimal operator A_0 of $\mathfrak{S}$, viz.,

$$A_0 x = \int_{-\infty}^{\omega_0} \lambda dP(\lambda)x \tag{14}$$

where $D(A_0)$, the domain of A_0, coincides with the set of all $x \in \bar{X}_0$ for which the integral in (14) exists. Also $D(A_0)$ is dense in $\bar{X}_0$ and A_0 is an unbounded spectral operator of real scalar type on $\bar{X}_0$.

Proof. That the semi-group $\mathfrak{S}$ is of finite type ω_0 follows from the facts that (i) W is an abelian B^*-algebra in the norm $||\cdot||$ (ii) $|| T(\xi) ||$ is lower-semi-continuous and (iii) $\mathfrak{S}$ is a nontrivial semi-group. Also it follows that $|| T(\xi) || = e^{\omega_0\xi}$.

Since the maximal ideal space $\mathfrak{M}$ of W is extremally disconnected and $|| T(\xi) || = e^{\omega_0\xi}$ the argument on p. 589 of Hille and Phillips [11] can be applied verbatim, taking W for $\mathfrak{B}$ there. Defining $\mathfrak{W}, \mathfrak{U}$ and $\alpha(m)$ in the same way as in [11] and putting $P(\lambda) = P(E_\lambda) = P\{m: \alpha(m) \leqq \lambda\}$ it is easy to see that $[P(\lambda)]$ generates a resolution of the identity relative to the subspace $\mathfrak{R}[P(\mathfrak{W})]$; i.e., $P(\lambda)$ satisfies conditions (ψ) in § 3 except that $\bigvee P(\lambda) = P(\mathfrak{W}) = J$. J may differ from I. Then arguing as in [11] we have

$$T(\xi)x = \int_{\mathfrak{M}} T(\xi)(m) d_m P(E)x = \int_{-\infty}^{\omega_0} e^{\lambda\xi} dP(\lambda)x \ . \tag{15}$$

Consequently,

$$x^* e^{-\omega_0 \xi} T(\xi)x = \int_{-\infty}^{0} e^{\lambda\xi} x^* dP(\lambda + \omega_0 \xi)x$$

for each $x \in X$ and $x^* \in X^*$. Since $x^* P(\lambda)x$ is continuous in the right for fixed x^* and x, by arguing in the same way as on p. 590 of [11] it can be shown that $[P(\lambda)]$ is uniquely determined by $\mathfrak{S}$. Now

$$\begin{aligned}
\left\| \int_{-\infty}^{\beta} e^{\lambda\xi} dP(\lambda)x \right\| &= \left\| \int_{\mathfrak{W}_\beta} e^{\xi\alpha(m)} d_m P(E)x \right\| \\
&= \left\| \int_{\mathfrak{W}} e^{\xi\alpha(m)} d_m P(E) P(\mathfrak{W}_\beta)x \right\| \\
&= \left\| \int_{\mathfrak{M}} T(\xi)(m) d_m P(E) P(\mathfrak{W}_\beta)x \right\|
\end{aligned}$$

where $\mathfrak{W}_\beta = \{m : m \in \mathfrak{M}, \alpha(m) \leqq \beta\}$.
Hence by Lemma 6 of Dunford [8] and Lemma 1 of § 2,

$$\begin{aligned}
\left\| \int_{-\infty}^{\beta} e^{\lambda\xi} dP(\lambda)x \right\| &\leqq 4 \sup_{m \in \mathfrak{M}} | T(\xi)(m) | \, \| P(\mathfrak{W}_\beta)x \| \\
&= 4 e^{\omega_0 \xi} \| P(\mathfrak{W}_\beta)x \| \\
&< 4 \max \{1, e^{\omega_0 t}\} \| P(\beta)x \| \\
&< \varepsilon
\end{aligned}$$

for ξ in $[0, t]$ if $\beta < -N(\varepsilon)$, as $\| P(\beta)x \| \to 0$ as $\beta \to -\infty$. Hence $\lim_{\beta \to -\infty} \int_{-\infty}^{\beta} e^{\lambda\xi} dP(\lambda)x = 0$ uniformly for ξ in $[0, t]$ so that the double limit

$$\lim_{\substack{\xi \to 0^+ \\ \beta \to -\infty}} \int_{\beta}^{\omega_0} e^{\lambda\xi} dP(\lambda)x$$

exists and hence the two iterated limits exist and are equal. Thus

$$\begin{aligned}
\lim_{\xi \to 0^+} T(\xi)x &= \lim_{\xi \to 0^+} \lim_{\beta \to -\infty} \int_{\beta}^{\omega_0} e^{\lambda\xi} dP(\lambda)x \\
&= \lim_{\beta \to -\infty} \lim_{\xi \to 0^+} \int_{\beta}^{\omega_0} e^{\lambda\xi} dP(\lambda)x = Jx
\end{aligned}$$

for $x \in X$. Hence the equation (13).

The argument on p. 590 in [11] holds here to show that $\mathfrak{R}(J) = \bar{X}_0$ and $JT(\xi) = T(\xi)J = T(\xi)$. Further, the part concerning the holomorphic extension can be proved arguing in the same way as in [11], replacing $(P(\lambda)x, y)$ there by $x^* P(\lambda)x$, $x \in X$ and $x^* \in X^*$.

Define the operator A as follows.

$$Ax = \lim_{n\to\infty}\int_{e_n} \alpha(m)d_m P(E)Jx \tag{16}$$

where $e_n = \{m: m \in \mathfrak{W}, |\alpha(m)| \leqq n\}$, for all x in $\bar{X}_0$ for which the limit in (16) exists. Since $P(\bigcup_{n=1}^{\infty} e_n)J = P(\mathfrak{W})J = J$, A is an unbounded spectral operator of real scalar type on $\bar{X}_0$ and

$$Ax = \int_{-\infty}^{w_0} \lambda dP(\lambda)x \tag{16$'$}$$

and $[P(\lambda)]$ generates the resolution of the identity of A with respect to $\bar{X}_0$ (see Remark 6 of § 3). Further the set $D(A)$ of all x in $\bar{X}_0$ for which the limit in (16) exists is dense in $\bar{X}_0$ by Theorem 4 of § 3.

If A_0 is the infinitesimal generator of $\mathfrak{S}$ with domain $D(A_0)$ then arguing as in [11] it can be shown that $D(A_0) = D(A)$ and $A_0 = A$. Hence from (16′), the equation (14) of the theorem follows.

This completes the proof of the theorem.

LEMMA 6. *Let $\mathfrak{S} \equiv [T(\xi): \xi > 0]$ be a semi-group of real scalar type operators belonging to a $W^*(||\cdot||)$-algebra W on X. Let $||T(\xi)||$ be bounded on every compact subset of $(0, \infty)$. Then $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$.*

Proof. The argument on p. 591 of [11] in the proof of Lemma 22.3.2 holds here verbatim because of the representation theory in Theorem 7. Also defining $S(\xi) = e^{-\omega_0\xi}T(\xi)$ as on p. 591 of [11] we see that $S(\xi)(m)$ is continuous and nonincreasing in ξ for each $m \in \mathfrak{M}$. Thus, as the set of hermitian elements in $\mathfrak{M}$ is a conditionally complete lattice,

$$\begin{aligned}\lim_{\xi\to\xi_0^-} S(\xi)(m) &= \bigwedge_{\xi<\xi_0} S(\xi)(m) = S(\xi_0)(m)\\ &= \bigvee_{\xi>\xi_0} S(\xi)(m) = \lim_{\xi\to\xi_0^+} S(\xi)(m)\ .\end{aligned}$$

Consequently by Theorem 3, § 2, it follows that $\lim_{\xi\to\xi_0^-} S(\xi)x = \bigwedge_{\xi<\xi_0} S(\xi)x = S(\xi_0)x$ and $\lim_{\xi\to\xi_0^+} S(\xi)x = \bigvee_{\xi>\xi_0} S(\xi)x = S(\xi_0)x$. Hence $S(\xi)$ and therefore $T(\xi)$ is strongly continuous at $\xi = \xi_0 > 0$. Hence the lemma.

The above lemma is applied to prove an ergodic theorem for semi-groups of real scalar type operators.

THEOREM 8. *Let $\mathfrak{S} \equiv [T(\xi): \xi > 0]$ be a semi-group of real scalar type operators of class X^* on X and let the members of $\mathfrak{S}$ belong to a $W^*(||\cdot||)$-algebra W. Let $\mathfrak{S}$ be of type ω_0 and $\omega_0 \leqq 0$.*

Then

$$\lim_{\xi_n\to\infty} T(\xi_n)x = [P(0) - P(0^-)]x,\ x \in X$$

where $[P(\lambda)]$ *generates the resolution of the identity relative to* $\bar{X}_0$ *for the infinitesimal operator* A_0 *of* $\mathfrak{S}$.

Proof. Since $T(\xi)$ are in the $W^*(||\cdot||)$-algebra W $||T^n(\xi)|| = ||T(\xi)||^n$. Hence arguing as on p. 592 of [11] we have

$$||T(\xi)|| = e^{\omega_0\xi} \leqq 1$$

for all ξ in $(0, \infty)$ as $\omega_0 \leqq 0$. Thus $||T(\xi)||$ is uniformly bounded in $(0, \infty)$ and hence by Lemma 6 $\mathfrak{S}$ is continuous in the strong operator topology, for $\xi > 0$. Now as $\mathfrak{S}$ satisfies the hypothesis of Theorem 7, making use of the representation given in that theorem we have

$$T(\xi)x = \int_{-\infty}^{0} e^{\lambda\xi} dP(\lambda)x,\ x \in X .$$

Let $P = P(0) - P(0^-) = P\{m: m \in \mathfrak{W}$ and $\alpha(m) = 0\}$ in the terminology of the proof of Theorem 7. Then we have

$$\begin{aligned} T(\xi)x - Px &= \int_{\mathfrak{W}} e^{\xi\alpha(m)} d_m P(E)x - \int_{\mathfrak{W}_P} d_m P(E)x \\ &= \int_{-\infty}^{0-} e^{\lambda\xi} dP(\lambda)x \\ &= \lim_{\beta\to 0-} \int_{-\infty}^{\beta} e^{\lambda\xi} dP(\lambda)x \end{aligned}$$

where $\mathfrak{W}_P = \{m: m \in \mathfrak{W}$ and $\alpha(m) = 0\}$. Now if $f_n(\lambda) = e^{\lambda\xi_n}$ where $\{\xi_n\}$ is an increasing sequence of positive numbers tending to ∞ and λ ranges in $(-\infty, \beta)$, $\beta < 0$, then $f_n(\lambda) \to 0$ as $\xi_n \to \infty$ for each λ. Further $f_1(\lambda) > f_n(\lambda)$. Hence by the generalized Lebesgue bounded convergence theorem on vector valued measures

$$\lim_{n\to\infty} \left|\left| \int_{-\infty}^{\beta} e^{\lambda\xi_n} dP(\lambda)x \right|\right| = 0 . \tag{17}$$

Thus, in view of (17), Lemma 6 of Dunford [8] and Lemma 1 of § 2 we have

$$\begin{aligned} &\limsup_{\xi_n\to\infty} \left|\left| T(\xi_n)x - Px \right|\right| \\ &\leqq \limsup_{\xi_n\to\infty} \left(\left|\left| \int_{-\infty}^{\beta} e^{\lambda\xi_n} dP(\lambda)x \right|\right| + \left|\left| \int_{\beta}^{0-} e^{\lambda\xi_n} dP(\lambda)x \right|\right| \right) \\ &= \limsup_{\xi_n\to\infty} \left|\left| \int_{\beta}^{0-} e^{\lambda\xi_n} dP(\lambda)x \right|\right| \\ &\leqq \limsup_{\xi_n\to\infty} 4 \sup_{\lambda\in[\beta,0)} e^{\lambda\xi_n} \left|\left| P(0^-)x - P(\beta)x \right|\right| \\ &= 4\, ||(P(0^-) - P(\beta))x|| \\ &< \varepsilon \end{aligned}$$

if β is sufficiently near to zero from below. Hence the theorem.

5. Semi-groups of scalar type operators. In this section we study the spectral representation of semi-groups of scalar type operators when the spectra of the members are not necessarily real.

DEFINITION 10. We shall donote by $\mathfrak{R}_1$[4] the set $\{R(\lambda, A): \lambda \in \rho(A)\}$ where A is the infinitesimal generator of a semi-group $\mathfrak{S}$ and $\rho(A)$ is the resolvent set of A.

LEMMA 7. *If $\mathfrak{S}$ is a semi-group of class A* (see p. 321 of [11]) *then the commutant of $\mathfrak{S}$ contains the set $\mathfrak{R}_1$ and $\mathfrak{S}^c = \mathfrak{R}_1^c$ where $\mathfrak{S}^c$ and $\mathfrak{R}_1^c$ are the commutants of $\mathfrak{S}$ and $\mathfrak{R}_1$ respectively.*

Proof. By Theorem 16.2.1 of [11], $\mathfrak{S}^c = \mathfrak{R}_1^c$. Since $\mathfrak{R}_1$ is abelian, $\mathfrak{R}_1 \subseteqq \mathfrak{R}_1^c = \mathfrak{S}^c$. Hence the lemma.

LEMMA 8. *Let $Q(\cdot)$ be an X^*-countably additive spectral measure over the Borel sets of the complex plane and let $Q(\Delta) = I$ where $\Delta = \{\lambda: \operatorname{Re}\lambda \leqq \omega_0\}$. Suppose A is a closed operator having its spectrum contained in Δ. If μ is a complex number such that* $\operatorname{Re}\mu > \omega_0$ *and if*

$$R(\mu, A) = \int_\Delta (\mu - \lambda)^{-1} d_\lambda Q(E)$$

then

$$Ax = \int_\Delta \lambda d_\lambda Q(E)x \tag{18}$$

(*the integral in* (18) *being understood as* $\lim_{n\to\infty}\int_{\sigma_n} \lambda d_\lambda Q(E)x$ *where $\{\sigma_n\}$ is an increasing sequence of Borel sets of Δ such that $Q(\bigcup_{n=1}^\infty \sigma_n) = I$*). *Further the integral in* (18) *exists if and only if x belongs to the domain of A. Further A is a spectral operator of scalar type with $Q(\cdot)$ as its resolution of the identity and is unbounded if $\sigma(A)$ is unbounded.*

Proof. Since $\operatorname{Re}\mu > \omega_0$, $g(\lambda) = (\mu - \lambda)^{-1}$ is a bounded measurable function defined on the set Δ and hence by Lemma 6 of Dunford [8]

$$R(\mu, A) = \int_\Delta (\mu - \lambda)^{-1} d_\lambda Q(E)$$

is a bounded scalar type operator of class X^* with its resolution of the identity given by

[4] However, in [11] this is denoted by $\mathfrak{R}$.

$$Q_g(\cdot) = Q(g^{-1}(\cdot)) \,. \tag{19}$$

Define $f(\lambda) = \mu - 1/\lambda$. Then let

$$f(R(\mu, A))x = \lim_{n\to\infty} \int_{e_n} f(\lambda) d_\lambda Q_g(E)x$$

where $e_n = \{\lambda : |\lambda| \leqq n, \text{dist.}\,(\lambda, 0) \geqq (1/n)\}$. Then as

$$Q_g(0) = Q\{\lambda : g(\lambda) = 0\} = Q(\varphi) = 0$$

the increasing sequence $\{e_n\}$ of bounded Borel sets of the complex plane is such that $Q_g(\bigcup_{n=1}^{\infty} e_n) = I$. Hence by Corollary under Theorem 4 of § 3 it follows that $f(R(\mu, A))$ is a spectral operator of scalar type.

Since $f(\lambda) = (\mu - 1/\lambda)$ it is clear that $f(R(\mu, A)) = f((\mu I - A)^{-1}) = A$. Thus

$$Ax = \lim_{n\to\infty} \int_{e_n} f(\lambda) d_\lambda Q_g(E)x$$

the limit existing if and only if $x \in D(A)$, where $D(A)$ denotes the domain of A.

