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In this paper a method is developed for determining best constants in inequalities of the following form:

$$\int_a^b |y|^p |y^{(n)}|^q w(x)dx \leqq K\left\{\int_a^b |y^{(n)}|^r m(x)dx\right\}^{(p+q)/r},$$

where $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0$ and $y^{(n-1)}$ is absolutely continuous.

It is first shown that for a certain class of m and w, equality can be attained in the inequality. Applying variational techniques reduces the determination of the best constant to a nonlinear eigenvalue problem for an integral operator. If m and w are sufficiently smooth this reduces further to a boundary value problem for a differential equation. The method is illustrated by determining the best constants in case (a, b) is a finite interval, $m(x) \equiv w(x) \equiv 1$, and $n = 1$.

A number of special cases of the inequality have been studied but usually without obtaining best constants. An exception to this is the case $n = 1, q = 0, p = r$ which was studied very thoroughly by Beesack [1], who gave a direct method for determining best constants. The method of [1] was modified by Boyd and Wong [5] to apply to the case $n = 1, q = 1, r = p + 1$. Recently Beesack and Das [2] obtained constants for the case $n = 1, r = p + q$ but these were not in general best possible.

We shall state our result only for $n = 1$ although it will be clear that the analogous result for $n > 1$ is valid. In our closing remarks we indicate a number of other inequalities to which the method of this paper applies.

1. **Preliminaries.** Throughout we assume that p, q, r, a, b are real numbers satisfying $p > 0, r > 1, 0 \leqq q < r$ and $-\infty \leqq a < b \leqq \infty$. The functions m and w are measurable and positive almost everywhere. We write $d\mu(x) = m(x)dx$ and

$$\|f\|_s = \left\{\int_a^b |f|^s d\mu\right\}^{1/s} \quad \text{for} \quad 0 < s < \infty .$$

The space L_m^s is the set of functions with $\|f\|_s < \infty$, with the usual identification. We shall use the notation $f_n \to f$ if $\|f_n - f\|_s \to 0$, and if $s \geqq 1$ so L_m^s is a Banach space, we write $f_n \xrightarrow{w} f$ for weak convergence in L_m^s. We denote the dual of L_m^s by $L_m^{s'}$ so for $s > 1, s' = s/(s-1)$.

We shall consider integral operators of the type

$$(1) \qquad Tf(x) = \int_a^b k(x, t)f(t)d\mu(t) ,$$

where $k(x, t) \geqq 0$ a.e. A function f is in the domain of T if

$$T|f|(x) < \infty \quad \text{a.e.}$$

For Theorem 1, the operator T becomes

$$(2) \qquad T_1f(x) = w(x)^{1/p}m(x)^{-1/p}\int_a^x f(t)dt ,$$

so that $k(x, t) = w(x)^{1/p}m(x)^{-1/p}m(t)^{-1}\chi_{[a,x]}(t)$. A necessary and sufficient condition for the domain of T_1 to contain L_m^r is that

$$\int_a^x m(t)^{-1(r-1)}dt < \infty \quad \text{for} \quad a \leqq x < b .$$

This follows from Hölder's inequality and its converse.

If T maps $L_m^r \to L_m^s$, where $s = pr/(r - q)$, with norm $\| T \| < \infty$, then we can define the functional J on L_m^r by

$$(3) \qquad J(f) = \int_a^b | Tf |^p | f |^q d\mu .$$

It then follows from Hölder's inequality that

$$(4) \qquad J(f) \leqq \| T \|^p \| f \|_r^{p+q} .$$

2. Main results.

THEOREM 1. *Suppose that* $w, m \in C^1(a, b)$, *that* $w(x) > 0$ a.e. *and* $m(x) > 0$ *for* $a < x < b$, *that* $p > 0, r > 1, 0 \leqq q < r$, *and that the operator* T_1 *defined by* (2) *is compact from* $L_m^r \to L_m^s (s = pr/(r - q))$. *Then the following eigenvalue problem* (P) *has solutions* (y, λ) *with* $y \in C^2(a, b)$ *and* $y(x) > 0, y'(x) > 0$ *in* (a, b).

