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AN EXPLICIT FORMULA FOR THE UNITS OF AN ALGEBRAIC NUMBER FIELD OF DEGREE $n \geqq 2$

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An infinite set of algebraic number fields is constructed; they are generated by a real algebraic irrational w, which is the root of an equation $f(w) = 0$ with integer rational coefficients of degree $n \geqq 2$. In such fields polynomials $P_s(w) = a_0w^s + a_1w^{s-1} + \cdots + a_{s-1}w + a_s$ and

$$Q_s(w) = b_0w^s + b_1w^{s-1} + \cdots + b_{s-1}w + b_s$$

($s = 1, \cdots, n-1$; a_k, b_k rational integers) are selected so that the Jacobi-Perron algorithm of the $n-1$ numbers

$$P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$$

carried out in this decreasing order of the polynomials, and of the $n-1$ numbers

$$Q_1(w), Q_2(w), \cdots, Q_{n-1}(w)$$

carried out in this increasing order of the polynomials both become periodic.

It is further shown that $n-1$ different Modified Algorithms of Jacobi-Perron, each carried out with $n-1$ polynomials $P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$ yield periodicity. From each of these algorithms a unit of the field $K(w)$ is obtained by means of a formula proved by the authors is a previous paper.

It is proved that the equation $f(x) = 0$ has n real roots when certain restrictions are put on its coefficients and that, under further restrictions, the polynomial $f(x)$ is irreducible in the field of rational numbers. In the field $K(w)$ $n-1$ different units are constructed in a most simple form as polynomials in w; it is proved in the Appendix that they are independent; the authors conjecture that these $n-1$ independent units are basic units in $K(w)$.

1. Algorithm of $n-1$ numbers. An ordered $(n-1)$-tuple

$$(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)}) \,, \qquad (n \geqq 2) \tag{1}$$

of given numbers, real or complex, among whom there is at least one irrational, will be called a basic sequence; the infinitely many $(n-1)$-tuples

$$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}) \,, \qquad (v = 0, 1, \cdots) \tag{2}$$

will be called supporting sequences. We shall denote by

(3) $$A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$$

the following algorithm connecting the components of the basic sequence with those of the supporting sequences:

(4)
$$a_k^{(v+1)} = \frac{a_{k+1}^{(v)} - b_{k+1}^{(v)}}{a_1^{(v)} - b_1^{(v)}}, \quad (k = 1, \cdots, n-2;\ v = 0, 1, \cdots);$$
$$a_{n-1}^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}};\ a_1^{(v)} \neq b_1^{(v)}; \qquad (v = 0, 1, \cdots).$$

The $(n-1)$-tuples $(a_1^{(v)}, a_2^{(v)}, \cdots, a_{n-11}^{(v)})$, $(v = 0, 1, \cdots)$ will be called generating sequences of the algorithm. $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ is called periodic, if there exist nonnegative integers s and natural numbers t such that

(5) $$a_i^{(v+t)} = a_i^{(v)}, \quad (i = 1, \cdots, n-1;\ v = s, s+1, \cdots).$$

Let be

(6) $$\min s = S; \qquad \min t = T;$$

then the S supporting sequences

(7) $$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}), \qquad (v = 0, 1, \cdots, S-1)$$

are called the primitive preperiod of the algorithm and S is called the length of the preperiod; the T supporting sequences

(8) $$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}), \quad (v = S, S+1, \cdots, S+T-1)$$

are called the primitive period of the algorithm, T is called the length of the period; $S + T$ is called the length of the algorithm. If $S = 0$, the algorithm is called purely periodic.

Two crucial questions emerge from a first look at such an algorithm:

(a) can a formation law be defined by whose help the supporting sequences could be obtained from the basic sequences and the generating sequences?

(b) under what condition is $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ periodic; what is then the nature of the basic sequence and what is the corresponding formation law for the supporting sequences?

For $n = 3$ an algorithm $A(a_1^{(0)}, a_2^{(0)})$ was first introduced by Jacobi [17] and a profound theory of an algorithm of $n-1$ numbers for $n \geqq 2$ was later developed by Oskar Perron [18]; in honor of these great mathematicians the first author of this paper called $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ the algorithm of Jacobi-Perron; they both used the following formation law for the supporting sequences: let $a_i^{(v)}$ be the components of the generating sequences; then

$$b_i^{(v)} = [a_i^{(v)}]\,, \qquad (i = 1, \cdots, n-1\,;\; v = 0, 1, \cdots) \tag{9}$$

where $[x]$ denotes, as customary, the greatest integer not exceeding x. For $n = 2$ the algorithm of Jacobi-Perron becomes the usual Euclidean algorithm.

One of Perron's [18] most significant results is the following

THEOREM. *Let the supporting sequences* $b_i^{(v)}$ $(i = 1, \cdots, n-1\,;$ $v = 0, 1, \cdots)$ *be obtained from the basic sequence* $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ *of real numbers by the formation law* (9). *If the nonnegative integers* $A_i^{(v)}$ *are formed by the recursion formula*

$$\begin{cases} A_i^{(i)} = 1\,;\; A_i^{(v)} = 0\,; & (i \neq v\,;\; i, v = 0, \cdots, n-1) \\ A_i^{(v+n)} = A_i^{(v)} + \sum_{j=1}^{n-1} b_j^{(v)} A_i^{(v+j)}\,, & (i = 0, \cdots, n-1,\; v = 0, 1, \cdots) \end{cases} \tag{10}$$

then $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ *converges in the sense that*

$$a_i^{(0)} = \lim_{v\to\infty} \frac{A_i^{(v)}}{A_0^{(v)}}\,. \qquad (i = 1, \cdots, n-1)\,. \tag{11}$$

Moreover, this theorem can be generalized, as was done by the First author ([8], [10], [11], [12[) in the following way:

Let the supporting sequences be obtained from the basic sequence by any formation law; if the $a_i^{(v)}$, $b_i^{(v)}$ are real numbers such that

$$\begin{cases} a_i^{(v)} > 0\,; \quad (i = 1, \cdots, n-1) \\ b_i^{(v)} \geqq 0\,; \quad (i = 1, \cdots, n-2) \quad 0 < b_{n-1}^{(v)} \leqq C\,; \\ b_i^{(v)}/b_{n-1}^{(v)} \leqq C\,;\; C \text{ a positive constant}, \qquad (v = 0, 1, \cdots) \end{cases} \tag{12}$$

and the numbers $A_i^{(v)}$ (here not necessary integers) are formed as in (10), then $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ converges in the sense of (11).

2. Previous results of the first author. Perron [18] has proved that if $A(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)})$ becomes periodic then the $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ belong to an algebraic number field of degree $\leqq n$. However, he did not succeed to construct, in a general way, algebraic fields K and to select out of K such $n-1$ numbers whose algorithm would become periodic. This was achieved by the first author for an infinite set of algebraic number fields $K(w)$, w being a real irrational root of an algebraic equation $f(w) = 0$ with rational coefficients. In his papers ([1]-[7]) he used (9) for the formation law of the supporting sequences, thus operating with the algorithm of Jacobi-Perron, though heavy restrictions had to be imposed on the coefficients of $f(w)$ in order to achieve periodicity. The first author succeeded to remove these restrictions by introducing a new formation law that generalizes (9) and is defined in the following way:

The $a_i^{(0)}$ and, subsequently, the $a_i^{(v)}$, $(i=1, \cdots, n-1;\ v=0,1,\cdots)$ being numbers of the field $K(w)$ have, generally, he form

$$a_i^{(v)} = a_i^{(v)}(w)\ , \qquad (i = 1, \cdots, n-1; v = 0, 1, \cdots) \tag{13}$$

as long as the $b_i^{(v)}$ are rationals. Let be

$$[w] = D\,; \tag{14}$$

then the formation law of the supporting sequences is given by the formula

$$b_i^{(v)} = a_i^{(v)}(D)\ , \qquad (i, v \text{ as in } (14))\ . \tag{15}$$

In previous papers of the authors the $a_i^{(0)}$ had the form

$$a_i^{(0)} = P_i(w)\ , \qquad (i = 1, \cdots, n-1)\ , \tag{16}$$

thus being polynomials in w with rational coefficients; now the second author of this paper asked the question, whether the algorithm of Jacobi-Perron or any other algorithm

$$A(P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w))$$

of polynomials of decreasing order would yield periodicity, too. This challenging problem could not be solved at first, with the exception of a very few numerical examples, w being a rather simple cubic irrational. Only recently the first author ([13], [14]) could give an affirmative answer. He achieved this by means of a highly complicated formation law for the supporting sequences. But while the new model works well for an infinite set of algebraic number fields $K(w)$; and though in certain cases it is identical with the Jacobi-Perron algorithm—its application does not, at least in this initial stage, seem to go beyond narrow limitations.

In this paper an algebraic number field $K(w)$ is constructed where w is a real algebraic irrational of highly complex nature; but just here it is possible to select polynomials in w such that the algorithms of Jacobi-Perron, viz. for the given $(n-1)$-tuples

$$(P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w))\ ,$$
$$(Q_{n-1}(w), Q_{n-2}(w), \cdots, Q_1(w))$$

both become periodic.

3. The generating polynomial. We shall call the polynomial of degree $n \geqq 2$, viz.

$$
(17) \qquad \begin{aligned} & f(x) = (x - D)(x - D_1)(x - D_2) \cdots (x - D_{n-1}) - d\,; \\ & D, D_i, d \text{ rational integers}\,;\ d \geqq 1\,; \\ & D > D_i\,;\ d \mid (D - D_i),\ (i = 1, \cdots, n-1)\,, \end{aligned}
$$

a Generating Polynomial, to be denoted by *GP*.

In what follows we shall need two theorems regarding the roots of the *GP*.

THEOREM 1. *The GP has one and only one real root w in the open interval $(D, +\infty)$. This root lies in the open interval $(D, D+1)$.*

Proof. The two assertions are immediate consequences of the following three inequalities which follow from the conditions in (17):

$$
\begin{aligned} & f(D) = -d < 0\,, \\ & f'(x) = (f(x) + d)\left(\frac{1}{x - D} + \frac{1}{x - D_1} + \cdots + \frac{1}{x - D_{n-1}}\right) > 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad x > D\,, \\ & f(D+1) = (D + 1 - D_1)(D + 1 - D_2) \cdots (D + 1 - D_{n-1}) - d \\ & \qquad \geqq (d+1)^{n-1} - d \geqq (d+1) - d = 1 > 0\,. \end{aligned}
$$

THEOREM 2. *Let the integers D, D_i occuring in the GP satisfy, in addition to* (17), *the conditions*

$$
(18) \qquad D = D_0 > D_1 > \cdots > D_{n-1}\,,
$$

and in the special case $d = 1$ moreover

$$
(19) \qquad \begin{cases} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4, \text{ for } n = 3\,; \\ \begin{cases} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \geqq 3 \text{ or} \\ D_0 - D_1, D_2 - D_3 \geqq 2, \text{ for } n = 4\,. \end{cases} \end{cases}
$$

Then the GP has exactly n different real roots. Of these lie

1 *in the open interval $(D_0, +\infty)$, more exactly in the open interval $(D_0, D_0 + 1)$,*

2 *in each of the open intervals (D_{2i}, D_{2i-1}), more exactly 1 in the open left half, 1 in the open right half of these intervals with $2 \leqq 2i \leqq n - 1$, 1 in the open interval $(-\infty, D_{n-1})$ if n is even.*

Proof. Since the total number of roots asserted in the latter three statements is exactly equal to the degree n of the *GP*, it suffices to prove the existence of at least 1, 2, 1 roots respectively within the indicated open intervals. For the first interval this has been done in Theorem 1. For the other intervals it suffices, besides the obvious facts

$$f(D_i) = -d < 0 \qquad (i = 0, 1, \cdots, n-1)$$

and

$$\lim_{x\to-\infty} f(x) = +\infty \quad \text{if} \quad n \text{ is even} ,$$

to verify the inequalities

$$f(c_i) > 0 \qquad (2 \leqq 2i \leqq n-1) ,$$

i.e.,

$$f(c_i) + d = (c_i - D_0)(c_i - D_1) \cdots (c_i - D_{n-1}) > d ,$$

with $2 \leqq 2i \leqq n-1$ and $c_i = (D_{2i-1} + D_{2i})/2$. Now according to (18)

$$\begin{aligned} c_i - D_j &< 0 \quad \text{for} \quad j = 0, 1, \cdots, 2i-1 , \\ c_i - D_j &> 0 \quad \text{for} \quad j = 2i, 2i+1, \cdots, n-1 , \end{aligned}$$

and as the j in the first line are in even number, certain at least

$$f(c_i) + d > 0 .$$

According to (17) and the obvious consequence $d \mid (D_i - D_j)$ one has more precisely

$$|c_i - D_j| \geqq d + \frac{d}{2} = \frac{3}{2}d \quad \text{for} \quad j \neq 2i-1, 2i ,$$

$$|c_i - D_j| \geqq \frac{1}{2}d \quad \text{for} \quad j = 2i-1, 2i ,$$

and hence

$$f(c_i) + d \geqq (3d/2)^{n-2}(d/2)^2 = \frac{3^{n-2}}{2}(d/2)^{n-1}d .$$

Observing that $2 \leqq 2i \leqq n-1$ implies $n \geqq 3$, one obtains thus for $d \geqq 2$ the desired inequalities

$$f(c_i) + d \geqq 3d/2 > d .$$

In the special case $d = 1$ still more precise lower estimates are required, viz.,

$$|c_i - D_j| \geqq (2i-1-j)d + \frac{d}{2} = 2i-1-j+\frac{1}{2} \text{ for } j = 0, 1, \cdots, 2i-1,$$

$$|c_i - D_j| \geqq (j-2i)d + \frac{d}{2} = j - 2i + \frac{1}{2} \quad \text{for} \quad j = 2i, \cdots, n-1.$$

The lower bounds have values from the sequence $1/2, 3/2, 5/2, \cdots$ For each relevant i two values $1/2$ and, if $n \geqq 5$, at least two values $3/2$ and one value $5/2$ occur. For $n \geqq 5$ therefore certainly

$$f(c_i) + 1 \geqq \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)^2\left(\frac{5}{2}\right) > 1.$$

In the remaining cases $d = 1$ with $n = 3,4$ there is only one relevant i, viz., $i = 1$. One verifies easily that the desired inequality

$$f(c_1) + 1 > 1$$

is true under the conditions (19).

We shall now rearrange $f(x)$ in powers of $x - D$. We shall first prove the formula

$$\begin{aligned} f^{(k)}(x) &= k!\sum(x - D_{i_1}) \cdots (x - D_{i_{n-k}}), \\ & 0 \leqq i_1 < i_2 < \cdots < i_{n-k} \leqq n-1, \\ & k = 1, \cdots, n-1. \end{aligned} \tag{20}$$

We shall denote

$$\begin{aligned} g(x) &= (x - D_0)(x - D_1) \cdots (x - D_{n-1}); \; f(x) = g(x) - d. \\ f'(x) &= g'(x) = g(x)\sum(1/(x - D_j)) \\ &= 1!\sum(x - D_{i_1})(x - D_{i_2}) \cdots (x - D_{i_{n-1}}) \\ & \qquad 0 \leqq i_1 < i_2 < \cdots < i_{n-1} \leqq n-1. \end{aligned} \tag{21}$$

Thus formula (20) is correct for $k = 1$. Let it be correct for $k = m$, namely

$$f^{(m)}(x) = m!\sum(x - D_{i_1})(x - D_{i_2}) \cdots (x - D_{i_{n-m}}),$$
$$0 \leqq i_1 < i_2 < i_3 < \cdots < i_{n-m} \leqq n-1$$

or, in virtue of (21)

$$\begin{cases} f^{(m)}(x) = m!\, g(x)\sum \dfrac{1}{(x - D_{j_1})(x - D_{j_2}) \cdots (x - D_{j_m})} \\ \qquad 0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1. \end{cases} \tag{22}$$

Differentiating (22) we obtain

$$\frac{1}{m!}f^{(m+1)}(x)$$

$$= g'(x) \sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}$$
$$0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1$$

$$+ g(x)\Big(\sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}\Big)'$$
$$0 \leqq j_1 < j_2 < j_3 < \cdots < j_m < n-1$$

$$= g(x)\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}$$
$$0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1$$

$$- g(x)\sum_{r=1}^{m}\sum \frac{1}{(x-D_{j_1})\cdots(x-D_{j_{r-1}})(x-D_{j_r})^2(x-D_{j_{r+1}})\cdots(x-D_{j_m})}\,.$$

But it is easily seen that

$$\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}$$
$$0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1$$

$$= \sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}$$
$$s \neq j_1, \cdots, j_m\,;\ 0 \leqq j_1 < j_2 < \cdots < j_m \leqq n-1$$

$$+ \sum \frac{1}{(x-D_{j_1})\cdots(x-D_{j_{r-1}})(x-D_{j_r})^2(x-D_{j_{r+1}})\cdots(x-D_{j_m})}$$
$$0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1\,.$$

Therefore

$$\frac{1}{m!}f^{(m+1)}(x)$$

$$= g(x)\sum_{s=0}^{n-1}\frac{1}{x-D_s}\sum \frac{1}{(x-D_{j_1})(x-D_{j_2})\cdots(x-D_{j_m})}$$
$$s \neq j_1, \cdots, j_m\,;\ 0 \leqq j_1 < j_2 < j_3 < \cdots < j_m \leqq n-1$$

$$= (m+1)g(x)\sum \frac{1}{(x-D_{t_1})(x-D_{t_2})\cdots(x-D_{t_{m+1}})}\,,$$
$$0 \leqq t_1 < t_2 < t_3 < \cdots < t_{m+1} \leqq n-1$$

$$f^{(m+1)}(x) = (m+1)!\,g(x)\sum \frac{1}{(x-D_{t_1})(x-D_{t_2})\cdots(x-D_{t_{m+1}})}$$
$$0 \leqq t_1 < t_2 < t_3 < \cdots < t_{m+1} \leqq n-1$$

$$= (m+1)!\sum (x-D_{i_1})(x-D_{i_2})\cdots(x-D_{i_{n-(m+1)}})$$
$$0 \leqq i_1 < i_2 < i_3 < \cdots < i_{n-(m+1)} \leqq n-1$$

which proves formula (20).

From (20) we obtain for $x = D_0 = D$, taking into account that $D - D_{i_1} = 0$ for $i_1 = 0$

$$(23) \qquad \begin{aligned} &f^{(k)}(D) = k! \sum (D - D_{i_1})(D - D_{i_2}) \cdots (D - D_{i_{n-k}}) , \\ &1 \leqq i_1 < i_2 < \cdots < i_{n-k} \leqq n - 1 , \\ &k = 1, \cdots, n - 1 . \end{aligned}$$

From (17) we obtain

$$(23.\ a) \qquad f(D) = -d \, ; \; f^{(n)}(D) = n! \, ,$$

and, combining (23), (23. a) and using Taylor's formula for developing $f(x)$ in powers of $x - D$,

$$(24) \qquad \begin{aligned} &f(x) = (x - D)^n + \left(\sum_{s=1}^{n-1} k_s (x - D)^{n-s}\right) - d , \\ &k_s = \sum (D - D_{i_1})(D - D_{i_2}) \cdots (D - D_{i_s}) , \\ &1 \leqq i_1 < i_2 < \cdots < i_s \leqq n - 1 . \end{aligned}$$

4. **Inequalities.** In this chapter we shall prove the inequalities needed for carrying out the Algorithm of Jacobi-Perron with a basic sequence $a_i^{(0)}$ $(i = 1, \cdots, n - 1)$ chosen from the field $K(w)$.

We obtain from Theorem 1 and $D < w < D + 1$

$$(25) \qquad [w] = D \, .$$

In the sequel we shall find the following notations useful

$$(26) \qquad \begin{cases} P_{i,i} = P_i = w - D_i \, , & (i = 1, \cdots, n - 1) \\ P_{i,k} = P_i P_{i+1} \cdots P_k \, ; & 1 \leqq i \leqq k \leqq n - 1 \, . \end{cases}$$

One of the basic inequalities needed in the following

$$(27) \qquad \begin{cases} [(w - D)P_{i_1} P_{i_2} \cdots P_{i_k}] = 0 \, , \\ 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n - 2 \, . \end{cases}$$

To prove (27) we have to verify

$$(28) \qquad 0 < (w - D)P_{i_1} P_{i_2} \cdots P_{i_k} < 1 \, .$$

From (25), (26) we obtain

$$P_i = w - D_i > D - D_i > 0 \, .$$

Thus the left-hand inequality of (28) is proved. From (17) we obtain

$$(29) \qquad w - D = \frac{d}{P_{1,n}} \, .$$

Therefore

$$(w - D)P_{i_1}P_{i_2} \cdots P_{i_k} = dP_{i_1}P_{i_2} \cdots P_{i_k}/P_{1,n}$$
$$= d/P_{i_{k+1}}P_{i_{k+2}} \cdots P_{i_{n-1}} < d/(D - D_{i_{k+1}})(D - D_{i_{k+2}}) \cdots (D - D_{i_{n-1}}) ;$$

but, as was proved before, $D - D_{i_j} \geqq d$; $(j = 1, \cdots, n - 1)$ therefore

$$(w - D)P_{i_1}P_{i_2} \cdots P_{i_k} < \frac{d}{d^{n-k-1}} \leqq \frac{d}{d} = 1 ,$$

which proves the right-hand inequality of (28).

From (27) we obtain easily, since $d \geqq 1$

$$\begin{cases} [(w - D)P_{i_1}P_{i_2} \cdots P_{i_k}] = 0 ; \\ 1 \leqq i_1 < i_2 < \cdots i_k \leqq n - 2 . \end{cases} \tag{30}$$

We further obtain, in virtue of (25)

$$[P_i] = D - D_i \qquad (i = 1, \cdots, n - 1) . \tag{31}$$

From (31) we obtain, since $d \mid D - D_i$,

$$[P_i/d] = (D - D_i)/d \qquad (i = 1, \cdots, n - 1) . \tag{32}$$

5. Jacobi-perron algorithm for polynomials of decreasing order.

DEFINITION. An $(n - 1)$ by $(n - 1)$ matrix of the form

$$\left\{ \begin{Vmatrix} 0 & 0 & \cdots & 0 & A_1 \\ 0 & 0 & \cdots & 0 & A_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & A_{n-1} \end{Vmatrix} \right. \tag{33}$$

will be called a *fugue*; the last column vector

$$\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{matrix}$$

will be called the *generator* of the fugue.

THEOREM 3. *Let $f(x)$ be the GP from* (17) *and w its only real root in the open interval* $(D, D + 1)$. *The Jacobi Perron Algorithm of the decreasing order polynomials*

$$(34)\qquad \begin{cases} a_s^{(0)} = \dfrac{1}{d}(w - D)P_{1,1}P_{2+s,n-1}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(0)} = \dfrac{1}{d}(w - D)P_{1,1}\,, & \\ a_{n-1}^{(0)} = P_{1,1}\,, & \end{cases}$$

is purely periodic and its primitive length is $T = n(n - 1$ *for* $d \neq 1$, *and* $T = n - 1$ *for* $d = 1$. *The period of length* $n(n - 1)$ *consist of* n *fugues. The generator of the first fugue has the form*

$$(35)\qquad \begin{matrix} D - D_1 \\ D - D_2 \\ \vdots \\ D - D_{n-1}\,. \end{matrix}$$

The generator of the $r + 1$ *-th fugue* $(r = 1, \cdots, n - 1)$ *has the form*

$$(36)\qquad \begin{cases} D - D_1 \\ D - D_2 \\ \vdots \\ \dfrac{D - D_r}{d} \\ \vdots \\ D - D_{n-1} \end{cases} .$$

The period of length $n - 1$ *consists of one fugue whose generator has the form* (35).