The resolution of the identity $F(\cdot)$ of A is given by

$$F(\cdot) = Q_g(f^{-1}(\cdot)) \tag{20}$$

so that from (19) and (20) it follows that

$$F(\cdot) = Q(\cdot) \,. \tag{21}$$

Thus from the relation (21) we have

$$\begin{aligned} Ax &= \lim_{n\to\infty} \int_{e_n} f(\lambda) d_\lambda Q_g(E)x \\ &= \lim_{n\to\infty} \int_{\overline{f(e_n)}} \lambda d_\lambda Q_{gf}(E)x \\ &= \lim_{n\to\infty} \int_{\overline{f(e_n)}} \lambda d_\lambda Q(E)x \end{aligned}$$

where $Q_{gf}(\cdot) = Q_g(f^{-1}(\cdot))$. Now replacing in the above $\overline{f(e_n)}$ by $\overline{f(e_n)} \cap \Delta = \sigma_n$, it is easy to check that $\{\sigma_n\}$ is an increasing sequence of bounded Borel sets such that $Q(\bigcup_{n=1}^{\infty} \sigma_n) = I$ and hence

$$Ax = \int_{\Delta} \lambda d_\lambda Q(E)x \,.$$

Since $\sigma(A) \subseteqq \Delta$, λ is $Q(\cdot)$-essentially unbounded if $\sigma(A)$ is unbounded and hence in this case A is an unbounded spectral operator of scalar type.

This completes the proof of the lemma.

LEMMA 9. *Let $\mathfrak{S} \equiv [S(\xi): \xi > 0]$ be a strongly measurable semi-group of scalar type operators of class X^* on X and let the members of $\mathfrak{S}$ belong to a $W^*(||\cdot||)$-algebra W. Then $\mathfrak{S}$ is of finite type ω_0 and*

$$\lim_{\xi \to 0+} S(\xi)x = Jx,\ x \in X \tag{22}$$

where J is a projection such that $JX = \bar{X}_0$ and

$$JS(\xi) = S(\xi)J = S(\xi) \tag{23}$$

for all $\xi > 0$. If $\bar{X}_0 = X$ then $\mathfrak{S}$ is (c_0)-summable.

Proof. Since W is an abelian B^*-algebra, and $\mathfrak{S}$ is strongly continuous, as it is strongly measurable, the argument on p. 594 of [11] holds here to show that

$$||\, S(\xi)\, || = e^{\omega_0 \xi}$$

so that except in the trivial case $\mathfrak{S} \equiv (0)$, ω_0 is finite.

Now let $S^*(\xi)$ be the Vidav adjoint of $S(\xi)$ in W. Since the $*$-operation in W is strongly continuous by Theorem 1 of § 2, $\mathfrak{S}^* \equiv [S^*(\xi): \xi > 0]$ is also a strongly continuous semi-group of scalar type operators of class X^*.

Let $T(\xi) = S^*((\xi/2))S((\xi/2))$. $T(\xi)$ is scalar type of class X^* as $T(\xi) \in W$. Since $||\, T(\xi)\, ||$ is uniformly bounded in any finite interval in $(0, \infty)$, from Lemma 6 of § 4 it follows that $T(\xi)$ is also strongly continuous for $\xi > 0$. Since $T(\xi)$ belongs to W, by Theorem 7 we have a projection J on X such that

$$JX = \overline{\bigcup_{\xi > 0} T(\xi)X}$$

and

$$JT(\xi) = T(\xi)J = T(\xi)$$

for all $\xi > 0$. Further J is in W.

Now if $\mathfrak{M}$ is the maximal ideal space of W, then for $m \in \mathfrak{M}$

$$|\, T(\xi)(m)\, | = S^*\Big(\frac{\xi}{2}\Big)(m) S\Big(\frac{\xi}{2}\Big)(m) = \Big| S\Big(\frac{\xi}{2}\Big)(m)\Big|^2$$

so that

$$(T(\xi)J)(m) = T(\xi)(m) = \Big| S\Big(\frac{\xi}{2}\Big)(m)\Big|^2 . \tag{24}$$

From (24) it follows that $S((\xi/2))(m) = 0$ if $J(m) = 0$. If $J(m) \neq 0$ then $J(m) = 1$ so that

$$S\left(\frac{\xi}{2}\right)(m) = S\left(\frac{\xi}{2}\right)(m)J(m) .$$

Hence in all cases we have

$$(S(\xi)J)(m) = S(\xi)(m) = (JS(\xi))(m)$$

from which the relation (23) follows.

From (23) we have $\bar{X}_0 \subseteqq \Re(J)$. Since

$$T(\xi)x = S\left(\frac{\xi}{2}\right)S^*\left(\frac{\xi}{2}\right)x$$

$$\bigcup_{\xi>0} T(\xi)X \subseteqq \bigcup_{\xi>0} S(\xi)X = \bar{X}_0$$

and hence we have

$$\overline{\bigcup_{\xi>0} T(\xi)X} = \Re(J) \subseteqq \bar{X}_0 .$$

Hence $\Re(J) = \bar{X}_0$.

The part of the lemma concerning the relation (22) can be proved by arguing as on pp. 317–18 of [11].

When $\bar{X}_0 = X$ then $J = I$ and hence $\mathfrak{S}$ is (c_0)-summable.

This concludes the proof of the lemma.

THEOREM 9. *Let $\mathfrak{S} \equiv [S(\xi): \xi > 0]$ be a strongly measurable semi-group of scalar type operators of class X^*. Let the members of $\mathfrak{S}$ belong to a $W^*(\|\cdot\|)$-algebra W. If $\bar{X}_0 = X$, then $\mathfrak{S}$ is (c_0)-summable and is of finite type ω_0 (say).*

Further suppose $\Re_1 \subseteqq W$ (see Definition 7 for $\Re_1$). *Then there exists a unique integral representation of $\mathfrak{S}$ in the form*

$$S(\xi)x = \int_\Delta e^{\lambda\xi} d_\lambda Q(E)x,\ x \in X$$

where $Q(\cdot)$ is the resolution of the identity on the Borel sets of the half plane $\Delta = \{\lambda: \mathrm{Re}\,\lambda \leqq \omega_0\}$ for the infinitesimal generator A, where A is given by

$$Ax = \int_\Delta \lambda d_\lambda Q(E)x \tag{25}$$

and the domain $D(A)$ of A is precisely the set of those x in X for which the right hand side of (25) *exists.* (The integral in (25) is to be understood in the sense given in Lemma 8). *Also A is an unbounded spectral operator of scalar type.*

Proof. From Lemma 9 it follows that $\mathfrak{S}$ is (c_0)-summable. Since

W is a $W^*(\|\cdot\|)$-algebra, it is a B^*-algebra and it contains $\mathfrak{S} \cup \mathfrak{R}_1$ by hypothesis. Also the spectrum of any element in W with respect to $\mathfrak{B}(X)$ is the same as that with respect to W. Hence by the remark following Theorem 16.2.2 of [11] the algebra W can be substituted for $\mathfrak{B}$ throughout the Chapter XVI of [11]. Arguing in the same way as on p. 595 of [11] and noting that $\| S(\xi) \| = e^{\omega_0 \xi}$ so that ω_0 is finite unless $\mathfrak{S} \equiv (0)$ and replacing $(P\psi(a_n)x, y)$ there by $x^*P\psi(a_n)x$ for x in X and x^* in X^* so that the last line on p.595 of [11] becomes

$$0 = x^*P\psi(a_n)x = P'x^*\psi(a_n)x \longrightarrow P'x^*x$$

as $n \longrightarrow \infty$ (where P' is the adjoint operator of P) and hence $0 = x^*Px$ for $x \in X$ and $x^* \in X^*$, we see that the projection P defined on p. 595 of [11] becomes the zero operator. Also arguing exactly in the same way as on p. 596 of [11] we obtain

$$S(\xi) = \int_{\Delta} e^{\lambda\xi} d_\lambda Q(E)$$

the integral converging in the uniform operator topology and $\Delta = \{\lambda : \operatorname{Re} \lambda \leqq \omega_0\}$.

Now since $\mathfrak{S}$ is (c_0)-summable, the resolvent of the infinitesimal generator A, which is closed, at the point μ $(\operatorname{Re} \mu > \omega_0)$ is given by

$$R(\mu, A)x = \int_0^\infty e^{-\mu\xi} S(\xi) d(\xi)x, \ x \in X .$$

Therefore

$$x^*R(\mu, A)x = \int_0^\infty e^{-\mu\xi}\left[\int_{\Delta} e^{\lambda\xi} x^* d_\lambda Q(E)x\right] d\xi$$

and interchanging the order of integration we get

$$x^*R(\mu, A)x = \int_{\Delta} (\mu - \lambda)^{-1} x^* d_\lambda Q(E)x, \ x \in X \text{ and } x^* \in X^* . \tag{26}$$

Since $(\mu - \lambda)^{-1}$ is a bounded Borel measurable function over Δ, $\int_{\Delta} (\mu - \lambda)^{-1} d_\lambda Q(E)$ exists in the uniform operator topology and hence from (26) we have

$$R(\mu, A) = \int_{\Delta} (\mu - \lambda)^{-1} d_\lambda Q(E) .$$

Now all the conditions of Lemma 9 are satisfied by $\mu, \Delta, Q(\cdot)$ and A and hence by Lemma 9 the relation (25) follows, with the domain of A being precisely the set of those x in X for which the right hand side of (25) exists.

Also $\sigma(A) = \alpha(\mathfrak{W})$ (see p. 596 of [11] for the definitions of $\alpha(m)$

and $\mathfrak{W}$) is unbounded if $\mathfrak{S}$ is strongly continuous and not uniformly continuous. Hence, again by Lemma 9 we conclude that A is an unbounded spectral operator of scalar type with $Q(\cdot)$ as its resolution of the identity.

For the uniqueness of the resolution of the identity of A the same argument in [11] holds here if we replace $(N(\xi)x, N(\eta)y)$ there by $x^*S^*(\eta)S(\xi)x$ where $S^*(\eta)$ is the Vidav adjoint of $S(\eta)$ in W and $x \in X$ and $x^* \in X^*$.

This completes the proof of the theorem.

6. A generalization of Stone's theorem.

In this section we give a generalization of Stone's theorem on one parameter group of unitary operators on Banach spaces, when the group is strongly measurable and its operators belong to a $W^*(||\cdot||)$-algebra. In [14] we gave the generalized notion of a unitary operator on a Banach space and obtained there an extension of Stone's theorem when the group was uniformly continuous and the underlying Banach space was reflexive.

DEFINITION 11. A spectral operator U of class X^* is said to be unitary in the equivalent norm $||\cdot||$ of X if U is an onto isometry in the norm $||\cdot||$.

LEMMA 10. *Let $\mathfrak{S} \equiv [U(\xi): \xi > 0]$ be a strongly measurable semi-group of scalar type operators of class X^* and let the members of $\mathfrak{S}$ belong to a $W^*(||\cdot||)$-algebra W. Let Σ be the* B.A. *of all projections in W. Then $U(\xi)$ are invariant on $\mathfrak{M}(x)$ for each $x \in X$ where $\mathfrak{M}(x)$ is the closed subspace spanned by the set $\{Ex: E \in \Sigma\}$. If the restriction of $U(\xi)$ on $\mathfrak{M}(x)$ be denoted by $U_x(\xi)$ then $\mathfrak{S}_x \equiv [U_x(\xi): \xi > 0]$ is again a semi-group of scalar type operators of class $\{\mathfrak{M}(x)\}^*$ on $\mathfrak{M}(x)$ and is strongly continuous on $\mathfrak{M}(x)$. If A_x is the infinitesimal generator of $\mathfrak{S}_x$ and A is that of $\mathfrak{S}$ then*

$$A/\mathfrak{M}(x) = A_x .$$

Proof. If $E_\xi(\cdot)$ is the resolution of the identity of $U(\xi)$ then it is easy to verify that $U(\xi)$ is invariant on $\mathfrak{M}(x)$ and $E_{\xi x}(\cdot) = E_\xi(\cdot)/\mathfrak{M}(x)$ is the resolution of the identity of the restriction $U_x(\xi)$ on $\mathfrak{M}(x)$. Also $U_x(\xi)$ is a scalar type operator on $\mathfrak{M}(x)$ of class $\{\mathfrak{M}(x)\}^*$ as the set function $E_{\xi x}(\cdot)y$, $y \in \mathfrak{M}(x)$ is countably additive. The strong continuity of $\mathfrak{S}_x$ is a consequence of that of $\mathfrak{S}$. It is easy to establish that A_x is the restriction of A to $\mathfrak{M}(x)$. Hence the lemma.

THEOREM 10. (A generalization of Stone's theorem on one-parameter group of unitary operators to Banach spaces). *Let*

$$\mathfrak{S} \equiv [U(\xi): -\infty < \xi < \infty]$$

be a group of scalar type operators of class X^* on X. Let $U(\xi)$ belong to a $W^*(\|\cdot\|)$-algebra W and let their spectrum lie on the unit circle. Further let $\mathfrak{S}$ be strongly measurable.

Then $\mathfrak{S}$ is continuous in the strong operator topology and $U(\xi)$ are unitary in the norm $\|\cdot\|$. Further there exists a unique representation for $U(\xi)$ of the form

$$U(\xi)x = \int_{-\infty}^{\infty} e^{i\lambda\xi} d_\lambda Q'(E)x,\ x \in X$$

with $Q'(E) = Q\{\lambda: i\lambda \in E\}$ for Borel sets E of the complex plane where $Q(\cdot)$ is the resolution of the identity for the infinitesimal generator A given by

$$Ax = i\int_{-\infty}^{\infty} \lambda d_\lambda Q'(E)x \tag{27}$$

and the domain $D(A)$ of A is precisely the set of all x in X for which the integral in (27) *exists.*

Proof. That $U(\xi)$ is strongly continuous follows from the hypothesis that $U(\xi)$ is strongly measurable. Since $U(\xi)$ belongs to the $W^*(\|\cdot\|)$-algebra W and since the spectrum of $U(\xi)$ lies on the unit circle it is easy to show that $U(\xi) = e^{iR_\xi}$ for some operator R_ξ in W, which is hermitian in the norm $\|\cdot\|$. Hence $\| U(\xi) \| = \| e^{iR_\xi} \| = 1$ for $-\infty < \xi < \infty$ by Lemma 1 of Vidav [18]. Also

$$\| U^{-1}(\xi) \| = \| U(-\xi) \| = 1$$

and hence $U(\xi)$ are onto isometry in $\|\cdot\|$. But, as $U(\xi)$ are scalar type $U(\xi)$ are unitary in $\|\cdot\|$.

All the conditions of Theorem 9 will be satisfied by $\mathfrak{S}$ if we show that $\mathfrak{R}_1 \subseteqq W$. In view of Theorems 4.3 and 4.5 of Bade [3] to prove $\mathfrak{R}_1 \subseteqq W$ it suffices to show that the resolvent operator $R(\lambda, A)$ of the infinitesimal generator A leaves invariant each $\mathfrak{M}(x)$ for $x \in X$, where $\mathfrak{M}(x)$ is the closed subspace spanned by $\{Ex: E \in \Sigma\}$, Σ being the B.A. of all projections in W. Now to prove this, let $U_x(\xi)$ be the restriction of $U(\xi)$ on $\mathfrak{M}(x)$. Then as said in the proof of Lemma 10, $U_x(\xi)$ are scalar type operators on $\mathfrak{M}(x)$, with their resolutions of the identity being given by $E_{\xi x}(\cdot) = E_\xi(\cdot)/\mathfrak{M}(x)$. Hence it will follow that $U_x(\xi)$ are onto isometries of $\mathfrak{M}(x)$ in the norm $\|\cdot\|$. Therefore, the set $\Lambda = \{\lambda: \mathrm{Re}\,\lambda \neq 0\}$ is dense in the resolvent set $\rho(A_x)$ of the infinitesimal generator A_x of $\mathfrak{S}_x \equiv [U_x(\xi): -\infty < \xi < \infty]$. Thus for a $\lambda \in \Lambda$, $(\lambda I - A_x)^{-1}$ is an everywhere defined operator on $\mathfrak{M}(x)$, as A_x is closed. As $\mathfrak{S}$ is a group of isometries on X, Λ is also dense in the resolvent set $\rho(A)$

and hence for $\lambda \in \Lambda$, $(\lambda I - A)^{-1}$ exists as an everywhere defined operator on X and is an extension of $(\lambda I - A_x)^{-1}$ because of Lemma 10. Thus $(\lambda I - A)^{-1} = (\lambda I - A_x)^{-1}$ on $\mathfrak{M}(x)$. But $(\lambda I - A_x)^{-1}\mathfrak{M}(x) \subseteqq \mathfrak{M}(x)$ so that $(\lambda I - A)^{-1}\mathfrak{M}(x) \subseteqq \mathfrak{M}(x)$. Since Λ is dense in $\rho(A)$, it follows that $R(\lambda, A)\mathfrak{M}(x) \subseteqq \mathfrak{M}(x)$ for all $\lambda \in \rho(A)$.