$$(P)\begin{cases} \text{(i)} & \dfrac{d}{dx}(r\lambda y'^{r-1}m - qy^p y'^{q-1}w) + py^{p-1}y'^q w = 0 \\ \text{(ii)} & \lim\limits_{x\to a} y(x) = 0 \text{ and } \lim\limits_{x\to b}(r\lambda y'^{r-1}m - qy^p y'^{q-1}w) = 0 \\ \text{(iii)} & \| y' \|_r = 1 . \end{cases}$$

There is a largest value λ *such that* (P) *has a solution and if* λ^* *denotes this value, then for any* $f \in L_m^r$,

$$(5) \qquad \int_a^b \left| \int_a^x f \right|^p | f |^q w(x)dx \leqq \frac{r\lambda^*}{p + q}\left\{\int_a^b | f |^r m(x)dx\right\}^{(p+q)/r} .$$

Equality holds in (5) *if and only if* $f = cy'$ a.e. *where* y *is a solution*

of (P) corresponding to $\lambda = \lambda^*$, *and* c *is any constant.*

The proof will require two lemmas which we state in reasonable generality.

LEMMA 1. *Suppose that* $p > 0, r > 1, 0 \leqq q < r$, *and that* T, *as defined by* (1) *is a compact operator from* $L_m^r \to L_m^s$, $(s = pr/(r-q))$. *Let* J *be defined by* (3), *and*

$$K^* = \sup \{J(f): \|f\|_r \leqq 1\} . \tag{6}$$

Then, there is an element $f_0 \in L_m^r$ *with* $\|f_0\|_r = 1$ *such that* $J(f_0) = K^*$.

Proof. Since $J(f) < J(|f|)$ unless f is of constant sign a.e., we can restrict consideration in (6) to $f \geqq 0$. Let $\{f_n\} \in L_m^r$ be a sequence with $f_n \geqq 0, \|f_n\| \leqq 1$ such that $J(f_n) \to K^*$. We begin by assuming $q > 0$ so that $1 < r/q < \infty$. By the weak sequential compactness of the unit balls of L_m^r and $L_m^{r/q}$ ([7], p. 68), and by the compactness of T, we may assume that there are functions

$$f \in L_m^r, h \in L_m^{r/q}, g \in L_m^s$$

such that $f_n \xrightarrow{w} f, f_n^q \xrightarrow{w} h, Tf_n \to g$ in the appropriate spaces; clearly $Tf = g$. Furthermore, by the uniform convexity of L_m^r and $L_m^{r/q}$ we may assume that f_n and f_n^q are strongly $(C, 1)$-summable to their weak limits ([7], p. 462), so that

$$\hat{f}_n = n^{-1} \sum_{k=1}^{n} f_k \longrightarrow f \quad \text{and} \quad \hat{h}_n = n^{-1} \sum_{k=1}^{n} f_k^q \longrightarrow h .$$

Now, we have

$$J(f_n) - \int_a^b g^p h d\mu = \int_a^b ((Tf_n)^p - g^p) f_n^q + \int_a^b g^p (f_n^q - h) . \tag{7}$$

Now, since $f_n^q \xrightarrow{w} h$ in $L_m^{r/q}$ and since $g^p \in L_m^{s/p} = L_m^{(r/q)'}$, the second integral in the right member of (7) tends to zero as $n \to \infty$. To show that the first integral tends to zero we consider separately $0 \leqq p < 1$ and $1 \leqq p < \infty$. If $0 \leqq p < 1$, we use the inequality $|A^p - B^p| \leqq |A - B|^p$ for $A \geqq 0, B \geqq 0$ to obtain

$$\begin{aligned} \left| \int_a^b (Tf_n)^p f_n^q - \int_a^b g^p f_n^q \right| &\leqq \int_a^b |Tf_n - g|^p f_n^q \\ &\leqq \|Tf_n - g\|_s^p \|f_n\|_r^q \leqq \|Tf_n - g\|_s^p . \end{aligned} \tag{8}$$

The second step follows from Hölder's inequality with exponents $s/p = r/(r-q)$ and r/q. The final term in (8) tends to zero since $Tf_n \to g$ in L_m^s.