Proof. In the sequel we shall use the notation

$$(37)\qquad \begin{cases} u\,;\ v = u(n-1+)\,v\,; & (u = 0, 1, \cdots;\ v = 0, 1, \cdots, n-2) \\ u\,;\ n-1 = u+1\,;\ 0\,. \end{cases}$$

Because of (26) the formula holds

$$(38)\qquad P_{i,s}/P_{i,k} = 1/P_{s+1,k}\,; \qquad 1 \leqq i \leqq s < k \leqq n - 1\,.$$

Since, from (17),

$$(w - D)(w - D_1)(w - D_2) \cdots (w - D_{n-1}) - d = 0\,,$$

we obtain

$$(39)\quad \begin{cases} \dfrac{1}{d}(w-D)P_{1,1}P_{2+s,n-1} = \dfrac{1}{P_{2,1+s}}\,; & (s=1,\cdots,n-3) \\ \dfrac{1}{d}(w-D)P_{1,1} = \dfrac{1}{P_{2,n-1}}\,. \end{cases}$$

We shall substitute these values for $a_s^{(0)}$ in (34), so that

$$(40)\quad \begin{cases} a_s^{(0)} = \dfrac{1}{P_{2,1+s}}\,; & (s=1,\cdots,n-2) \\ a_{n-1}^{(0)} = P_{1,1}\,. \end{cases}$$

We obtain from (34), in virtue of (30), (31)

$$(41)\quad b_s^{(0)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(0)} = D - D_1\,.$$

We obtain from (31)

$$(42)\quad \begin{cases} P_{i,i} - [P_{i,i}] = w - D\,; \\ \dfrac{P_{i,i}}{d} - \left[\dfrac{P_{i,i}}{d}\right] = \dfrac{w-D}{d} & (i=1,\cdots,n-1)\,. \end{cases}$$

From (40)—(42) we obtain

$$\begin{cases} a_s^{(0)} - b_s^{(0)} = \dfrac{1}{P_{2,1+s}}\,, & (s=1,\cdots,n-2) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = w - D\,; \\ a_1^{(0)} - b_1^{(0)} = \dfrac{1}{P_{2,2}}\,; \\ a_{1+s}^{(0)} - b_{1+s}^{(0)} = \dfrac{1}{P_{2,2+s}}\,; & (s=1,\cdots,n-3) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = w - D\,, \end{cases}$$

so that, in virtue of (4)

$$\begin{cases} a_s^{(1)} = \dfrac{P_{2,2}}{P_{2,2+s}}\,, & (s=1,\cdots,n-3) \\ a_{n-2}^{(1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(1)} = P_{2,2}\,. \end{cases}$$

From these formulas we obtain, in virtue of (40)

$$(43)\quad \begin{cases} a_s^{(1)} = \dfrac{1}{P_{3,2+s}}\,, & (s=1,\cdots,n-3) \\ a_{n-2}^{(1)} = (w-D)P_{2,2}\,, \\ a_{n-1}^{(1)} = P_{2,2} \end{cases}$$

Since

$$1/P_{3,2+s} = \frac{1}{d}(w - D)P_1P_2 ,$$

we obtain, from (43) and in virtue of (30), (27), (31)

$$b_s^{(1)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = D - D_2 ,$$

and from (43), (44), in virtue of (42)

$$\text{(45)} \qquad \begin{cases} a_1^{(1)} - b_1^{(1)} = \dfrac{1}{P_{3,3}} , \\ a_{1+s}^{(1)} - b_{1+s}^{(1)} = \dfrac{1}{P_{3,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-2}^{(1)} - b_{n-2}^{(1)} = (w - D)P_{2,2} , \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = w - D . \end{cases}$$

Frow (45) we obtain, in virtue of (4) and (38)

$$\begin{cases} a_s^{(2)} = \dfrac{P_{3,3}}{P_{3,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = (w - D)P_{2,2}P_{3,3} , \\ a_{n-2}^{(2)} = (w - D)P_{3,3} , \\ a_{n-1}^{(2)} = P_{3,3} ; \end{cases}$$

$$\text{(46)} \qquad \begin{cases} a_s^{(2)} = \dfrac{1}{P_{4,3+s}} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = (w - D)P_{2,3} , \\ a_{n-2}^{(2)} = (w - D)P_{3,3} , \\ a_{n-1}^{(2)} = P_{3,3} . \end{cases}$$

We shall now prove the formula

$$\text{(47)} \qquad \begin{cases} a_s^{(k)} = 1/P_{k+2,k+1+s} , & (s = 1, \cdots, n-k-2) \\ a_{n-k-2+i}^{(k)} = (w - D)P_{1+i,k+1} , \\ a_{n-1}^{(k)} = P_{k+1,k+1} , \\ k = 2, \cdots, n-3 . \end{cases}$$

Formula (47) is valid for $k = 2$ in virtue of (46). We shall prove its validity for $k + 1$. Since

$$1/P_{k+2,k+1+s} = \frac{1}{d}(w - D)P_{1,k+1} ,$$

we obtain from (47), in virtue of (30), (27), (31)

$$\text{(48)} \qquad b_j^{(k)} = 0 ; \quad (j = 1, \cdots, n-2) \qquad b_{n-1}^{(k)} = D - D_{k+1} ,$$

and from (47), (48), in virtue of (42)

$$\begin{cases} a_s^{(k)} - b_s^{(k)} = 1/P_{k+2,k+1+s} , & (s = 1, \cdots, n-k-2) \\ a_{n-k-2+i}^{(k)} - b_{n-k-2+i}^{(k)} = (w-D)P_{1+i,k+1} , & (i = 1, \cdots, k) \\ a_{n-1}^{(k)} - b_{n-1}^{(k)} = w - D ; \end{cases}$$

$$\begin{cases} a_1^{(k)} - b_1^{(k)} = 1/P_{k+2,k+2} , \\ a_{1+s}^{(k)} - b_{1+s}^{(k)} = 1/P_{k+2,k+2+s} , & (s = 1, \cdots, n-k-3) \\ a_{n-k-2+i}^{(k)} - b_{n-k-2+i}^{(k)} = (w-D)P_{1+i,k+1} , & (i = 1, \cdots, k) \\ a_{n-1}^{(k)} - b_{n-1}^{(k)} = w - D , \end{cases}$$

so that, in virtue of (4)

$$\begin{cases} a_s^{(k+1)} = P_{k+2,k+2}/P_{k+2,k+2+s} , & (s = 1, \cdots, n-k-3) \\ a_{n-k-3+i}^{(k+1)} = (w-D)P_{1+i,k+1}P_{k+2,k+2} , & (i = 1, \cdots k) \\ a_{n-2}^{(k+1)} = (w-D)P_{k+2,k+2} , \\ a_{n-1}^{(k+1)} = P_{k+2,k+2} , \end{cases}$$

and, in virtue of (42),

$$(49) \qquad \begin{cases} a_s^{(k+1)} = 1/P_{k+3,k+2+s} , & (s = 1, \cdots, n-k-3) \\ a_{n-k-3+i}^{(k+1)} = (w-D)P_{1+i,k+2} , & (i = 1, \cdots, k+1) \\ a_{n-1}^{(k+1)} = P_{k+2,k+2} . \end{cases}$$

With (49) formula (47) is proved.

We now obtain from (47) for $k = n-3$

$$(50) \qquad \begin{cases} a_1^{(n-3)} = 1/P_{n-1,n-1} , \\ a_{1+i}^{(n-3)} = (w-D)P_{1+i,n-2} , & (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-3)} = P_{n-2,n-2} . \end{cases}$$

From (50) we obtain, in virtue of (30), (27), (31)

$$(51) \qquad b_s^{(n-3)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3)} = D - D_{n-2} ,$$

and from (50), (51), in virtue of (42)

$$(52) \qquad \begin{cases} a_1^{(n-3)} - b_1^{(n-3)} = 1/P_{n-1,n-1} , \\ a_{1+i}^{(n-3)} - b_{1+i}^{(n-3)} = (w-D)P_{1+i,n-2} , & (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-3)} - b_{n-1}^{(n-3)} = w - D . \end{cases}$$

From (52) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(n-2)} = (w-D)P_{1+i,n-2}P_{n-1,n-1} , & (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-2)} = (w-D)P_{n-1,n-1} , \\ a_{n-1}^{(n-2)} = P_{n-1,n-1} ; \end{cases}$$

or

(53)
$$\begin{cases} a_s^{(n-2)} = (w - D)P_{1+i,n-1} , & (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-2)} = P_{n-1,n-1} . \end{cases}$$

From (53) we obtain, in virtue of (27), (31),

(54)
$$b_s^{(n-2)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2)} = D - D_{n-1} ,$$

and from (53), (54), in virtue of (42),

$$\begin{cases} a_s^{(n-2)} - b_s^{(n-2)} = (w - D)P_{1+s,n-1} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D ; \end{cases}$$

$$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = (w - D)P_{2,n-1} , \\ a_{1+s}^{(n-2)} - b_{1+s}^{(n-2)} = (w - D)P_{2+s,n-1} , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D , \end{cases}$$

so that, in virtue of (4),

$$\begin{cases} a_s^{(n-1)} = P_{2+s,n-1}/P_{2,n-1} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(n-1)} = 1/P_{2,n-1} , \\ a_{n-1}^{(n-1)} = 1/(w - D)P_{2,n-1} ; \end{cases}$$

but, from (39) we obtain

$$1/(w - D)P_{2,n-1} = P_{1,1}/d ;$$

therefore,

$$\begin{cases} a_s^{(n-1)} = 1/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-1)} = P_{1,1}/d ; \end{cases}$$

thus, with the notation of (37),

(55)
$$\begin{cases} a_s^{(1;0)} = 1/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(1;0)} = P_{1,1}/d . \end{cases}$$

From (55) we obtain, in virtue of (30), (32), and since

$$1/P_{2,1+s} = \frac{1}{d}(w - D)P_{2+s,n-1}$$

(56)
$$b_s^{(1;0)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;0)} = \frac{D - D_1}{d} ,$$

and from (55), (56), in virtue of (42)

$$\begin{cases} a_s^{(1;0)} - b_s^{(1;0)} = 1/P_{2,1+s} , & (s = 1, \cdots, n-2) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = \dfrac{w-D}{d} ; \end{cases}$$

or

$$\begin{cases} a_1^{(1;0)} - b_1^{(1;0)} = 1/P_{2,2} , \\ a_{1+s}^{(1;0)} - b_{1+s}^{(1;0)} = 1/P_{2,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = \dfrac{w-D}{d} ; \end{cases}$$

thus, in virtue of (4),

$$\begin{cases} a_s^{(1;1)} = P_{2,2}/P_{2,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = (w-D)P_{2,2}/d , \\ a_{n-1}^{(1;1)} = P_{2,2} ; \end{cases}$$

or

$$\text{(57)} \qquad \begin{cases} a_s^{(1;1)} = 1/P_{3,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = (w-D)P_{2,2}/d , \\ a_{n-1}^{(1;1)} = P_{2,2} . \end{cases}$$

From (57) we obtain, as before,

$$\text{(58)} \qquad b_s^{(1;1)} = 0 ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1;1)} = D - D_2 ,$$

and from (57), (58), in virtue of (42),

$$\begin{cases} a_s^{(1;1)} - b_s^{(1;1)} = 1/P_{3,2+s} , & (s = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = (w-D)P_{2,2}/d , \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = w - D ; \end{cases}$$

or

$$\text{(59)} \qquad \begin{cases} a_1^{(1;1)} - b_1^{(1;1)} = 1/P_{3,3} , \\ a_{1+s}^{(1;1)} - b_{1+s}^{(1;1)} = 1/P_{3,3+s} , & (s = 1, \cdots, n-4) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = (w-D)P_{2,2}/d , \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = w - D . \end{cases}$$

From (59) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(1;2)} = P_{3,3}/P_{3,3+s} , & (s = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = (w-D)P_{2,2}P_{3,3}/d , \\ a_{n-2}^{(1;2)} = (w-D)P_{3,3} , \\ a_{n-1}^{(1;2)} = P_{3,3} , \end{cases}$$

or

$$(60)\qquad \begin{cases} a_s^{(1;2)} = 1/P_{4,3+s}\,, & (s = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = (w - D)P_{2,3}/d\,, & \\ a_{n-2}^{(1;2)} = (w-D)P_{3,3}\,, & \\ a_{n-1}^{(1;2)} = P_{3,3}\,. & \end{cases}$$

We shall now prove the formula

$$(61)\qquad \begin{cases} a_2^{(1;k)} = 1/P_{k+2,k+1+s}\,, & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d\,, & \\ a_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1}\,, & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} = P_{k+1,k+1}\,, & \\ k = 2, \cdots, n-3\,. & \end{cases}$$

Formula (61) is correct for $k = 2$, in virtue of (60). We shall prove by induction that it is correct for $k+1$.

We obtain from (61), as before,

$$(62)\qquad b_s^{(1;k)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(k)} = D - D_{k+1}\,,$$

and from (61), (62), in virtue of (42)

$$\begin{cases} a_s^{(1;k)} - b_s^{(1;k)} = 1/P_{k+2,k+1+3}\,, & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(1;k)} - b_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d\,, & \\ a_{n-k-1+i}^{(1;k)} - b_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1} & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} - b_{n-1}^{(1;k)} = w - D\,; & \end{cases}$$

or

$$(63)\qquad \begin{cases} a_1^{(1;k)} - b_1^{(1;k)} = 1/P_{k+2,k+2}\,, & \\ a_{1+s}^{(1;k)} - b_{1+s}^{(1;k)} = 1/P_{k+2,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-1}^{(1;k)} - b_{n-k-1}^{(1;k)} = (w - D)P_{2,k+1}/d\,, & \\ a_{n-k-1+i}^{(1;k)} - b_{n-k-1+i}^{(1;k)} = (w - D)P_{2+i,k+1}\,, & (i = 1, \cdots, k-1) \\ a_{n-1}^{(1;k)} - b_{n-1}^{(1;k)} = w - D\,. & \end{cases}$$

From (63) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(1;k+1)} = P_{k+2,k+2}/P_{k+2,k+2+s}\,, & (s = 1, \cdots, n-k-3) \\ a_{n-k-2}^{(1;k+1)} = (w-D)P_{2,k+1}P_{k+2,k+2}/d\,, & \\ a_{n-k-2+i}^{(1;k+1)} = (w - D)P_{2+i,k+1}P_{k+2,k+2}\,, & (i = 1, \cdots, k-1) \\ a_{n-2}^{(1;k+1)} = (w - D)P_{k+2,k+2}\,, & \\ a_{n-1}^{(1;k+1)} = P_{k+2,k+2}\,; & \end{cases}$$

or

$$(64)\quad \begin{cases} a_s^{(1;k+1)} = 1/P_{k+3,k+2+s}\,, & (s=1,\cdots,n-k-3)\\ a_{n-k-2}^{(1;k+1)} = (w-D)P_{2,k+2}\,, & \\ a_{n-k-2+i}^{(1;k+1)} = (w-D)P_{2+i,k+2}\,, & (i=1,\cdots,k)\\ a_{n-1}^{(1;k+1)} = P_{k+2,k+2}\,. & \end{cases}$$

With (64) formula (61) is proved.

We now obtain from (61) for $k = n-3$

$$(65)\quad \begin{cases} a_1^{(1;n-3)} = 1/P_{n-1,n-1}\,, & \\ a_2^{(1;n-3)} = (w-D)P_{2,n-2}/d\,, & \\ a_{2+i}^{(1;n-3)} = (w-D)P_{2+i,n-2}\,, & (i=1,\cdots,n-4)\\ a_{n-1}^{(1;n-3)} = P_{n-2}\,. & \end{cases}$$

From (65) we obtain, as before,

$$(66)\qquad b_s^{(1;n-3)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(1;n-3)} = D - D_{n-2}\,.$$

From (65), (66) we obtain as before

$$(67)\quad \begin{cases} a_1^{(1;n-3)} - b_1^{(1;n-3)} = 1/P_{n-1,n-1}\,, & \\ a_2^{(1;n-3)} - b_2^{(1;n-3)} = (w-D)P_{2,n-2}/d\,, & \\ a_{2+i}^{(1;n-3)} - b_{2+i}^{(1;n-3)} = (w-D)P_{2+i,n-2}\,, & (i=1,\cdots,n-4)\\ a_{n-1}^{(1;n-3)} - b_{n-i}^{(1;n-3)} = w-D\,; & \end{cases}$$

and from (67), in virtue of (4),

$$\begin{cases} a_1^{(1;n-2)} = (w-D)P_{2,n-2}P_{n-1,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} = (w-D)P_{2+i,n-2}P_{n-1,n-1}\,, & (i=1,\cdots,n-4)\\ a_{n-2}^{(1;n-2)} = (w-D)P_{n-1,n-1}\,, & \\ a_{n-1}^{(1;n-2)} = P_{n-1,n-1}\,; & \end{cases}$$

or

$$(68)\quad \begin{cases} a_1^{(1;n-2)} = (w-D)P_{2,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} = (w-D)P_{2+i,n-1}\,, & (i=1,\cdots,n-3)\\ a_{n-1}^{(1;n-2)} = P_{n-1,n-1}\,. & \end{cases}$$

From (68) we obtain, as before,

$$(69)\qquad b_s^{(1;n-2)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(1;n-2)} = D - D_{n-1}\,,$$

and from (68), (69), in virtue of (42)

$$(70)\quad \begin{cases} a_1^{(1;n-2)} - b_1^{(1;n-2)} = (w-D)P_{2,n-1}/d\,, & \\ a_{1+i}^{(1;n-2)} - b_{1+i}^{(1;n-2)} = (w-D)P_{2+i,n-1}\,, & (i=1,\cdots,n-3)\\ a_{n-1}^{(1;n-2)} - b_{n-1}^{(1;n-2)} = w-D\,. & \end{cases}$$

From (70) we obtain, in virtue of (4) and (39)

$$\begin{cases} a_s^{(2;0)} = dP_{2+s,n-1}/P_{2,n-1}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;0)} = d/P_{2,n-1}\,, & \\ a_{n-1}^{(2;0)} = P_{1,1}\,; & \end{cases}$$

or

$$(71) \qquad \begin{cases} a_s^{(2;0)} = d/P_{2,1+s}\,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(2;0)} = P_{1,1}\,. & \end{cases}$$

From (71) we obtain, as before,

$$(72) \qquad b_s^{(2;0)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;0)} = D - D_1\,,$$

and from (71), (72), in virtue of (42)

$$\begin{cases} a_s^{(2;0)} - b_s^{(2;0)} = d/P_{2,1+s}\,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(2;0)} - b_{n-1}^{(2;0)} = w - D\,; & \end{cases}$$

or

$$(73) \qquad \begin{cases} a_1^{(2;0)} - b_1^{(2;0)} = d/P_{2,2}\,, & \\ a_{1+s}^{(2;0)} - b_{1+s}^{(2;0)} = d/P_{2,2+s}\,, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(2;0)} - b_{n-1}^{(2;0)} = w - D\,. & \end{cases}$$

From (73) we obtain, in virtue of (4)

$$\begin{cases} a_s^{(2;1)} = P_{2,2}/P_{2,2+s}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} = (w - D)P_{2,2}/d\,, & \\ a_{n-1}^{(2;1)} = P_{2,2}/d\,; & \end{cases}$$

or

$$(74) \qquad \begin{cases} a_s^{(2;1)} = 1/P_{3,2+s}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} = (w - D)P_{2,2}/d\,, & \\ a_{n-1}^{(2;1)} = P_{2,2}/d\,; & \end{cases}$$

and from (74), as before,

$$(75) \qquad b_s^{(2;1)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;1)} = (D - D_2)/d\,.$$

From (74), (75) we obtain, in virtue of (42)

$$\begin{cases} a_s^{(2;1)} - b_s^{(2;1)} = 1/P_{3,2+s}\,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(2;1)} - b_{n-2}^{(2;1)} = (w - D)P_{2,2}/d\,, & \\ a_{n-1}^{(2;1)} - b_{n-1}^{(2;1)} = (w - D)/d\,; & \end{cases}$$

or

$$(76)\quad \begin{cases} a_1^{(2;1)} - b_1^{(2;1)} = 1/P_{3,3}\,, & \\ a_{1+s}^{(2;1)} - b_{1+s}^{(2;1)} = 1/P_{3,3+s}\,, & (s = 1, \cdots, n-4) \\ a_{n-2}^{(2;1)} - b_{n-2}^{(2;1)} = (w - D)P_{2,2}/d\,, & \\ a_{n-1}^{(2;1)} - b_{n-1}^{(2;1)} = (w - D)/d\,. & \end{cases}$$

From (76) we obtain, in virtue of (4),

$$\begin{cases} a_s^{(2;2)} = P_{3,3}/P_{3,3+s}\,, & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} = (w - D)P_{2,2}P_{3,3}/d\,, & \\ a_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\,, & \\ a_{n-1}^{(2;2)} = P_{3,3}\,; & \end{cases}$$

or

$$(77)\quad \begin{cases} a_s^{(2;2)} = 1/P_{4,3+s}\,, & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\,, & \\ a_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\,, & \\ a_{n-1}^{(2;2)} = P_{3,3}\,. & \end{cases}$$

From (77) we obtain, as before,

$$(78)\qquad b_s^{(2;2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;2)} = D - D_3\,,$$

and from (77), (78) and in virtue of (42),

$$\begin{cases} a_s^{(2;2)} - b_s^{(2;2)} = 1/P_{4,3+s}\,, & (s = 1, \cdots, n-4) \\ a_{n-3}^{(2;2)} - b_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\,, & \\ a_{n-2}^{(2;2)} - b_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\,, & \\ a_{n-1}^{(2;2)} - b_{n-1}^{(2;)} = w - D\,, & \end{cases}$$

or

$$(79)\quad \begin{cases} a_1^{(2;2)} - b_1^{(2;2)} = 1/P_{4,4}\,, & \\ a_{1+s}^{(2;2)} - b_{1+s}^{(2;2)} = 1/P_{4,4+s}\,, & (s = 1, \cdots, n-5) \\ a_{n-3}^{(2;2)} - b_{n-3}^{(2;2)} = (w - D)P_{2,3}/d\,, & \\ a_{n-2}^{(2;2)} - b_{n-2}^{(2;2)} = (w - D)P_{3,3}/d\,, & \\ a_{n-1}^{(2;2)} - b_{n-1}^{(2;2)} = w - D\,. & \end{cases}$$

From (79) we obtain, in virtue of (4), and carrying out cancellation and multiplication as before,

$$(80)\quad \begin{cases} a_s^{(2;3)} = 1/P_{5,4+s}\,, & (s = 1, \cdots, n-5) \\ a_{n-4}^{(2;3)} = (w - D)P_{2,4}/d\,, & \\ a_{n-3}^{(2;3)} = (w - D)P_{3,4}/d\,, & \\ a_{n-2}^{(2;3)} = (w - D)P_{4,4}\,, & \\ a_{n-1}^{(2;3)} = P_{4,4}\,. & \end{cases}$$

We shall now prove the formula

$$(81)\quad \begin{cases} a_s^{(2;k)} = 1/P_{k+2,k+1+s}\,, & (s = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(2;k)} = (w-D)P_{2,k+1}/d\,, & \\ a_{n-k}^{(2;k)} = (w-D)P_{3,k+1}/d\,, & \\ a_{n-k+i}^{(2;k)} = (w-D)P_{3+i,k+1}\,, & (i = 1, \cdots, k-2) \\ a_{n-1}^{(2;k)} = P_{k+1,k+1}\,; & k = 3, \cdots, n-3\,. \end{cases}$$

The proof of (81) is by induction like that of formula (61) or (47). First we see that (81) is correct for $k = 3$; then we show that it is correct for $k + 1$.