Thus the semi-group $\mathfrak{S}^1 \equiv [U(\xi): \xi > 0]$ satisfies the hypothesis of Theorem 9 and hence the proof of Theorem 9 holds verbatim. As $U(\xi)$ are in W with their spectrum on the unit circle, it follows that the range of $U(\xi)(m)$ is the unit circle so that $|U(\xi)(m)| = 1$ for $m \in \mathfrak{M}$, where $\mathfrak{M}$ is the maximal ideal space of W. Thus the function $\alpha(m)$ defined in the proof of Theorem 9 is purely imaginary and $\alpha(\mathfrak{W}) = \sigma(A)$ reduces to a subset of the imaginary axis. Hence we have

$$U(\xi)x = \int_{\Delta} e^{\lambda\xi} d_\lambda Q(E)x \tag{28}$$
$$= \int_{-\infty}^{\infty} e^{i\lambda\xi} d_\lambda Q'(E)x$$

where $Q'(E) = Q[\lambda: i\lambda \in E]$ and

$$Ax = \int_{\Delta} \lambda' d_{\lambda'} Q(E)x$$
$$= i\int_{-\infty}^{\infty} \lambda d_\lambda Q'(E)x \ .$$

Further the domain of A is precisely the set of those x in X for which the integral in (27) exists. Also $Q(\cdot)$ is the resolution of the identity of A.

For $\xi < 0$,

$$U(\xi) = U^*(-\xi) \text{ (Vidav adjoint of } U(-\xi))$$
$$= \left[\int_{-\infty}^{\infty} e^{-i\lambda\xi} d_\lambda Q'(E)\right]^*$$
$$= \int_{-\infty}^{\infty} e^{i\lambda\xi} d_\lambda Q'(E)$$

as the spectral projections $Q'(E)$ are in W and hence are hermitian in $||\cdot||$ by Remark 5 of § 2. For $\xi = 0$ the representation clearly holds. This completes the proof of the theorem.

REMARK 7. In all the above theorems on semi-groups of operators, the hypothesis that the members of the semi-group belong to a $W^*(||\cdot||)$-algebra can be replaced by an equivalent hypothesis that the B.A. Σ determined by the resolutions of the identity of the members of the semi-group is σ-complete. When the Banach space X is weakly complete, it suffices to assume that Σ is bounded. Here in Theorem 10 we have proved the extension of Stone's theorem to arbitrary Banach spaces. On the other hand in [6] Berkson has obtained the extension but for weakly complete Banach spaces with a weaker hypothesis that there is a uniform bound on the resolutions of the identity of all operators to $U(\xi)$. The fact that these different resolutions of the identity generate a bounded B.A. is obtained as a corollary of his method of proof.

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CONCERNING THE INFINITE DIFFERENTIABILITY OF SEMIGROUP MOTIONS

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Let S be a real Banach space. Let C denote the infinitesimal generator of a strongly continuous semigroup T of bounded linear transformations on S. This paper presents a construction which proves that for each $b > 1$ there is a dense subset $D(b)$ of S so that if p is in $D(b)$, then

(A) p is in the domain of C^n for all positive integers n and

(B) $\lim_{n\to\infty} \| C^n p \| (n!)^{-b} = 0$.

Condition (B) will be used in § 3 to obtain series solutions to the partial differential equations $U_{12} = CU$ and $U_{11} = CU$.

Suppose G is a strongly continuous one-parameter group of bounded linear transformations on S which has the property that there is a positive number K so that $|G(x)| < K$ for all numbers x. Let A denote the infinitesimal generator of G. In 1939, Gelfand [1] presented a construction which showed there is a dense subset R of S so that if p is in R, then

(C) p is in the domain of A^n for all positive integers n and

(D) $\lim_{n\to\infty} \| A^n p \| (n!)^{-1} = 0$.

Hille and Phillips, in their work on Semigroups [2], used Gelfand's construction to prove there is a dense subset R of S which satisfies condition (A) with respect to the operator C. Hille and Phillips, however, do not present estimates on the size of $\| C^n p \|$. Also, this author has not been able to use their construction to obtain estimates on the size of $\| C^n p \|$.

2. Infinite differentiability of semigroup motions.

Let $b > 1$. Let a be a number so that $1 < a < b$. Let M be a positive number so that $|T(X)| < M$ for all nonnegative numbers x less than or equal $\sum_{n=1}^{\infty} n^{-a}$. For each point p in the domain of C (denoted by D_C) and each positive integer n, let $p(n+1, n) = p$. For each point p in D_C and each pair (k, n) of positive integers so that $k \leqq n$, let

$$p(k, n) = k^a \int_0^{k^{-a}} du\, T(u) p(k+1, n) \, .$$

THEOREM 1. *Suppose p is in D_C and each of k and n is a positive integer. Then*

$$\| p(k, k+n-1) \| \leqq M \| p \| \, .$$

Proof. Let $w = \prod_{j=0}^{n-1} (k+j)^a$. For each nonnegative integer j, let $r(j) = (k + j)^{-a}$. Then

$$\begin{aligned}
&|| p(k, k + n - 1) || \\
&= w \left|\left| \int_0^{r(0)} du_0 T(u_0) \int_0^{r(1)} du_1 T(u_1) \cdots \int_0^{r(n-1)} du_{n-1} T(u_{n-1}) p \right|\right| \\
&= w \left|\left| \int_0^{r(0)} du_0 \int_0^{r(1)} du_1 \cdots \int_0^{r(n-1)} du_{n-1} T(u_0 + u_1 + \cdots + u_{n-1}) p \right|\right| < M || p || .
\end{aligned}$$

THEOREM 2. *Suppose p is in D_C and k is a positive integer. Then*

$$|| p(k, k) - p || \leqq M || Cp || k^{-a} .$$

Proof. Theorem 2 follows from the definition of $p(k, k)$ and the fact that $T(x)p - p = \int_0^x du T(u) Cp$ for all $x > 0$.

THEOREM 3. *Suppose p is in D_C and each of k and n is a positive integer. Then*

$$|| p(k, k + n) - p(k, k + n - 1) || \leqq M^2 || Cp || (k + n)^{-a} .$$

Proof. Let w and $r(j)$ be defined as in the proof of Theorem 1. Then

$$\begin{aligned}
&|| p(k, k + n) - p(k, k + n - 1) || \\
&= (k + n)^a w \left|\left| \int_0^{r(0)} du_0 T(u_0) \cdots \int_0^{r(n-1)} du_{n-1} T(u_{n-1}) \left[\int_0^{r(n)} du_n (T(u_n) p - p) \right] \right|\right| \\
&= (k + n)^a w \left|\left| \int_0^{r(0)} du_0 \cdots \int_0^{r(n-1)} du_{n-1} T(u_0 + \cdots + u_{n-1}) \right.\right. \\
&\qquad \left.\left. \left[\int_0^{r(n)} du_n (T(u_n) p - p) \right] \right|\right| < M^2 || Cp || (k + n)^{-a} .
\end{aligned}$$

COROLLARY. *Suppose p is in D_C and k is a positive integer. Then the sequence*

$$S(p, k): p(k, k), p(k, k + 1), p(k, k + 2) ,$$

converges in S.

Proof. Theorem 3 and the fact that $\sum_{n=0}^{\infty} (k + n)^{-a}$ converges imply $S(p, k)$ is a cauchy sequence in S. Since S is complete, $S(p, k)$ will converge.

For each point p in D_C and each positive integer k, let the sequential limit point of $S(p, k)$ be denoted by p_k. Let

$$D(b): \{p_k \mid p \text{ is in } D_C \text{ and } k \text{ is a positive integer}\} .$$

THEOREM 4. *Suppose p_k is in $D(b)$. Then $p_k \leqq M || p ||$.*

Proof. Theorem 4 follows from Theorem 1 and the fact that p_k is the sequential limit point of $S(p, k)$.

THEOREM 5. *$D(b)$ is a dense subset of S.*

Proof. Suppose q is in S and q is not in $D(b)$. Let $\varepsilon > 0$. Since D_C is a dense subset of S, there is a point p in D_C so that

(1) $\| p - q \| < \varepsilon/3$.

Theorem 2 implies there is a positive integer k so that

(2) $\| p(k, k) - p \| < \varepsilon/3$ and

(3) $(M + 1)^2 \| Cp \| \sum_{n=0}^{\infty} (k + n)^{-a} < \varepsilon/3$.

Theorem 2, Theorem 3 and statement (3) imply there is a p_k in $D(b)$ so that

(4) $\| p_k - p(k, k) \| < \varepsilon/3$.

Statements (1), (2) and (4) imply $\| p_k - q \| < \varepsilon$. Thus, $D(b)$ is a dense subset of S.

THEOREM 6. *Suppose p_k is in $D(b)$. Then*

$$p_k = k^a \int_0^{k-a} du\, T(u) p_{k+1} \, .$$

Proof. Let $\varepsilon > 0$. Then there is a positive integer n so that

(1) $\| p(k, k + n) - p_k \| < \varepsilon/2$ and

(2) $\| p(k + 1, k + n) - p_{k+1} \| < \varepsilon/2M$.

Statement (2) implies

(3) $\left\| p(k, k + n) - k^a \int_0^{k-a} du\, T(u) p_{k+1} \right\| < \varepsilon/2$.

Theorem 6 now follows from statements (1) and (3).

THEOREM 7. *The elements of $D(b)$ satisfy conditions* (A) *and* (B).

Proof. Suppose p_k is an element of $D(b)$. Theorem 6 implies p_k is in the domain of C^n for all positive integers n and that

(1) $C^n p_k = \prod_{j=0}^{n-1} (k + j)^a \prod_{j=0}^{n-1} [T(1/(k + j)^a) - I] p_{k+n}$.

Thus, the elements of $D(b)$ satisfy condition (A). Statement (1) and Theorem 2 imply

(2) $\| C^n p_k \| \leqq [\prod_{j=0}^{n-1} (k + j)^a](M + 1)^{n+1} \| p \|$.

Statement (2) implies p_k satisfies condition (B). The proof of Theorem 7 is now complete.

3. Partial differential equations in a banach space. The results of § 2 will be used in this section to obtain series solutions to the partial differential equations $U_{12} = CU$ and $U_{11} = CU$. Solutions to these equations may be easily obtained if C is a bounded linear transformation. The transformation C, however, may be unbounded; that is, C may be discontinuous at each point where it is defined.

For each subset D of S, let $P(D)$ denote the set of all functions g for which there is a nonnegative integer n and a sequence $p_0, p_1, \cdots, p_n$ each term of which is in D so that

$$g(x) = \sum_{i=0}^{n} x^i p_i$$

if $x \geqq 0$. If D is a dense subset of S, it may be shown that $P(D)$ is a dense subset of the set of continuous functions from $[0, d](d > 0)$ to S.

THEOREM 8. *Let $d > 0$. Let b be a number so that $1 < b < 2$. Suppose each of g and h is a function in $P(D(b))$ so that $g(0) = h(0)$. Then there is a function U from $[0, d] \times [0, d]$ to S so that*

(i) $U_{12}(x, y) = CU(x, y)$ *for all* (x, y) in $[0, d] \times [0, d]$,

(ii) $U(x, 0) = g(x)$ *for all x in* $[0, d]$ *and*

(iii) $U(0, y) = h(y)$ *for all y in* $[0, d]$.

Proof. Suppose n is a nonnegative integer and $p_0, p_1, \cdots, p_n$ is a sequence each term of which is in $D(b)$ so that

$$g(x) = \sum_{i=0}^{n} x^i p_i$$

if $x \geqq 0$. Suppose m is a nonnegative integer and $q_0, q_1, \cdots, q_m$ is a sequence each term of which is in $D(b)$ so that

$$h(y) = \sum_{i=0}^{n} y^i q_i$$

if $y \geqq 0$. Let U be the function from $[0, d] \times [0, d]$ to S so that if (x, y) is in $[0, d] \times [0, d]$, then

$$\begin{aligned}(1)\quad U(x, y) = {} & \textstyle\sum_{i=1}^{n} x^i p_i + \sum_{i=0}^{m} y^i q_i \\ & + \textstyle\sum_{i=1}^{n} \sum_{k=1}^{\infty} (xy)^k x^i C^k p_i/(k!)(i+1) \cdots (i+k) \\ & + \textstyle\sum_{i=0}^{m} \sum_{k=1}^{\infty} (xy)^k y^i C^k q_i/(k!)(i+1) \cdots (i+k).\end{aligned}$$

Theorem 7 implies U is well defined on $[0, d] \times [0, d]$. Theorem 7 and the fact that C is a closed transformation imply $U_{12}(x, y) = CU(x, y)$ for all (x, y) in $[0, d] \times [0, d]$. Statement (1) implies $U(x, 0) = g(x)$ and $U(0, y) = h(y)$ for all (x, y) in $[0, d] \times [0, d]$.

THEOREM 9. *Let $d > 0$. Let b be a number so that $1 < b < 2$. Suppose each of g and h is a function in $P(D(b))$. Then there is a function U from $[0, d] \times [0, d]$ to S so that*

(i) $U_{11}(x, y) = CU(x, y)$ *for all* (x, y) *in* $[0, d] \times [0, d]$,

(ii) $U(0, y) = g(y)$ *if y is in* $[0, d]$ *and*

(iii) $U_1(0, y) = h(y)$ *if* y *is in* $[0, d]$.

Proof. Let each of g and h be defined as in the proof of Theorem 8. Then let U be the function from $[0, d] \times [0, d]$ to S so that for each (x, y) in $[0, d] \times [0, d]$,

$$(1) \quad U(x, y) = \sum_{i=0}^{n} y^i p_i + x \sum_{i=0}^{m} y^i q_i + \sum_{i=0}^{n} \sum_{k=1}^{\infty} x^{2k} y^i C^k p_i/((2k)!)(i+1) \cdots (i+k) + \sum_{i=0}^{m} \sum_{k=1}^{\infty} x^{2k+1} y^i C^k q_i/((2k+1)!)(i+1) \cdots (i+k).$$

An argument analogous to that used in Theorem 8 may be used to show U is well defined on $[0, d] \times [0, d]$ and that U satisfies conditions (i), (ii) and (iii) in the hypothesis of this theorem.

REMARKS. (1) The solution U to the Theorem 8 has the property that for each (x, y) in $[0, d] \times [0, d]$, is in the domain of C^n for all positive integers n. The same remark is true for the solution to the equation in Theorem 9.

(2) Theorem 5 implies there are solutions to $U_{12} = CU$ and $U_{11} = CU$ for a set of boundary functions which is dense in the set of continuous functions from $[0, d]$ to S.

(3) Theorem 9 and Theorem 5 imply there are solutions to the ordinary differential equation $y'' = Cy$ for a dense set of initial values for $y(0)$ and $y'(0)$.

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A NOTE ON CERTAIN DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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In this paper an exact solution is obtained for the dual series equations

$$(1)\qquad \sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\alpha+n+1)}L_n^{(\alpha)}(x)=f(x)\,,\qquad 0\leqq x<y\,,$$

$$(2)\qquad \sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\alpha+\beta+n)}L_n^{(\sigma)}(x)=g(x)\,,\qquad y<x<\infty\,,$$

where $\alpha+\beta+1>\beta>1-m$, $\sigma+1>\alpha+\beta>0$, m **is a positive integer,**

$$L_n^{(\alpha)}(x)=\binom{\alpha+n}{n}{}_1F_1[-n;\alpha+1;x]\,,$$

is the Laguerre polynomial and $f(x)$ **and** $g(x)$ **are prescribed functions.**

The method used is a generalization of the multiplying factor technique employed by Lowndes [4] to solve a special case of the above equations when

$$\sigma=\alpha,\ A_n=\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n)C_n,\ \alpha+\beta>0\quad\text{and}\quad 1>\beta>0\,.$$

In another paper by the present author [5] equations (1) and (2) have been solved by considering separately the equations when (i) $g(x)\equiv 0$, (ii) $f(x)\equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation. Indeed it is easy to verify that the solution obtained earlier [5] is in complete agreement with the one given in this paper.

2. The following results will be required in the analysis.

(i) The orthogonality relation for Laguerre polynomials given by [3, p. 292 (2)] and [3, p. 293 (3)]:

$$(3)\qquad \int_0^{\infty}e^{-x}x^{\alpha}L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)dx=\frac{\Gamma(\alpha+n+1)}{n!}\delta_{mn},\ \alpha>-1\,,$$

where δ_{mn} is the Kronecker delta.