In case $1 \leqq p < \infty$, we consider instead

$$(9) \qquad \left| \left\{ \int_a^b (Tf_n)^p f_n^q \right\}^{1/p} - \left\{ \int_a^b g^p f_n^q \right\}^{1/p} \right| \leqq \left\{ \int_a^b | Tf_n - g |^p f_n^q \right\}^{1/p}$$

by Minkowski's inequality. As in (8), the right member of (9) tends to zero. Thus, if $A_n = \int_a^b (Tf_n)^p f_n^q$ and $B_n = \int_a^b g^p f_n^q$, we have that

$$| A_n^{1/p} - B_n^{1/p} | \longrightarrow 0 .$$

But $\{A_n\}$ and $\{B_n\}$ are bounded sequences ($A_n = J(f_n) \leqq || T ||^p$ by (4), $B_n \leqq || g ||_s^p \leqq || T ||^p$ by Hölder's inequality and $Tf_n \longrightarrow g$), and thus $| A_n - B_n | \leqq p | A_n^{1/p} - B_n^{1/p} | \cdot || T ||^{p-1}$ shows that $A_n - B_n \longrightarrow 0$ as required. Hence, we have

$$(10) \qquad K^* = \int_a^b g^p h d\mu .$$

In case $q = 0$, (10) also holds with $h \equiv 1$, by a similar argument.

Now we show the existence of f_0 for which $J(f_0) = K^*$. The cases $0 \leqq q < 1$ and $1 \leqq q < r$ are considered separately. If $0 \leqq q < 1$, define $f_0 = f$. Since $\varphi(t) = t^q$ is concave, we have

$$(11) \qquad \hat{f}_n^q = (n^{-1} \sum_1^n f_k)^q \geqq n^{-1} \sum_1^n f_k^q = \hat{h}_n .$$

Now, since $\hat{f}_n \longrightarrow f_0$ in L_m^r, we have

$$(12) \qquad \begin{aligned} \left| \int_a^b g^p f_0^q - \int_a^b g^p \hat{f}_n^q \right| &\leqq \int_a^b g^p | f_0^q - \hat{f}_n^q | \\ &\leqq \int_a^b g^p | f_0 - \hat{f}_n |^q \leqq || g ||_s^p || f_0 - \hat{f}_n ||_r^q \longrightarrow 0 . \end{aligned}$$

Similarly, $\int_a^b g^p \hat{h}_n \longrightarrow \int_a^b g^p h$. Thus, combining (10), (11) and (12) we obtain

$$(13) \qquad \begin{aligned} J(f_0) = \int_a^b g^p f_0^q &= \lim \int_a^b g^p \hat{f}_n^q \\ &\geqq \lim \int_a^b g^p \hat{h}_n = \int_a^b g^p h = K^* . \end{aligned}$$

However $|| f_0 ||_r \leqq 1$ so $J(f_0) \leqq K^*$ and hence (13) implies $J(f_0) = K^*$ from which it is clear that $|| f_0 ||_r = 1$.