We now obtain from (81) for $k = n - 3$

$$(82)\quad \begin{cases} a_1^{(2;n-3)} = 1/P_{n-1,n-1}\,, & \\ a_2^{(2;n-3)} = (w-D)P_{2,n-2}/d\,, & \\ a_3^{(2;n-3)} = (w-D)P_{3,n-2}/d\,, & \\ a_{3+i}^{(2;n-3)} = (w-D)P_{3+i,n-2}\,, & (i = 1, \cdots, n-5) \\ a_{n-1}^{(2;n-3)} = P_{n-2,n-2}\,; & \end{cases}$$

and from (82), as before,

$$(83)\quad b_s^{(2;n-3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;n-3)} = D - D_{n-2}\,.$$

From (82), (83) we obtain, in virtue of (42),

$$(83)\quad \begin{cases} a_1^{(2;n-3)} - b_1^{(2;n-3)} = 1/P_{n-1,n-1}\,, \\ a_2^{(2;n-3)} - b_2^{(2;n-3)} = (w-D)P_{2,n-2}/d\,, \\ a_3^{(2;n-3)} - b_3^{(2;n-3)} = (w-D)P_{3,n-2}/d\,, \\ a_{3+i}^{(2;n-3)} - b_{3+i}^{(2;n-3)} = (w-D)P_{3+i,n-2}\,, \\ a_{n-1}^{(2;n-3)} - b_{n-1}^{(2;n-3)} = w - D\,. \end{cases}$$

From (83) we obtain, in virtue of (4) and carrying out multiplication as before,

$$(84)\quad \begin{cases} a_1^{(2;n-2)} = (w-D)P_{2,n-1}/d\,, & \\ a_2^{(2;n-2)} = (w-D)P_{3,n-1}/d\,, & \\ a_{2+i}^{(2;n-2)} = (w-D)P_{3+i,n-1}\,, & (i = 1, \cdots, n-4) \\ a_{n-1}^{(2;n-2)} = P_{n-1,n-1}\,. & \end{cases}$$

From (84) we obtain, as before,

$$(85)\quad b_s^{(2;n-2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(2;n-2)} = D - D_{n-1}\,,$$

and from (84), (85), in virtue of (42),

$$(85a)\qquad \begin{cases} a_1^{(2;n-2)} - b_1^{(2;n-2)} = (w-D)P_{2,n-1}/d\ , \\ a_2^{(2;n-2)} - b_2^{(2;n-2)} = (w-D)P_{3,n-1}/d\ , \\ a_{2+i}^{(2;n-2)} - b_{2+i}^{(2;n-2)} = (w-D)P_{3+i,n-1}\ , & (i=1,\cdots,n-4) \\ a_{n-1}^{(2;n-2)} - b_{n-1}^{(2;n-2)} = w-D\ . \end{cases}$$

From (85a) we obtain, in virtue of (4),

$$\begin{cases} a_1^{(3;0)} = P_{3,n-1}/P_{2,n-1}\ , \\ a_{1+i}^{(3;0)} = dP_{3+i,n-1}/P_{2,n-1}\ , & (i=1,\cdots,n-4) \\ a_{n-2}^{(3;0)} = d/P_{2,n-1}\ , \\ a_{n-1}^{(3;0)} = d/(w-D)P_{2,n-1}\ ; \end{cases}$$

or, after carrying out the necessary cancellation and multiplication

$$(86)\qquad \begin{cases} a_1^{(3;0)} = 1/P_{2,2}\ , \\ a_{1+i}^{(3;0)} = d/P_{2,2+i}\ , & (i=1,\cdots,n-3) \\ a_{n-1}^{(3;0)} = P_{1,1}\ . \end{cases}$$

From (86) we obtain, as before,

$$(87)\qquad b_s^{(3;0)} = 0\ ;\quad (s=1,\cdots,n-2)\qquad b_{n-1}^{(3;0)} = D - D_1\ ,$$

and from (86), (87), in virtue of (42)

$$(88)\qquad \begin{cases} a_1^{(3;0)} - b_1^{(3;0)} = 1/P_{2,2}\ , \\ a_{1+i}^{(3;0)} - b_{1+i}^{(3;0)} = d/P_{2,2+i}\ , & (i=1,\cdots,n-3) \\ a_{n-1}^{(3;0)} - b_{n-1}^{(3;0)} = w-D\ . \end{cases}$$

From (88) we obtain, in virtue of (4), and carrying out the necessary cancellation

$$(89)\qquad \begin{cases} a_i^{(3;1)} = d/P_{3,2+i}\ , & (i=1,\cdots,n-3) \\ a_{n-2}^{(3;1)} = (w-D)P_{2,2}\ , \\ a_{n-1}^{(3;1)} = P_{2,2}\ ; \end{cases}$$

and from (89), as before,

$$(90)\qquad b_s^{(3;1)} = 0\ ;\quad (s=1,\cdots,n-2)\qquad b_{n-1}^{(3;1)} = D - D_2\ .$$

From (89), (90) we obtain, in virtue of (42),

$$(91)\qquad \begin{cases} a_1^{(3;1)} - b_1^{(3;1)} = d/P_{3,3}\ , \\ a_{1+i}^{(3;1)} - b_{1+i}^{(3;1)} = d/P_{3,3+i}\ , & (i=1,\cdots,n-4) \\ a_{n-2}^{(3;1)} - b_{n-2}^{(3;1)} = (w-D)P_{2,2}\ , \\ a_{n-1}^{(3;1)} - b_{n-1}^{(3;1)} = w-D\ . \end{cases}$$

From (91) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(92)\qquad \begin{cases} a_i^{(3;2)} = 1/P_{4,3+i}\,, & (i = 1, \cdots, n-4) \\ a_{n-3}^{(3;2)} = (w-D)P_{2,3}/d\,, \\ a_{n-2}^{(3;2)} = (w-D)P_{3,3}/d\,, \\ a_{n-1}^{(3;2)} = P_{3,3}/d\,; \end{cases}$$

and from (92), as before,

$$(93)\qquad b_s^{(3;2)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;2)} = \frac{D - D_3}{d}\,.$$

From (92), (93) we obtain, in virtue of (42)

$$(93a)\qquad \begin{cases} a_1^{(3;2)} - b_1^{(3;2)} = 1/P_{4,4}\,, \\ a_{1+i}^{(3;2)} - b_{1+i}^{(3;2)} = 1/P_{4,4+i}\,, & (i = 1, \cdots, n-5) \\ a_{n-3}^{(3;2)} - b_{n-3}^{(3;2)} = (w-D)P_{2,3}/d\,, \quad a_{n-2}^{(3;2)} - b_{n-2}^{(3;2)} = \dfrac{(w-D)P_{3,3}}{d}\,, \\ a_{n-1}^{(3;2)} - b_{n-1}^{(3;2)} = (w-D)/d\,. \end{cases}$$

From (93a) we obtain, in virtue of (4),

$$(94)\qquad \begin{cases} a_i^{(3;3)} = 1/P_{5,4+i}\,, & (i = 1, \cdots, n-5) \\ a_{n-4}^{(3;3)} = (w-D)P_{2,4}/d\,, \\ a_{n-3}^{(3;3)} = (w-D)P_{3,4}/d\,, \\ a_{n-2}^{(3;3)} = (w-D)P_{4,4}/d\,, \\ a_{n-1}^{(3;3)} = P_{4,4}\,. \end{cases}$$

From (94) we obtain, as before,

$$(95)\qquad b_s^{(3;3)} = 0\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;3)} = D - D_4\,,$$

and from (94), (95), in virtue of (42),

$$(96)\qquad \begin{cases} a_1^{(3;3)} - b_1^{(3;3)} = 1/P_{5,5}\,, \\ a_{1+i}^{(3;3)} - b_{1+i}^{(3;3)} = 1/P_{5,5+i}\,, & (i = 1, \cdots, n-6) \\ a_{n-4}^{(3;3)} - b_{n-4}^{(3;3)} = (w-D)P_{2,4}/d\,, \\ a_{n-3}^{(3;3)} - b_{n-3}^{(3;3)} = (w-D)P_{3,4}/d\,, \\ a_{n-2}^{(3;3)} - b_{n-2}^{(3;3)} = (w-D)P_{4,4}/d\,, \\ a_{n-1}^{(3;3)} - b_{n-1}^{(3;3)} = w - D\,. \end{cases}$$

From (96) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(97)\qquad \begin{cases} a_i^{(3;4)} = 1/P_{6,5+i}\,, & (i = 1, \cdots, n-6) \\ a_{n-5}^{(3;4)} = (w-D)P_{2,5}/d\,, \\ a_{n-4}^{(3;4)} = (w-D)P_{3,5}/d\,, \\ a_{n-3}^{(3;4)} = (w-D)P_{4,5}/d\,, \\ a_{n-2}^{(3;4)} = (w-D)P_{5,5}\,, \\ a_{n-1}^{(3;4)} = P_{5,5}\,. \end{cases}$$

We shall now prove the formula

$$(98)\qquad \begin{cases} a_i^{(3;k)} = 1/P_{k+2,k+1+i}\,, & (i = 1, \cdots, n-k-2) \\ a_{n-k-1}^{(3;k)} = (w-D)P_{2,k+1}/d\,, \\ a_{n-k}^{(3;k)} = (w-D)P_{3,k+1}/d\,, \\ a_{n-k+1}^{(3;k)} = (w-D)P_{4,k+1}/d\,, \\ a_{n-k+1+s}^{(3;k)} = (w-D)P_{4+s,k+1}\,, & (s = 1, \cdots, k-3) \\ a_{n-1}^{(3;k)} = P_{k+1,k+1}\,, \\ k = 4, \cdots, n-3\,. \end{cases}$$

Formula (98) is correct for $k = 4$ because of (97). We then prove as before, that it is correct for $k+1$, so that (98) is verified. We obtain from (98), as before,

$$(99)\qquad b_s^{(3;k)} = 0\,;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;k)} = D - D_{k+1}\,,$$

and again from (98), for $k = n-3$,

$$(100)\qquad \begin{cases} a_1^{(3;n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+i}^{(3;n-3)} = (w-D)P_{1+i,n-2}/d\,, & (i = 1, 2, 3) \\ a_{4+s}^{(3;n-3)} = (w-D)P_{4+s,n-2}\,, & (s = 1, \cdots, n-6) \\ a_{n-1}^{(3;n-3)} = P_{n-2,n-2}\,. \end{cases}$$

From (100) we obtain, as before,

$$(101)\qquad b_s^{(3;n-3)} = 0\,;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(3;n-3)} = D - D_{n-2}\,,$$

and from (100), (101), in virtue of (42)

$$(102)\qquad \begin{cases} a_1^{(3;n-3)} - b_1^{(3;n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+i}^{(3;n-3)} - b_{1+i}^{(3;n-3)} = (w-D)P_{1+i,n-2}/d\,, & (i = 1, 2, 3) \\ a_{4+s}^{(2;n-3)} - b_{4+s}^{(3;n-3)} = (w-D)P_{4+s,n-2}\,, & (s = 1, \cdots, n-6) \\ a_{n-1}^{(3;n-3)} - b_{n-1}^{(3;n-3)} = w - D\,. \end{cases}$$

From (102) we obtain in virtue of (4), and carrying out the necessary cancellation and multiplication

$$(103)\qquad \begin{cases} a_i^{(3;n-2)} = (w-D)P_{1+i,n-1}/d\,, & (i=1,2,3) \\ a_{3+s}^{(3;n-2)} = (w-D)P_{4+s,n-1}\,, & (s=1,\cdots,n-5) \\ a_{n-1}^{(3;n-2)} = P_{n-1,n-1}\,, & \end{cases}$$

and from (103), as before,

$$(104)\qquad b_s^{(3;n-2)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(3;n-2)} = D - D_{n-1}\,.$$

From (103), (104) we obtain, in virtue of (42),

$$(105)\qquad \begin{cases} a_1^{(3;n-2)} - b_1^{(3;n-2)} = (w-D)P_{2,n-1}/d\,, & \\ a_2^{(3;n-2)} - b_2^{(3;n-2)} = (w-D)P_{3,n-1}/d\,, & \\ a_3^{(3;n-2)} - b_3^{(3;n-2)} = (w-D)P_{4,n-1}/d\,, & \\ a_{3+s}^{(3;n-2)} - b_{3+s}^{(3;n-2)} = (w-D)P_{4+s,n-1}\,, & (s=1,\cdots,n-5) \\ a_{n-1}^{(3;n-2)} - b_{n-1}^{(3;n-2)} = w-D\,, & \end{cases}$$

and from (105), in virtue of (4),

$$(106)\qquad \begin{cases} a_1^{(4;0)} = 1/P_{2,2}\,, & \\ a_2^{(4;0)} = 1/P_{2,3}\,, & \\ a_{2+s}^{(4;0)} = d/P_{2,3+s}\,, & (s=1,\cdots,n-4) \\ a_{n-1}^{(4;0)} = P_{1,1}\,. & \end{cases}$$

The reader will easily verify, on ground of previous formulas, that the $4(n-1)$ supporting sequences

$$b_1^{(i;k)},\, b_{2i}^{(i;k)},\, \cdots,\, b_{n-1}^{(i;k)} \qquad (k=0,\cdots,n-2;\ i=0,1,2,3)$$

generate the first four fugues whose form is that as demanded by Theorem 3.

The complete proof of Theorem 3 is based on the following

LEMMA 1. *Let the generating sequence*

$$a_s^{(k;0)} \qquad (s=1,\cdots,n-1;\ k=3,\cdots,n-2)$$

have the form

$$(107)\qquad \begin{cases} a_i^{(k;0)} = 1/P_{2,1+i}\,, & (i=1,\cdots,k-2) \\ a_{k-2+s}^{(k,0)} = d/P_{2,k-1+s}\,, & (s=1,\cdots,n-k) \\ a_{n-1}^{(k;0)} = P_{1,1}\,; & \end{cases}$$

then the $n-1$ supporting sequences

$$b_1^{(k;0)},\, b_2^{(k;0)},\, \cdots,\, b_{n-1}^{(k;0)}\,, \qquad (i=0,\cdots,n-2)$$

generate a fugue which has the form of the $k+1$-th fugue as demanded *by Theorem* 3, *and the generating sequence* $a_s^{(k+1;0)}$ $(s = 1, \cdots, n-1)$ *has the form of* (107), *where* k *is to be substituted by* $k+1$.

Proof. In virtue of formula (86), the generating sequence

$$a_1^{(k;0)}, a_2^{(k;0)}, \cdots, a_{n-1}^{(k;0)}$$

has the form as in (107) for $k = 3$. The $n-1$ supporting sequences $b_1(3;0), b_2(3;0), \cdots, b(3;0)$ form the fourth fugue of the period as demanded by Theorem 3. The generating sequence

$$a_1^{(k+1;0)}, a_2^{(k+1;0)}, \cdots, a_{n-1}^{(k+1;0)}$$

too, has the form as in (107) for $k = 3$, in virtue of formula (106). Thus the lemma is correct for $k = 3$. Let it be correct for $k = m$. That means that the n $-$ 1 supporting sequences

$$b_1^{(m;i)}, b_2^{(m;i)}, \cdots, b_{n-1}^{(m;i)}, \qquad (i = 0, \cdots, n-2)$$

form the $m+1$ -th fugue as demanded by Theorem 3, and that the generating sequence

$$a_1^{(m+1;0)}, a_2^{(m+1;0)}, \cdots, a_{n-1}^{(m+1;0)}$$

has the form

$$(108) \qquad \begin{cases} a_i^{(m+1;0)} = 1/P_{2,1+i}, & (i = 1, \cdots, m-1) \\ a_{m-1+s}^{(m+1;0)} = d/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-1}^{(m+1;0)} = P_{1,1}. & \end{cases}$$

From (108) we obtain, as before,

$$(109) \qquad b_s^{(m+1;0)} = 0; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;0)} = D - D_1$$

and from (108), (109), in virtue of (42),

$$(110) \qquad \begin{cases} a_1^{(m+1;0)} - b_1^{(m+1;0)} = 1/P_{2,2}, & \\ a_{1+i}^{(m+1;0)} - b_{1+i}^{(m+1;0)} = 1/P_{2,2+i}, & (i = 1, \cdots, m-2) \\ a_{m-1+s}^{(m+1;0)} - b_{m-1+s}^{(m+1;0)} = d/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-1}^{(m+1;0)} - b_{n-1}^{(m+1;0)} = w - D. & \end{cases}$$

From (110) we obtain, in virtue of (4)

$$\begin{cases} a_i^{(m+1,1)} = P_{2,2}/P_{2,2+i}, & (i = 1, \cdots, m-2) \\ a_{m-2+s}^{(m+1;1)} = dP_{2,2}/P_{2,m+s}, & (s = 1, \cdots, n-m-1) \\ a_{n-2}^{(m+1;1)} = (w-D)P_{2,2}, & \\ a_{n-1}^{(m+1;1)} = P_{2,2}; & \end{cases}$$

or

$$
(111) \quad \begin{cases} a_i^{(m+1;1)} = 1/P_{3,2+i}\ , & (i = 1, \cdots, m-2) \\ a_{m-2+s}^{(m+1;1)} = d/P_{3,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-2}^{(m+1;1)} = (w - D)P_{2,2}\ , & \\ a_{n-1}^{(m+1;1)} = P_{2,2}\ . & \end{cases}
$$

We shall now prove the formula

$$
(112) \quad \begin{cases} a_i^{(m+1;t)} = 1/P_{t+2,t+1+i}\ , & (i = 1, \cdots, m-t-1) \\ a_{m-t-1+s}^{(m+1;t)} = d/P_{t+2,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-t-2+j}^{(m+1;t)} = (w - D)P_{1+j,t+1}\ , & (j = 1, \cdots, t) \\ a_{n-1}^{(m+1;t)} = P_{t+1,t+1}\ , & \\ t = 1, \cdots, m-2\ . & \end{cases}
$$

Formula (112) is correct for $t = 1$, in virtue of formula (111). We shall prove that, being correct for t, it is correct for $t + 1$. From (112) we obtain, as before,

$$
(113) \quad b_s^{(m+1;t)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;t)} = D - D_{t+1}\ ,
$$

and from (112), (113), in virtue of (42)

$$
(114) \quad \begin{cases} a_1^{(m+1;t)} - b_1^{(m+1;t)} = 1/P_{t+2,t+2}\ , & \\ a_{1+i}^{(m+1;t)} - b_{1+i}^{(m+1;t)} = 1/P_{t+2,t+2+i}\ , & (i = 1, \cdots, m-t-2) \\ a_{m-t-1+s}^{(m+1;t)} - b_{m-t-1+s}^{(m+1;t)} = d/P_{t+2;m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-t-2+j}^{(m+1;t)} - b_{n-t-2+j}^{(m+1,t)} = (w - D)P_{1+j,t+1}\ , & (j = 1, \cdots, t) \\ a_{n-1}^{(m+1;t)} - b_{n-1}^{(m+1,t)} = w - D\ . & \end{cases}
$$

From (114) we obtain, in virtue of (4)

$$
(115) \quad \begin{cases} a_i^{(m+1,t+1)} = P_{t+2,t+2}/P_{t+2,t+2+i}\ , & (i = 1, \cdots, m-t-2) \\ a_{m-t-2+s}^{(m+1;t+1)} = dP_{t+2,t+2}/P_{t+2,m+S}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-t-3+j}^{(m+1;t+1)} = (w - D)P_{1+j,t+1}P_{t+2,t+2}\ , & (j = 1, \cdots, t) \\ a_{n-2}^{(m+1;t+1)} = (w - D)P_{t+2,t+2}\ , & \\ a_{n-1}^{(m+1;t+1)} = P_{t+2,t+2}\ , & \end{cases}
$$

and from (115), carrying out the necessary cancellation and multiplication

$$
(116) \quad \begin{cases} a_i^{(m+1;t+1)} = 1/P_{t+3,t+2+i}\ , & (i = 1, \cdots, m-t-2) \\ a_{m-t-2+s}^{(m+1;t+1)} = d/P_{t+3,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-t-3+j}^{(m+1;t+1)} = (w - D)P_{1+j,t+2} & (j = 1, \cdots, t+1) \\ a_{n-1}^{(m+1;t+1)} = P_{t+2,t+2}\ . & \end{cases}
$$

With (116) formula (112) is proved. We now obtain from (112), for $t = m - 2$,

$$(117)\quad \begin{cases} a_1^{(m+1;m-2)} = 1/P_{m,m}\ , & \\ a_{1+s}^{(m+1;m-2)} = d/P_{m,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-m+j}^{(m+1;m-2)} = (w-D)P_{1+j,m-1}\ , & (j = 1, \cdots, m-2) \\ a_{n-1}^{(m+1;m-2)} = P_{m-1,m-1}\ ; & \end{cases}$$

and from (117), as before,

$$(118)\quad b_s^{(m+1;m-2)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-2}^{(m+1;m-2)} = D - D_{m-1}\ .$$

From (117), (118) we obtain, in view of (42)

$$(119)\quad \begin{cases} a_1^{(m+1;m-2)} - b_1^{(m+1;m-2)} = 1/P_{m,m}\ , & \\ a_{1+s}^{(m+1;m-2)} - b_{1+s}^{(m+1;m-2)} = d/P_{m,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-m+j}^{(m+1;m-2)} - b_{n-m+j}^{(m+1;m-2)} = (w-D)P_{1+j,m-1}\ , & (j = 1, \cdots, m-2) \\ a_{n-1}^{(m+1;m-2)} - b_{n-1}^{(m+1;m-2)} = w - D\ ; & \end{cases}$$

and from (119), in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(120)\quad \begin{cases} a_s^{(m+1;m-1)} = d/P_{m+1,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\ , & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} = P_{m,m}\ . & \end{cases}$$

From (120) we obtain, as before,

$$(121)\quad b_s^{(m+1;m-1)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1,m-1)} = D - D_m\ ,$$

and from (120), (121), in virtue of (42)

$$\begin{cases} a_s^{(m+1;m-1)} - b_s^{(m+1;m-1)} = d/P_{m+1,m+s}\ , & (s = 1, \cdots, n-m-1) \\ a_{n-m-1+j}^{(m+1;m-1)} - b_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\ , & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} - b_{n-1}^{(m+1;m-1)} = w - D\ ; & \end{cases}$$

or

$$(122)\quad \begin{cases} a_1^{(m+1;m-1)} - b_1^{(m+1;m-1)} = d/P_{m+1,m+1}\ , & \\ a_{1+s}^{(m+1;m-1)} - b_{1+s}^{(m+1;m-1)} = d/P_{m+1,m+1+s}\ , & (s = 1, \cdots, n-m-2) \\ a_{n-m-1+j}^{(m+1;m-1)} - b_{n-m-1+j}^{(m+1;m-1)} = (w-D)P_{1+j,m}\ , & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1;m-1)} - b_{n-1}^{(m+1;m-1)} = w - D\ . & \end{cases}$$

From (122) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication

$$(123)\quad \begin{cases} a_s^{(m+1;m)} = 1/P_{m+2,m+1+s}\ , & (s = 1, \cdots, n-m-2) \\ a_{n-m-1+j}^{(m+1;m)} = (w-D)P_{1+j,m+1}/d\ , & (j = 1, \cdots, m) \\ a_{n-1}^{(m+1;m)} = P_{m+1,m+1}/d\ . & \end{cases}$$

From (123) we obtain, as before,

$$(124)\quad b_s^{(m+1;m)} = 0\ ;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;m)} = \frac{D - D_{m+1}}{d},$$

and from (123), (124), in virtue of (42),

$$(125)\quad \begin{cases} a_1^{(m+1;m)} - b_1^{(m+1;m)} = 1/P_{m+2,m+2}\ , \\ a_{1+s}^{(m+1;m)} - b_{1+s}^{(m+1;m)} = 1/P_{m+2,m+2+s}\ , & (s = 1, \cdots, n-m-3) \\ a_{n-m-2+j}^{(m+1;m)} - b_{n-m-2+j}^{(m+1;m)} = (w - D)P_{1+j,m+1}/d\ , & (j = 1, \cdots, m) \\ a_{n-1}^{(m+1;m)} - b_{n-1}^{(m+1;m)} = (w - D)/d\ . \end{cases}$$

From (125) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication,

$$(126)\quad \begin{cases} a_s^{(m+1;m+1)} = 1/P_{m+3,m+2+s}\ , & (s = 1, \cdots, n-m-3) \\ a_{n-m-3+j}^{(m+1;m+1)} = (w - D)P_{1+j,m+2}/d\ , & (j = 1, \cdots, m+1) \\ a_{n-1}^{(m+1;m+1)} = P_{m+2,m+2}\ . \end{cases}$$

From (126) we obtain, as before,

$$(127)\quad b_s^{(m+1;m+1)} = 0\ ;\quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(m+1;m+1)} = D - D_{m+2}\ ,$$

and from (126), (127), in virtue of (42)

$$(128)\quad \begin{cases} a_1^{(m+1;m+1)} - b_1^{(m+1;m+1)} = 1/P_{m+3,m+3}\ , \\ a_{1+s}^{(m+1;m+1)} - b_{1+s}^{(m+1;m+1)} = 1/P_{m+3,m+3+s}\ , & (s = 1, \cdots, n-m-4) \\ a_{n-m-3+j}^{(m+1;m+1)} - b_{n-m-3+j}^{(m+1;m+1)} = (w - D)P_{1+j,m+2}/d\ , & (j = 1, \cdots, m+1) \\ a_{n-1}^{(m+1;m+1)} - b_{n-1}^{(m+1;m+1)} = w - D\ . \end{cases}$$

From (128) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(129)\quad \begin{cases} a_s^{(m+1;m+2)} = 1/P_{m+4,m+3+s}\ , & (s = 1, \cdots, n-m-4) \\ a_{n-m-4+j}^{(m+1;m+2)} = (w - D)P_{1+j,m+3}/d\ , & (j = 1, \cdots, m+1) \\ a_{n-2}^{(m+1;m+2)} = (w - D)P_{m+3,m+3}\ , \\ a_{n-1}^{(m+1;m+2)} = P_{m+3,m+3}\ . \end{cases}$$

We shall now prove the formula

$$(130)\quad \begin{cases} a_s^{(m+1;m+k)} = 1/P_{m+k+2,m+k+1+s}\ , & (s = 1, \cdots, n-m-k-2) \\ a_{n-m-k-2+j}^{(m+1;m+k)} = (w - D)P_{1+j,m+k+1}/d\ , & (j = 1, \cdots, m+1) \\ a_{n-k-1+t}^{(m+1;m+k)} = (w - D)P_{m+2+t,m+k+1}\ , & (t = 1, \cdots, k-1) \\ a_{n-1}^{(m+1;m+k)} = P_{m+k+1,m+k+1}\ , \\ k = 2, \cdots, n-m-3\ . \end{cases}$$

Formula (130) is correct for $k=2$, in virtue of (129). Presuming it is correct for k, we shall prove its correctness for $k+1$.