(ii) The formula (27), p. 190 of [2] in the form:

$$(4)\qquad \frac{d^m}{dx^m}\{x^{\alpha+m}L_n^{(\alpha+m)}(x)\}=\frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)}x^{\alpha}L_n^{(\alpha)}(x)\,.$$

(iii) The following forms of the known integrals [2, p. 191 (30)] and [3, p. 405 (20)]:

$$(5)\qquad \int_0^\xi x^\alpha(\xi-x)^{\beta-1}L_n^{(\alpha)}(x)dx = \frac{\Gamma(\alpha+n+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+n+1)}\xi^{\alpha+\beta}L_n^{(\alpha+\beta)}(\xi)\ ,$$

where $\alpha > -1, \beta > 0$, and

$$(6)\qquad \int_\xi^\infty e^{-x}(x-\xi)^{\beta-1}L_n^{(\alpha)}(x)dx = \Gamma(\beta)e^{-\xi}L_n^{(\alpha-\beta)}(\xi)\ ,$$

where $\alpha + 1 > \beta > 0$.

3. **Solution of the equations.** Multiplying equation (1) by $x^\alpha(\xi-x)^{\beta+m-2}$, where m is a positive integer, equation (2) by $e^{-x}(x-\xi)^{\sigma-\alpha-\beta}$, and integrating with respect to x over $(0, \xi)$, (ξ, ∞) respectively we find, on using (5) and (6), that

$$(7)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n)} L_n^{(\alpha+\beta+m-1)}(\xi)\\ &\quad= \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)}\int_0^\xi x^\alpha(\xi-x)^{\beta+m-2}f(x)dx\ ,\end{aligned}$$

where $0 < \xi < y, \alpha > -1, \beta + m > 1$, and

$$(8)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n)} L_n^{(\alpha+\beta-1)}(\xi)\\ &\quad= \frac{e^\xi}{\Gamma(\sigma-\alpha-\beta+1)}\int_\xi^\infty e^{-x}(x-\xi)^{\sigma-\alpha-\beta}g(x)dx\ ,\end{aligned}$$

where $y < \xi < \infty, \sigma + 1 > \alpha + \beta > 0$.

If we now multiply equation (7) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ and use the formula (4) we see that it becomes

$$(9)\qquad \begin{aligned}&\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n)} L_n^{(\alpha+\beta-1)}(\xi)\\ &\quad= \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta+m-1)}\frac{d^m}{d\xi^m}\int_0^\xi x^\alpha(\xi-x)^{\beta+m-2}f(x)dx\ ,\end{aligned}$$

where $0 < \xi < y, \alpha > -1$, and $\beta + m > 1$.

The left-hand sides of equations (8) and (9) are now identical and an application of the orthogonality relation (3) yields the solution of equations (1) and (2) in the form

$$(10)\qquad \begin{aligned}A_n &= \frac{n!}{\Gamma(\beta+m-1)}\int_0^y e^{-\xi}L_n^{(\alpha+\beta-1)}(\xi)F(\xi)d\xi\\ &\quad+ \frac{n!}{\Gamma(\sigma-\alpha-\beta+1)}\int_y^\infty \xi^{\alpha+\beta-1}L_n^{(\alpha+\beta-1)}(\xi)G(\xi)d\xi\ ,\\ &\quad n = 0, 1, 2, 3, \cdots\ ,\end{aligned}$$

where

$$F(\xi) = \frac{d^m}{d\xi^m}\int_0^\xi x^\alpha(\xi - x)^{\beta+m-2}f(x)dx \tag{11}$$

and

$$G(\xi) = \int_\xi^\infty e^{-x}(x - \xi)^{\sigma-\alpha-\beta}g(x)dx\ , \tag{12}$$

provided that $\alpha + \beta + 1 > 1 - m$ and $\sigma + 1 > \alpha + \beta > 0$, m being a positive integer.

When $\sigma = \alpha$, $A_n = \Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n)C_n$, the above equations provide the solution to Lowndes' equations for

$$\alpha + \beta > 0, 1 > \beta > 1 - m\ ,$$

and when $m = 1$ the results are in complete agreement (see [4], p. 124). Note also that the dual equations considered recently by Askey [1, p. 683, Th. 3] are essentially the same as Lowndes' equations.

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A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS

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Let A and B be associative algebras and define the Frattini subalgebra of $A, \phi(A)$, to be the intersection of all maximal subalgebras of A if maximal subalgebras of A exist and as A otherwise. Conditions on B will be found such that B cannot be an ideal of A contained in $\phi(A)$.

Hobby in [2] has shown that a nonabelian group G cannot be the Frattini subgroup of any p-group if the center of G is cyclic. Chao in [1] has shown that a nonabelian Lie algebra L can not be the Frattini subalgebra of any nilpotent Lie algebra if the center of L is one dimensional. In this note, we find a similiar result in the theory of associative algebras. However, in this case, it is not necessary to place any restrictions on the containing algebra.

Let A be an associative algebra over a field F and let B be an ideal of A. If $x \in A$, then x induces an endomorphism of the additive group of B by $L_x(b) = xb$ for all $b \in B$. Let $E(B, A)$ be the collection of all endomorphisms of this type. Then $E(B, A)$ is a subspace of the vector space of all linear transformations from B into B and is an associative algebra under the compositions $L_x + L_y = L_{x+y}, \alpha L_x = L_{\alpha x}$ and $L_x L_y = L_{xy}$ for all $x, y \in A$ and all $\alpha \in F$. Clearly $E(B, B)$ is an ideal of $E(B, A)$. If C is an ideal of A contained in B, then let $E(B, A, C) = \{E \in E(B, A); E(c) = 0 \text{ for all } c \in C\}$. Then $E(B, A, C)$ is an ideal of $E(B, A)$ and $E(B, A)/E(B, A, C)$ is isomorphic to $E(C, A)$. Note that the mapping from A onto $E(B, A)$ which assigns to each $a \in A$ the element L_a is an algebra homomorphism. We define the right annihilating series of B inductively. Let $r_1(B) = \{c \in B; bc = 0$ for all $b \in B\}$ and let $r_j(B)$ be the ideal of B such that $r_j(B)/r_{j-1}(B)$ $r_1(B/r_{j-1}(B))$ for $j > 1$. Since B is an ideal in $A, r_i(B)$ is an ideal in A for all i.

The following lemma is immediate.

LEMMA. *If A and A' are associative algebras and π is a homomorphism from A onto A', then $\pi(\phi(A)) \subseteq \phi(\pi(A))$. Furthermore, if the kernel of π is contained in $\phi(A)$, then $\pi(\phi(A)) = \phi(\pi(A))$.*

THEOREM. *Let B be an associative algebra such that* $\dim r_1(B) = 1$ *and* $\dim r_2(B) = k$ *where* $1 < k < \infty$. *Then B cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

Proof. Suppose that to the contrary B is an ideal contained in the Frattini subalgebra of the associative algebra A. Then

$$E(B, B) \subseteq \phi(E(B, A)) .$$

For if T is the mapping from A onto $E(B, A)$ defined by $T(a) = L_a$ for all $a \in A$, then, by the lemma,

$$E(B, B) = T(B) \subseteq T(\phi(A)) \subseteq \phi(T(A)) = \phi(E(B, A)) .$$

Let $z_1, \cdots, z_k$ be a basis for $r_2(B)$ such that z_k is a basis $r_1(B)$. For notational convenience, let $r_i = r_i(B)$ for all i. Let π be the natural homomorphism from $E(B, A)$ onto $E(r_2, A)$. Since

$$\begin{aligned} E(B, B) + E(B, A, r_2)/E(B, A, r_2) &\simeq E(B, B)/E(B, A, r_2) \cap E(B, B) \\ &= E(B, B)/E(B, B, r_2) \simeq E(r_2, B) \end{aligned}$$

it follows that

$$E(r_2, B) \simeq \pi(E(B, B)) \subseteq \pi(\phi(E(B, A))) \subseteq \phi(E(r_2, A)) .$$

We now show that $E(r_2, B) \not\subseteq \phi(E(r_2, A))$ by showing that $E(r_2, B)$ is complemented in $E(r_2, A)$. For $i = 1, \cdots, k-1$, define linear transformations e_i from r_2 onto r_1 by

$$e_i(z_j) = \begin{cases} \delta_{ij} z_k & \text{for } j = 1, \cdots, k-1 \\ 0 & \text{for } j = k \end{cases}$$

where δ_{ij} is the Kronecker delta. Let $S = ((e_1, \cdots, e_{k-1}))$. We claim that $S = E(r_2, B)$. Since $r_1 = ((z_k))$ and $B \cdot r_2 \subseteq r_1, E(r_2, B) \subseteq S$. To show that $S = E(r_2, B)$, we shall show that $\dim E(r_2, B) = k - 1 = \dim S$. For each $x \in B, L_x$ induces a linear transformation from r_2 into $r_1 \simeq F$, where F is the ground field. Therefore, we may consider each $L_x, x \in B$ as a linear functional on r_2. That is, $E(r_2, B) \subseteq (r_2)^*$ where $(r_2)^*$ is the dual space of r_2. Consequently, $\dim E(r_2, B) = \dim r_2 - \dim r_2^B$ where $r_2^B = \{z \in r_2; L_x(z) = 0 \text{ for all } x \in B\}$. Clearly $r_2^B = r_1$. Then, since $\dim r_2 = k$ and $\dim r_1 = 1, \dim E(r_2, B) = k - 1$ and $S = E(r_2, B)$.

We now show that S is complemented in $E(r_2, A)$. Let

$$M = \{E \in E(r_2, A); E(z_i) = \textstyle\sum_{j=1}^{k-1} \lambda_{ij} z_j, \lambda_{ij} \in F, i = 1, \cdots, k-1$$

and $E(z_k) = \lambda_k z_k, \lambda_k \in F\}$. M is clearly a subalgebra of $E(r_2, A)$ and $M \cap S = 0$. We claim that $M + S = E(r_2, A)$. Let $E \in E(r_2, A)$. Then $E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij} z_j + \lambda_{ik} z_k$ for $i = 1, \cdots, k-1$ and $E(z_k) = \lambda_k z_k$. However $E = E - \sum_{i=1}^{k-1} \lambda_{ik} e_i + \sum_{i=1}^{k-1} \lambda_{ik} e_i$ where $E - \sum_{i=1}^{k-1} \lambda_{ik} e_i \in M$ and $\sum_{i=1}^{k-1} \lambda_{ik} e_i \in S$. Therefore $M + S = E(r_2, A)$. We claim that $M \neq 0$. If $M = 0$, then $E(r_2, A) = E(r_2, B)$ which contradicts

$$E(r_2, B) \subseteq \phi(E(r_2, A)) \subset E(r_2 A) .$$

Consequently, S is complemented in $E(r_2, A)$, contradicting $S \subseteq \phi(E(r_2, A))$. This contradiction establishes the result.

COROLLARY. *Let B be a finite dimensional nontrivial nilpotent associative algebra with* $\dim r_1(B) = 1$. *Then B cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

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MARTINGALES OF VECTOR VALUED SET FUNCTIONS

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This paper is concerned with the norm convergence of Banach space valued martingales in Orlicz spaces whose underlying measure is (possibly) only finitely additive. Because of the possible incompleteness of these Orlicz spaces of measurable point functions, this subject will be treated in the setting of Orlicz spaces of set functions V^{Φ} rather than the corresponding spaces L^{Φ} of measurable point functions. First, a conditional expectation P_B, operating on finitely additive set functions, is introduced and related to the usual conditional expectation E^B operating on L^1 by the equality

$$(*) \qquad P_B(F)(E) = \int_E E^B(f)d\mu \qquad E \in \Sigma$$

where (Ω, Σ, μ) is a measure space, B is a sub σ-field of Σ and $F(E) = \int_E f d\mu$ for $E \in \Sigma$.

Then, with the use of P_B martingales of set functions are defined and their convergence in appropriate V^{Φ} spaces is investigated. In addition, in the countably additive case, the results obtained for martingales of set functions are related to martingales of measurable point functions and extensions of certain results of Scalora, Chatterji, and Helms are obtained.

The study of finitely additive set functions appears to have begun during the close of the last century with such notions as Jordan Content. Through the first half of this century, with the introduction of the Lebesgue theory, most effort was concentrated on countably additive set functions. Recently, however, certain work, such as representations of linear functionals of the space of bounded functions has demanded the employment of finitely additive set functions. More important is the fact that finitely additive set functions provide considerable flexibility in applications and are sometimes no more untractable than their countably additive counterparts.

In their new approach to probability theory, Dubins and Savage [6] have noted that countable additivity is sometimes unnecessarily restrictive and have dropped it. In the study of the classical function spaces L^p, Bochner [1] and Leader [12] find it "natural to consider" the L^p spaces of finitely additive set functions. More recently in [16, 17] Bochner and Leader's groundwork was placed in the Orlicz space setting. In various ways, each of these papers present the argument that certain classical results can be handled more easily with the set function approach and perhaps more importantly, that new results may be obtained by employing this approach.

The purpose of the present paper is to treat the theory of norm convergence of martingales in Orlicz spaces, not in the classical manner, but rather to treat this theory in the setting of finitely additive set functions in the context of [16] and [17]. Here again, the goal will be to reduce to a minimum the limiting processes needed in the study of mean martingale convergence.

In the first section, preliminaries including relevant facts about the $V^{\Phi}(X)$[16, 17] spaces are given and collected for ready reference. The second section introduces a generalized conditional expectation operator which operates on vector valued finitely additive set functions. Properties of this generalized conditional expectation are exploited in the third section where martingales of finitely additive set functions are defined and studied. Here extensions of certain known results of Scalora [15], Chatterji [4], Helms [9], and Krickeberg and Pauc [11] are obtained.

1. **Preliminaries.** Throughout this paper Ω is a point set: Σ is a field of subsets of Ω, and μ is a finitely additive (extended) real valued nonnegative set function defined on Σ. $\Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure. X is a Banach space. Φ is a Young's function [18] with complementary function Ψ.

By $V^{\Phi}(\Sigma, X)$ is meant the linear space of all finitely additive, μ-continuous[1], X-valued set functions F defined on Σ_0 which satisfy

$$(1.1) \qquad I_{\Phi}(F/k) = \sup_{\pi} \sum_{\pi} \Phi(\|F(E_n)\|/k\mu(E_n))\mu(E_n) \leqq 1$$

for some $k > 0$, where the supremum is taken over all partitions π consisting of a finite collection $\{E_n\}$ of disjoint members Σ_0 and the convention $0/0 = 0$ is observed. Upon the introduction of the norm N_{Φ} defined for $F \in V^{\Phi}(\Sigma, X)$ by

$$(1.2) \qquad N_{\Phi}(F) = \inf\{k > 0 : I_{\Phi}(F/k) \leqq 1\} ,$$

$V^{\Phi}(\Sigma, X)$ becomes a Banach space [16, 17].

A partition π is a finite collection $\{E_n\}$ of disjoint members of Σ_0. The partitions are partially ordered by defining $\pi_1 \leqq \pi_2$ if each members of π_1 can be written as a union of members of π_2. Corresponding to each to each $F \in V^{\Phi}(\Sigma, X)$ and each partition $\pi = \{E_n\}$ is the function

* F is μ-continuous if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies $\|F(E)\| < \varepsilon$.

$$F_\pi = \sum_\pi \frac{F(E_n)}{\mu(E_n)} \mu \cdot E_n \tag{1.3}$$

where $\mu \cdot E_n$ is the set function defined for $E \in \Sigma_0$ by $\mu \cdot E_n(E) = \mu(E \cap E_n)$. A set function of the form E_π will be termed a step function. The introduction of F_π allows us to single out a (possibly proper) closed subspace of $V^\Phi(\Sigma, X)$. By $S^\Phi(\Sigma, X)$ is meant the collection of all $F \in V^\Phi(\Sigma, X)$ such that $\lim_\pi N_\Phi(F_\pi - F) = 0$ where the limit is taken in the Moore-Smith sense through all partitions π.

THEOREM 1.1. *If X is reflexive and Φ obeys the Δ_2-condition ($\Phi(2x) \leqq K\Phi(x)$ for some K and all x), then $S^\Phi(\Sigma, X) = V^\Phi(\Sigma, X)$.*

The proof of this theorem may be found in [17, IV. 7]. If, $\Phi(x) = |x|$, then the corresponding $V^\Phi(\Sigma, X)$ and $S^\Phi(\Sigma, X)$ will be denoted by $V^1(\Sigma, X)$ and $S^1(\Sigma, X)$ respectively.