In case $1 \leqq q < r$, let $f_0 = h^{1/q}$. Now, instead of (11), we have $\hat{f}_n^q \leqq \hat{h}_n$. Since $|| \hat{h}_n - h ||_{r/q} \longrightarrow 0$, and since $| \hat{h}_n^{1/q} - h^{1/q} | \leqq | \hat{h}_n - h |^{1/q}$ we have

$$\begin{aligned} || \hat{h}_n^{1/q} - h^{1/q} ||_r &= \left\{ \int_a^b | \hat{h}_n^{1/q} - h^{1/q} |^r \right\}^{1/r} \\ &\leqq \left\{ \int_a^b | \hat{h}_n - h |^{r/q} \right\}^{1/r} = || \hat{h}_n - h ||_{r/q}^{1/q} . \end{aligned}$$

Thus $\hat{h}_n^{1/q} \to h^{1/q} = f_0$ in L_m^r and since T is continuous, $T\hat{h}_n^{1/q} \to Tf_0$ in L_m^r. However $\hat{f}_n \leqq \hat{h}_n^{1/q}$ and $k(x, t) \geqq 0$ a.e. so $T\hat{f}_n \leqq T\hat{h}_n^{1/q}$, a.e. and thus $Tf = g \leqq Tf_0$ a.e. Thus (10) implies $K^* \leqq J(f_0)$, which again means that $J(f_0) = K^*$ and $\| f_0 \|_r = 1$.

REMARK. A simple sufficient condition for T to be compact from $L_m^r \to L_m^s$ is that k have finite (r', s)-double norm. That is

$$(14) \qquad ||| T ||| = \left\{ \int_a^b \left[\int_a^b k(x, t)^{r'} d\mu(t) \right]^{s/r'} d\mu(x) \right\}^{1/s} < \infty$$

(see [9], p. 319; the proof there applies even if $0 < s < 1$).

Using (4), we see that $K^* \leqq \| T \|^p \leqq ||| T |||^p$ so (14) also supplies an upper bound for K^* (rarely the best).

For the operator T_1 given by (2) one may calculate that

$$(15) \qquad ||| T_1 |||^s = \int_a^b w(x)^{r/(r-q)} m(x)^{-q/(r-q)} \left[\int_a^x m(t)^{-1/(r-1)} dt \right]^{s/r'} dx \, .$$

In the paper of Beesack and Das [2], the following inequality is proved: If $pq > 0$, $p + q > 1$, $y(a) = 0$ and y is absolutely continuous, then

$$(16) \qquad \int_a^b | y |^p | y' |^q w(x) dx \leqq K_1(b, p, q) \int_a^b | y' |^{p+q} m(x) dx \, ,$$

where $K_1(b, p, q)$ is explicitly given. The constant $K_1(b, p, q)$ equals the best constant K^* if and only if for some $c \geqq 0$

$$(17) \qquad w(x) = c m(x)^{(q-1)/(r-1)} \left(\int_a^x m(t)^{-1/(r-1)} dt \right)^{p(1-q)/q} (r = p + q) \, .$$

The constant $K_1(b, p, q)$ given there is in fact equal to $(q/r)^{q/r} ||| T_1 |||^p$, so, unless (17) holds we have

$$(18) \qquad K^* < K_1(b, p, q) < ||| T_1 |||^p \, .$$

LEMMA 2. *Suppose that T is given by* (1), *and that $k(x, t) > 0$ for almost all (x, t) with $a \leqq t \leqq x \leqq b$. Let $p > 0, r > 1, 0 \leqq q < r$, and suppose T is a bounded operator from $L_m^r \to L_m^s$. Let J be defined by* (3), *K^* by* (6). *Let f satisfy $\| f \|_r = 1$ and $J(f) = K^*$. Then f is of constant sign* a.e. *and*

(a) $f \neq 0$ a.e.

(b) *f satisfies* a.e. *the equation*

$$(19) \qquad r\lambda f^{r-1}(x) - q(Tf)^p(x) f^{q-1}(x) - p \int_a^b k(t, x)(Tf)^{p-1}(t) f^q(t) d\mu(t) = 0 \, ,$$

where $\lambda = \lambda^* = K^*(p + q)/r$. Furthermore λ^* is the largest value of

λ *for which* (19) *has a solution* f *with* $\|f\|_r = 1$.