From (130) we obtain, as before,

$$(131)\quad b_s^{(m+1;m+k)} = 0\,;\quad (s=1,\cdots,n-2)\qquad b_{n-1}^{(m+1;m+k)} = D - D_{m+k+1}$$

and from (130), (131), in virtue of (42)

$$(132)\quad \begin{cases} a_1^{(m+1;m+k)} - b_1^{(m+1;m+k)} = 1/P_{m+k+2,m+k+2}\,, \\ a_{1+s}^{(m+1;m+k)} - b_{1+s}^{(m+1;m+k)} = 1/P_{m+k+2,m+k+2+s}\,, & (s=1,\cdots,n-m-k-3) \\ a_{n-m-k-2+j}^{(m+1;m+k)} - b_{n-m-k-2+j}^{(m+1;m+k)} = (w-D)P_{1+j,m+k+1}/d\,, & (j=1,\cdots,m+1) \\ a_{n-k-1+t}^{(m+1;m+k)} - b_{n-k-1+t}^{(m+1;m+k)} = (w-D)P_{m+2+t,m+k+1}\,, & (t=1,\cdots,k-1) \\ a_{n-1}^{(m+1;m+k)} - b_{n-1}^{(m+1;m+k)} = w-D\,. \end{cases}$$

From (132) we obtain, in virtue of (4) and after carrying out the necessary cancellation and multiplication

$$\begin{cases} a_s^{(m+1;m+k+1)} = 1/P_{m+k+3,m+k+2+s}\,, & (s=1,\cdots,n-m-k-3) \\ a_{n-m-k-3+j}^{(m+1;m+k+1)} = (w-D)P_{1+j,m+k+2}/d\,, & (j=1,\cdots,m+1) \\ a_{n-k-2+t}^{(m+1;m+k+1)} = (w-D)P_{m+2+t,m+k+2}\,, & (t=1,\cdots,k) \\ a_{n-1}^{(m+1;m+k+1)} = P_{m+k+2,m+k+2}\,, \end{cases}$$

which is formula (130) with k being replaced by $k+1$; this proves formula (130).

We now obtain from (130) for $k=n-m-3$

$$(134)\quad \begin{cases} a_1^{(m+1;n-3)} = 1/P_{n-1;n-1}\,, \\ a_{1+j}^{(m+1;n-3)} = (w-D)P_{1+j,n-2}/d\,, & (j=1,\cdots,m+1) \\ a_{m+2+t}^{(m+1;n-3)} = (w-D)P_{m+2+t,n-2}\,, & (t=1,\cdots,n-m-4) \\ a_{n-1}^{(m+1;n-3)} = P_{n-2,n-2}\,, \end{cases}$$

and from (134), as before,

$$(135)\quad b_s^{(m+1;n-3)} = 0\,;\quad (s=1,\cdots,n-2)\qquad b_{n-1}^{(m+1;n-3)} = D - D_{n-2}\,,$$

From (134), (135) we obtain, in virtue of (42),

$$(136)\quad \begin{cases} a_1^{(m+1;n-3)} - b_1^{(m+1;n-3)} = 1/P_{n-1,n-1}\,, \\ a_{1+j}^{(m+1;n-3)} - b_{1+j}^{(m+1;n-3)} = (w-D)P_{1+j,n-2}/d\,, & (j=1,\cdots,m+1) \\ a_{m+2+t}^{(m+1;n-3)} - b_{m+2+t}^{(m+1;n-3)} = (w-D)P_{m+2+t,n-2}\,, & (t=1,\cdots,n-m-4) \\ a_{n-1}^{(m+1;n-3)} - b_{n-1}^{(m+1;n-3)} = w-D\,, \end{cases}$$

and from (136), in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(137)\quad \begin{cases} a_j^{(m+1;n-2)} = (w-D)P_{1+j,n-1}/d\,, & (j=1,\cdots,m+1) \\ a_{m+1+t}^{(m+1;n-2)} = (w-D)P_{m+2+t,n-1}\,, & (t=1,\cdots,n-m-3) \\ a_{n-1}^{(m+1;n-2)} = P_{n-1,n-1}\,. \end{cases}$$

From (137) we obtain, as before,

$$(138)\quad b_s^{(m+1;n-2)} = 0\,; \quad (s=1,\cdots,n-2) \qquad b_{n-1}^{(m+1;n-2)} = D - D_{n-1}\,,$$

and from (137), (138), in virtue of (4),

$$(139)\quad \begin{cases} a_s^{(m+1;n-2)} - b_1^{(m+1;n-2)} = (w-D)P_{2,n-1}/d\,, \\ a_{1+j}^{(m+1;n-2)} - b_{1+j}^{(m+1;n-2)} = (w-D)P_{2+j,n-1}/d\,, & (j=1,\cdots,m) \\ a_{m+1+t}^{(m+1;n-2)} - b_{m+1+t}^{(m+1;n-2)} = (w-D)P_{m+2+t,n-1}\,, & (t=1,\cdots,n-m-3) \\ a_{n-1}^{(m+1,n-2)} - b_{n-1}^{(m+1;n-2)} = w-D\,. \end{cases}$$

From (139) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(140)\quad \begin{cases} a_j^{(m+2;0)} = 1/P_{2,1+j}\,, & (j=1,\cdots,m) \\ a_{m+1+t}^{(m+2;0)} = d/P_{2,m+1+t}\,, & (t=1,\cdots,n-m-2) \\ a_{n-1}^{(m+2;0)} = P_{1,1}\,. \end{cases}$$

According to formula (109) (one line of the period), formula (113) ($m-2$ lines of the period), formula (121) (one line of the period), formula (124) (one line of the period), formula (127) (one line of the period), (131), ($n-m-4$ lines of the period) and formula (138) (one line of the period-totally $1+m-2+1+1+1+n-m-4+1=n-1$) the $m-2$ -th fugue has the form as demanded by Theorem 3. Since (140) is formula (107) for $k=m+2$, the Lemma 1 is completely proved.

In view of the Lemma 1 we obtain that the $(n-5)(n-1)$ lines

$$b_1^{(k;0)},\, b_2^{(k,0)},\, \cdots,\, b_{n-1}^{(k;0}\,, \qquad (k=4,\cdots,n-2)$$

form $n-5$ fugues, beginning with the fifth fugue, as demanded by Theorem 3; we further obtain, applying the lemma for $k=n-2$, $k+1=n-1$, that the generating sequence $a_i^{(n-1;0)}$, $(i=1,\cdots,n-1)$ has the form, following (108)

$$(141)\quad \begin{cases} a_i^{(n-1;0)} = 1/P_{2,1+i}\,, & (i=1,\cdots,n-3) \\ a_{n-2}^{(n-1;0)} = d/P_{2,n-1}\,, \\ a_{n-1}^{(n-1;0)} = P_{1,1}\,. \end{cases}$$

From (141) we obtain, as before,

$$(142)\qquad b_s^{(n-1;0)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;0)} = D - D_1\ ,$$

and from (141), (142), in virtue of (42)

$$(143)\qquad \begin{cases} a_1^{(n-1;0)} - b_1^{(n-1;0)} = 1/P_{2,2}\ , \\ a_{1+i}^{(n-1;0)} - b_{1+i}^{(n-1;0)} = 1/P_{2,2+i}\ , & (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = d/P_{2,n-1}\ , \\ a_{n-1}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = w - D\ . \end{cases}$$

From (143) we obtain, in virtue of (4), and after carrying out the necessary cancellation and multiplication

$$(144)\qquad \begin{cases} a_i^{(n-1;1)} = 1/P_{3,2+i}\ , \\ a_{n-3}^{(n-1;1)} = d/P_{3,n-1}\ , \\ a_{n-2}^{(n-1;1)} = (w - D)P_{2,2}\ , \\ a_{n-1}^{(n-1;1)} = P_{2,2}\ . \end{cases}$$

We shall now prove the formula

$$(145)\qquad \begin{cases} a_i^{(n-1;k)} = 1/P_{k+2,k+1+i}\ , & (i = 1, \cdots, n-k-3) \\ a_{n-k-2}^{(n-1;k)} = d/\ P_{k+2,n-1}\ , \\ a_{n-k-2+s}^{(n-1;k)} = (w - D)P_{1+s,1+k}\ , & (s = 1, \cdots, k) \\ a_{n-1}^{(n-1;k)} = P_{k+1,k+1}\ , \\ k = 1, \cdots, n-4\ . \end{cases}$$

In virtue of (144) formula (145) is correct for $k = 1$. We prove, by completely analogous methods used to prove previous, similar formulae that it is correct for $k + 1$, thus verifying its correctness. We now obtain from (145), as before,

$$(146)\qquad b_s^{(n-1;k)} = 0\ ; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;k)} = D - D_{k+1}\ ,$$

and again from (145), for $k = n - 4$,

$$(147)\qquad \begin{cases} a_1^{(n-1,n-4)} = 1/P_{n-2,n-2}\ , \\ a_2^{(n-1;n-4)} = d/P_{n-2,n-1}\ , \\ a_{2+s}^{(n-1;n-4)} = (w - D)P_{1+s,n-3}\ , & (s = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-4)} = P_{n-3,n-3}\ . \end{cases}$$

From (147) and (146) (for $k = n - 4$) we obtain, in virtue of (42),

$$(148)\qquad \begin{cases} a_1^{(n-1;n-4)} - b_1^{(n-1;n-4)} = 1/P_{n-2,n-2}\ , \\ a_2^{(n-1;n-4)} - b_2^{(n-1;n-4)} = d/P_{n-2,n-1}\ , \\ a_{2+s}^{(n-1;n-4)} - b_{2+s}^{(n-1;n-4)} = (w - D)P_{1+s,n-3}\ , & (s = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-4)} - b_{n-1}^{(n-1;n-4)} = w - D\ , \end{cases}$$

and from (148), in virtue of (4), and carrying out the necessary cancellation and multiplication,

$$
\text{(149)} \qquad \begin{cases} a_1^{(n-1;n-3)} = d/P_{n-1,n-1} \,, \\ a_{1+s}^{(n-1;n-3)} = (w-D)P_{1+s,n-2} \,, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-3)} = P_{n-2,n-2} \,. \end{cases}
$$

From (149) we obtain, as before,

$$
\text{(150)} \qquad b_s^{(n-1;n-3)} = 0 \,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-3)} = D - D_{n-2} \,,
$$

and from (149), (150), in virtue of (42)

$$
\text{(151)} \qquad \begin{cases} a_1^{(n-1;n-3)} - b_1^{(n-1;n-3)} = d/P_{n-1,n-1} \,, \\ a_{1+s}^{(n-1;n-3)} - b_{1+s}^{(n-1;n-3)} = (w-D)P_{1+s,n-2} \,, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-3)} - b_{n-1}^{(n-1;n-3)} = w - D \,. \end{cases}
$$

From (151) we obtain, in virtue of (4), and carrying out the necessary cancellation and multiplication

$$
\text{(152)} \qquad \begin{cases} a_s^{(n-1;n-2)} = (w-D)P_{1+s,n-1}/d \,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n-1;n-2)} = P_{n-1,n-1}/d \,, \end{cases}
$$

and from (152), as before,

$$
\text{(153)} \qquad b_s^{(n-1,n-2)} = 0 \,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-2)} = \frac{D - D_{n-1}}{d} \,.
$$

From (152), (153) we obtain, in virtue of (42),

$$
\text{(154)} \qquad \begin{cases} a_1^{(n-1;n-2)} - b_1^{(n-1;n-2)} = (w-D)P_{2,n-1}/d \,, \\ a_{1+s}^{(n-1;n-2)} - b_{1-s}^{(n-1;n-2)} = (w-D)P_{2+s,n-1}/d \,, & (s = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} - b_{n-1}^{(n-1;n-2)} = (w-D)/d \,, \end{cases}
$$

and from (154), in virtue of (4),

$$
\begin{cases} a_s^{(n;0)} = P_{2+s,n-1}/P_{2,n-1} = 1/P_{2,1+s} \,, & (s = 1, \cdots, n-3) \\ a_{n-2}^{(n;0)} = 1/P_{2,n-1} \,, \\ a_{n-1}^{(n;0)} = d/(w-D)P_{2,n-1} = (w-D)P_{1,n-1}/(w-D)P_{2,n-1} = P_{1,1} \,. \end{cases}
$$

Thus

$$
\text{(155)} \qquad \begin{cases} a_s^{(n;0)} = 1/P_{2,1+s} \,, & (s = 1, \cdots, n-2) \\ a_{n-1}^{(n;0)} = P_{1,1} \,. \end{cases}
$$

Comparing (40) with (155) we see that

$$a_s^{(n;0)} = a_s^{(0)} , \qquad (s = 1, \cdots, n-1)$$

i.e.,

$$a_s^{(n(n-1))} = a_s^{(0)} \qquad (s = 1, \cdots, n-1) \tag{156}$$

which proves that, in case $d \neq 1$, the Jacobi-Perron Algorithm of the basic sequence $a_s^{(0)}$ $(s = 1, \cdots, n-1)$ from (34) is purely periodic and its length $T = (n-1)n$. Since, in virtue of (142), (146), (150), (153) the $n-1$ supporting sequences

$$b_1^{(n-1;k)}, b_2^{(n-1;k)}, \cdots, b_{n-1}^{(n-1;k)} \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which is the n -th fugue of the period, we see that this last fugue, together with the $4 + (n-5) = n-1$ preceding ones form the n fugues of the period, as demanded by Theorem 3.

In case $d = 1$, we obtain from (55)

$$a_s^{(1;0)} = 1/P_{2,1+s} , \quad (s = 1, \cdots, n-2) ; \qquad a_{n-1}^{(1;0)} = P_{1,1} , \tag{157}$$

so that, comparing (157) with (40), we obtain

$$a_s^{(n-1)} = a_s^{(0)} , \qquad (s = 1, \cdots, n-1) \tag{158}$$

so that the length of the period is here $T = n - 1$; from (41), (44), (48) (54) we obtain that in the case $d = 1$ the period has the form as demanded by Theorem 3.

The reader should note that proving case $d \neq 1$ we presumed $n \geqq 6$. The special cases $n = 2, 3, 4, 5$ are proved analogously.

We shall now give a few numeric examples. Let the generating polynomial be

$$f(x) = x^5 - 15x^4 + 54x^3 - 3 = 0 ,$$

which can be easily rearranged into

$$f(x) = (x-9)(x-6)x^3 - 3 = 0$$

and has the form (17) with

$$D = 9,\ D_1 = 6,\ D_2 = D_3 = D_4 = 0 ;\ d = 3 ;$$
$$9 < w < 10 ;\ (w-9)(w-6)w^3 - 3 = 0 .$$

The Jabobi-Perron Algorithm of the basic sequence

$$\frac{(w-9)(w-6)w^2}{3}, \frac{(w-9)(w-6)w}{3}, \frac{(w-9)(w-6)}{3}, w-6$$

or

$$\frac{w^4-15w^3+54w^2}{3}, \frac{w^3-15w^2+54w}{3}, \frac{w^2-15w+54}{3}, w-6$$

is purely periodic with period length $T = 20$. The period has the form

$$\begin{array}{cccc}
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
\\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
\\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
\\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
\\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 3\,.
\end{array}$$

Let the generating polynomial be

$$f(x) = x^6 - 3x^5 - 5x^4 + 15x^3 + 4x^2 - 12x - 1 = 0\,,$$

which is easily rearranged into

$$f(x) = (x-3)(x-2)(x-1)x(x+1)(x+2) - 1 = 0$$

and has the form (17) with

$$D = 3,\ D_1 = 2,\ D_2 = 1,\ D_3 = 0,\ D_4 = -1,\ D_5 = -2\,;\ d = 1\,;$$
$$3 < w < 4\,,$$
$$(w-3)(w-2)(w-1)(w+1)(w+2)w - 1 = 0\,.$$

The Jacobi-Perron algorithm of the basic sequence

$$\begin{aligned}
a_1^{(0)} &= (w-3)(w-2)w(w+1)(w+2) &&= w^5 - 2w^4 - 7w^3 + 8w^2 + 12w\,,\\
a_2^{(0)} &= (w-3)(w-2)(w+1)(w+2) &&= w^4 - 2w^3 - 7w^2 + 8w + 12\,,\\
a_3^{(0)} &= (w-3)(w-2)(w+2) &&= w^3 - 3w^2 - 4w + 12\,,\\
a_4^{(0)} &= (w-3)(w-2) &&= w^2 - 5w + 6\,,\\
a_5^{(0)} &= w-2 &&= w - 2\,,
\end{aligned}$$

is purely periodic and the period length is $T = 5$. The period has the form

$$\begin{matrix}
0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 0 & 3\\
0 & 0 & 0 & 0 & 4\\
0 & 0 & 0 & 0 & 5\,.
\end{matrix}$$

Let the generating polynomial be

$$f(x) = x^3 - 16x - 2 = 0\,,$$

which is easily rearranged into

$$f(x) = (x-4)x(x+4) - 2 = 0$$

and has the form (17) with

$$D = 4,\ D_1 = 0,\ D_2 = -4\,;\ d = 2\,;$$
$$4 < w < 5\,,$$
$$(w-4)(w+4)w - 2 = 0\,.$$

The Jacobi-Perron algorithm of the basic sequence

$$\frac{(w-4)w}{2} = \frac{w^2 - 4w}{2}\,,\ w$$

is purely periodic and the period length $T = 6$; the period has the form

$$\begin{matrix}
0 & 4\\
0 & 8\\
0 & 2\\
0 & 8\\
0 & 4\\
0 & 4\,.
\end{matrix}$$

6. The Jacobi-perron algorithm for polynomial of increasing order. In this section we shall show that, by imposing further conditions on the coefficients of the GP from (17), one can select increasing order polynomials from the algebraic number field $K(w)$ generated by $f(w) = 0$, $D < w < D + 1$, such that their Jacobi-Perron algorithm is purely periodic. This result is stated in

THEOREM 4. *Let the coefficients of the GP in addition to* (17) *fulfil the inequalities* $D - D_i \geqq 2d(n-1)$, *i.e., altogether*

$$\begin{aligned} &D,\ D_i,\ d \text{ rational integers};\ d \geqq 1;\ n \geqq 2; \\ &d \mid (D - D_i);\ D - D_i \geqq 2d(n-1); \qquad (i = 1, 2, \cdots, n-1); \end{aligned} \tag{159}$$

Let w be the only real root in the open interval $(D; D+1)$. Then the Jacobi-Perron algorithm of the basic sequence

$$\begin{aligned} &a_i^{(0)}(w) = \textstyle\sum_{j=0}^{i} k_s (w - D)^{i-s}, \qquad (s = 1, \cdots, n-1);\ k_s = 1; \\ &k_s = \textstyle\sum (D - D_{j_1})(D - D_{j_2}) \cdots (D - D_{j_s}), \quad (s = 1, \cdots, n-1), \\ &1 \leqq j_1 < j_2 < \cdots < j_s \leqq n-1 \end{aligned} \tag{160}$$

is purely periodic and its length $T = n$ for $d > 1$, and $T = 1$ for $d = 1$. The period has the form

$$\begin{aligned} &b_i^{(0)} = k_i && (i = 1, \cdots, n-1); \\ &\begin{cases} b_i^{(s)} = k_i & (i = 1, \cdots, n-1-s), \\ b_i^{(s)} = k_i/d & (i = n-s, \cdots, n-1;\ s = 1, \cdots, n-2); \end{cases} \\ &b_i^{(n-1)} = k_i/d && (i = 1, \cdots, n-1); \\ &d > 1. \end{aligned} \tag{161}$$

$$b_i^{(0)} = k_i \qquad (i = 1, \cdots, n-1);\ d = 1. \tag{161a}$$

Proof. This is essentially based on the simple formula

$$[a_i^{(0)}(w)] = k_i \qquad (i = 1, \cdots, n-1). \tag{162}$$

Since, as will be proved later, w is irrational under the conditions (159), we have to verify the two inequalities

$$k_i < a_i^{(0)}(w) < k_i + 1 \qquad (i = 1, \cdots, n-1), \tag{163}$$

or, in virtue of (160)

$$0 < (w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D) < 1. \tag{164}$$

The left-hand inequality of (164) follows from $w > D$ and $k_i > 0$. We shall prove the right-hand inequality

$$(w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D) < 1. \tag{165}$$

Since $0 < w - D < 1$, we obtain $(w - D)^i \leqq w - D$, and we shall prove, since

$$
\begin{aligned}
&(w - D)^i + k_1(w - D)^{i-1} + \cdots + k_{i-1}(w - D) \\
&\leqq (w - D) + k_1(w - D) + \cdots + k_{i-1}(w - D) ,
\end{aligned}
$$

$$(w - D)(1 + k_1 + k_2 + \cdots + k_{i-1}) < 1 . \tag{166}$$

From $w > D$, $(w - D)(w - D_1) \cdots (w - D_{n-1}) - d = 0$, we obtain

$$
\begin{aligned}
w - D &= d/((w - D_1)(w - D_2) \cdots (w - D_{n-1})) \\
&< d/((D - D_1(D - D_2) \cdots (D - D_{n-1})) .
\end{aligned} \tag{167}
$$

We shall now prove the inequality

$$k_s(w - D) < 2^{-(n-1-s)} , \qquad (s = 1, \cdots, n - 2) . \tag{168}$$

Let the D_i be arranged in nondecreasing order, so that

$$D - D_1 \geqq D - D_2 \geqq \cdots \geqq D - D_{n-1} . \tag{169}$$

In virtue of (169), and taking into account the values of k_s from (160) we obtain

$$
\begin{aligned}
&k_s(w - D) \leqq (w - D) \sum (D - D_1)(D - D_2) \cdots (D - D_s) \\
&= \binom{n-1}{s}(w - D)(D - D_1)(D - D_2) \cdots (D - D_s) \\
&< \frac{\binom{n-1}{s}(D - D_1)(D - D_2) \cdots (D - D_s)d}{(D - D_1)(D - D_2) \cdots (D - D_{n-1})} ,
\end{aligned}
$$

in virtue of (17). Therefore

$$k_s(w - D) < \frac{\binom{n-1}{s}d}{(D - D_{s+1})(D - D_{s+2}) \cdots (D - D_{n-1})} . \tag{170}$$

But $D - D_i \geqq 2d(n - 1)$; therefore we obtain from (170)

$$
\begin{aligned}
&k_s(w - D) < \frac{\binom{n-1}{s}d}{(2d(n-1))^{n-s-1}} \\
&= \frac{\binom{n-1}{s}}{2^{n-s-1}(n-1)^{n-s-1}d^{n-2-s}} \leqq \frac{\binom{n-1}{s}}{2^{n-s-1}(n-1)^{n-s-1}} = \frac{\binom{n-1}{n-s-1}}{2^{n-s-1}(n-1)^{n-s-1}} \\
&= \frac{1}{2^{n-s-1}} \cdot \frac{n-1}{n-1} \cdot \frac{n-2}{2(n-1)} \cdot \cdots \cdot \frac{s+1}{(n-s-1)(n-1)} \leqq \frac{1}{2^{n-s-1}} ,
\end{aligned}
$$

which proves formula (168).

We further obtain from (167)

$$w - D < \frac{d}{(2d(n-1))^{n-1}} = \frac{1}{2^{n-1}(n-1)^{n-1}d^{n-2}} \leqq \frac{1}{2^{n-1}} .$$

In virtue of this result and of (168), we now obtain from (166)

$$(w - D)(1 + k_1 + \cdots + k_{i-1}) < \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^{n-i}}$$

$$\leqq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2} = 1 - \frac{1}{2^{n-1}} < 1 .$$

Thus (162) is proved.