As usual, the Orlicz space $L^\Phi(\Sigma, X)$ is the space of all totally μ-measurable X-valued functions f which satisfy

$$\int_\Omega \Phi(\|f\|/k) d\mu \leqq 1 \tag{1.4}$$

for some k where the integral here and thoughout this paper is that of [7, Chap. III]. With functions which differ only on a μ-null set [7, Chap. III] identified, $L^\Phi(\Sigma, X)$ becomes a normed linear space (complete if μ is countably additive) under the norm N_Φ defined for $f \in L^\Phi(\Sigma, X)$ by

$$N_\Phi(f) = \inf \left\{ k > 0 \colon \int_\Omega \Omega(\|f\|/k) d\mu \leqq 1 \right\} . \tag{1.5}$$

The use of identical symbols for the $V^\Phi(\Sigma, X)$ and the $L^\Phi(\Sigma, X)$ norms will be justified in the next result. No confusion should arise since set functions will normally be denoted by upper case letters, while point functions will be denoted by lower case letters.

A nonnegative set function G defined on Σ is said to have the finite subset property if $E \in \Sigma, G(E) = \infty$ implies the existence of $E_0 \subset E, E_0 \in \Sigma$ such that $0 < G(E_0) < \infty$.

THEOREM 1.2. *Suppose μ has the finite subset property. The mapping $\lambda \colon L^\Phi(\Sigma, X) \to V^\Phi(\Sigma, X)$ defined for $f \in L^\Phi(\Sigma, X)$ by $\lambda f(E) = \int_E f d\mu, E \in \Sigma_0$, is an isometric injection of $L^\Phi(\Sigma, X)$ into $V^\Phi(\Sigma, X)$. If μ is countably additive, Σ is a σ-field, X is reflexive, and Φ obeys the Δ_2-condition, then the range of λ is all of $V^\Phi(\Sigma, X)$.*

The proof of this theorem may be synthesized from [17, II. 5], [17, IV. 8], and the fact that if μ is countably additive, $L^{\Phi}(\Sigma, X)$ is complete.

2. A generalized conditional expectation. The purpose of this section is to define and explore the properties of a generalized conditional expectation operator operating on finitely additive set functions. An attempt will be made to relate this operator and its properties to the usual conditional expectation [15] operating on point functions.

DEFINITION 2.1. A class of sets $B \subset \Sigma$ is a subfield of Σ if and only if B is a ring and $\Omega \in B$. A partition π_B $\{E_n\}$ is a B-partition if $\{E_n\} \subset B$.

DEFINITION 2.2. Let B be a subfield of Σ. A set function $F \in V^{\Phi}(\Sigma, X)$ is termed B-measurable if for each $E \in \Sigma_0$,

$$F(E) = \lim F_{\pi_B}(E) \tag{2.1}$$

where the limit is taken in the Moore-Smith sense through all B-partitions π_B.

The following result establishes the existence of an operator analogous to the usual conditional expectation [13, 15].

THEOREM 2.3. *Let Φ obey the Δ_2-condition and B be a subfield of Σ. Then for each $F \in S^{\Phi}(\Sigma, X)$ there exists a B-measurable set function $F_B \in S^{\Phi}(\Sigma, X)$ such that*

(i) $F_B(E) = F(E)$ *for all* $E \in B \cap \Sigma_0$,

(ii) $N_\Phi(F_B) \leqq N_\Phi(F)$,

and

(iii) $\lim_{\pi_B} N_\Phi(F_B - (F_B)_{\pi_B}) = 0$,

where the limit is taken in the Moore-Smith sense through all B-partitions π_B.

Proof. Let $F \in S^{\Phi}(\Sigma, X)$ be arbitrary and consider the mapping $\theta: S^{\Phi}(\Sigma, X) \to V^{\Phi}(B, X)$ defined by $\theta(F) = F \mid B$ where $F I B$ is the restriction of F to $B \cap \Sigma_0$. The linearity of θ is clear. Moreover for any $k > 0$,

$$\begin{aligned} I_\Phi(\theta F/K) &= \sup_{\pi B} \sum_{\pi B} \Phi(\| F(E_n) \|/k\mu(E_n))\mu(E_n) \\ &\leqq \sup_{\pi} \sum_{\pi} \Phi(\| F(E_n) \|/k\mu(E_n))\mu(E_n) \leqq I_\Phi(F/k) . \end{aligned} \tag{2.2}$$

From inequality (2.2) and the definition of the N_Φ norm, it follows immediately that $N_\Phi(\theta F) \leqq N_\Phi(F)$. Thus θ is a linear contraction.

Next we shall show that the range of Φ is contained in $S^\varphi(B, X)$ (which is possibly strictly contained in $V^\varphi(\Sigma, X)$). Because θ is a contradiction, it suffices to show θ maps step functions (i.e., functions of the form F_π) into $S^\varphi(B, X)$. From the linearity of θ, we can infer that this reduces to showing that $\theta(x\mu\cdot E) \in S^\varphi(B, X)$ for each $x \in X$ and all $E \in \Sigma_0$. Thus, let $x \in X$, $E \in \Sigma_0$, and $\pi_B = \{E_n\}$ be a B-partition. A brief computation yields

$$(2.3)\quad N_\Phi(\theta(x\mu\cdot E) - (\theta(x\mu\cdot E))_{\pi_B}) = \|x\|\, N_\Phi(\mu\cdot E \mid B - (\mu\cdot E \mid B)_{\pi_B})$$

where the last norm is taken in $V^\varphi(B, R)(R = \text{reals})$. Since $\mu\cdot E \mid B$ is $\mu \mid B$-continuous and satisfies $I_\Phi(\mu\cdot E \mid B) < \infty$, and Φ obeys the Δ_2-condition, Theorem 1.1 implies $\mu\cdot E \mid B \in S^\varphi(B, R)$. Thus

$$\lim_{\pi_B} N_\Phi(\mu\cdot E \mid B - (\mu\cdot E \mid B)_{\pi_B}) = 0 .$$

In view of this, the definition of $S^\varphi(B, X)$ and (2.3), we have $\theta(x\mu\cdot E) \in S^\varphi(B, X)$. This proves $\theta(S^\varphi(\Sigma, X)) \subset S^\varphi(B, X)$.

Leaving, for the time being, the problem of projecting $S^\varphi(\Sigma, X)$ into $S^\varphi(B, X)$, we shall now consider the opposite problem: the extension of members of $S^\varphi(B, X)$ to members of $S^\varphi(\Sigma, X)$. Let $G \in S^\varphi(B, X)$ and π_B be a B-partition. Then for $E \in B \cap \Sigma_0$,

$$G_{\pi_B}(E) = \sum_{\pi_B} \frac{G(E_n)}{\mu \mid B(E_n)} (\mu\cdot E_n \mid B)(E) .$$

Clearly G_{π_B} has a "natural" extension to all of Σ_0-namely

$$\sum_{\pi_B} \frac{G(E_n)}{\mu(E_n)} \mu\cdot E_n ,$$

which is defined for all $E \in \Sigma_0$. Denote this extension by $\rho(G_{\pi_B})$. Then evidently $\rho(G_{\pi_B}) \in S^\varphi(\Sigma, X)$, and clearly ρ is linear. Moreover, as a brief computation shows, $N_\Phi(G_{\pi_B}) = N_\Phi(\rho(G_{\pi_B}))$.

Now, for $G \in S_\Phi(B, X)$, we have

$$\lim_{\Delta_B, \pi_B} N_\Phi(G_{\Delta_B} - G_{\pi_B}) = 0 .$$

Hence

$$\lim_{\Delta_B, \pi_B} N_\Phi(\rho(G_{\Delta_B}) - \rho(G_{\pi_B})) = 0 .$$

This and the completeness of $S^\varphi(\Sigma, X)$ assures the existence of $\rho(G) \in S^\varphi(\Sigma, X)$ such that

$$(2.4)\qquad \lim_{\pi_B} N_\Phi(\rho(G) - \rho(G_{\pi_B})) = 0 .$$

Moreover,

$$N_\Phi(G) = \lim_{\pi_B} N_\Phi(G_{\pi_B}) = \lim_{\pi_B} N_\Phi(\rho(G_{\pi_B})) = N_\Phi(\rho(G)) .$$

Also note that if $E \in B \cap \Sigma_0$, then $G(E) = \lim_{\pi_B} G_{\pi_B}(E) = \lim_{\pi_B} \rho(G_{\pi_B})(E)$, by the definition of $\rho(G_{\pi_B}) = \rho(G)(E)$, since norm convergence in V^Φ implies setwise convergence for sets in Σ_0. Therefore we have

$$\rho(G) \mid B = G , \tag{2.5}$$

$$\rho(G)_{\pi_B} = \rho(G_{\pi_B}) \tag{2.6}$$

and

$$\lim_{\pi_B} N_\Phi(\rho(G) - \rho(G)_{\pi_B}) = 0 , \tag{2.7}$$

Now, to prove the theorem, let $F \in S^\Phi(\Sigma, X)$ and consider $F_B = \rho(\theta(F))$. By the definition of θ and (2.5) (with $\theta(F) = G$), $F_B(E) = F(E)$, $E \in B \cap \Sigma_0$, and (i) is satisfied. Since θ is a contraction and ρ is norm preserving, $N_\Phi(F_B) \leqq N_\Phi(F)$, and (ii) is satisfied. (iii) follows immediately from (2.7).

A corollary of the proof of Theorem 2.3 is given below for use later.

COROLLARY 2.4. *Let Φ obey the Δ_2-condition and B be a subfield of Σ. Then there exists a "natural" isometric embedding ρ of $S^\Phi(B, X)$ into $S^\Phi(\Sigma, X)$. The image of $S^\Phi(B, X)$ under ρ consists* of *all B-measurable members of $S^\Phi(\Sigma, X)$.*

Proof. The assertions of this corollary are all clear form the proof of Theorem 2.3 with the possible exception of the linearity of ρ. It is clear from ρ's definition that ρ is linear on the step functions. Since step functions are dense in $S^\Phi(B, X)$, ρ is linear on all of $S^\Phi(B, \Sigma)$.

The above corollary allows us to think of $S^\Phi(B, X)$ as a subspace of $S^\Phi(\Sigma, X)$ in very much the same way that the B-measurable members of $L^\Phi(B, X)$ constitute a subspace of $L^\Phi(\Sigma, X)$. With the aid of theorem 2.3, an operator P_B which will be called a generalized conditional expectation and which is a genuine generalization of Kolmogrorov's classical concept of probability theory (cf. Theorem 2.7) can be defined.

DEFINITION 2.5. Let Φ obey the Δ_2-condition and B be a subfield of Σ. For $F \in S^\Phi(\Sigma, X)$, the operator P_B is defined by

$$P_B(F) = F_B \tag{2.8}$$

where $F \to F_B$ in the sense of Theorem 2.3.

The following theorem is an immediate consequence of Theorem 2.3.

THEOREM 2.6. *If Φ obeys the Δ_2-condition and B is a subfield of Σ, then*

(i) *P_B on $S^\Phi(\Sigma, X)$ is linear and contractive,*

(ii) *$P_B(F) \mid B = F \mid B$,*

and

(iii) $\lim_{\pi_B} N_\Phi(P_B(F) - F_{\pi_B}) = 0$.

The relationship between the operator P_B and the usual conditional expectation operator E^B[15, pp. 353-356] which operates on point functions is clarified in the next result.

THEOREM 2.7. *Suppose Σ is a σ-field, B is a sub-σ-field of Σ, and μ is a countably additive finite measure on Σ. If E^B is the usual conditional expectation operator on $L^1(\Omega, \Sigma, \mu, X)$ then $\lambda E^B(f) = P_B(\lambda f)$ for all f $L^1(\Omega, \Sigma, \mu, X)$ or equivalently,*

$$\int_E E^B(f)d\mu = P_B\Big(\int_{(\cdot)} fd\mu\Big)(E)$$

for all $E \in \Sigma$ where λ is the isometric isomorphism of L^1 into V^1 of Theorem 1.2.

Proof. Since simple functions are dense in $L^1(\Omega, \Sigma, \mu, X)(L^1(\Sigma, X))$, it suffices to prove the statement for all simple functions f. The linearity of E^B, P_B, and the integral allow us to reduce this problem to the problem of showing $E^B(f) = P_B(\lambda f)$ for all f of the form $f = x\chi_E$ where $x \in X, E \in \Sigma$, and χ_E is the characteristic or indicator function of E. The definition of E^B on $L^1(\Sigma, X)$ [15] and [7, IV. 8. 17] imply

$$E^B(x\chi_E) = xE^B(\chi_E) = \lim_{\pi_B} \sum_B \frac{\int_{E_n} xE^B(\chi_E)d\mu}{\mu(E_n)} \chi_{E_n}$$

strongly in $L^1(B, X)$ and therefore in $L^1(\Sigma, X)$. Hence by the continuity of λ,

$$\lambda(E^B(x\chi_E)) = \lim_{\pi_B} \lambda\left(\sum_{\pi_B} \frac{\int_{E_n} xE^B(\chi_E)d\mu}{\mu(E_n)} \chi_{E_n}\right),$$

$$\lim_{\pi_B} \sum_{\pi_B} \frac{\int_{E_n} x\chi_E d\mu}{\mu(E_n)} \mu\cdot E_n\ , \quad \text{since} \quad E_n \in B\ ,$$

$$\lim_{\pi_B} \sum_{\pi_B} \frac{\lambda(x\lambda_E)(E_n)}{\mu(E_n)} \mu \cdot E_n \ ,$$
$$\lim_{\pi_B} \sum_{\pi_B} \frac{\lambda f(E_n)}{\mu(E_n)} \mu \cdot E_n = P_B(\lambda f) \ ,$$

strongly in $V^1(\Sigma, X)$ by Theorem 2.6.

The crux of Theorem 2.7 is that the operator P_B is a genuine extension of the classical conditional expectation operator E^B. Indeed, the definition of P_B does not depend directly on the Radon-Nikodym theorem (which is not available in usable form), while the definition of E^B depends crucially on the Radon-Nikodym theorem. But, as shown above, P_B coincides with E^B whenever the Radon-Nikodym theorem is applicable. Another property common to P_B and E^B is contained in the next result.

THEOREM 2.8. *Let Φ obey the Δ_2-condition. If $B_1 \subset B_2$ are subfields of Σ, then*

$$P_{B_1}(P_{B_2}) = P_{B_2}(P_{B_1}) = P_{B_1}$$

on $S^\Phi(\Sigma, X)$. Consequently P_B is a contractive projection of $S^\Phi(\Sigma, X)$ into $S^\Phi(\Sigma, X)$.

Proof. If $G \in S^\Phi(\Sigma, X)$ is arbitrary, then according to Theorem 2.6,

$$P_{B_1}(G) = \lim_{\pi_{B_1}} G_{\pi_{B_1}} \quad \text{strongly in} \quad V^\Phi(\Sigma, X) \ .$$

Hence if $F \in S^\Phi(\Sigma, X)$,

$$P_{B_1}(P_{B_2}(F)) = \lim_{\pi_{B_1}} (P_{B_2}(F))_{\pi_{B_1}} \qquad \text{in } N_\Phi \text{ norm} \ .$$

Since $P_{B_2}(F)$ agrees with F on B_2-sets and $B_1 \subset B_2$, $P_{B_2}(F)$ agrees with F on B_1-sets. Therefore

$$(P_{B_2}(F))_{\pi_{B_1}} = F_{\pi_{B_1}}$$

for each B_1-partition π_{B_1}, and

$$P_{B_1}(P_{B_2}(F)) = \lim_{\pi_{B_1}} (P_{B_2}(F))_{\pi_{B_1}} = \lim_{\pi_{B_1}} F_{\pi_{B_1}} = P_{B_1}(F)$$

strongly in $V^\Phi(\Sigma, X)$. Hence $P_{B_1}(P_{B_2}) = P_{B_1}$.

To prove $P_{B_2}(P_{B_1}) = P_{B_1}$, the boundedness of P_{B_1} and P_{B_2} will be used. If $F \in S^\Phi(\Sigma, X)$, then Theorem 2.6 implies $P_{B_1}(F) = \lim_{\pi_{B_1}} F_{\pi_{B_1}}$ strongly in $U^\Phi(\Sigma, X)$. The boundedness of P_{B_2} yields

$$P_{B_2}(P_{B_1}(F)) = \lim_{\pi_{B_1}} P_{B_2}(F_{\pi_{B_1}}) \,.$$

But each B_1-partition π_{B_1} is a B_2-partition since $B_1 \subset B_2$. Therefore $P_{B_2}(F_{\pi_{B_1}}) = F_{B_1}$ and

$$P_{B_2}(P_{B_1}(F)) = \lim_{\pi_{B_1}} F_{\pi_{B_1}} = P_{B_1}(F)$$

by Theorem 2.6.