Proof. (a) We have seen that f is of constant sign a.e. so we assume $f \geqq 0$ a.e. Let $E = \{x: f(x) = 0\}$; we must show that E is a null set. First choose a function $h \in L^r_m$ such that $h(x) \geqq 0$, and $h(x) > 0$ if and only if $x \in E$. Such h exist: if $\mu(E) < \infty$, take $h = \chi_E$, while if $\mu(E) = \infty$, let

$$E_n = E \cap [-n, n] \cap \{x: m(x) \leqq n\} ,$$

so $\mu(E_n) < \infty$, and define $h = \sum \gamma_n \chi_{E_n}$ where $\{\gamma_n\}$ is chosen so $\gamma_n > 0$ and $\sum \gamma_n^r \mu(E_n) < \infty$.

For $\varepsilon > 0$, define $f_\varepsilon = f + \varepsilon h$, and let $F = Tf$, $F_\varepsilon = Tf_\varepsilon$, $H = Th$. since $J(f)/\|f\|_r^{p+q}$ is maximal, we have

$$\begin{aligned} 0 &\leqq J(f_\varepsilon) - J(f) \leqq (\|f_\varepsilon\|_r^{p+q} - 1)J(f) \\ &= \{(1 + \varepsilon^r \|h\|_r^r)^{(p+q)/r} - 1\}J(f) \\ &= \varepsilon^r \|h\|_r^r \gamma \xi^{\gamma-1} J(f), \text{ (where } \gamma = (p+q)/r, \text{ and } 1 < \xi < 1 + \varepsilon^r \|h\|_r^r \\ &= 0(\varepsilon^r) \text{ as } \varepsilon \downarrow 0. \end{aligned} \tag{20}$$

First assume that $q > 0$, so if $CE = [a, b] \backslash E$, we may write

$$J(f_\varepsilon) - J(f) = \varepsilon^q \int_E F_\varepsilon^p h^q + \int_{CE} (F_\varepsilon^p - F^p) f^q . \tag{21}$$

From (20) and (21) we immediately deduce that

$$0 \leqq \int_E F^p h^q \leqq \int_E F_\varepsilon^p h^q = 0(\varepsilon^{r-q}) \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0 .$$

Thus, $F(x) = 0$ a.e. on E so $k(x, t) = 0$ a.e. on $E \times CE$.

Next, we note that $F(x) > 0$ a.e. on CE, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$. Thus, for almost all x in CE, we have $(d/d\varepsilon)F_\varepsilon^p f^q = pF_\varepsilon^{p-1} Hf^q$. Hence, if $0 < \varepsilon < \varepsilon_0$ we have.

$$pF^{p-1} Hf^q \leqq \varepsilon^{-1}(F_\varepsilon^p - F^p) f^q \text{ a.e. on } CE, p \geqq 1 \tag{22}$$

$$pF_{\varepsilon_0}^{p-1} Hf^q \leqq \varepsilon^{-1}(F_\varepsilon^p - F^p) f^q \text{ a.e. on } CE, 0 < p < 1 . \tag{23}$$

Thus, if $p \geqq 1$, (20), (21) and (22) imply that

$$0 \leqq \int_{CE} pF^{p-1} Hf^q \leqq \varepsilon^{-1} \int_{CE} (F_\varepsilon^p - F^p) f^q = 0(\varepsilon^{r-1}) \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0 . \tag{24}$$

Thus, since $F(x) \neq 0$, we have $H(x) = 0$ a.e. on CE. A similar argument using (20), (21) and (23) proves $H(x) = 0$ a.e. on CE, if $0 < p < 1$. Thus $k(x, t) = 0$ a.e. on $CE \times E$, and hence on $(E \times CE) \cup (CE \times E)$. But, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$, the last sentence implies that $ExCE$ has plane measure zero and so either $\mu(E) = 0$ or $\mu(CE) = 0$. However, $\mu(CE) = 0$ implies that $f = 0$ a.e. contradicting $J(f) = K^* \neq 0$.