In virtue of (163), we obtain the inequalities

$$\frac{k_i}{d} < \frac{a_i^{(0)}(w)}{d} < \frac{k_i + 1}{d} \leqq \frac{k_i}{d} + 1 ,$$

so that

(171) $$\left[\frac{a_i(w)}{d}\right] = \frac{k_i}{d} .$$

(162) and (171) provide the key to our proof of Theorem 4. The further course of the proof is similar to methods used in previous papers ([10], [12]) and we shall, therefore, give here only a very general outline of same. Denoting in the sequel

(173) $$a_i^{(0)}(w) = a_i^{(0)} , \qquad (i = 1, \cdots, n-1)$$

we obtain from (160), (162)

$$a_{i+1}^{(0)} = (w - D)a_i^{(0)} + k_{i+1} , \qquad (i = 0, \cdots, n-2) a_0^{(0)} = 1 ,$$

$$a_{i+1}^{(0)} - b_{i+1}^{(0)} = a_{i+1}^{(0)} - k_{i+1} ,$$

(174) $$a_{i+1}^{(0)} - b_{i+1}^{(0)} = (w - D)a_i^{(0)} \qquad (i = 0, \cdots, n-2) .$$

We further obtain from (24), for $f(w) = 0$,

$$(w - D)^n + k_1(w - D)^{n-1} + k_2(w - D)^{n-2} + \cdots + k_{n-1}(w - D) - d = 0 ,$$

$$\frac{1}{w - D} = \frac{(w - D)^{n-1} + k_1(w - D)^{n-2} + \cdots + k_{n-1}}{d} = \frac{a_{n-1}^{(0)}}{d} ;$$

since, from (174), $a_1^{(0)} - b_1^{(0)} = w - D$, we obtain

(175) $$\frac{1}{a_1^{(0)} - b_1^{(0)}} = \frac{a_{n-1}^{(0)}}{d} .$$

We shall now carry out the Jacobi-Perron algorithm of the basic sequence (160) and obtain from (162)

$$b_s^{(0)} = k_s\,, \qquad (s = 1, \cdots, n-1) \tag{176}$$

and from (174), (175), in virtue of (4)

$$\begin{cases} a_1^{(0)} - b_1^{(0)} = w - D\,, \\ a_{i+1}^{(0)} - b_{i+1}^{(0)} = (w - D)a_i^{(0)}\,, & (i = 1, \cdots, n-2) \end{cases}$$

$$\begin{cases} a_i^{(1)} = a_i^{(0)}\,, & (i = 1, \cdots, n-2) \\ a_{n-1}^{(1)} = a_{n-1}^{(0)}/d\,. \end{cases} \tag{177}$$

From (177) we obtain, in virtue of (162), (171)

$$b_s^{(1)} = k_s\,; \quad (s = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = k_{n-1}/d\,, \tag{178}$$

and from (177), (178), in virtue of (174), (175)

$$\begin{cases} a_1^{(1)} - b_1^{(1)} = w - D\,, \\ a_{1+i}^{(1)} - b_{1+i}^{(1)} = (w - D)a_i^{(0)}\,, & (i = 1, \cdots, n-3) \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = (w - D)a_{n-2}^{(0)}/d\,, \end{cases}$$

$$\begin{cases} a_i^{(2)} = a_i^{(0)}\,, & (i = 1, \cdots, n-3) \\ a_{n-2}^{(2)} = a_{n-2}^{(0)}/d\,, \\ a_{n-1}^{(2)} = a_{n-1}^{(0)}/d\,. \end{cases} \tag{178a}$$

It will now be easy to prove formula

$$\begin{cases} a_i^{(s)} = a_i^{(0)}\,, & (i = 1, \cdots, n-s-1) \\ a_{n-s-1+j}^{(s)} = a_{n-s-1+j}^{(s)}/d\,, & (j = 1, \cdots, s) \\ s = 1, \cdots, n-2\,. \end{cases} \tag{179}$$

Formula (179) is correct for $s = 1, 2$ in virtue of formulas (177), and (178a). It is then presumed that it is correct for $s = m$ and proved that it is correct for $s = m + 1$.

We now obtain from (179), in virtue of (162), (171)

$$\begin{aligned} &b_i^{(s)} = k_i\,; && (i = 1, \cdots, n-s-1) \\ &b_{n-s-1+j}^{(s)} = \frac{k_{n-s-1+j}}{d}\,; && (j = 1, \cdots, s) \\ &s = 1, \cdots, n-2\,. \end{aligned} \tag{180}$$

We further obtain from (179), (180) for $s = n - 2$

$$\begin{aligned} &a_1^{(n-2)} = a_1^{(0)}\,; \quad a_{1+j}^{(n-2)} = a_{1+j}^{(0)}/d\,; && (j = 1, \cdots, n-2) \\ &b_1^{(n-2)} = k_1\,; \quad b_{1+j}^{(n-2)} = k_{1+j}/d\,, && (j = 1, \cdots, n-2) \end{aligned}$$

so that, in virtue of (174), (175), (4)

$$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = w - D\,, \\ a_{1+j}^{(n-2)} - b_{1+j}^{(n-2)} = (w - D)a_j^{(0)}/d\,, & (j = 1, \cdots, n-2) \end{cases}$$

$$a_i^{(n-1)} = a_i^{(0)}/d \qquad (i = 1, \cdots, n-1)\,. \tag{181}$$

From (181) we obtain, in virtue of (171)

$$b_i^{(n-1)} = k_i/d\,, \qquad (i = 1, \cdots, n-1) \tag{182}$$

and from (181), (182), in virtue of (174)

$$\begin{cases} a_1^{(n-1)} - b_1^{(n-1)} = (w - D)/d\,, \\ a_{1+i}^{(n-1)} - b_{1+i}^{(n-1)} = (w - D)a_i^{(0)}/d\,, & (i = 1, \cdots, n-2) \end{cases}$$

so that, in virtue of (4) and (175)

$$a_i^{(n)} = a_i^{(0)}\,, \qquad (i = 1, \cdots, n-1) \tag{183}$$

which proves that the Jacobi-Perron algorithm of the basic sequence $a_i^{(0)}$ $(i = 1, \cdots, n-1)$ is purely periodic and its length $T = n$ for $d > 1$. We further obtain from (177), for $d = 1$,

$$a_i^{(1)} = a_i^{(0)}\,, \qquad (i = 1, \cdots, n-1)$$

so that in this case the Jacobi-Perron algorithm is purely periodic and its length $T = 1$.

From (176), (180), (182) we conclude that the period of the algorithm has the form as demanded by Theorem 4, for $d \geqq 1$.

We shall take up the numeric examples of § 5 to illustrate Theorem 4.

1. $f(x) = x^5 - 15x^4 + 54x^3 - 3 = (x - 9)(x - 6)x^3 - 3 = 0\,.$

Developing $f(x)$ in powers of $x - 9$ we obtain

$$\begin{aligned} f(x) = {} & (x - 9)^5 + 30(x - 9)^4 + 324(x - 9)^3 \\ & + 1458(x - 9)^2 + 2187(x - 9) - 3 = 0\,. \end{aligned}$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w - 9) + 30 = w + 21\,; \\ a_2^{(0)} &= (w - 9)^2 + 30(w - 9) + 324 = w^2 + 12w + 135\,; \\ a_3^{(0)} &= (w - 9)^3 + 30(w - 9)^2 + 324(w - 9) + 1458 \\ &= w^3 + 3w^2 + 27w + 243\,; \\ a_4^{(0)} &= (w - 9)^4 + 30(w - 9)^3 + 324(w - 9)^2 + 1458(w - 9) + 2187 \\ &= w^4 - 6w^3\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

$$\begin{array}{cccc} 30 & 324 & 1458 & 2187 \\ 30 & 324 & 1458 & 729 \\ 30 & 324 & 486 & 729 \\ 30 & 108 & 486 & 729 \\ 10 & 108 & 486 & 729\,. \end{array}$$

2. $$\begin{aligned} f(x) &= x^6 - 3x^5 - 5x^4 + 15x^3 + 4x^2 - 12x - 1 \\ &= (x-3)(x-2)(x-1)x(x+1)(x+2) - 1 = 0\,. \end{aligned}$$

Developing $f(x)$ in powers of $x - 3$ we obtain

$$\begin{aligned} f(x) = (x-3)^6 + 15(x-3)^5 + 85(x-3)^4 + 225(x-3)^3 \\ + 274(x-3)^2 + 120(x-3) - 1 = 0\,. \end{aligned}$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w-3) + 15 = w + 12\,; \\ a_2^{(0)} &= (w-3)^2 + 15(w-3) + 85 = w^2 + 9w + 49\,; \\ a_3^{(0)} &= (w-3)^3 + 15(w-3)^2 + 85(w-3) + 225 \\ &= w^3 + 6w^2 + 22w + 78\,; \\ a_4^{(0)} &= (w-3)^4 + 15(w-3)^3 + 85(w-3)^2 + 225(w-3) + 274 \\ &= w^4 + 3w^3 + 94w^2 - 258w + 40\,; \\ a_5^{(0)} &= (w-3)^5 + 15(w-3)^4 + 85(w-3)^3 + 225(w-3)^2 \\ &\quad + 274(w-3) + 120 = w^5 - 5w^3 + 4w\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

$$15 \qquad 85 \qquad 225 \qquad 274 \qquad 120\,.$$

3. $$f(x) = x^3 - 16x - 2 = (x-4)x(x+4) - 2 = 0\,.$$

Developing $f(x)$ in powers of $x - 4$ we obtain

$$f(x) = (x-4)^3 + 12(x-4)^2 + 32(x-4) - 2 = 0\,.$$

The basic sequence has the form

$$\begin{aligned} a_1^{(0)} &= (w-4) + 12 = w + 8\,; \\ a_2^{(0)} &= (w-4)^2 + 12(w-4) + 32 = w^2 + 4w\,. \end{aligned}$$

The period of the Jacobi-Perron algorithm of these numbers has the form

$$\begin{array}{rr} 12 & 32 \\ 12 & 16 \\ 6 & 16 \,. \end{array}$$

We shall now return to formula (11) in order to calculate w and obtain for Theorem 3:

$$a_{n-1}^{(0)} = w - D_1 = \lim_{v\to\infty} (A_{n-1}^{(v)}/A_0^{(v)}) ,$$

for Theorem 4:

$$a_1^{(0)} = w - D + k_1 = \lim_{v\to\infty} (A_1^{(v)}/A_0^{(v)}) ,$$

where the $A_0^{(v)}$, $A_{n-1}^{(v)}$ from Theorem 3 are not the same as $A_0^{(v)}$, $A_1^{(v)}$ from Theorem 4. Yet, as the first author has proved, there are always indices v_3 for the $A_i^{(v)}$ from Theorem 3 and indices v_4 for the $A_i^{(v)}$ from Theorem 4 such that

$$A_i^{(v_3)} = A_i^{(v_4)} .$$

7. **Units of the field $K(w)$.** Let the coefficients of the *GP*

$$f(x) = (x - D_0)(x - D_1) \cdots (x - D_{n-1}) - d$$

now fulfil the conditions (17), (18), (19) from Theorems 1, 2 and the supplementary inequalities from Theorem 3, i.e., altogether

$$\begin{aligned} &D_i,\ d \text{ rational integers}\,;\ d \geqq 1\,;\ n \geqq 2\,; \\ &D_0 > D_1 > \cdots > D_{n-1}\,;\ d \mid (D_0 - D_i)\ , \\ &D_0 - D_i \geqq 2d(n-1)\ ,\ (i = 1, \cdots n-1)\,; \end{aligned}$$

(184) and in the special case $d = 1$ moreover

$$\begin{aligned} &D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4 \text{ for } n = 3\ , \\ &D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \ngeqq 3 \text{ or} \\ &D_0 - D_1,\ D_2 - D_3 > 2 \text{ for } n = 4\ , \end{aligned}$$

and let be

$$f(w) = (w - D_0)(w - D_1) \cdots (w - D_{n-1}) - d = 0\,; \qquad D_0 < w < D_0 + 1\,. \tag{185}$$

Perron [18] has proved the following important theorem:

If the supporting sequences of the Jacobi-Perron algorithm fulfil the conditions

$$b_{n-1}^{(v)} \geqq n + b_1^{(v)} + b_2^{(v)} + \cdots b_{n-2}^{(v)}\ , \qquad (v = 0, 1, \cdots) \tag{186}$$

then $f(w)$ is irreducible in the rational number field.

We shall apply Theorem 3. Here

$$b_1^{(v)} = b_2^{(v)} = \cdots = b_{n-2}^{(v)} = 0\,;\ b_{n-1}^{(v)} = D_0 - D_i \text{ or } \frac{D_0 - D_i}{d}\,.$$

In order to verify (186), we thus have to prove $D_0 - D_i \geqq nd$. But in virtue of (184) we have, indeed,

$$D_0 - D_i \geqq 2d(n-1) \geqq nd, \text{ since } n \geqq 2\,, \quad (i = 1, \cdots, n-1)\,.$$

Thus $f(w)$ is irreducible in the field of rational numbers, which is true already under the conditions (159), and w, as well as the other roots of $f(x)$ are algebraic irrationals of degree n. Thus, in virtue of Theorem 2 and the conditions (184), $f(x)$ has n different real roots which are all algebraic irrationals of degree n. According to the famous Dirichlet theorem, the exact number of (independent) basic units of the field $K(w)$ is $N = r_1 + r_2 - 1$, where

r_1 is the number of real roots of $f(x)$,

r_2 is the number of pairs of conjugate complex roots of $f(x)$.

In our case $r_1 = n$; $r_2 = 0$, so that $N = n - 1$. We shall now prove

THEOREM 5. *Under the conditions* (184) *the n algebraic irrationals*

$$e_k = \frac{(w - D_k)^n}{d}\,, \qquad (k = 0, 1, \cdots, n-1) \tag{188}$$

are n different units of the field $K(w)$.

That the numbers (188) are all different follows from $D_i \neq D_j$, $(i \neq j\,; i, j = 0, 1, \cdots, n-1)$. We further note that one of the numbers (188), for instance

$$e_{n-1} = (w - D_{n-1})^n/d$$

can be expressed by the other $n - 1$ numbers. We obtain from (185)

$$d/(w - D_{n-1}) = (w - D_0)(w - D_1) \cdots (w - D_{n-2})\,,$$
$$d^n/(w - D_{n-1})^{+n} = (w - D_0)^n(w - D_1)^n \cdots (w - D_{n-2})^n\,,$$

and from this

$$\frac{d}{(w - D_{n-1})^n} = \frac{(w - D_0)^n}{d} \cdot \frac{(w - D_1)^n}{d} \cdots \frac{(w - D_{n-2})^n}{d}\,,$$

so that

$$e_{n-1}^{-1} = e_0 e_1 \cdots e_{n-2} . \tag{189}$$

There is a simple algebraic method to prove that the e_k are all units (see the Appendix by H. Hasse); for this purpose, in view of (189), it suffices to show that the e_k are algebraic integers. This, however, does not disclose the more organic connection between a unit of a field and the periodic algorithm of a basis of the field; after a unit of a field has been found by some device, it is easy to verify that it is one, indeed. The problem of calculating a unit in a quadratic field $K(\sqrt{m})$ is entirely solved by developing $\sqrt{m}$ in a periodic continuous fraction by Euclid's algorithm.

In a joint paper with Helmut Hasse [16] it was proved that in the case of a periodic Jacobi-Perron algorithm carried out on a basis $w, w^2, \cdots, w^{n-1}$ of an algebraic field $K(w)$, $w = (D^n + d)^{1/n}$; d, D natural numbers, $d \mid D$, a unit of the field is given by the formula

$$e^{-1} = a_{n-1}^{(S)} a_{n-1}^{(S+1)} \cdots a_{n-1}^{(S+T-1)} , \tag{190}$$

where S and T (see (6)) denote the length of the preperiod and the period of the algorithm respectively.[1]

Turning to Theorem 3, we obtain $S = 0$, $T = n(n-1)$ for $d \neq 1$, and formula (190) takes the form

$$e^{-1} = \prod_{v=0}^{n(n-1)-1} a_{n-1}^{(v)} = \prod_{i=0}^{n-1} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} . \tag{191}$$

Following up the various stages of the proof of Theorem 3, one can easily varify the relations

$$\prod_{k=0}^{n-2} a_{n-1}^{(k)} = P_{1,1} P_{2,2} \cdots P_{n-1,n-1} , \tag{192}$$

$$\prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)-k} = d^{-1} P_{1,1} P_{2,2} \cdots P_{n-1,n-1}, \qquad (i = 1, \cdots n-1) . \tag{193}$$

In virtue of (192), (193) we obtain from (191)

$$e^{-1} = d^{-(n-1)} (P_{1,1} P_{2,2} \cdots P_{n-1,n-1})^n . \tag{194}$$

From (39) we obtain

$$P_{1,1} P_{2,2} \cdots P_{n-1,n-1} = \frac{d}{w - D} , \tag{195}$$

and from (194), (195)

$$e^{-1} = \frac{d}{(w-D)^n} , \quad e = \frac{(w-D)^n}{d} , \tag{196}$$

[1] Formula (190) holds for any algebraic irrational w.

which proves Theorem 5 for $k = 0$, since $D = D_0$. Yet it is rather complicated to prove the remaining statement of Theorem 5, namely that the other e_k $(k = 1, \cdots, n-2)$ are units of $K(w)$ which can be derived from a periodic algorithm like e_0. We say deliberately periodic algorithm and not periodic Jacobi-Perron algorithm, which has its good reasons in the following observation: if one reads the author's joint paper with Professor Helmut Hasse carefully enough, he will soon realize that in order to prove formula (190) two presumptions are necessary—first that the numbers $b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}$ $(v = 0, 1, \cdots)$ be all integers; second that the algorithm be periodic, while the formation law by which the $b_i^{(v)}$ are derived from the $a_i^{(v)}$ is altogether not essential. In this chapter we shall define a new formation law for the $b_i^{(v)}$ and obtain, on ground of it, a periodic algorithm for $n-1$ polynomials chosen from the field $K(w)$. In this algorithm the $b_i^{(v)}$ will all be rational integers so that formula (190) can be applied. These results are laid down in Theorem 6. Before we state this theorem, we shall explain the new formation law for the $b_i^{(v)}$ and introduce, to this end, a few more notations.

DEFINITION. Let w be the only real root in the open interval $(D_0, D_0 + 1)$ of equation (185), so that

$$(w - D_0)(w - D_1) \cdots (w - D_{n-1}) - d = 0 .$$

Let the elements of the basic sequence of an algorithm G be polynomials in w with rational coefficients, i.e.,

$$a_s^{(0)} = a_s^{(0)}(w) = \textstyle\sum_{i=0}^{s} C_i w^{s-i} \qquad (s = 1, \cdots, n-1) ; \tag{197}$$

if the $b_s^{(v)}$ $(s = 1, \cdots, n-1;\ v = 0, 1, \cdots)$ are rationals, then, in virtue of (4), the $a_s^{(v)}$, too, are polynomials in w with rational coefficients for all s, v, i.e.,

$$a_s^{(v)} = a_s^{(v)}(w) = \textstyle\sum_{i=0}^{s} C_i^{(v)} w^{s-i} \quad (s = 1, \cdots, n-1;\ v = 0, 1, \cdots) \tag{198}$$

G is called the Modified Algorithm of Jacobi-Perron, if the $b_i^{(v)}$ are obtained from the $a_i^{(v)}$ by the formation law

$$b_s^{(v)} = a_s^{(v)}(D_k) \qquad (s, v \text{ as in } (198)) . \tag{199}$$

Here D_k is one of the numbers $D_0, D_1, \cdots, D_{n-1}$; D_k remains the same during the process of G.

We shall now introduce the following notations

$$R_{i,i} = w - D_{i,i}\,;\ D_{i,i} \text{ any of the numbers } D_0, \cdots, D_{n-1}\,;$$
$$(200)\quad R_{i,i} \neq R_{j,j} \text{ for } i \neq j\,;$$
$$R_{i,j} = R_{i,i}R_{i+1,i+1} \cdots R_{j,j}\,, \qquad (0 \leqq i \leqq j \leqq n-1).$$

From (185) and (200) we obtain

$$(201)\quad \begin{aligned} &R_{0,n-1} = d\,; \\ &1/R_{i,j} = R_{0,i-1}R_{j+1,n-1}/d\,, && (0 < i \leqq j < n-1) \\ &1/R_{0,j} = R_{j+1,n-1}/d\,, && (0 \leqq j < n-1) \\ &1/R_{i,n-1} = R_{0,i-1}/d\,, && (0 < i \leqq n-1)\,. \end{aligned}$$

We are now able to state

THEOREM 6. *Under the conditions* (186) *let*

$$(202)\qquad R_{1,1}, R_{2,2}, \cdots, R_{n-2,n-2}$$

be any $n-2$ *of the* $n-1$ *polynomials*

$$(203)\qquad P_{0,0}, \cdots, P_{k-1,k-1}, P_{k+1,k+1}, \cdots P_{n-1,n-1}\,; \quad (k = 1, \cdots, n-2)$$

then the Modified Algorithm of Jacobi-Perron of the basis

$$(204)\qquad a_i^{(0)} = R_{1,n-1-i}P_{k,k}\,; \quad (i = 1, \cdots, n-2) \qquad a_{n-1}^{(0)} = R_{1,1}$$

is purely periodic; the length of the period is $T = n(n-1)$ *for* $d > 1$ *and* $T = n-1$ *for* $d = 1$. *The period of length* $T = n-1$ *consists of one fugue; its generator has the form*

$$(205)\qquad \begin{cases} D_k - D_{1,1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots\cdots \\ D_k - D_{2,2}\,. \end{cases}$$

The period of length $n(n-1)$ *consists of* n *fugues; the generator of the first fugue has the form*

$$(206a)\qquad \begin{cases} D_k - D_{1,1} \\ (D_k - D_{n-1,n-1})d^{-1} \\ D_k - D_{n-2,n-2} \\ D_k - D_{n-3,n-3} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ D_k - D_{2,2} \end{cases}$$

The generator of the i *-th fugue* $(i = 2, \cdots, n-3)$ *has the form*

$$\text{(206b)} \qquad \begin{cases} D_k - D_{1,1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots\cdots \\ D_k - D_{n-(i-1),n-(i-1)} \\ (D_k - D_{n-i,n-i})d^{-1} \\ D_k - D_{n-(i+1),n-(i+1)} \\ \cdots\cdots\cdots\cdots \\ D_k - D_{2,2} \,. \end{cases}$$

The generator of the $n-2$ *-th fugue has the form*

$$\text{(206c)} \qquad \begin{cases} D_k - D_{1,1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots \\ D_k - D_{3,3} \\ (D_k - D_{2,2})d^{-1} \,. \end{cases}$$

The generator of the $n-1$ *-th fugue has the form* (205) ; *the generator of the* n *-th fugue has the form*

$$\text{(206d)} \qquad \begin{cases} (D_k - D_{1,1})d^{-1} \\ D_k - D_{n-1,n-1} \\ D_k - D_{n-2,n-2} \\ \cdots\cdots\cdots \\ D_k - D_{2,2} \,. \end{cases}$$

The reader should note that the generators (205) and (206a)-(206d) consist of rational integers only. The differences $D_k - D_{i,i}$ $(i = 1, \cdots, n-1)$ are algebraic sums of natural numbers ; and since $d \mid D_k$, $d \mid D_{i,i}$, so is $d \mid D_k - D_{i,i}$. One further notes that these generators contain no zeros, since $P_{k,k} \neq R_{i,i}$ and therefore $D_k \neq D_{i,i}$, $(i = 1, \cdots, n-1)$.