That P_B is a contractive projection follows from $P_B^2 = P_B$ and Theorem 2.6.

According to [17, I. 13], $V^{\Phi}(\Sigma, X) \subset V^1(\Sigma, X)$ for all Young's functions provided $\mu(\Omega) < \infty$, it is natural consider P_B applied to functions in $V^{\Phi}(\Sigma, X)$ which belong to $S^1(\Sigma, X)$. The (closed) subspace of such functions will be denoted by $R^{\Phi}(\Sigma, X)$. According to [17, I. 13], N_{Φ} dominates N_1, the variation norm of V^1, when $\mu(\Omega) < \infty$; hence $S^{\Phi}(\Sigma, X) \subset R^{\Phi}(\Sigma, X) \subset V^{\Phi}(\Sigma, X) \subset V^1(\Sigma, X)$. In the case $\mu(\Omega) = \infty$, as the preceding definitions and theorems show, P_B is directly defined on S^{Φ} only if Φ obeys the Δ_2-condition. Because stronger hypotheses on Φ are needed when $\mu(\Omega) = \infty$, some of the following theorems will be stated in two parts- the first part dealing with the case $\mu(\Omega) < \infty$ and the second part dealing with the case $\mu(\Omega) = \infty$.

Some properties of P_B on $R^{\Phi}(\Sigma, X)$ are collected below:

THEOREM 2.9. *Let $\mu(\Omega)$ be finite and B be a subfield of Σ: then*

(i) *$P_B: R^{\Phi}(\Sigma, X) \to R^{\Phi}(\Sigma, X)$ is a contractive projection.*

(ii) *$P_B(F)(E) = \lim_{\pi_B} F_{\pi_B}(E)$ in X for each F in $R^{\Phi}(\Sigma, X)$ and $E \in \Sigma$.*

(iii) *$P_B(F) \mid B = F \mid B$ for all $F \in R^{\Phi}(\Sigma, X)$.*

Proof. Let $F \in R^{\Phi}(\Sigma, X)$. (iii) is simply Theorem 2.6 applied to $S^1(\Sigma, X)$. (ii) follows directly from Theorem 2.6 (iii) applied to $S^1(\Sigma, X)$. To prove (i), note that P_B maps $S^1(\Sigma, X)$ into $S^1(\Sigma, X)$. Therefore, to show P_B is a contractive projection on $R^{\Phi}(\Sigma, X)$, it suffices to show $N_{\Phi}(P_B(F)) \leqq N_{\Phi}(F)$ for $F \in R^{\Phi}(\Sigma, X)$. (This has already been established in the case Φ obeys the Δ_2-condition. A separate proof is furnished for the general Φ). From (ii) and the "lower semi-continuity" of I_{Φ}[17, I. 7], it follows that for each $k > 0$,

$$I_{\Phi}(P_B(F)/k) \leqq \liminf_{\pi_B} I_{\Phi}(F_{\pi_B}/k) \leqq I_{\Phi}(F/k) \,.$$

Hence $N_{\Phi}(P_B(F) \leqq N_{\Phi}(F)$

3. Martingales of additive set functions. The study of martingales of point functions in the finitely additive context appears to be somewhat intractable because of the possible incompleteness of the

$L^\varphi(\Sigma, X)$ spaces when the underlying measure μ is only finitely additive. However, even when μ is only finitely additive, the corresponding $V^\varphi(\Sigma, X)$ spaces are complete. Primarily for this reason we shall deal with martingales from a set function standpoint. In addition, as will be seen later, the assumption of only finite additivity will not present any special difficulties. Using a definition equivalent to that of Krickeberg and Pauc [11] we now define a martingale of additive set functions. This is a generalization of the classical concept of Doob [5] to the setting of finitely additive set functions (cf. Theorem 3.2).

DEFINITION 3.1. Let $\{B_\tau, \tau \in T\}$ be an increasing net of subfields of Σ (i.e., $B_1 \subset B_2$ if $\tau_1 \leqq \tau_2$). A $V^\varphi(\Sigma, X)(S^\varphi(\Sigma, X), R^\varphi(\Sigma, X))$-martingale is a net of finitely additive set functions $\{F_\tau, B_\tau, \tau \in T\}$ such that $F_\tau \in V^\varphi(\Sigma, X)(S^\varphi(\Sigma, X)\, R^\varphi(\Sigma, X))$ for each $\tau \in T$ and $P_{B_{\tau}}(F_{\tau_2}) = F_{\tau_1}$ for $\tau_1 \leqq \tau_2$.

In some important cases, martingales of set functions and martingales of measurable point functions [15, p. 358] can be indentified under the isometric isomorphism λ: $L^\varphi(\Sigma, X) \to V^\varphi(\Sigma, X)$ of Theorem 1.2. This is made precise in the following result.

THEOREM 3.2. *Let Σ be a σ-field and $\{B_\tau, \tau \in T\}$ be an increasing net of sub-σ-fields of Σ. If μ is a countably additive finite measure on Σ, then $\{F_\tau, B_\tau, \tau \in T\}$ is an $R^\varphi(\Sigma, X)$-martingale if and only if $\{\lambda^{-1}(F_\tau), B_\tau, \tau \in T\}$ is a martingale (of point functions) in $L^\varphi(\Sigma, X)$. This means if $f_\tau = \lambda^{-1}(F_\tau)$*

(i) *Each f_τ is B_τ-measurable,*

and

(ii) $\int_E f_{\tau_1} d\mu = \int_E f_{\tau_2} d\mu$ for all $\tau_2 \geqq \tau_1$ *and* $E \in B_{\tau_1}$

Proof. (Necessity). Because of the definition of $R^\varphi(\Sigma, X)$, it may and will be assumed that $R^\varphi(\Sigma, X) = S^1(\Sigma, X)$. The hypothesis guarantees $\lambda(L^1(\Sigma, X)) = S^1(\Sigma, X)$; so that $\lambda^{-1}(F_\tau) = f_\tau \in L^1(\Sigma, X)$ for each $\tau \in T$. From Theorem 2.6 and the definition of an $R^\varphi(\Sigma, X)$-martingale, it follows that

$$\lim_{\pi_{B_\tau}} N_1(F_\tau - (F_\tau)_{\pi_{B_\tau}}) = 0 .$$

Hence

$$\lim_{B} N_1(\lambda^{-1}(F_\tau) - \lambda^{-1}((F_\tau)_{\pi_{B_\tau}})) = 0 .$$

Since the $\lambda^{-1}(F_\tau)_{\pi_{B_\tau}}$ are B_τ-simple functions, it follows that $f_\tau = \lambda^{-1}(F_\tau)$ is B_τ-measurable for each τ. This establishes (i). (ii) follows directly from Definition 3.1; if $\lambda^{-1}(F_\tau) = f_\tau$, then for each B_{τ_1} and $\tau_2 \geqq \tau_1$, we have

$$\int_E f_{\tau_1} d\mu = F_{\tau_1}(E) = F_{\tau_2}(E) = \int_E f_{\tau_2} d\mu ,$$

for $E \in B_\tau$.

(Sufficiency). If $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^1(\Sigma, X)$ and $F_\tau(\cdot) = \lambda f_\tau(\cdot) = \int_{(\cdot)} f_\tau d\mu$ for each $\tau \in T$, (i) implies $F_{\tau_1}(E) = F_{\tau_2}(E)$ for each $E \in B_{\tau_1}$ and all $\tau_2 \geqq \tau_1$. Hence

$$F_{\tau_1} = \lambda f_{\tau_1} = \lambda E^{B_{\tau_1}}(f_{\tau_2}) = P_{B_{\tau_1}}(\lambda f_{\tau_2}) = P_{B_{\tau_1}}(F_{\tau_2})$$

by Theorem 2.7. Hence $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $S^1(\Sigma, X)$.

We shall now begin a study of the norm or mean convergence of martingales of set functions. The main result dealing with this problem is contained in the following theorem.

THEOREM 3.3. (i) *Let $\mu(\Omega) < \infty$ and $\{F_\tau, B_\tau, \tau \in T\}$ be an $R^\varphi(\Sigma, X)$ martingale. If $\Sigma_1 = \bigcup_\tau B_\tau$ and the function F_1 defined by $\lim_\tau F_\tau(E) = F_1(E)$ for $E \in \Sigma_1$ belongs to $S^\Phi(\Sigma_1, X)$, then the net $\{F_\tau, B_\tau, \tau \in T\}$ converges in the $V^\Phi(\Sigma, X)$ norm.*

(ii) *If $\mu(\Omega) = \infty$, and Φ obeys the Δ_2-condition, the same conclusion holds if $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $S^\Phi(\Sigma, X)$ and the function F_1 defined by $\lim_\tau F_\tau(E) = F_1(E)$ for $E \in \Sigma_1 \cap \Sigma_0$ belongs to $S^\Phi(\Sigma_1, X)$ where Σ_0 is the ring of sets in Σ of finite μ-measure.*

Proof. (i) Since $\{B_\tau, \tau \in T\}$ is an increasing net of subfields of Σ, it is clear that $\Sigma_1 = \bigcup_\tau B_\tau$ is a subfield of Σ. From Definition 3.1, it follows immediately that $F_1(E) = \lim_\tau F_\tau(E)$ exists for each $E \in \Sigma_1$. Now consider the net $\{F_\tau \mid \Sigma_1, \tau \in T\}$ in $V^\Phi(\Sigma, X)$. By hypothesis F_1 belongs to $S^\Phi(\Sigma_1, X)$. Therefore if $\varepsilon > 0$ is given, there exists a Σ_1-partition π such that $N_\Phi(F_1 - F_{1_\pi}) < \varepsilon/2$. By virtue of the facts that $\Sigma_1 = \bigcup_\tau B_\tau$ and $\{B_\tau, \tau \in T\}$ is an increasing net of fields, there exists a B_{τ_0} such that $\pi \subset B_\tau$ for all $\tau \geqq \tau_0$. Moreover Corollary 2.4 guarantees the existence of a Σ_1-measurable function F defined on all of Σ such that $N_\Phi(F - F_\pi) < \varepsilon/2$. In addition, from the definition of P_{B_τ}, we have $P_{B_\tau}(F) = F_\tau$ for all $\tau \in T$ and $P_{B_\tau}(F_\pi) = F_\pi$ for all $\tau \geqq \tau_0$. Therefore for $\tau \geqq \tau_0$, the triangle inequality yields

$$N_\Phi(F - F_\tau) \leqq N_\Phi(F - F_\pi) + N_\Phi(F_\pi - F_{B_\tau})$$
$$N_\Phi(F - F_\pi) + N_\Phi(P_{B_\tau}(F - F_\pi)) \leqq 2N_\Phi(F - F_\pi) < \varepsilon ,$$

since P_{B_τ} is a contraction. Consequently $\lim_\tau N_\Phi(F - F_\tau) = 0$.

The proof of (ii) is the same with a few obvious modifications.

Because of its generality, the hypothesis of Theorem 3.3 may be somewhat difficult to verify. The utilization of Theorem 1.1 permits us to state some corollaries which are more easily applied.

COROLLARY 3.4. *Let X be reflexive and Φ obey the Δ_2-condition. If $\{F_\tau, B_\tau, \tau \in T\}$ is a $V^\Phi(\Sigma, X)$-martingale which satisfies*

(i) *$N_\Phi(F_\tau) \leqq M < \infty$ for some M and all $\tau \in T$ and*

(ii) *$\nu(F_\tau, \cdot)$ are uniformly μ-continuous (but not necessarily finite) where $\nu(F_\tau, E)$ is the variation of F_τ on E for $E \in \Sigma_0$, then*

(a) *the net $\{F_\tau, \tau \in T\}$ converges in $V^\Phi(\Sigma, X)$ norm,*

and

(b) *If Ψ, the complementary function to Φ, is also continuous* (ii) *of the hypothesis may be dropped.*

Proof. (a) Since Φ obeys the Δ_2-condition and X is reflexive, Theorem 1.1 implies $S^\Phi(\Sigma, X) = V^\Phi(\Sigma, X)$; therefore Theorem 3.3 is applicable. According to this theorem, it need only be shown that the set function F_1 defined on $\Sigma_1 \cap \Sigma_0$, where $\Sigma_1 = \bigcup_\tau B_\tau$, by $F_1(E) = \lim_\tau F_\tau(E)$ belongs to $S^\Phi(\Sigma, X)(= V^\Phi(\Sigma, X))$. The "lower semi-continuity" of $I_\Phi(\cdot)$ [17, I. 7] yields for each $k > M$

$$I_\Phi(F_1/k) = \liminf_\tau I_\Phi(F_\tau \mid \Sigma_1/k) \leqq \sup_\tau \{I_\Phi(F_\tau/k)\} \leqq 1 ,$$

since $N_\Phi(F_\tau) \leqq M$ for each $\tau \in T$. In addition, from the "lower semi-continuity" of the variation ν, one has $\nu(F_1, E) \leqq \liminf \nu(F_\tau, E)$ for all $E \in \Sigma_1$. Since the $\nu(F, \cdot)$ are uniformly μ-continuous, it follows that $\nu(F_1, \cdot)$ is $\mu \mid \Sigma_1$-continuous and hence that F_1 is $\mu \mid \Sigma_1$-continuous. Therefore $F_1 \in V^\Phi(\Sigma_1, X) = S(\Sigma_1, X)$. This proves (a).

(b) If Ψ is continuous, then the fact that $I_\Phi(F_1/k) < \infty$ for some k guarantees F_1 is $\mu \mid \Sigma_1$-continuous by [17, I. 17]. Hence (ii) may be dropped.

A specialization of Corollary 3.4 to martingales of point functions yields the following result.

COROLLARY 3.5. *Let Σ be a σ-field and μ be countably additive and finite on Σ. If X is reflexive, Φ satisfies the Δ_2-condition and $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$ such that*

(i) *$N_\Phi(f_\tau) \leqq M < \infty$ for some M and all $\tau \in T$,*

and

(ii) *the functions $\|f_\tau\|_X$ are uniformly integrable,*

then

(a) *the net $\{f_\tau, \tau \in T\}$ converges in the L^Φ norm.*

(b) *If Ψ, the complementary function to Φ, is continuous, then*

(ii) *of the hypothesis may be dropped.*

Proof. (a) Let $F_\tau = \lambda f_\tau$. According to Theorem 3.2, $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale of set functions. Since λ is an isometry, $N_\Phi(F_\tau) = N_\Phi(f_\tau) \leqq M$ for all $\tau \in T$. From (ii) $\nu(F_\tau, \cdot) = \int_{(\cdot)} \|f_\tau\|_X d\mu$ are uniformly μ-continuous because the $\|f_\tau\|_X$ are uniformly integrable. An application of Corollary 3.4 shows $\{F_\tau, \tau \in T\}$ converges in $V^\Phi(\Sigma, X)$ norm. Since the current hypothesis guarantees that $L^\Phi(\Sigma, X)$ is complete, and λ is an isometry, it follows that $\{f_\tau, \tau \in T\}$ converges in $L^\Phi(\Sigma, X)$.

(b) This follows directly from the above and Corollary 3.4.

Corollary 3.5 extends one of the main results of Chatterji [4, Th. 3] in two ways: The index set T is possibly uncountable, and the convergence is in $L^\Phi(\Sigma, X)$ while in [4], the convergence is in $L^p(\Sigma, X)$, $1 \leqq \rho < \infty$. Furthermore the methods of proof in [4] do not seem to apply to the more general setting of Corollary 3.5.