Proof of Theorem 6. We first make the following observation : since, in virtue of (202), (203), we can have either

$$P_{k,k} = R_{0,0} \text{ or } P_{k,k} = R_{n-1,n-1}$$

we shall choose

$$\text{(207)} \qquad P_{k,k} = R_{0,0} \,.$$

We shall now carry out the Modified Algorithm of Jacobi-Perron for the basic sequence (204). We obtain from (204), since every factor $a_i^{(0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(208) \qquad b_i^{(0)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(0)} = D_k - D_{1,1} \, .$$

From (204), (208) we obtain, since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$

$$(209) \qquad \begin{cases} a_1^{(0)} - b_1^{(0)} = R_{1,n-2}P_{k,k} \, , & \\ a_{1+i}^{(0)} - b_{1+i}^{(0)} = R_{1,n-2-i}P_{k,k} \, , & (i = 1, \cdots, n-3) \\ a_{n-1}^{(0)} - b_{n-1}^{(0)} = P_{k,k} \, , & \end{cases}$$

and from (209), in virtue of (4) and (201), (207)

$$(210) \qquad \begin{cases} a_i^{(1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d \, , & (i = 1, \cdots, n-3) \\ a_{n-2}^{(1)} = R_{n-1,n-1}P_{k,k}/d \, , & \\ a_{n-1}^{(1)} = R_{n-1,n-1}/d \, . & \end{cases}$$

From (210) we obtain, since every $a_i^{(1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(211) \qquad b_i^{(1)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1)} = (D_k - D_{n-1,n-1})d^{-1} \, ,$$

and from (210), (211), since

$$(R_{n-1,n-1}/d) - (D_k - D_{n-1,n-1})d^{-1} = (w - D_k)d^{-1} = P_{k,k}d^{-1} \, ,$$

$$(211) \qquad \begin{cases} a_1^{(1)} - b_1^{(1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}d^{-1} \, , & \\ a_{1+i}^{(1)} - b_{1+i}^{(1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}d^{-1} \, , & (i = 1, \cdots, n-4) \\ a_{n-2}^{(1)} - b_{n-2}^{(1)} = R_{n-1,n-1}P_{k,k}d^{-1} \, , & \\ a_{n-1}^{(1)} - b_{n-1}^{(1)} = P_{k,k}d^{-1} \, . & \end{cases}$$

From (211) we obtain, in virtue of (4) and (201), (207),

$$(212) \qquad \begin{cases} a_i^{(2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d \, , & (i = 1, \cdots, n-4) \\ a_{n-3}^{(2)} = R_{n-2,n-1}P_{k,k}/d \, , & \\ a_{n-2}^{(2)} = R_{n-2,n-2}P_{k,k}/d \, , & \\ a_{n-1}^{(2)} = R_{n-2,n-2} \, . & \end{cases}$$

From (212) we obtain, since every $a_i^{(2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(213) \qquad b_i^{(2)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(2)} = D_k - D_{n-2,n-2} \, ,$$

and from (212), (213), since $R_{n-2,n-2} - (D_k - D_{n-2,n-2}) = w - D_k = P_{k,k}$

$$(214)\quad \begin{cases} a_1^{(2)} - b_1^{(2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d\ , \\ a_{1+i}^{(2)} - b_{1+i}^{(2)} = R_{1,n-4-i}R_{n-2,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-5) \\ a_{n-3}^{(2)} - b_{n-3}^{(2)} = R_{n-2,n-1}P_{k,k}/d\ , \\ a_{n-2}^{(2)} - b_{n-2}^{(2)} = R_{n-2,n-2}P_{k,k}/d\ , \\ a_{n-1}^{(2)} - b_{n-1}^{(2)} = P_{k,k} \end{cases}$$

From (214) we obtain, in virtue of (4) and (201), (207)

$$(215)\quad \begin{cases} a_i^{(3)} = R_{1,n-4-i}R_{n-3,n-1}P_{k,k}/d\ , \qquad (i = 1, \cdots, n-5) \\ a_{n-4}^{(3)} = R_{n-3,n-1}P_{k,k}/d\ , \\ a_{n-3}^{(3)} = R_{n-3,n-2}P_{k,k}/d\ , \\ a_{n-2}^{(3)} = R_{n-3,n-3}P_{k,k}\ , \\ a_{n-1}^{(3)} = R_{n-3,n-3}\ . \end{cases}$$

We shall now prove the formula

$$(216)\quad \begin{cases} a_i^{(t)} = R_{1,n-1-t-i}R_{n-t,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-2-t) \\ a_{n-1-t}^{(t)} = R_{n-t,n-1}P_{k,k}/d\ , \\ a_{n-t}^{(t)} = R_{n-t,n-2}P_{k,k}/d\ , \\ a_{n-t+j}^{(t)} = R_{n-t,n-2-j}P_{k,k}\ , \qquad (j = 1, \cdots, t-2) \\ a_{n-1}^{(t)} = R_{n-t,n-t}\ , \\ t = 3, \cdots, n-t\ . \end{cases}$$

Formula (216) is correct for $t = 3$, in virtue of formula (215). Let it be correct for $t = m$ $(m = 3, \cdots, n-4)$. From (216) we obtain, for $t = m$, since every $a_i^{(m)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(217)\qquad b_i^{(m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m)} = D_k - D_{n-m,n-m}\ ,$$

and from (216) (for $t = m$) and (217), since

$$R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}$$

$$(218)\quad \begin{cases} a_1^{(m)} - b_1^{(m)} = R_{1,n-2-m}R_{n-m,n-1}P_{k,k}/d\ , \\ a_{1+i}^{(m)} - b_{1+i}^{(m)} = R_{1,n-2-m-i}R_{n-m,n-1}P_{k,k}/d\ , \quad (i = 1, \cdots, n-3-m) \\ a_{n-1-m}^{(m)} - b_{n-1-m}^{(m)} = R_{n-m,n-1}P_{k,k}/d\ , \\ a_{n-m}^{(m)} - b_{n-m}^{(m)} = R_{n-m,n-2}P_{k,k}/d\ , \\ a_{n-m+j}^{(m)} - b_{n-m+j}^{(m)} = R_{n-m,n-2-j}P_{k,k}\ , \qquad (j = 1, \cdots, m-2) \\ a_{n-1}^{(m)} - b_{n-1}^{(m)} = P_{k,k}\ . \end{cases}$$

From (218) we obtain, in virtue of (4) and (201), (207)

$$(219)\quad \begin{cases} a_i^{(m+1)} = R_{1,n-2-m-i}R_{n-m-1,n-1}P_{k,k}/d\,, & (i = 1, \cdots, n-3-m) \\ a_{n-2-m}^{(m+1)} = R_{n-m-1,n-1}P_{k,k}/d\,, & \\ a_{n-1-m}^{(m+1)} = R_{n-m-1,n-2}P_{k,k}/d\,, & \\ a_{n-1-mj}^{(m+1)} = R_{n-m-1,n-2-j}P_{k,k}\,, & (j = 1, \cdots, m-1) \\ a_{n-1}^{(m+1)} = R_{n-m-1,n-m-1}\,. & \end{cases}$$

But (219) is formula (216) for $t = m + 1$, which completes the proof of this formula. We now obtain from (216) for $t = n - 3$

$$(220)\quad \begin{cases} a_1^{(n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, & \\ a_2^{(n-3)} = R_{3,n-1}P_{k,k}/d\,, & \\ a_3^{(n-3)} = R_{3,n-2}P_{k,k}/d\,, & \\ a_{3+j}^{(n-3)} = R_{3,n-2-j}P_{k,k}\,, & (j = 1, \cdots, n-5) \\ a_{n-1}^{(n-3)} = R_{3,3}\,. & \end{cases}$$

From (220) we obtain, since every $a_i^{(n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(221)\qquad b_i^{(n-3)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3)} = D_k - D_{3,3}$$

and from (220), (221), since $P_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$

$$(222)\quad \begin{cases} a_1^{(n-3)} - b_1^{(n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, & \\ a_2^{(n-3)} - b_2^{(n-3)} = R_{3,n-1}P_{k,k}/d\,, & \\ a_3^{(n-3)} - b_3^{(n-3)} = R_{3,n-2}P_{k,k}/d\,, & \\ a_{3+j}^{(n-3)} - b_{3+j}^{(n-3)} = R_{3,n-2-j}P_{k,k}\,, & (j = 1, \cdots, n-5) \\ a_{n-1}^{(n-3)} - b_{n-1}^{(n-3)} = P_{k,k}\,. & \end{cases}$$

From (222) we obtain, in virtue of (4) and (201), (207),

$$(223)\quad \begin{cases} a_1^{(n-2)} = R_{2,n-1}P_{k,k}/d;\ a_2^{(n-2)} = R_{2,n-2}P_{k,k}/d\,, \\ a_{2+j}^{(n-2)} = R_{2,n-2-j}P_{k,k},\ (j = 1, \cdots, n-4),\ a_{n-1}^{(n-2)} = R_{2,2}\,, \end{cases}$$

and from (223), since every $a_i^{(n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199);

$$(224)\qquad b_i^{(n-2)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2)} = D_k - D_{2,2}\,.$$

From (223), (224) we obtain, since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(225)\quad \begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = R_{2,n-1}P_{k,k}/d\,, & \\ a_2^{(n-2)} - b_2^{(n-2)} = R_{2,n-2}P_{k,k}/d\,, & \\ a_{2+j}^{(n-2)} - b_{2+j}^{(n-2)} = R_{2,n-2-j}P_{k,k}\,, & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = P_{k,k}\,, & \end{cases}$$

and from (225), in virtue of (4) and (201), (207)

$$(226)\qquad \begin{cases} a_1^{(1;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+j}^{(1;0)} = R_{1,n-2-j}P_{k,k}\,, & (j = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} = R_{1,1}\,. \end{cases}$$

Here we are making use of the notation (37) $u; v = u(n-1) + v$. In virtue of formulae (208), (211), (213), (217), (224) the first $n-1$ supporting sequences of the algorithm form a fugue which has the form of the first fugue as demanded by Theorem 6.

From (226) we obtain, since every $a_i^{(1;0)}$ $(i = 1, \cdots, n-2)$ has the factor $P_{k,k}$, and in virtue of (199)

$$(227)\qquad b_i^{(1;0)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;0)} = D_k - D_{1,1}\,,$$

and from (226), (227), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(228)\qquad \begin{cases} a_1^{(1;0)} - b_1^{(1;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+j}^{(1;0)} - b_{1+j}^{(1;0)} = R_{1,n-2-j}P_{k,k}\,, & (j = 1, \cdots, n-3) \\ a_{n-1}^{(1;0)} - b_{n-1}^{(1;0)} = P_{k,k}\,. \end{cases}$$

From (228) we obtain, in virtue of (4) and (201), (207),

$$(229)\qquad \begin{cases} a_j^{(1;1)} = R_{1,n-2-j}R_{n-1,n-1}P_{k,k}\,, & (j = 1, \cdots, n-3) \\ a_{n-2}^{(1;1)} = R_{n-1,n-1}P_{k,k}\,, \\ a_{n-1}^{(1;1)} = R_{n-1,n-1}\,, \end{cases}$$

and from (229), since every $a_i^{(1;1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(230)\qquad b_i^{(1;1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;1)} = D_k - D_{n-1,n-1}\,.$$

From (229), (230) we obtain, since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$

$$(231)\qquad \begin{cases} a_1^{(1;1)} - b_1^{(1;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}\,, \\ a_{1+j}^{(1;1)} - b_{1+j}^{(1;1)} = R_{1,n-3-j}R_{n-1,n-1}P_{k,k}\,, & (j = 1, \cdots, n-4) \\ a_{n-2}^{(1;1)} - b_{n-2}^{(1;1)} = R_{n-1,n-1}P_{k,k}\,, \\ a_{n-1}^{(1;1)} - b_{n-1}^{(1;1)} = P_{k,k}\,, \end{cases}$$

and from (231), in virtue of (4) and (201), (207)

$$(232)\qquad \begin{cases} a_j^{(1;2)} = R_{1,n-3-j}R_{n-2,n-1}P_{k,k}/d\,, & (j = 1, \cdots, n-4) \\ a_{n-3}^{(1;2)} = R_{n-2,n-1}P_{k,k}/d\,, \\ a_{n-2}^{(1;2)} = R_{n-2,n-2}P_{k,k}/d\,, \\ a_{n-1}^{(1;2)} = R_{n-2,n-2}/d\,. \end{cases}$$

From (232) we obtain, since every $a_i^{(1;2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(233) $\quad b_i^{(1;2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;2)} = (D_k - D_{n-2,n-2})/d$,

and from (232), (233), since

$$(R_{n-2,n-2}/d) - ((D_k - D_{n-2,n-2})/d) = (w - D_k)/d = P_{k,k}/d ,$$

$$(234) \quad \begin{cases} a_1^{(1;2)} - b_1^{(1;2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d , \\ a_{1+j}^{(1;2)} - b_{1+j}^{(1;2)} = R_{1,n-4-j}R_{n-2,n-1}P_{k,k}/d , \quad (j = 1, \cdots, n-5) \\ a_{n-3}^{(1;2)} - b_{n-3}^{(1;2)} = R_{n-2,n-1}P_{k,k}/d , \\ a_{n-2}^{(1;2)} - b_{n-2}^{(1;2)} = R_{n-2,n-2}P_{k,k}/d , \\ a_{n-1}^{(1;2)} - b_{n-1}^{(1;2)} = P_{k,k}/d . \end{cases}$$

From (234) we obtain, in virtue of (4) and (201), (207),

$$(235) \quad \begin{cases} a_j^{(1;3)} = R_{1,n-4-j}R_{n-3,n-1}P_{k,k}/d , & (j = 1, \cdots, n-5) \\ a_{n-5+s}^{(1;3)} = R_{n-3,n-s}P_{k,k}/d , & (s = 1, \cdots, 3) \\ a_{n-1}^{(1;3)} = R_{n-3,n-3} . \end{cases}$$

From (235) we obtain, since every $a_i^{(1;3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(236) $\quad b_i^{(1;3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;3)} = D_k - D_{n-3,n-3}$,

and from (235), (236), since $R_{n-3,n-3} - (D_k - D_{n-3,n-3}) = w - D_k = P_{k,k}$

$$(237) \quad \begin{cases} a_1^{(1;3)} - b_1^{(1;3)} = R_{1,n-5}R_{n-3,n-1}P_{k,k}/d , \\ a_{1+j}^{(1;3)} - b_{1+j}^{(1,3)} = R_{1,n-5-j}R_{n-3,n-1}P_{k,k}/d , & (j = 1, \cdots, n-6) \\ a_{n-5+s}^{(1;3)} - b_{n-5+s}^{(1;3)} = R_{n-3,n-s}P_{k,k}/d , & (s = 1, 2, 3) \\ a_{n-1}^{(1;3)} - b_{n-1}^{(1;3)} = P_{k,k} . \end{cases}$$

From (237) we obtain, in virtue of (4) and (201), (207),

$$(238) \quad \begin{cases} a_j^{(1;4)} = R_{1,n-5-j}R_{n-4,n-1}P_{k,k}/d , & (j = 1, \cdots, n-6) \\ a_{n-6+s}^{(1;4)} = R_{n-4,n-s}P_{k,k}/d , & (s = 1, 2, 3) \\ a_{n-2}^{(1;4)} = R_{n-4,n-4}P_{k,k} , \\ a_{n-1}^{(1;4)} = R_{n-4,n-4} . \end{cases}$$

We shall now prove the formula

$$(239) \quad \begin{cases} a_j^{(1,t)} = R_{1,n-1-t-j}R_{n-t,n-1}P_{k,k}/d , & (j = 1, \cdots, n-t-2) \\ a_{n-t-2+s}^{(1;t)} = R_{n-t,n-s}P_{k,k}/d , & (s = 1, 2, 3) \\ a_{n-t+1+u}^{(1;t)} = R_{n-t,n-3-u}P_{k,k} , & (u = 1, \cdots, t-3) \\ a_{n-1}^{(1;t)} = R_{n-t,n-t} , \\ t = 4, \cdots, n-3 . \end{cases}$$

Formula (239) is correct for $t = 4$, in virtue of (238). We presume (239) is correct for $m \geqq 4$, i.e.,

$$(240)\quad \begin{cases} a_j^{(1;m)} = R_{1,n-1-m-j}R_{n-m,n-1}P_{k,k}/d\ , & (j = 1, \cdots, n-m-2) \\ a_{n-m-2+s}^{(1;m)} = R_{n-m,n-s}P_{k,k}/d\ , & (s = 1, 2, 3) \\ a_{n-m+1+u}^{(1;m)} = R_{n-m,n-3-u}P_{k,k}\ , & (u = 1, \cdots, m-3) \\ a_{n-1}^{(1;m)} = R_{n-m,n-m}\ . \end{cases}$$

From (240) we obtain, since every $a_i^{(1;m)}$ contains the factor $P_{k,k}$ $(i = 1, \cdots, n-2)$, and in virtue of (199)

$$(241)\quad b_i^{(1;m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;m)} = D_k - D_{n-m,n-m}\ ,$$

and from (240), (241), since

$$R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}\ ,$$

$$(242)\quad \begin{cases} a_1^{(1;m)} - b_1^{(1;m)} = R_{1,n-m-2}R_{n-m,n-1}P_{k,k}/d\ , \\ a_{1+j}^{(1;m)} - b_{1+j}^{(1;m)} = R_{1,n-m-2-j}R_{n-m,n-1}P_{k,k}/d\ , & (j = 1, \cdots, n-m-3) \\ a_{n-m-2+s}^{(1;m)} - b_{n-m-2+s}^{(1;m)} = R_{n-m,n-s}P_{k,k}/d & (s = 1, 2, 3) \\ a_{n-m+1+u}^{(1;m)} - b_{n-m+1+u}^{(1;m)} = R_{n-m,n-3-u}P_{k,k}\ , & (u = 1, \cdots, m-3) \\ a_{n-1}^{(1;m)} - b_{n-1}^{(1;m)} = P_{k,k}\ . \end{cases}$$

From (242) we obtain, in virtue of (4) and (201), (207),

$$(243)\quad \begin{cases} a_j^{(1;m+1)} = R_{1,n-m-2-j}R_{n-m,n-1}P_{k,k}/d\ , & (j = 1, \cdots, n-m-3) \\ a_{n-m-3+s}^{(1;m+1)} = R_{n-m-1,n-s}P_{k,k}/d\ , & (s = 1, 2, 3) \\ a_{n-m+u}^{(1;m+1)} = R_{n-m-1,n-3-u}P_{k,k}\ , & (u = 1, \cdots, m-2) \\ a_{n-1}^{(1;m+1)} = R_{n-m-1,n-m-1}\ . \end{cases}$$

Substituting $m + 1$ for t in formula (239) we obtain formula (243) which completes the proof of (239).

From (239) we now obtain for $t = n - 3$,

$$(244)\quad \begin{cases} a_1^{(1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\ , \\ a_{1+s}^{(1;n-3)} = R_{3,n-s}P_{k,k}/d\ , & (s = 1, 2, 3) \\ a_{4+u}^{(1;n-3)} = R_{3,n-3-u}P_{k,k}\ , & (u = 1, \cdots, n-6) \\ a_{n-1}^{(1;n-3)} = R_{3,3}\ , \end{cases}$$

and from (244), since every $a_i^{(1;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(246)\quad \begin{cases} a_1^{(1;n-3)} - b_1^{(1;n-3)} = R_{1,1}R_{3,n-1}R_{k,k}/d\ , \\ a_{1+s}^{(1;n-3)} - b_{1+s}^{(1;n-3)} = R_{3,n-s}P_{k,k}/d\ , & (s = 1, 2, 3) \\ a_{4+u}^{(1;n-3)} - b_{4+u}^{(1;n-3)} = R_{3,n-3-u}P_{k,k}\ , & (u = 1, \cdots, n-6) \\ a_{n-1}^{\prime 1;n-3)} - b_{n-1}^{(1;n-3)} = P_{k,k}\ . \end{cases}$$

From (246) we obtain, in virtue of (4) and (201), (207)

$$
(247)\qquad \begin{cases} a_1^{(1;n-2)} = R_{2,n-1}P_{k,k}/d\ , \\ a_2^{(1;n-2)} = R_{2,n-2}P_{k,k}/d\ , \\ a_3^{(1;n-2)} = R_{2,n-3}P_{k,k}/d\ , \\ a_{3+u}^{(1;n-2)} = R_{2,n-3-u}P_{k,k}\ , & (u = 1, \cdots, n-5) \\ a_{n-1}^{(1;n-2)} = R_{2,2}\ . \end{cases}
$$

From (247) we obtain, since every $a_i^{(1;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$
(248)\qquad b_i^{(1;n-3)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(1;n-2)} = D_k - D_{2,2}\ ,
$$

and from (247), (248), since

$$
R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}
$$

$$
(249)\qquad \begin{cases} a_1^{(1;n-2)} - b_1^{(1;n-2)} = R_{2,n-1}P_{k,k}/d\ , \\ a_2^{(1;n-2)} - b_2^{(1;n-2)} = R_{2,n-2}P_{k,k}/d\ , \\ a_3^{(1;n-2)} - b_3^{(1;n-2)} = R_{2,n-3}P_{k,k}/d\ , \\ a_{3+i}^{(1;n-2)} - b_{3+i}^{(1;n-2)} = R_{2,n-3-i}P_{k,k}\ , \\ a_{n-1}^{(1;n-2)} - b_{n-1}^{(1;n-2)} = P_{k,k}\ . \end{cases}
$$

From (249) we obtain, in virtue of (4) and (201), (207) ,

$$
(250)\qquad \begin{cases} a_1^{(2;0)} = R_{1,n-2}P_{k,k}/d\ , \\ a_2^{(2;0)} = R_{1,n-3}P_{k,k}/d\ , \\ a_{2+i}^{(2;0)} = R_{1,n-3-i}P_{k,k}\ , & (i = 1, \cdots, n-4) \\ a_{n-1}^{(2;0)} = R_{1,1}\ . \end{cases}
$$

In virtue of formulae (227), (230), (233), (236), (241), (248), the $n-1$ supporting sequences, starting with the n -th sequence of the algorithm, form a fugue which has the form of the second fugue as demanded by Theorem 6.

The proof of Theorem 6 is essentially based on the following

LEMMA 2. *If the generating sequence*

$$
a_i^{(t;0)};\ (i = 1, \cdots, n-1;\ t = 1, \cdots, n-4)
$$

has the form

$$
(251)\qquad \begin{cases} a_i^{(t;0)} = R_{1,n-1-i}P_{k,k}/d\ , & (i = 1, \cdots, t) \\ a_{t+j}^{(t;0)} = R_{1,n-1-t-j}P_{k,k}\ , & (j = 1, \cdots, n-t-2) \\ a_{n-1}^{(t;0)} = R_{1,1}\ , \end{cases}
$$

then the $n-1$ supporting sequences

$$b_1^{(t;s)}, b_2^{(t;s)}, \cdots, b_{n-1}^{(t;s)} \qquad (s = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the $t+1$ *-th fugue as demanded by Theorem* 6, *and the generating sequence*

$$a_1^{(t+1;0)}, a_2^{(t+1;0)}, \cdots, a_{n-1}^{(t+1;0)}$$

has the form (251) *where* t *is to be replaced by* $t+1$.