A set function martingale version of a theorem of Krickeberg and Pauc [11, Th. 6, p. 500] is contained in

THEOREM 3.6. (a) *Suppose $\mu(\Omega) < \infty$ and $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $R^\Phi(\Sigma, X)$. Then the following are equivalent*:

(i) *$\{F_\tau, \tau \in T\}$ converges in the strong topology of $V^\Phi(\Sigma, X)$.*

(ii) *$\{F_\tau, \tau \in T\}$ converges in the weak topology of $V^\Phi(\Sigma, X)$.*

(iii) $\lim_\tau x^*(F_\tau(E)) = x^*(z_E)$ *for each E some $x_E \in X$, and all $x^* \in X^*$, the conjugate space to X. The function $F_\infty(E) = x_E$ for $E \in \Sigma$, is a member of $V^\Phi(\Sigma, X)$ and for each $\varepsilon > 0$, there exists a parameter $\tau_\varepsilon \in T$ and a B_{τ_ε}-measurable function $G_{\tau_\varepsilon} \in R^\Phi(\Sigma, X)$ such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon$.*

(b) *If $\mu(\Omega) = \infty$, thetheorem remains true provided Φ obeys the Δ_2-condition, R^Φ is replaced by S^Φ and the limit in* (iii) *is taken only for $E \in \Sigma_0$.*

Proof. (i) $\rightarrow$ (ii) is obvious. (ii) $\rightarrow$ (iii): Let $x^* \in X^*$ and $E \in \Sigma$. It is easily seen that the functional l defined for $F \in V^\Phi(\Sigma, X)$ by $l(F) = x^*(F(E))$ is a bounded linear functional on $V^\Phi(\Sigma, X)$. By (ii), the net $\{F_\tau, \tau \in T\}$ converges in the weak topology to some $H \in V^\Phi(\Sigma, X\}$. Therefore $x^*(H(E)) = \lim_\tau x^*(F_\tau(E))$ for each $E \in \Sigma_0$ and $x^* \in X^*$. This is the first part of (iii) with $H = F_\infty$. Now let M be the collection of all $V^\Phi(\Sigma, X)$ functions which are B_τ-measurable for some $\tau \in T$. From the fact that $\{B_\tau, \tau \in T\}$ is an increasing net of subfields of Σ, it follows that M is a linear submanifold of $V^\Phi(\Sigma, X)$. But $\{F_\tau, \tau \in T\}$ converges weakly to F_∞ and each $F_\tau \in M$. Hence F_∞ belongs to the strong closure of M since the weak and strong closure of a linear manifold are identical. Consequently for each $\varepsilon > 0$, there exists a parameter $\tau \in T$ and a B_τ-measurable function G_{τ_ε} such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon$. This proves (ii) $\rightarrow$ (iii).

(iii) $\rightarrow$ (i). Let $\varepsilon > 0$ be given. From (iii), there exists a $\tau_2 \in T$ and a B_{τ_ε}-measurable function G_{τ_ε} such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon/2$. Since ε is arbitrary and G_{τ_ε} belongs to the closed subspace $R^\Phi(\Sigma, X)$ of $V^\Phi(\Sigma, X)$, we have $F_\infty \in R^\Phi(\Sigma, X)$. Hence $P_{B_\tau}(F_\infty)$ is defined for each $\tau \in T$. In addition, $P_{B_\tau}(F_\infty)$ and F_τ are both B_τ-measurable and agree on B_τ-sets. It follows that $P_{B_\tau}(F_\infty) = F_\tau$. Moreover since G_{τ_ε} is B_{τ_ε}-measurable and belongs to $R\ (\Sigma, X)$, $P_{B_\tau}(G_{\tau_\varepsilon}) = G_{\tau_\varepsilon}$ for all $\tau \geqq \tau_\tau$. Thus for $\tau \geqq \tau_\varepsilon$,

$$\begin{aligned} N_\Phi(F_\infty - F_\tau) &\leqq N_\Phi(F_\infty - G_{\tau_\varepsilon}) + N_\Phi(C_{\tau_\varepsilon} - F_\tau) \\ &= N_\Phi(F_\infty - G_{\tau_\varepsilon}) + N_\Phi(P_{B_\tau}(G_{\tau_\varepsilon} - F_\infty)) \\ &\leqq 2N_\Phi(F_\infty - F_{\tau_\varepsilon}) < \varepsilon\ , \end{aligned}$$

since P_{B_τ} is a contraction. This proves (a).

(b) With a few evident modifications, the proof is the same.

As a corollary to Theorem 3.6, the following extension of [11, Th. 6, p. 500] for vector-valued functions can be given.

COROLLARY 3.7. *Let Σ be a σ-field and μ be a countably additive finite measure on Σ. If $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$, the following conditions are equivalent*:

(i) *$\{f_\tau, \tau \in T\}$ converges in the strong topology of $L^\Phi(\Sigma, X)$.*

(ii) *$\{f_\tau, \tau \in T\}$ converges in the weak topology of $L^\Phi(\Sigma, X)$.*

(iii) *There exists a function $f_\infty \in L^\Phi(\Sigma, X)$* such that

$$\lim x^*\Big(\int_E f_\tau d\mu\Big) = x^*\Big(\int_E f_\infty d\mu\Big)$$

for each $E \in \Sigma$ and all $x^* \in X^*$, and for each $\varepsilon > 0$ there exists a parameter $\tau_\varepsilon \in T$ and a B_{τ_ε}-measurable function g_{τ_ε} in $L^\Phi(\Sigma, X)$ such that $N_\Phi(g_{\tau_\varepsilon} - f_\infty) < \varepsilon$.

Proof. If $\lambda: L^\Phi(\Sigma, X) \rightarrow V^\Phi(\Sigma, X)$ is the isometric isomorphism of Theorem 1.2, since $L^\Phi(\Sigma, X) \subset L^1(\Sigma, X)$, we have $\lambda(L^\Phi(\Sigma, X)) \subset R^\Phi(\Sigma, X)$. The proof now follows directly from Theorem 3.6 after an application of Theorem 3.2 and the isometric isomorphism λ.

The next theorem and its corollary are the final results of this section.

THEOREM 3.8. *Let X be reflexive, Φ and its complementary* *function Ψ be continuous and Φ obey the Δ_2-condition. If $\{F_\tau, B_\tau, \tau \in T\}$ is a $V^\Phi(\Sigma, X)$-martingale, then the following statements are equivalent.*

(i) *$\{F_\tau, B_\tau, \tau \in T\}$ converges weakly in $V^\Phi(\Sigma, X)$.*

(ii) *$\{F_\tau, B_\tau, \tau \in T\}$ converges strongly in $V^\Phi(\Sigma, X)$.*

(iii) *There exists F_∞ in $V^\Phi(\Sigma, X)$ such that $F_\tau = P_{B_\tau}(F_\infty)$ for all $\tau \in T$.*

(iv) *The set $\{F_\tau, \tau \in T\}$ is (strongly) bounded.*

Proof. (i) $\rightarrow$ (ii) is Theorem 3.6. (ii) $\rightarrow$ (iii): Let F_∞ be the strong limit of $\{F_\tau, \tau \in T\}$. Then $F_\infty \in V^\Phi(\Sigma, X)$ and clearly $F_\tau = P_{B_\tau}(F_\infty)$ for all $\tau \in T$. (iii) $\rightarrow$ (iv): Since P_{B_τ} is a contraction,

$$N_\Phi(F_\tau) = N_\Phi(P_{B_\tau}(F_\infty)) \leqq N_\Phi(F_\infty) .$$

(iv) $\rightarrow$ (i) follows from Corollary 3.4.

A formulation of Theorem 3.8 for martingales of point functions is

COROLLARY 3.9. *Let X be reflexive, Φ and its complementary function Ψ be continuous and Φ obey the Δ_2-condition. If Σ is a σ-field and μ is a countably additive finite measure on Σ and $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$, then the following statements are equivalent.*

(i) *$\{f_\tau, \tau \in T\}$ converges in the weak topology of $L^\Phi(\Sigma, X)$.*

(ii) *$\{f_\tau, \tau \in T\}$ converges in the strong topology of $L^\Phi(\Sigma, X)$.*

(iii) *There exists $f_\infty \in L^\Phi(\Sigma, X)$ such that $f_\tau = E^{B_\tau}(f_\infty)$ for all $\tau \in T$.*

(iv) *The set $\{f_\tau, \tau \in T\}$ is (strongly) bounded.*

Proof. The proof follows from an application of the isometric isomorphism λ of $L^\Phi(\Sigma, X)$ onto (in this case) $V^\Phi(\Sigma, X)$ and theorems 2.7, 3.2 and 3.8.

It should be noted that Corollary 3.9 subsumes some of the main results of [9]. In addition, it extends these results to the vector-valued case and to the Orlicz space setting.

Furthermore, it should be noted that some of the preceding results for finitely additive set functions can be deduced from known results by use of the isomorphism theorems of [7, IV. 9]. On the other hand, the method of approach of this paper seems more direct and seems to yield more insight than the indirect method using known results and the isomorphism theorems of [7, IV. 9].

Finally, we note that the above results do not, in general, admit extensions to proving pointwise convergence theorems for martingales of point function. Indeed, the properties of the integrals of the functions involved enables these methods to work by avoiding the need for consideration of the (non-) measurability of the limits of nets of (point) functions. Thus, little information about pointwise convergence can be deduced from this approach. However, it is too much to expect to be able to deduce such results from the properties of finitely additive set functions, and a different approach is needed in such a study.

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CONDITIONS FOR A MAPPING TO HAVE THE SLICING STRUCTURE PROPERTY

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Let $p: E \to B$ be a fibering in the sense of Serre. As is well known the fibering need not be a fibering in any stronger sense. However it is expected that if certain conditions are placed on E, p or B then p might be a fibration in a stronger sense. This paper gives such conditions.

The main result of this paper is:

THEOREM 1. Let p be an n-regular perfect map from a complete metric space (E, d) onto a locally equiconnected space B. If $\dim E \times B \leqq n$ then p has the slicing structure property (in particular p is a Hurewicz fibration).

The following definitions will be needed.

DEFINITION 1. A space X is *locally equiconnected* if for each point x, there exists a neighborhood U_x of x and a map

$$N: U_x \times U_x \times I \to X$$

satisfying $N(a, b, 0) = a$, $N(a, b, 1) = b$, and $N(a, a, t) = a$.

DEFINITION 2. A map p from E to B is n-regular if it is open and satisfies the following property: given any x in E and any neighborhood U of x there exists a neighborhood V of x such that if $f: S^m \to V \cap p^{-1}(y)$ for some $y \in B$ $(m \leqq n)$ then there exists

$$F: B^{m+1} \to U \cap p^{-1}(y)$$

which is an extension of f.

DEFINITION 3. A family $\mathscr{S}$ of sets of Y is equi-LC^n if for every $y \in S \in \mathscr{S}$ and every neighborhood U of y in Y there exists a neighborhood V of y such that for every $S \in \mathscr{S}$, every continuous image of an m-sphere $(m \leqq n)$ in $S \cap V$ is contractible in $S \cap U$.

Note 1. If $p: E \to B$ is n-regular then the collection $\{p^{-1}(b) \mid b \in B\}$ is equi-LC^n.

DEFINITION 4. A family $\mathscr{S}$ of sets of a metric space (Y, d) is *uniformly* equi-LC^n with respect to d if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $f: S^m \to S \cap N(x, \delta)$ $(m \leqq n$ and $S \in \mathscr{S})$ then there exists $F: B^{m+1} \to S \cap N(x, \varepsilon)$ which is an extension of f.

DEFINITION 5. A map $p: E \to B$ has the *covering homotopy pro-*

perty for a class of spaces if given any space X in the class and maps $F: X \times I \to B$ and $g: X \to E$ such that $F(x, 0) = pg(x)$ then there exists a map: $G: X \times I \to E$ such that $pG = F$ and $G(x, 0) = g(x)$.

DEFINITION 6. A map $p: E \to B$ is a *Serre fibration* if p has the covering homotopy property for the class of polyhedra. It is a *Hurewicz fibration* if it has the covering homotopy property for all spaces.

DEFINITION 7. A map $p: E \to B$ has the *slicing structure property* (SSP) if for each point $b \in B$ there exists a neighborhood U_b of b and a map $\psi_b: p^{-1}(U_b) \times U_b \to p^{-1}(U_b)$ such that (1) $\psi_b(e, p(e)) = e$ and (2) $p\psi_b = \pi_2$ (the projection onto U_b).

DEFINITION 8. A function $\varphi: X \to 2^Y$ (Y metric) is continuous if given $\varepsilon > 0$; every $x_0 \in x$ has a neighborhood U such that for every $x \in U$, $\varphi(x_0) \subset N_\varepsilon(\varphi(x))$ and $\varphi(x) \subset N_\varepsilon(\varphi(x_0))$.

DEFINITION 9. A selection for a function $\varphi: X \to 2^Y$ is a map $g: X \to Y$ such that $g(x) \in \varphi(x)$.

A mapping is a continuous function. All spaces will be Hausdorff. The n-dimensional sphere will be denoted by S^n and the ball which it bounds B^{n+1}. If f is a mapping $\mathrm{Gr}(f)$ will denote the graph of f.

The following theorem of Michael will be needed:

THEOREM M. *Let Z be paracompact, let $X = Z \times I$ and let Y be a complete metric space with metric ρ. Let $\mathscr{S} \subset 2^Y$ be uniformly* equi-LC^n *with respect to ρ and let $\varphi: X \to \mathscr{S}$ be continuous with respect to ρ. Let* dim $Z \leqq n$ *and let $A = (Z \times 0) \cup (C \times I)$ where C is closed in Z. Then every selection for $\varphi \mid A$ can be extended to a* selection for φ.

2. Proof of Theorem 1 and its consequences.

Proof. Let $b_0 \in B$. Since B is locally equiconnected at b_0 there exists a neighborhood U of b_0 and a map $N_U: U \times U \times I \to B$ such that $N_U(x, y, 0) = x$, $N_U(x, y, 1) = y$, and $N_U(x, x, t) = x$. Let $P_U = p \mid p^{-1}(U)$ and define $g: \mathrm{Gr}(p_U) \to p^{-1}(U)$ by $g(e, p(e)) = e$. Also define $F: p^{-1}(U) \times B \to B$ by $F(e, b) = b$ and

$$H: p^{-1}(U) \times U \times I \to p^{-1}(U) \times B$$

by $H(e, b, t) = (e, N_U(p(e), b, t))$. Note $H(e, b, 0) = (e, N_U(p(e), b, 0)) = (e, p(e))$ and $H(e, b, 1) = (e, N_U(p(e), b, 1)) = (e, b)$. Further define

$$g'\colon (p^{-1}(U) \times U \times 0) \cup (\mathrm{Gr}(P_U) \times I) \to E$$

by $g'(e, b, t) = e$ and $K\colon p^{-1}(U) \times U \times I \to B$ by

$$K(e, b, t) = F(H(e, b, t))$$

and note that $pg' = K \mid (p^{-1}(U) \times U \times 0) \cup \mathrm{Gr}(P_U) \times I$.

Therefore we have the following commutative diagram.

$$\begin{tikzcd}
 & & E \arrow[d, "p"] \\
(p^{-1}(U) \times U \times 0) \cup \mathrm{Gr}(p_U) \times I \arrow[urr, bend left, "g'"] \arrow[r, phantom, "\subset"] & p^{-1}(U) \times U \times I \arrow[r, "K"] \arrow[ur, dotted, "G"] & B
\end{tikzcd}$$

Now Theorem M will be applied. Let $Z = p^{-1}(U) \times U$, $Y = E$, and $\varphi\colon Z \times I \to \mathscr{S} \subset 2^Y$ be defined by $\varphi(z, t) = p^{-1}K(z, t)$ and let $C = \mathrm{Gr}(p_U)$. Note Z is paracompact and φ is continuous since p is perfect. Since p is n-regular $\{p^{-1}(b)\}$ in equi-LC^n and by Proposition 2.1 [3] there exists a metric σ on E agreeing with the topology such that $\sigma \geqq d$ and $\{p^{-1}(b)\}$ is uniformly equi-LC^n. Since $\sigma \geqq d$, (E, σ) is a complete metric space. It should also be noted that $\dim Z \leqq n$ and that g' is a selection for $\varphi \mid (Z \times 0) \cup (C \times I)$. Hence by Theorem M, g' could be extended to a selection G for φ (i.e.,

$$G\colon p^{-1}(U) \times U \times I \to E$$

in such a way that the above diagram will still be commutative with the addition of G).

Define $\varphi_U\colon p^{-1}(U) \times U \to p^{-1}(U)$ by $\varphi_U(e, b) = G(e, b, 1)$. Note if $(e, b) \in p^{-1}(U) \times U$ then

$$\begin{aligned} G(e, b, 1) \in p^{-1}K(e, b, 1) &= p^{-1}FH(e, b, 1) = p^{-1}F(e, N_U(p(e), b, 1)) \\ &= p^{-1}F(e, b) = p^{-1}(b) \in p^{-1}(U)\ . \end{aligned}$$

Hence the range of φ_U is as stated. It is now easy to see that φ_U satisfies the conditions to be a slicing function. This completes the proof.

Note 2. The hypothesis that p be perfect was used only to show that $\{p^{-1}(b) \mid b \in B\}$ is a continuous collection and that B is paracompact. Hence if this could be shown some other way a stronger theorem will be obtained.

COROLLARY 1. *If* $p: E \to B$ *is a Serre fibration and* E *and* B *are finite dimensional compact* ANR's *then* p *has the SSP.*

Proof. It is well known that ANR's are locally equiconnected. It also follows from [2] that p is n-regular for all n. Hence the proof follows from Theorem 1.