Proof. The Lemma 2 is correct for $t = 1$, as can be easily verified by the formulae (226), (250) and the remark following formula (250). We shall presume that the Lemma 2 is correct for $t = m-1$ $(m \geqq 2)$ and shall prove its correctness for $t+1 = m$. We obtain from (251), on ground of the second statement of the Lemma 2 (viz. for $t+1 = m$)

$$(252) \qquad \begin{cases} a_i^{(m;0)} = R_{1,n-1-i}P_{k,k}/d\,, & (i = 1, \cdots, m) \\ a_{m+j}^{(m;0)} = R_{1,n-1-m-j}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-1}^{(m;0)} = R_{1,1}\,. \end{cases}$$

From (252) we obtain, since every $a_i^{(m;0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(253) \qquad b_i^{(m;0)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;0)} = D_k - D_{1,1}\,,$$

and from (252), (253), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(254) \qquad \begin{cases} a_1^{(m;0)} - b_1^{(m;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+i}^{(m;0)} - b_{1+i}^{(m;0)} = R_{1,n-2-i}P_{k,k}/d\,, & (i = 1, \cdots, m-1) \\ a_{m+j}^{(m;0)} - b_{m+j}^{(m;0)} = R_{1,n-1-m-j}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-1}^{(m;0)} - b_{n-1}^{(m;0)} = P_{k,k}\,. \end{cases}$$

From (254) we obtain, in virtue of (4) and (201), (207),

$$(255) \qquad \begin{cases} a_i^{(m,1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-1) \\ a_{m-1+j}^{(m;1)} = R_{1,n-1-m-j}R_{n-1,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-2}^{(m;1)} = R_{n-1,n-1}P_{k,k}\,, \\ a_{n-1}^{(m;1)} = R_{n-1,n-1}\,. \end{cases}$$

From (255) we obtain, since every $a_i^{(m;1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(256) \qquad b_i^{(m;1)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;1)} = D_k - D_{n-1,n-1}\,,$$

and from (255), (256), since

$$R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}\,,$$

$$(257)\quad \begin{cases} a_1^{(m;1)} - b_1^{(m;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(m;1)} - b_{1+i}^{(m;1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-2) \\ a_{m-1+j}^{(m;1)} - b_{m-1+j}^{(m;1)} = R_{1,n-1-m-j}R_{n-1,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-2}^{(m;1)} - b_{n-2}^{(m;1)} = R_{n-1,n-1}P_{k,k}\,, \\ a_{n-1}^{(m;1)} - b_{n-1}^{(m;1)} = P_{k,k}\,. \end{cases}$$

From (257) we obtain, in virtue of (4) and (201), (207),

$$(258)\quad \begin{cases} a_i^{(m;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-2) \\ a_{m-2+j}^{(m;2)} = R_{1,n-1-m-j}R_{n-2,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-3}^{(m;2)} = R_{n-2,n-1}P_{k,k}\,, \\ a_{n-2}^{(m;3)} = R_{n-2,n-2}P_{k,k}\,, \\ a_{n-1}^{(m;3)} = R_{n-2,n-2}\,. \end{cases}$$

We shall now prove the formula

$$(259)\quad \begin{cases} a_i^{(m;t)} = R_{1,n-1-t-i}R_{n-t,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-t) \\ a_{m-t+j}^{(m;t)} = R_{1,n-1-m-j}R_{n-t,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-t-2+u}^{(m;t)} = R_{n-t,n-u}P_{k,k}\,, & (u = 1, \cdots, t) \\ a_{n-1}^{(m;t)} = R_{n-t,n-t}\,, \\ t = 1, \cdots, m-1\,. \end{cases}$$

Formula (259) is correct for $t = 1, 2$, in virtue of formulae (255), (258). Let it be correct for $t = s \geqq 2$, i.e.,

$$(260)\quad \begin{cases} a_i^{(m;s)} = R_{1,n-1-s-i}R_{n-s,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-s) \\ a_{m-s+j}^{(m;s)} = R_{1,n-1-m-j}R_{n-s,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-s-2+u}^{(m;s)} = R_{n-s,n-u}P_{k,k}\,, & (u = 1, \cdots, s) \\ a_{n-1}^{(m;s)} = R_{n-s,n-s}\,. \end{cases}$$

From (260) we obtain, since every $a_i^{(m;s)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(261)\quad b_i^{(m;s)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;s)} = D_k - D_{n-s,n-s}\,,$$

and from (260), (261), since $R_{n-s,n-s} - (D_k - D_{n-s,n-s}) = w - D_k = P_{k,k}$,

$$(262)\quad \begin{cases} a_1^{(m;s)} - b_1^{(m;s)} = R_{1,n-2-s}R_{n-s,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(m;s)} - b_{1+i}^{(m;s)} = R_{1,n-2-s-i}R_{n-s,n-1}P_{k,k}/d\,, & (i = 1, \cdots, m-s-1) \\ a_{m-s+j}^{(m;s)} - b_{m-s+j}^{(m;s)} = R_{1,n-1-m-j}R_{n-s,n-1}P_{k,k}\,, & (j = 1, \cdots, n-m-2) \\ a_{n-s-2+u}^{(m;s)} - b_{n-s-2+u}^{(m;s)} = R_{n-s,n-u}P_{k,k}\,, & (u = 1, \cdots, s) \\ a_{n-1}^{(m;s)} - b_{n-1}^{(m;s)} = P_{k,k}\,. \end{cases}$$

From (262) we obtain, in virtue of (4) and (201), (207)

$$(263)\quad \begin{cases} a_i^{(m;s+1)} = R_{1,n-2-s-i}R_{n-1-s,n-1}P_{k,k}/d\ , & (i = 1, \cdots, m-s-1) \\ a_{m-1-s+j}^{(m;s+1)} = R_{1,n-1-m-j}R_{n-1-s,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-s-3+u}^{(m;s+1)} = R_{n-1-s,n-u}P_{k,k}\ , & (u = 1, \cdots, s+1) \\ a_{n-1}^{(m;s+1)} = R_{n-1-s,n-1-s}\ . \end{cases}$$

But (263) is formula (260) where s is to be replaced by $s+1$; this completes the proof of formula (259),

We now obtain from (259), for $t = m-1$,

$$(264)\quad \begin{cases} a_1^{(m;m-1)} = R_{1,n-m-1}R_{n-m+1,n-1}P_{k,k}/d\ , \\ a_{1+j}^{(m;m-1)} = R_{1,n-1-m-j}R_{n-m+1,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-m-1+u}^{(m;m-1)} = R_{n-m+1,n-u}P_{k,k}\ , & (u = 1, \cdots, m-1) \\ a_{n-1}^{(m;m-1)} = R_{n-m+1,n-m+1}\ . \end{cases}$$

From (264) we obtain, since every $a_i^{(m;m-1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(265)\quad b_i^{(m;m-1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m-1)} = D_k - D_{n-m+1,n-m+1}\ ,$$

and from (264), (265), since

$$R_{n-m+1,n-m+1} - (D_k - D_{n-m+1,n-m+1}) = w - D_k = P_{k,k}\ ,$$

$$(266)\quad \begin{cases} a_1^{(m;m-1)} - b_1^{(m;m-1)} = R_{1,n-m-1}R_{n-m+1,n-1}P_{k,k}/d\ , \\ a_{1+j}^{(m;m-1)} - b_{1+j}^{(m;m-1)} = R_{1,n-1-m-j}R_{n-m+1,n-1}P_{k,k}\ , \\ \qquad\qquad (j = 1, \cdots, n-m-2) \\ a_{n-m-1+u}^{(m;m-1)} - b_{n-m-1+u}^{(m;m-1)} = R_{n-m+1,n-u}P_{k,k}\ , & (u = 1, \cdots, m-1) \\ a_{n-1}^{(m;m-1)} - b_{n-1}^{(m;m-1)} = P_{k,k}\ . \end{cases}$$

From (266) we obtain, in virtue of (4) and (201), (207),

$$(267)\quad \begin{cases} a_j^{(m;m)} = R_{1,n-1-m-j}R_{n-m,n-1}P_{k,k}\ , & (j = 1, \cdots, n-m-2) \\ a_{n-m-2+u}^{(m;m)} = R_{n-m,n-u}P_{k,k}\ , & (u = 1, \cdots, m) \\ a_{n-1}^{(m;m)} = R_{n-m,n-m}\ . \end{cases}$$

From (267) we obtain, since every $a_i^{(m;m)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(268)\quad b_i^{(m;m)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m)} = D_k - D_{n-m,n-m}\ ,$$

and from (267), (268), since $R_{n-m,n-m} - (D_k - D_{n-m,n-m}) = w - D_k = P_{k,k}$,

$$(269) \quad \begin{cases} a_1^{(m;m)} - b_1^{(m;m)} = R_{1,n-2-m}R_{n-m,n-1}P_{k,k} \,, \\ a_{1+j}^{(m;m)} - b_{1+j}^{(m;m)} = R_{1,n-2-m-j}R_{n-m,n-1}P_{k,k} \,, & (j = 1, \cdots, n-m-3) \\ a_{n-m-2+u}^{(m;m)} - {}_{n-m-2+u}^{(m;m)} = R_{n-m,n-u}P_{k,k} \,, & (u = 1, \cdots, m) \\ a_{n-1}^{(m;m)} - b_{n-1}^{(m;m)} = P_{k,k} \,. \end{cases}$$

From (269) we obtain, in virtue of (4) and (201), (207) ,

$$(270) \quad \begin{cases} a_j^{(m;m+1)} = R_{1,n-2-m-j}R_{n-m-1,n-1}P_{k,k}/d \,, & (j = 1, \cdots, n-m-3) \\ a_{n-m-3+u}^{(m;m+1)} = R_{n-m-1,n-u}P_{k,k}/d \,, & (u = 1, \cdots, m+1) \\ a_{n-1}^{(m;m+1)} = R_{n-m-1,n-m-1}/d \,, \end{cases}$$

and from (270), since every $a_i^{(m;m+1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(271) \quad b_i^{(m;m+1)} = 0; \; (i = 1, \cdots, n-2) \quad b_{n-1}^{(m;m+1)} = (D_k - D_{n-m-1,n-m-1})/d \,.$$

From (270), (271) we obtain, since

$$(R_{n-m-1,n-m-1}/d) - ((D_k - D_{n-m-1,n-m-1})/d) = (w - D_k)/d = P_{k,k}/d \,,$$

$$(272) \quad \begin{cases} a_1^{(m;m+1)} - b_1^{(m;m+1)} = R_{1,n-3-m}R_{n-1-m,n-1}P_{k,k}/d \,, \\ a_{1+j}^{(m;m+1)} - b_{1+j}^{(m;m+1)} = R_{1,n-3-m-j}R_{n-1-m,n-1}P_{k,k}/d \,, \\ & (j = 1, \cdots, n-m-4) \\ a_{n-3-m+u}^{(m;m+1)} - b_{n-3-m+u}^{(m;m+1)} = R_{n-m-1,n-u}P_{k,k}/d \,, & (u = 1, \cdots, m+1) \\ a_{n-1}^{(m;m+1)} - b_{n-1}^{(m;m+1)} = P_{k,k}/d \,, \end{cases}$$

and from (272), in virtue of (4) and (201), (207)

$$(273) \quad \begin{cases} a_j^{(m;m+2)} = R_{1,n-3-m-j}R_{n-2-m,n-1}P_{k,k}/d \,, & (j = 1, \cdots, n-m-4) \\ a_{n-m-4+u}^{(m;m+2)} = R_{n-m-2,n-u}P_{k,k}/d \,, & (u = 1, \cdots, m+2) \\ a_{n-1}^{(m;m+2)} = R_{n-m-2,n-m-2} \,. \end{cases}$$

From (273) we obtain, since every $a_i^{(m,m+2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(274) \quad b_i^{(m;m+2)} = 0; \; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;m+2)} = D_k - D_{n-m-2,n-m-2} \,,$$

and from (273), (274), since

$$R_{n-m-2,n-m-2} - (D_k - D_{n-m-2,n-m-2}) = w - D_k = P_{k,k} \,,$$

$$(275) \quad \begin{cases} a_1^{(m;m+2)} - b_1^{(m;m+2)} = R_{1,n-m-4}R_{n-m-2,n-1}P_{k,k}/d \,, \\ a_{1+j}^{(m;m+2)} - b_{1+j}^{(m;m+2)} = R_{1,n-m-4-j}R_{n-m-2,n-1}P_{k,k}/d \,, \\ & (j = 1, \cdots, n-m-5) \\ a_{n-m-4+u}^{(m;m+?)} - b_{n-m-4+u}^{(m;m+2)} = R_{n-m-2,n-u}P_{k,k}/d \,, & (u = 1, \cdots, m+2) \\ a_{n-1}^{(m;m+2)} - b_{n-1}^{(m;m+2)} = P_{k,k} \,. \end{cases}$$

From (275) we obtain, in virtue of (4) and (201), (207),

$$(276)\quad \begin{cases} a_j^{(m;m+3)} = R_{1,n-m-4-j}R_{n-m-3,n-1}P_{k,k}/d\ , & (j=1,\cdots,n-m-5) \\ a_{n-m-5+u}^{(m;m+3)} = R_{n-m-3,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{n-2}^{(m;m+3)} = R_{n-m-3,n-m-3}P_{k,k}\ , & \\ a_{n-1}^{(m;m+3)} = R_{n-m-3,n-m-3}\ . & \end{cases}$$

We shall now prove the formula

$$(277)\quad \begin{cases} a_j^{(m;m+t)} = R_{1,n-m-1-t-j}R_{n-m-t,n-1}P_{k,k}/d\ , & (j=1,\cdots,n-m-2-t) \\ a_{n-m-2-t+u}^{(m;m+t)} = R_{n-m-t,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{n-t+i}^{(m;m+t)} = R_{n-m-t,n-m-2-i}P_{k,k}\ , & (i=1,\cdots,t-2) \\ a_{n-1}^{(m;m+t)} = R_{n-m-t,n-m-t}\ , & \\ t=3,\cdots,n-m-3 & \end{cases}$$

Formula (277) is correct for $t=3$, in virtue of (276). As before, (277) is proved by induction.

We now obtain, from (277), since every $a_i^{(m;m+t)}$ $(i=1,\cdots,n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(278)\quad b_i^{(m;m+t)} = 0;\ (i=1,\cdots,n-2) \qquad b_{n-1}^{(m;m+t)} = D_k - D_{n-m-t,n-m-t}\ .$$

We further obtain from (278), for $t=n-m-3$

$$(279)\quad \begin{cases} a_1^{(m;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\ , & \\ a_{1+u}^{(m;n-3)} = R_{3,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{m+3+i}^{(m;n-3)} = R_{3,n-m-2-i}P_{k,k}\ , & (i=1,\cdots,n-m-5) \\ a_{n-1}^{(m;n-3)} = R_{3,3}\ . & \end{cases}$$

From (279) we obtain, since every $a_i^{(m;n-3)}$ $(i=1,\cdots,n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(280)\quad b_i^{(m;n-3)} = 0;\ (i=1,\cdots,n-2) \qquad b_{n-1}^{(m;n-3} = D_k - D_{3,3}\ ,$$

and from (279), (280), since $R_{3,3}-(D_k-D_{3,3}) = w - D_k = P_{k,k}$,

$$(281)\quad \begin{cases} a_1^{(m;n-3)} - b_1^{(m;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\ , & \\ a_{1+u}^{(m;n-3)} - b_{1+u}^{(m;n-3)} = R_{3,n-u}P_{k,k}/d\ , & (u=1,\cdots,m+2) \\ a_{m+3+i}^{(m;n-3)} - b_{m+3+i}^{(m;n-2)} = R_{3,n-m-2-i}P_{k,k}\ , & (i=1,\cdots,n-m-5) \\ a_{n-1}^{(m;n-3)} - b_{n-1}^{(m;n-3)} = P_{k,k} & \end{cases}$$

From (281) we obtain, in virtue of (4) and (201), (207) ,

$$(282)\quad \begin{cases} a_i^{(m;n-2)} = R_{2,n-i}P_{k,k}/d\ , & (i=1,\cdots,m+2) \\ a_{m+2+j}^{(m;n-2)} = R_{2,n-m-2-j}P_{k,k}\ , & (j=1,\cdots,n-m-4) \\ a_{n-1}^{(m;n-2)} = R_{2,2}\ . & \end{cases}$$

From (282) we obtain, since every $a_i^{(m;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(283) \qquad b_i^{(m;n-2)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(m;n-2)} = D_k - D_{2,2} \,,$$

and from (282), (283), since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(284) \quad \begin{cases} a_1^{(m;n-2)} - b_1^{(m;-2)} = R_{2,n-1} P_{k,k}/d \,, & \\ a_{1+i}^{(m;n-2)} - b_{1+i}^{(m;n-2)} = R_{2,n-1-i} P_{k,k}/d \,, & (i = 1, \cdots, m+1) \\ a_{m+2+j}^{(m;n-2)} - b_{m+2+j}^{(m;n-2)} = R_{2,n-m-2-j} P_{k,k} \,, & (j = 1, \cdots, n-m-4) \\ a_{n-1}^{(m;n-2)} - b_{n-1}^{(m;n-2)} = P_{k,k} \,. & \end{cases}$$

From (284) we obtain, in virtue of (4) and (201), (207)

$$(285) \quad \begin{cases} a_i^{(m+1;0)} = R_{1,n-1-i} P_{k,k}/d \,, & (i = 1, \cdots, m+1) \\ a_{m+1+j}^{(m+1;0)} = R_{1,n-m-2-j} P_{k,k} \,, & (j = 1, \cdots, n-m-3) \\ a_{n-1}^{(m+1;0)} = R_{1,1} \,. & \end{cases}$$

We note that formula (285) is obtained from formula (252) replacing in the latter m by $m+1$.
We further note that the $n-1$ supporting sequences

$$b_1^{(m;s)}, b_2^{(m;s)}, \cdots, b_{n-1}^{(m;s)} \qquad (s = 0, 1, \cdots, n-2)$$

generate a fugue which has the form of the $m+1$ -th fugue, as demanded by Theorem 6. Thus the Lemma 2 is completely proved.

We now obtain, on ground of the lemma, and in virtue of formula (251) for $t = n-3$, since (251) is correct for $t+1$, too,

$$(286) \quad \begin{cases} a_i^{(n-3;0)} = R_{1,n-1-i} P_{k,k}/d \,, & (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-3;0)} = R_{1,1} P_{k,k} \,, & \\ a_{n-1}^{(n-3;0)} = R_{1,1} \,. & \end{cases}$$

from (286) we obtain, since every $a_i^{(n-3;0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$ and in virtue of (199)

$$(287) \qquad b_i^{(n-3;0)} = 0; \ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;0)} = D_k - D_{1,1} \,,$$

and from (286), (287), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(288) \quad \begin{cases} a_1^{(n-3;0)} - b_1^{(n-3;0)} = R_{1,n-2} P_{k,k}/d \,, & \\ a_{1+i}^{(n-3;0)} - b_{1+i}^{(n-3;0)} = R_{1,n-2-i} P_{k,k}/d \,, & (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-3;0)} - b_{n-2}^{(n-3;0)} = R_{1,1} P_{k,k} \,, & \\ a_{n-1}^{(n-3;0)} - b_{n-1}^{(n-3;0)} = P_{k,k} \,. & \end{cases}$$

From (288) we obtain, in virtue of (4) and (201), (207),

$$(289)\quad \begin{cases} a_i^{(n-3;1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-4) \\ a_{n-3}^{(n-3;1)} = R_{1,1}R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;1)} = R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-1}^{(n-3;1)} = R_{n-1,n-1}\ , & \end{cases}$$

and from (289), since every $a_i^{(n-3;1)}$ contains the factor $P_{k,k}$ $(i = 1, \cdots, n-2)$, and in virtue of (199)

$$(290)\quad b_i^{(n-3;1)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;1)} = D_k - D_{n-1,n-1}\ .$$

From (289), (290) we obtain, since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$

$$(291)\quad \begin{cases} a_1^{(n-3;1)} - b_1'^{(n-3;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d\ , & \\ a_{1+i}^{(n-3;1)} - b_{1+i}^{(n-3;1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-5) \\ a_{n-3}^{(n-3;1)} - b_{n-3}^{(n-3;1)} = R_{1,1}R_{n-1n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;1)} - b_{n-2}^{(n-3;1)} = R_{n-1,n-1}P_{k,k}\ , & \\ a_{n-1}^{(n-3;1)} - b_{n-1}^{(n-3;1)} = P_{k,k}\ , & \end{cases}$$

and from (291), in virtue of (4) and (201), (207),

$$(292)\quad \begin{cases} a_i^{(n-3;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-5) \\ a_{n-4}^{(n-3;2)} = R_{1,1}R_{n-2,n-1}P_{k,k};\ a_{n-3}^{(n-3;2)} = R_{n-2,n-1}P_{k,k}\ , & \\ a_{n-2}^{(n-3;2)} = R_{n-2,n-2}P_{k,k};\ a_{n-1}^{(n-3;2)} = R_{n-2,n-2}\ . & \end{cases}$$

It is now easy to prove the formula

$$(293)\quad \begin{cases} a_i^{(n-3;t)} = R_{1,n-t-1-i}R_{n-t,n-1}P_{k,k}/d\ , & (i = 1, \cdots, n-t-3) \\ a_{n-t-2}^{(n-3;t)} = R_{1,1}R_{n-t,n-1}P_{k,k}\ , & \\ a_{n-t-2+j}^{(n-3;t)} = R_{n-t,n-j}P_{k,k}\ , & (j = 1, \cdots, t) \\ a_{n-1}^{(n-3;t)} = R_{n-t,n-t}\ , & \\ t = 1, \cdots, n-4\ . & \end{cases}$$

Formula (293) is true for $t = 1, 2$, in virtue of (289), (292). It is then presumed that (293) is true for $m \geqq 1$ and proved, as before, that it is correct for $m + 1$, too, which completes the proof of (293). From (293) we obtain, since every $a_i^{(n-3;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(294)\quad b_i^{(n-3;t)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;t)} = D_k - D_{n-t,n-t}\ ,$$

and further for $t = n - 4$,

$$(295)\quad \begin{cases} a_1^{(n-3;n-4)} = R_{1,2}R_{4,n-1}P_{k,k}/d\ , & \\ a_2^{(n-3,n-4)} = R_{1,1}R_{4,n-1}P_{k,k}\ , & \\ a_{2+j}^{(n-3;n-4)} = R_{4,n-j}P_{k,k}\ , & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-3;n-4)} = R_{4,4}\ . & \end{cases}$$

From (295) we obtain, since every $a_i^{(n-3;n-4)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(296) $$b_i^{(n-3,n-4)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-4)} = D_k - D_{4,4} ,$$

and from (295), (296), since $R_{4,4} - (D_k - D_{4,4}) = w - D_k = P_{k,k}$,

(297) $$\begin{cases} a_1^{(n-3;n-4)} - b_1^{(n-3;n-4)} = R_{1,2}R_{4,n-1}P_{k,k}/d , \\ a_2^{(n-3;n-4)} - b_2^{(n-3;n-4)} = R_{1,1}R_{4,n-1}P_{k,k} , \\ a_{2+j}^{(n-3;n-4)} - b_{2+j}^{(n-3;n-4)} = R_{4,n-j}P_{k,k} , \qquad (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-3;n-4)} - b_{n-1}^{(n-3;n-4)} = P_{k,k} . \end{cases}$$

From (297) we obtain, in virtue of (4) and (201), (207),

(298) $$\begin{cases} a_1^{(n-3;n-3)} = R_{1,1}R_{3,n-1}P_{k,k} , \\ a_{1+j}^{(n-3;n-3)} = R_{3,n-j}P_{k,k} , \qquad (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-3)} = R_{3,3} . \end{cases}$$

From (298) we obtain, since every $a_i^{(n-3;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(299) $$b_i^{(n-3;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-3)} = D_k - D_{3,3} ,$$

and from (298), (299), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$,

(299a) $$\begin{cases} a_1^{(n-3;n-3)} - b_1^{(n-3;n-3)} = R_{1,1}R_{3,n-1}P_{k,k} , \\ a_{1+j}^{(n-3;n-3)} - b_{1+j}^{(n-3;n-3)} = R_{3,n-j}P_{k,k} , \qquad (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-3)} - b_{n-1}^{(n-3;n-3)} = P_{k,k} . \end{cases}$$

From (299a) we obtain, in virtue of (4) and (201), (207) ,

(300) $$\begin{cases} a_j^{(n-3;n-2)} = R_{2,n-j}P_{k,k}/d , \qquad (j = 1, \cdots, n-2) \\ a_{n-1}^{(n-3;n-2)} = R_{2,2}/d , \end{cases}$$

and from (300), since every $a_j^{(n-3;n-2)}$ $(j = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(301) $$b_i^{(n-3;n-2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-3;n-2)} = (D_k - D_{2,2})/d .$$

From (300), (301) we obtain, since $(R_{2,2}/d) - ((D_k - D_{2,2})/d) = P_{k,k}/d$,

(302) $$\begin{cases} a_1^{(n-3;n-2)} - b_1^{(n-3;n-2)} = R_{2,n-1}P_{k,k}/d , \\ a_{1+j}^{(n-3;n-2)} - b_{1+j}^{(n-3;n-2)} = R_{2,n-1-j}P_{k,k}/d , \qquad (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-3;n-2)} - b_{n-1}^{(n-3;n-2)} = P_{k,k}/d , \end{cases}$$

and from (302), in virtue of (4) and (201), (207),

(303) $$\begin{cases} a_j^{(n-2;0)} = R_{1,n-1-j}P_{k,k}/d , \qquad (j = 1, \cdots, n-2) \\ a_{n-1}^{(n-2;0)} = R_{1,1} . \end{cases}$$

Formulae (287), (290), (294), (299), (301) show that the $n-1$ supporting sequences

$$b_1^{(n-3;k)}, b_2^{(n-3;k)}, \cdots, b_{n-1}^{(n-3;k)} \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the $n-2$ -th fugue as demanded by Theorem 6.

From (303) we obtain, since every $a_j^{(n-2;0)}$ $(j = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199)

$$(304) \qquad b_i^{(n-2;0)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;0)} = D_k - D_{1,1} ,$$

and from (303), (304), since $R_{1,1} - (D_k - D_{1,1}) = w - D_k = P_{k,k}$,

$$(305) \qquad \begin{cases} a_1^{(n-2;0)} - b_1^{(n-2;0)} = R_{1,n-2}P_{k,k}/d , \\ a_{1+j}^{(n-2;0)} - b_{1+j}^{(n-2;0)} = R_{1,n-2-j}P_{k,k}/d , \qquad (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;0)} - b_{n-1}^{(n-2;0)} = P_{k,k} . \end{cases}$$

From (305) we obtain, in virtue of (4) and (107), (108) ,

$$(306) \qquad \begin{cases} a_j^{(n-2;1)} = R_{1,n-2-j}R_{n-1,n-1}P_{k,k}/d , \qquad (j = 1, \cdots, n-3) \\ a_{n-2}^{(n-2;1)} = R_{n-1,n-1}P_{k,k} , \\ a_{n-1}^{(n-2;1)} = R_{n-1,n-1} . \end{cases}$$

It is now easy to prove the formula

$$(307) \qquad \begin{cases} a_j^{(n-2;t)} = R_{1,n-1-t-j}R_{n-t,n-1}P_{k,k}/d , \qquad (j = 1, \cdots, n-2-t) \\ a_{n-2-t+i}^{(n-2;t)} = R_{n-t,n-i}P_{k,k} , \qquad (i = 1, \cdots, t) \\ a_{n-1}^{(n-2;t)} = R_{n-t,n-t} , \\ t = 1, \cdots, n-3 . \end{cases}$$

Formula (307) is correct for $t = 1$, in virtue of formula (306), (307) is then proved by induction.
From (307) we obtain, since every $a_i^{(n-2;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(308) \qquad b_i^{(n-2;t)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;t)} = D_k - D_{n-t,n-t} ,$$

and further from (307), for $t = n-3$,

$$(309) \qquad \begin{cases} a_1^{(n-2;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d , \\ a_{1+i}^{(n-2;n-3)} = R_{3,n-i}P_{k,k} , \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-3)} = R_{3,3} . \end{cases}$$

From (309) we obtain, since every $a_i^{(n-2;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$ and in virtue of (199),

$$(310) \qquad b_i^{(n-2;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;n-3)} = D_k - D_{3,3} ,$$

and from (309) (310), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = P_{k,k}$,

$$(311) \quad \begin{cases} a_1^{(n-2;n-3)} - b_1^{(n-2;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(n-2;n-3)} - b_{1+i}^{(n-2;n-3)} = R_{3,n-i}P_{k,k}\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-3)} - b_{n-1}^{(n-2;n-3)} = P_{k,k}\,. \end{cases}$$

From (311) we obtain, in virtue of (4) and (201), (207),

$$(312) \quad \begin{cases} a_i^{(n-2;n-2)} = R_{2,n-i}P_{k,k}\,, \qquad (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-2;n-2)} = R_{2,2}\,, \end{cases}$$

and from (312), since every $a_i^{(n-2,n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(313) \quad b_i^{(n-2;n-2)} = 0;\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-2;n-2)} = D_k - D_{2,2}\,.$$

From (312), (313) we obtain, since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(314) \quad \begin{cases} a_1^{(n-2;n-2)} - b_1^{(n-2;n-2)} = R_{2,n-1}P_{k,k}\,, \\ a_{1+i}^{(n-2;n-2)} - b_{1+i}^{(n-2;n-2)} = R_{2,n-1-i}P_{k,k}\,, \quad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-2;n-2)} - b_{n-1}^{(n-2;n-2)} = P_{k,k}\,, \end{cases}$$

and from (314), in virtue of (4) and (201), (207),

$$(315) \quad \begin{cases} a_i^{(n-1;0)} = R_{1,n-1-i}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-2) \\ a_{n-1}^{(n-1;0)} = R_{1,1}/d\,. \end{cases}$$

Formula (304), (308), (313) show that the $n-1$ supporting sequences $b_i^{(n-2,k)}$ $(i = 1, \cdots, n-1;\ k = 0, 1, \cdots, n-2)$ form a fugue which has the form as demanded by Theorem 6.

From (315) we obtain, since every $a_i^{(n-1;0)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(316) \quad b_i^{(n-1;0)} = 0,\ (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;0)} = (D_k - D_{1,1})/d\,,$$

and from (315), (316), since $(R_{1,1}/d) - ((D_k - D_{1,1})/d) = (w - D_k)/d = P_{k,k}/d$,

$$(317) \quad \begin{cases} a_1^{(n-1;0)} - b_1^{(n-1;0)} = R_{1,n-2}P_{k,k}/d\,, \\ a_{1+i}^{(n-1;0)} - b_{1+i}^{(n-1;0)} = R_{1,n-2-i}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;0)} - b_{n-1}^{(n-1;0)} = P_{k,k}/d\,. \end{cases}$$

From (317) we obtain, in virtue of (4) and (201), (207),

$$(318) \quad \begin{cases} a_i^{(n-1;1)} = R_{1,n-2-i}R_{n-1,n-1}P_{k,k}/d\,, \qquad (i = 1, \cdots, n-3) \\ a_{n-2}^{(n-1;1)} = R_{n-1,n-1}P_{k,k}/d\,, \\ a_{n-1}^{(n-1;1)} = R_{n-1,n-1}\,. \end{cases}$$

From (318) we obtain, since every $a_i^{(n-1;1)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(319) \qquad b_i^{(n-1;1)} = 0, (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;1)} = D_k - D_{n-1,n-1} ,$$

and from (318), (319), since $R_{n-1,n-1} - (D_k - D_{n-1,n-1}) = w - D_k = P_{k,k}$,

$$(320) \quad \begin{cases} a_1^{(n-1;1)} - b_1^{(n-1;1)} = R_{1,n-3}R_{n-1,n-1}P_{k,k}/d , \\ a_{1+i}^{(n-1;1)} - b_{1+i}^{(n-1)} = R_{1,n-3-i}R_{n-1,n-1}P_{k,k}/d , \qquad (i = 1, \cdots, n-4) \\ a_{n-2}^{(n-1;1)} - b_{n-2}^{(n-1;1)} = R_{n-1,n-1}P_{k,k}/d , \\ a_{n-1}^{(n-1;1)} - b_{n-1}^{(n-1;1)} = P_{k,k} . \end{cases}$$

From (320) we obtain, in virtue of (4) and (201), (207),

$$(321) \quad \begin{cases} a_i^{(n-1;2)} = R_{1,n-3-i}R_{n-2,n-1}P_{k,k}/d , \qquad (i = 1, \cdots, n-4) \\ a_{n-3}^{(n-1;2)} = R_{n-2,n-1}P_{k,k}/d , \\ a_{n-3}^{(n-1;2)} = R_{n-2,n-2}P_{k,k} , \\ a_{n-1}^{(n-1;2)} = R_{n-2,n-2} , \end{cases}$$

and from (321), since every $a_i^{(n-1;2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

$$(322) \qquad b_i^{(n-1;2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;2)} = D_k - D_{n-2,n-2} .$$

From (321), (322) we obtain, since $R_{n-2,n-2} - (D_k - D_{n-2,n-2}) = w - D_k = P_{k,k}$,

$$(323) \quad \begin{cases} a_1^{(n-1;2)} - b_1^{(n-1,2)} = R_{1,n-4}R_{n-2,n-1}P_{k,k}/d , \\ a_{1+i}^{(n-1;2)} - b_{1+i}^{(n-1;2)} = R_{1,n-4-i}R_{n-2,n-1}P_{k,k}/d , \qquad (i = 1, \cdots, n-5) \\ a_{n-3}^{(n-1;2)} - b_{n-3}^{(n-1;2)} = R_{n-2,n-1}P_{k,k}/d , \\ a_{n-2}^{(n-1;2)} - b_{n-2}^{(n-1;2)} = R_{n-2,n-2}P_{k,k} , \\ a_{n-1}^{(n-1;2)} - b_{n-1}^{(n-1;2)} = P_{k,k} , \end{cases}$$

and from (323), in virtue of (4) and (201), (207),

$$(324) \quad \begin{cases} a_i^{(n-1;3)} = R_{1,n-4-i}R_{n-3,n-1}P_{k,k}/d , \qquad (i = 1, \cdots, n-5) \\ a_{n-4}^{(n-1;3)} = R_{n-3,n-1}P_{k,k}/d , \\ a_{n-3}^{(n-1;3)} = R_{n-3,n-2}P_{k,k}; \ a_{n-2}^{(n-1;3)} = R_{n-3,n-3}P_{k,k} ; \\ a_{n-1}^{(n-1;3)} = R_{n-3,n-3} . \end{cases}$$

It is now easy to prove the formula

$$(325) \quad \begin{cases} a_i^{(n-1;t)} = R_{1,n-t-i}R_{n-t,n-1}P_{k,k}/d , \quad (i = 1, \cdots, n-2-t) \\ a_{n-1-t}^{(n-1;t)} = R_{n-t,n-1}P_{k,k}/d , \\ a_{n-1-t+j}^{(n-1;t} = R_{n-t,n-1-j}P_{k,k} , \qquad (j = 1, \cdots, t-1) \\ a_{n-1}^{(n-1;t)} = R_{n-t,n-t} , \\ t = 3, \cdots, n-3 . \end{cases}$$

Formula (325) is correct for $t = 3$, in virtue of formula (324), (325) is then proved by induction.

From (325) we obtain, since every $a_i^{(n-1;t)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(326) $\quad b_i^{(n-1;t)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;t)} = D_k - D_{n-t,n-t}$,

and further from (325), for $t = n - 3$,

$$(326a)\quad \begin{cases} a_1^{(n-1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, \\ a_2^{(n-1;n-3)} = R_{3,n-1}P_{k,k}/d\,, \\ a_{2+j}^{(n-1;n-3)} = R_{3,n-1-j}P_{k,k}\,, & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-1;n-3)} = R_{3,3}\,. \end{cases}$$

From (326a) we obtain, since every $a_i^{(n-1;n-3)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(327) $\quad b_i^{(n-1;n-3)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-3)} = D_k - D_{3,3}$,

and from (326), (327), since $R_{3,3} - (D_k - D_{3,3}) = w - D_k = D_{k,k}$,

$$(328)\quad \begin{cases} a_1^{(n-1\cdot n-3)} - b_1^{(n-1;n-3)} = R_{1,1}R_{3,n-1}P_{k,k}/d\,, \\ a_2^{(n-1;n-3)} - b_2^{(n-1;n-3)} = R_{3,n-1}P_{k,k}/d\,, \\ a_{2+j}^{(n-1;n-3)} - b_{2+j}^{(n-1;n-3)} = R_{3,n-1-j}P_{k,k}\,, & (j = 1, \cdots, n-4) \\ a_{n-1}^{(n-1,n-3)} - b_{n-1}^{(n-1,n-3)} = P_{k,k}\,. \end{cases}$$

From (328) we obtain, in virtue of (4) and (201), (207),

$$(329)\quad \begin{cases} a_1^{(n-1;n-2)} = R_{2,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(n-1;n-2)} = R_{2,n-1-j}P_{k,k}\,, & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} = R_{2,2}\,. \end{cases}$$

From (329) we obtain, since every $a_i^{(n-1;n-2)}$ $(i = 1, \cdots, n-2)$ contains the factor $P_{k,k}$, and in virtue of (199),

(330) $\quad b_i^{(n-1;n-2)} = 0; (i = 1, \cdots, n-2) \qquad b_{n-1}^{(n-1;n-2)} = D_k - D_{2,2}$,

and from (329), (330), since $R_{2,2} - (D_k - D_{2,2}) = w - D_k = P_{k,k}$,

$$(331)\quad \begin{cases} a_1^{(n-1;n-2)} - b_1^{(n-1;n-2)} = R_{2,n-1}P_{k,k}/d\,, \\ a_{1+i}^{(n-1;n-2)} - b_{1+i}^{(n-1;n-2)} = R_{2,n-1-j}P_{k,k}\,, & (j = 1, \cdots, n-3) \\ a_{n-1}^{(n-1;n-2)} - b_{n-1}^{(n-1;n-2)} = P_{k,k}\,. \end{cases}$$

From (331) we obtain, in virtue of (4) and (201), (207),

$$(332)\quad \begin{cases} a_i^{(n;0)} = R_{1,n-1-j}P_{k,k}\,, & (j = 1, \cdots, n-2) \\ a_{n-i}^{(n;0)} = R_{1,1}\,. \end{cases}$$

Comparing formula (332) with formula (204), we obtain

$$(333) \qquad a_i^{(0)} = a_i^{(n;0)} = a_i^{(n(n-1))}, \qquad (i = 1, \cdots, n-1),$$

so that the Modified Algorithm of Jacobi-Perron for the basic sequence (204) is indeed purely periodic with length of period $T = n(n-1)$ for $d > 1$.

For $d = 1$ we obtain, comparing formula (226) with (204),

$$(334) \qquad a_i^{(0)} = a_i^{(1;0)} = a_i^{(n-1)}, \qquad (i = 1, \cdots, n-1),$$

so that in this case the Algorithm is purely periodic with length of period $T = n - 1$.

Formulae (316), (319), (322), (326), (327), (330) show that the $n - 1$ supporting sequences

$$b_1^{(n-1.k)}, b_2^{(n-1;k)}, \cdots, b_{n-1}^{(n-1;k)}, \qquad (k = 0, 1, \cdots, n-2)$$

form a fugue which has the form of the n -th fugue as demanded by Theorem 6. Thus, for $d > 1$, and from what was proved before, the $n(n-1)$ supporting sequences of the Modified Algorithm of Jacobi-Perron form n fugues of the form (206a)—(206d). In case $d = 1$, they all have the form (205). By this Theorem 6 is completely proved.

The reader should note the necessity to presume $n > n_0$, (n_0 a constant) while carrying out the proof of Theorem 6. The cases $n = 2, \cdots, n_0$ are easily proved separately by the same mothods used for the proof of Theorem 6.

We shall now find units of the field $K(w)$ by means of the Modified Algorithm of Jacobi-Perron.

As Hasse and I have proved in our paper [16], a unit e of the field $K(w)$ is obtained from a periodic Jacobi-Perron Algorithm by means of formula (190), viz.

$$e^{-1} = \prod_{v=S}^{S+T-1} a_{n-1}^{(v)},$$

where S and T denote, as before, the lengths of the pre-period and period of the periodic Jacobi-Perron algorithm respectively.

It is one of the most striking and basic properties of *any* periodic algorithm G with integral supporting sequences

$$b_i^{(v)}, \qquad (i = 1, \cdots, n-1;\ v = 0, 1, \cdots)$$

$b_i^{(v)}$ rational integers, that formula (190) holds for this general case of the G. The proof of this statement is not too complicated and follows exactly the lines of the methods used in [16], though certain additional results are necessary (see, for example, my paper [12]).

We then obtain from (190), since in our case again $S = 0$, $T = n(n-1)$ for $d > 1$, as in (191),

$$e_k^{-1} = \prod_{v=0}^{n(n-1)-1} a_{n-1}^{(v)} = \prod_{i=0}^{n-1} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} .$$

Now it is not difficult to verify, following up the various stages of the proof of the Modified algorithm of Jacobi-Perron, that the relations hold

$$(335) \qquad \begin{cases} \prod_{k=0}^{n-2} a_{n-1}^{(i(n-1)+k)} = R_{1,n-1}/d , & (i = 0, 1, \cdots, n-3, n-1) \\ \prod_{k=0}^{n-2} a_{n-1}^{((n-2)(n-1)+k)} = R_{1,n-1} . \end{cases}$$

We thus obtain from (191), in virtue of (335),

$$(336) \qquad e_k^{-1} = (R_{1,n-1})^n / d^{n-1} .$$

From (201) we obtain $1/R_{1,n-1} = R_{0,0}/d$, and, since $R_{0,0} = R_{k,k}$,

$$(337) \qquad R_{1,n-1} = d/P_{k,k} .$$

From (336), (337) we now obtain

$$e_k^{-1} = d/(P_{k,k})^n ,$$

or

$$(338) \qquad e_k = \frac{(w - D_k)^n}{d} \qquad (k = 1, \cdots, n-1) ,$$

so that with (196), (338) Theorem 5. is now completely proved by means of the Modified Algorithm of Jacobi-Perron, since (338) includes the case $d = 1$, too.

The $n - 1$ units $e_0, e_1, \cdots, e_{n-2}$ are all different, since $D_k > D_{k+1}$ $(k = 0, 1, \cdots, n-2)$. It is proved below that they are independent (see the Appendix by Hasse) in the sense that there cannot exist an equation of the form

$$e_0^{a_0} e_1^{a_1} \cdots e_{n-2}^{a_{n-2}} = 1 ,$$

where the $a_0, a_1, \cdots, a_{n-2}$ are rational integers not all equal zero.

Concluding we shall illustrate (338) by a numeric example. Let the GP be a fourth degree polynomial

$$f(x) = (x - 10)(x - 6)(x - 2)(x + 4) - 2 = 0 ;$$
$$f(w) = 0; \; 10 < w < 11 ;$$
$$D_0 = 10; \; D_1 = 6; \; D_2 = 2; \; D_3 = -4; \; d = 2 ;$$
w is a fourth degree irrational.

We obtain from $f(w) = 0$:

$$w^4 - 14w^3 + 20w^2 + 248w - 482 = 0 ,$$
$$w^4 = 14w^3 - 20w^2 - 248w + 482 .$$

Thus

$$(w - 6)^4 = -10w^3 + 196w^2 - 1112w + 1778 ;$$
$$(w - 2)^4 = 6w^3 + 4w^2 - 280w + 498 ;$$
$$(w + 4)^4 = 30w^3 + 76w^2 + 8w + 738 .$$

Substituting these values in (338) we obtain the independent units

$$e_1 = 5w^3 - 98w^2 + 556w - 889 ;$$
$$e_2 = 3w^3 + 2w^2 - 140w + 249 ;$$
$$e_3 = 15w^3 + 38w^2 + 4w + 369 .$$

Appendix. (By Helmut HASSE, at present Honolulu (Hawaii)). In § 7 of this paper L. Bernstein, by applying a modified Jacobi-Perron algorithm to suitable bases of a certain type of totally real algebraic number-fields K of degree $n \geqq 2$, obtained a system of n algebraic units in K with product 1. I shall prove here under slightly stronger conditions that every $n - 1$ of these units are independent.

The fields K in question are generated by a root w of a polynomial of type

$$(1) \qquad f(x) = \prod_{v=0}^{n-1} (x - D_v) - d ,$$

where the D_v and d are rational integers, $d \geqq 1$, satisfying the conditions (184), viz.

$$(2) \qquad D_0 > D_1 > \cdots > D_{n-1} ,$$

$$(3) \qquad D_v \equiv D_0 \bmod. d ,$$

$$(4) \qquad D_0 - D_v \geqq 2d(n - 1) , \qquad (v = 1, \cdots, n - 1) ,$$

and in the special case $d = 1$ moreover the inequalities (19), viz.

$$(5) \qquad \begin{cases} D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 4 \text{ for } n = 3 , \\ D_1 - D_2 \geqq 2 \text{ or } D_0 - D_1 \geqq 3 \text{ or } D_2 - D_3 \geqq 3 \text{ or} \\ \qquad D_0 - D_1, D_2 - D_3 \geqq 2 \text{ for } n = 4 . \end{cases}$$

In addition to these conditions I shall have to presuppose the inequalities

$$(6) \qquad D_{2k-1} - D_{2k} \geqq 2 \qquad (2 \leqq 2k \leqq n - 1)$$

to be satisfied in the special case $d = 1$.

I shall prove

THEOREM. *Let w be a root of a polynomial of type* (1) *whose coefficients satisfy the conditions* (2), (3), (4), (5), (6). *Then the n algebraic numbers*

$$e_m = \frac{(w - D_m)^n}{d} \qquad (m = 0, 1, \cdots, n-1)$$

are algebraic units with product

$$\prod_{m=0}^{n-1} e_m = 1 \, ,$$

and every $n - 1$ of them are independent.

Proof. (a) By (3)

$$(w - D_m)^n \equiv \prod_{v=0}^{n-1} (w - D_v) \bmod. d \, ,$$

and by (1)

$$\prod_{v=0}^{n-1} (w - D_v) = f(w) + d = d \, .$$

Hence

$$(w - D_m)^n \equiv 0 \bmod. d \, ,$$

so that the e_m are algebraic integers.

(b) By (1) their product

$$\prod_{m=0}^{n-1} e_m = \prod_{m=0}^{n-1} d^{-1}(w - D_m)^n = \frac{(f(w) + d)^n}{d^n} = \frac{d^n}{d^n} = 1 \, .$$

Hence the e_m are algebraic units.

(c) According to Theorem 2, the generating polynomial $f(x)$ has n different real roots

$$w^{(0)} > w^{(1)} > \cdots > w^{(n-1)}$$

(each of which may take the place of the above w), and the relative position of these roots between and outside of the sequence (2) is such that, for every fixed v, in virtue of the congruences (3)

$$|\, w^{(v)} - D_m \,| > \begin{cases} d \text{ for all } m \neq v \text{ except possibly one} \\ \dfrac{1}{2} d \text{ for the possible exception } m \neq v \, . \end{cases}$$

The possible exception occurs for one of the two D_m which include $w^{(v)}$ (so far $v > 0$ and for even n also $v < n - 1$), and hence only for $n \geqq 3$ (since for $n = 2$ both roots $w^{(0)}, w^{(1)}$ are excluded by D_0, D_1). From these inequalities it follows that the units

$$e_m^{(v)} = \frac{(w^{(v)} - D_m)^n}{d}$$

for every fixed v satisfy the inequalities

$$|e_m^{(v)}| > \left\{\begin{array}{l} d^{n-1}/d = d^{n-2} \text{ for all } m \neq v \text{ except possibly one} \\ \dfrac{1}{2}d^{n-1}/d = \dfrac{1}{2}d^{n-2} \text{ for the possible exception } m \neq v \end{array}\right\} .$$

Since the exception does not occur for $n = 2$, and since in virtue of the presupposition (6) the factor 1/2 may be dropped in the special case $d = 1$, these inequalities imply throughout

$$|e_m^{(v)}| > 1 \quad \text{for} \quad m \neq v .$$

On the strength of the product relation then necessarily

$$|e_v^{(v)}| < 1 .$$

Now the polynomial $f(x)$ is irreducible, as Bernstein derived at the beginning of § 7 from Theorem 3. under the conditions (4). Hence for each fixed m the $e_m^{(v)}$ are the algebraic conjugates of e_m. Hence by a well-known theorem of Minkowski[1] the latter inequalities imply that for any fixed pair m_0, v_0 the determinant

$$|\log|e_m^{(v)}||_{m \neq m_0, v \neq v_0} \neq 0 .$$

From this it follows that every $n-1$ of the n units e_m are independent.

Note. In spite of this very simple theory of the unit system e_m, Bernstein's more lengthly subordination of these units under a modified Jacobi-Perron algorithm by means of Theorem 6. seems to me still to be of importance. "The more organic connection between a unit in a field K and a periodic algorithm of a basis of K", as Bernstein put it after Theorem 5, may be essential for attacking the important question whether those units are fundamental units of a ring (Dedekind order) in K. An answer to this question may lead to lower estimates of the class number h of K.[2]

[2] See H. Hasse, Zahlentheorie, 2. Aufl., Berlin 1963; 28, **2**, Hilfsatz.

[3] Compare for this: H. Hasse, Über mehrklassige, aber eingeschlechtige reelquadratische Zahlkoerper, Elem. d. Math. **20** (1965), 49–59.

REFERENCES

1. Leon Bernstein, *Periodical continued fractions of degree n by Jacobi's Algorithm*, J. Reine Angew. Math. **213** (1964), 31-38.

2. ———, *Representation of* $(D^n - d)^{1/n}$ *as a periodic continued fraction by Jacobi's Algorithm*, Math. Nachrichten **1019** (1965), 179-200.

3. ———, *Periodicity of Jacobi's Algorithm for a special type of Cubic Irrationals*, J. Reine Angew. Math. **213** (1964), 134-146.

4. ———, *Periodische Jacobische Algorithmen fuer eine unendliche Klasse Algebraischer Irrationalzahlen vom Grade n und einige Klassen Kubischer Irrationalzahlen*, J. Reine Angew. Math. **215/216** (1944), 76-83.

5. ———, *Periodische Jacobi-Perronsche Algorithmen*, Archiv der Mathematik **15** (1964), 421-429.

6. ———, *New infinite clasess of periodic Jacobi-Perron algorithms*, Pacific J. Math. **16** (1965), 1-31.

7. ———, *A periodic Jacobi-Perron Algorithm*, Canad. J. Math. **17** (1965), 933-945.

8. *Rational approximation of algebraic irrationals by means of a modified Jacobi-Perron Algorithm*, Duke Math. J. **32** (1965), 161-176.

9. ———, *Periodische Kettenbrueche beliebiger Periodenlaenge*, Math. Zein. **86**, 128-135.

10. ———, *A probability function for partitions*, submitted to Amer. Math. Monthly.

11. ———, *Rational approximation of algebraic numbers*, Proc. National Conference on Data Processing, REHOVOTH, 1964, IPA, 91-105 (Hebrew).

12. ———, *The modified algorithm of Jacobi-Perron*, Mem. Amer. Math. Soc. **67** (1966), 1-44.

13. ———, *Der B-Algorithmus und seine Anwendung*, J. Reine Angew. Math. **227** (1967), 150-177.

14. ———, *Ein Neuer Algorithmus fuer Absteigende Basispotenzen im Kubischen Koerper*, Math. Nach. **33** (1967), 257-272.

15. ———, *The generalized Pellian equation*, Trans. Amer. Math. Soc. **127** (1967), 76-89.

16. Leon Berstein and Helmut Hasse, *Einheitenberechung mittels des Jacobi-Pernschen Algorithmus*, J. Reine Angew. Math. **218** (1965), 51-69.

17. C. G. J. Jacobi, *Allgemeine Theorie der kettenbruchaehnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird*, J. Reine Angew. Math. **69** (1868).

18. Oskar Perron, *Grundlagen fuer eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann. **64** (1907).

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