Theorem 1 and Corollary 1 allow us to get the following generalizations of Raymond's results in [5].

COROLLARY 2. *Let* $p: E \to B$ *be a Serre fibration of a connected compact metric finite dimensional* ANR *onto a compact metric finite dimensional* ANR. *Suppose that* E *is an* n-*gm over* L (*a field or the integers*). *Then*:

(*a*) *each fiber* F_b *is a* k-*gm over* L

(*b*) B is an $(n\text{-}k) - gm$ *over* L.

COROLLARY 3. *Let* $p: E \to B$ *is a Serre fibration of a connected compact metric finite dimensional* ANR *onto a compact metric finite dimensional* ANR *base* B. *Suppose that* E *is a* (*generalized*) *manifold* (*over a principal ideal domain*) *and some fiber has a component of dimension* $\leqq 2$. *Then* p *is locally trivial.*

Another theorem which follows from Michael's Theorem 1.2 [3] is the following:

THEOREM 2. *Let* $p: E \to B$ *be an* n-*regular map from a complete metric space* E *onto a paracompact space* B. *Assume that*

$$\dim E \times B \leqq n + 1$$

and $p^{-1}(b)$ *is* C^n *for every* $b \in B$. *Then* p *has the* SSP *and the slicing structure could be chosen with only one slicing function.*

Proof. Define $g: \mathrm{Gr}(p) \to E$ by $g(e, p(e)) = e$ and $F: E \times B \to B$ by $F(e, b) = e$. The $\varphi(e, b) = p^{-1}F(e, b)$ is a carrier and g is a selection for $\varphi \mid \mathrm{Gr}(p)$. Hence by Theorem 1.2 [3] g could be extended to a selection G for φ. It is easily seen that G is the desired slicing function.

Note 3. Theorem 2 has corollaries similar to those of Theorem 1 and the author leaves them to the reader to develop.

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ON CONTINUOUS MAPPINGS OF METACOMPACT ČECH COMPLETE SPACES

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Under what may be thought of as a guise of a description of pathology are indicated here certain ways in which Čech completeness, Arhangel'skiĭ's p-space concept, and metacompactness enlarge on the respective concepts of metric absolute G_δ's, metrizability, and paracompactness. This is done through examination of certain aspects of the theory of multivalued mappings. It is taken as a point of orientation that the topic of Tychonoff locally bicompact spaces has a substantial mathematical interest. It is assumed obvious that such spaces are locally paracompact p-spaces. An underlying point of view is that the class of regular locally paracompact p-spaces extends along natural lines the class of regular locally metrizable spaces.

Let us observe these theorems: (1) A Hausdorff space is paracompact if and only if it is fully normal [13]. (2) A space is metacompact if and only if for every collection G of open sets covering it there exists a collection H of open sets covering it such that if P is a point, the collection of all members of H containing P refines a finite subcollection K of G [22]. (3) A T_1 space is metrizable if and only if it is fully normal and has a base of countable order (cf. definitions below) [3]. (4) A T_1 space has a uniform base (cf. definitions below) if and only if it is metacompact and has a base of countable order [27]. (5) Metacompactness is invariant under the action on a topological space of a closed continuous mapping [23]. We may then see that whether or not a metacompact T_1 topological space S has a perfect mapping onto a space with a uniform base depends only on whether S has a perfect mapping onto a space having a base of countable order. Similarly, since full normalcy of a topological space is also an invariant under the action of a closed continuous mapping [10], whether a fully normal T_1 space S has a perfect mapping onto a metrizable space depends only on whether S has a perfect mapping onto a space having a base of countable order. These reductions achieve heightened interest in view of the invariance of the base of countable order property under the actions of peripherally bicompact closed continuous mappings on T_1 spaces [21], the intimacy of its relation to the topic of interior mappings [16, 19], and certain work of Frolík and Arhangel'skiĭ which will now be described.

Frolík showed that a paracompact Hausdorff space is Čech complete (cf. definition below) if and only if it has a perfect mapping onto a complete metric space [7]. This fundamental contribution was enlarged by Arhangel'skiĭ, first of all by his exercise of an extraordinary ingenuity in isolating the concept of a p-space (cf. definition below), and secondly by his equating for the Hausdorff paracompact cases the property of being a p-space with the admitting of a perfect mapping onto a metrizable space [4]. Since Čech completeness is preserved under the action of a perfect mapping between Tychonoff spaces [8], we may now see that both Frolík's theorem and Arhangel'skiĭ's theorem may be interpreted as having in common the remarkable feature of pivoting on the existence of perfect mappings of the respective (paracompact) spaces onto spaces having bases of countable order.

We now observe that the first four theorems reviewed above put one in a position to interpret that for spaces satisfying the first Trennungsaxiom and having uniform bases, the distinction metrizable-nonmetrizable corresponds precisely to the distinction unity-finitude of order >1 of certain refined collections K. Can it be the case, one may ask with this preparation, that nevertheless there exists a metacompact Čech complete space admitting of no perfect mapping onto an absolute G_δ space with a uniform base or, equivalently (in view of the above theorems), onto a space having a base of countable order? An answer in the negative might be suspected to have profound structural implications.

Let us look at this in another way. H. H. Wicke and the author proposed at the last meeting of the International Congress of Mathematicians in Moscow the thesis that the base of countable order concept, especially when enriched by an appropriate notion of completeability, express substantially much of what is topologically fundamental in the concept of metrizability [18]. If this be valid, then heuristically one might conclude that either every Tychonoff p-space has a perfect mapping onto a space having a base of countable order or else there exists a metacompact Tychonoff p-space which cannot be so transformed. If the reader but put himself in a frame of mind receptive to this line of reasoning, he may sense a heuristic justification for the either/or conclusion. For radical as it may seem in the contemporary milieu the thesis carries with it the corollary that paracompactness-like properties are not naively a part of the essential content of metrizability. (In this connection, see [14], [17], [20], [24], [25].)

Insofar as technique and exposition can be distinguished in a work of this kind, the technical portion of this mémoire will be devoted to the demonstration of the existence of a first countable, regular T_0 locally bicompact, screenable, metacompact space S of power c which admits of no Lindelöfian continuous mapping whatsoever onto a Hausdorff space having a base of countable order. It follows that

S has no perfect mapping onto a space having a base of countable order.

Let us inquire now into the significance of the requirement of first countability of the example. One may say that owing to the definitional sacrifice by any such example of the uniform first countability of the base of countable order property any further interest in first countability is eclipsed to at best a peripheral position of interest. Therefore the underlying issue likely has to do with whether first countability in itself must in certain situations go a long way toward uniformity in this sense. This is in fact the point here. There exist Čech complete spaces Σ of ordinal numbers with respect to the order topology which have no perfect mappings onto spaces with bases of countable order. But every first countable subspace of such a Σ has a base of countable order since it contains no dense subspace [20]. The significance of this is reinforced by the behavior of the base of countable order property under the action of Cartesian products [15] and its hereditary character [27].

One might pursue the significance of the existence of such examples in considerable additional detail. Why the emphasis on the power of the space? Why the reference to σ-discrete refinements? Why the specific mention of screenability? Certain of these questions bear on the intimacy of the interplay between the topics of bases of countable order and paracompactness-like properties [22]. Suffice it here to say that it was felt virtually obligatory to resolve the above question prior to stating the general thesis [28, cf. also 17].

2. Definitions and notation.

Except that the null set convention is not employed, general terminology usually follows along the lines of [9]. As a technical point it is noted that *compactness* is taken in the Fréchet sense, though in the present context one might apply the theorem of [2] that T_1 compact metacompact spaces are bicompact. For *screenable space* and *development of a space*, see [5]; for *metacompact space*, see [2]. As in [11], if K is a collection of sets, K^* denotes the sum of the elements of K. If M is a point set, $\bar{M}$ denotes the (contextually implicit) closure of M. By an *arc* will be understood a bicompact connected Hausdorff point set having exactly two noncut points. By an endpoint of an arc α is meant a noncut point of α. A *perfect mapping* is a bicompact, closed continuous mapping. A *uniform base* for a space S is a base B for S such that if B' is an infinite subcollection of B and P belongs to all members of B', then B' is a base for S at P [1]. Note that a developable space has a uniform base if and only if it is metacompact [1].

A space is *Čech complete* if and only if it is a Tychonoff space S the set of all points of which is an inner limiting set in a Stone-Čech bicompactification $\beta(S)$ [6]. It follows that the set of all points of S is an inner limiting set of any Hausdorff space in which S can be densely embedded [6]. Note that all regular T_0 locally bicompact spaces are Čech complete. A T_1 space S is a *p-space* if and only if it is covered by each term of a sequence $G_1, G_2, \cdots$ of collections of open sets of a Wallman bicompactification ωS such that for each point P of S, all points common to the sets $st(G_n)_P$ belong to S [4]. For the Tychonoff cases, this requires such a sequence $G_1, G_2, \cdots$ with respect to S for any Hausdorff space in which S can be densely embedded. Note the analogy with developability [cf. 26].

A *base of countable order* for a space S is a base B for S such that if $D_1, D_2, \cdots$ is a sequence of distinct members of B each including its successor and P is a point common to all the sets D_n, then $\{D_1\} + \{D_2\} + \cdots$ is a base for S at P [3]. Particularly to be noted are the role of bases of countable order in the characterization of developability involving a paracompactness-like refinement condition [27], the close bearing of the concept on the topic of interior transformations [16, 19], and its tractability to appropriate completeness formulations [18, 19, 20].

3. The construction. The technique of construction utilizes in a rather straightforward manner classical theorems on transfinite cardinalities of a sort such as are developed in [12]. The proof of properties is designed to reduce the question of the existence of certain Lindelöfian mappings in effect to that of the existence of a perfect mapping onto a space having a base of countable order through utilization of restrictions to certain bicompact domains.

THEOREM. *There exists a metacompact screenable locally compact Hausdorff space S of the power of the continuum satisfying these conditions:* (1) *Any collection of open sets covering S is refined by a σ-discrete collection of point sets covering S.* (2) *No Lindelöfian continuous mapping with S as its domain has a Hausdorff space with a base of countable order as its range.* (3) *S is first countable.*

Proof. There exists a sequence $\alpha_1, \alpha_2, \cdots$ of mutually exclusive first countable arcs of cardinal number c such that for each n, there exists a collection Q_n of mutually exclusive subarcs of α_n satisfying these conditions: (1) No element of Q_n contains an endpoint *of* α_n. (2) Q_n^* is dense in α_n. (3) If q and q' are two elements of Q_n then (a) there exists a nonseparable subset Y of $\alpha_n - Q_n^*$ such that q

separates Y from one endpoint of α_n (in the sense of [11]) and q' separates Y from the other and (b) there exist c members of Q_n similarly separated from the endpoints of α_n by q and q'.

Let Γ denote the set of all sequences $J_1, J_2, \cdots$ such that each J_n is a subarc of α_n the endpoints of which belong to $\alpha_n - Q_n^*$. Let M denote a set of power c not intersecting $\alpha_1 + \alpha_2 + \cdots$. There exists a transformation θ of M onto a collection W of simple infinite sequences such that (1) the n^{th} term of each sequence in W belongs to Q_n, (2) no element of $Q_1 + Q_2 + \cdots$ is a term of two members of W, and (3) if $J_1, J_2, \cdots$ belongs to Γ, there exist c sequences $q_1, q_2, \cdots$ in W such that each J_n includes q_n. For each q in $Q_1 + Q_2 + \cdots$, let X_q denote a cut point of q. For each n, let α_n' denote the set of all points P of α_n such that either (1) $\alpha - Q_n^*$ contains P or (2) P is a noncut point of some member of Q_n or (3) P is X_q for some q in Q_n.

Let τ denote the collection to which an element belongs if and only if it is the sum of some sets D satisfying one of these conditions:

(1) For some n, D is an open set of α_n' (in the relative topology).

(2) For some n and element μ of M, D is

$$\{\mu\} + \{X_{q_n}\} + \{X_{q_{n+1}}\} + \cdots ,$$

where $q_1, q_2, \cdots$ denotes $\theta(\mu)$. Let S denote τ^*.

The first countable regular T_0 space (S, τ) is screenable, locally compact, and has the σ-discrete refinement property stated in the theorem. All regular spaces with this σ-discrete refinement property are countably metacompact. Moreover, every countably metacompact screenable space is metacompact. Hence (S, τ) is metacompact. Since (S, τ) is a regular T_0 locally bicompact space, it is Čech complete. Clearly, $\overline{\overline{S}} = c$.

Suppose there exists a Lindelöfian continuous mapping f of (S, τ) onto a Hausdorff space having a base of countable order.

(I) Each f/α_n' is closed and bicompact. Since having a base of countable order is an hereditary property for a space [27], $f(\alpha_n')$ has such a base [21]. Thus the bicompact Hausdorff space $f(\alpha_n')$ is metrizable [3].

For some n let G denote the decomposition of α_n' induced by f. There exists a meaning for the notation $U_{i,h}$, for positive integers i and subsets h of G, such that for some development $H_1, H_2, \cdots$ of G (with respect to the quotient topology) the terms of which are finite, these conditions are satisfied: (1) For each i and element h of H_i, $U_{i,h}$ is a finite collection of sets covering h^* any nondegenerate element of which is the common part of α_n' and some connected open subset of α_n. (2) For each i and element h of H_{i+1} there exists some h' in H_i including h such that the closure of each member of $U_{i+1,h}$ is a subset of some member of $U_{i,h'}$ and is covered by h'. For each i, let K_i denote the sum of all collections $U_{i,h}$ for elements h of H_i.

With application of König's lemma it may be seen that if P belongs to $\alpha_n - Q_n^*$ and g is the member of G containing P there exist sequences $h_1, h_2, \cdots$ and $D_1, D_2, \cdots$ of sets such that (1) each h_i belongs to H_i, contains g, and includes $\bar{h}_{i+1}$, (2) each D_i is a member of U_{k,h_i} containing P, and (3) each D_i includes $\bar{D}_{i+1}$. With use of the compactness of α'_n it may be seen that if $\{P\}$ is the common part of the sets D_i then $\{D_1\} + \{D_2\} + \cdots$ is a base for α'_n at P. Since $K_1 + K_2 + \cdots$ is countable and $\alpha_n - Q_n^*$ is nonseparable, there exist some P in $\alpha_n - Q_n^*$, $h_1, h_2, \cdots$ and $D_1, D_2, \cdots$ as above such that the common part L of the sets $\bar{D}_i$ is nondegenerate. With use of conditions (2) and (3) of the first paragraph of this proof it may be seen that L contains two points of $\alpha_n - Q_n^*$. Since $H_1, H_2, \cdots$ is a development for G and each h_i includes $\bar{h}_{i+1}$, it may be seen that $h_1^*, h_2^*, \cdots$ converges to the member g of G containing P. This requires that L be a subset of g.

(II) Let Δ denote the decomposition of S induced by f. Suppose there exist a member δ of Δ and a sequence $n_1, n_2, \cdots$ of increasing positive integers such that for each i, δ includes the common part of S and some subarc v_i of α_{n_i} the endpoints of which belong to $\alpha_{n_i} - Q_{n_i}^*$. Then there exists a sequence $J_1, J_2, \cdots$ belonging to Γ such that for each i, J_{n_i} is v_i. With the use of condition (3) of the definition of θ and the definition of τ it may be seen that there exists an uncountable closed and isolated subset of M which is on the boundary of

$$S\cdot(v_1 + v_2 + \cdots)$$

and which therefore must be included by the closed point set δ. But this involves a contradiction, for δ is Lindelöfian. With application of (I) above it follows that there exist a sequence $\delta_1, \delta_2, \cdots$ of distinct members of Δ and a sequence $n_1, n_2, \cdots$ of increasing positive integers suchthat for each i, α_{n_i} has a subarc v_i the endpoints of which belong to $\alpha_{n_i} - Q_{n_i}^*$ such that δ_i includes $v_i \cdot \alpha'_{n_i}$. Since no δ_i contains uncountably many elements of M, there exists a countable subset T of M such that $\delta_1 + \delta_2 + \cdots$ does not intersect $M - T$. Uncountably many points of M belong to the boundary of $S\cdot(v_1 + v_2 + \cdots)$. So there exists an element δ of Δ intersecting $M - T$ and an infinite subsequence σ of $\delta_1, \delta_2, \cdots$ such that $f(\sigma)$ converges uniquely to δ. But a contradiction is involved, for $T + \delta \cdot M$ is countable, and uncountably many points of M are limit points of the sum of the terms of σ.

It follows that there exists no Lindelöfian continuous mapping of (S, τ) onto a Hausdorff space having a base of countable order.

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