Thus $\mu(E) = 0$ as required.

In case $q = 0$, (21) no longer holds. In this case, let

$$A = \{x: F(x) = 0\}$$

so $k(x, t) = 0$ a.e. on $A \times CE$. Clearly $\mu(A \cap CE) = 0$, since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$. Instead of (21) we have

$$J(f_\varepsilon) - J(f) = \varepsilon^p \int_A H^p + \int_{CA} (F_\varepsilon^p - F^p) \,. \tag{25}$$

Proceeding as in (24), we use the second integral in (25) together with (20) to show that $H(x) = 0$ a.e. on CA, so $k(x, t) = 0$ a.e. on $CA \times E$. Now if $B = CA \cap E$ has $\mu(B) > 0$, we would have $k(x, t) = 0$ a.e. on $B \times B$ with contradicts $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$ and thus $\mu(B) = \mu(E \backslash A) = 0$. We already have shown that $\mu(A \backslash E) = 0$. Thus $k(x, t) = 0$ a.e. on $(A \times CE) \cup (CA \times E)$ means $k(x, t) = 0$ a.e. on $(E \times CE) \cup (CE \times E)$, which leads to a contradiction as before. (We note that if $p < r$, a simpler argument is available using the first integral in (25).)

(b) Consider the functional

$$I(f) = \lambda \| f \|_r^r - J(f) = \int_a^b [\lambda f^r - (Tf)^p f^q] d\mu \,.$$

We shall show that if $J(f) = K^*$, and if $|h| \leqq f$, then for $\lambda = \lambda^* = K^*(p + q)/r$, we have

$$\delta I(f; h) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(I(f + \varepsilon h) - I(f)) = 0 \,. \tag{26}$$

First, suppose that $|h| \leqq f$ and that $|\varepsilon| \leqq 1/2$. Now define $A(\varepsilon) = J(f + \varepsilon h)$ and $B(\varepsilon) = \| f + \varepsilon h \|_r^r$. Then A and B are differentiable at $\varepsilon = 0$, and

$$A'(0) = \int_a^b (pF^{p-1} f^q H + qF^p f^{q-1} h) d\mu \tag{27}$$

$$B'(0) = \int_a^b r f^{r-1} h d\mu \,. \tag{28}$$

To see this, note that $(d/d\varepsilon) F_\varepsilon^p f_\varepsilon^q = pF_\varepsilon^{p-1} H f_\varepsilon^q + qF_\varepsilon^p f_\varepsilon^{q-1} h$ a.e. since $f > 0$ a.e. by (a), and $F > 0$ a.e. since $k(x, t) > 0$ a.e. for $a \leqq t \leqq x \leqq b$, and thus $f_\varepsilon > 0$ a.e., $F_\varepsilon > 0$ a.e. for $|\varepsilon| \leqq \frac{1}{2}$ and $|h| \leqq f$.

But, we have

$$\int_a^b F^{p-1} H f^q \leqq \left\{ \int_a^b F^s \right\}^{(p-1)/s} \left\{ \int_a^b H^s \right\}^{1/s} \left\{ \int_a^b f^r \right\}^{q/r} \tag{29}$$

by Hölder's general inequality with exponents $s/(p - 1)$, s and r/q. Similarly, one shows $F^p f^{q-1} h$ is integrable. And, for $|\varepsilon| \leqq \frac{1}{2}$ one may bound $|(d/d\varepsilon) F_\varepsilon^p f_\varepsilon^q|$ in terms of $F^{p-1} H f^q$ and $F^p f^{q-1} h$. For example, if $p \geqq 1, q \geqq 1$, one has

$$\left|\frac{d}{d\varepsilon} F_\varepsilon^p f_\varepsilon^q\right| \leqq \left(\frac{3}{2}\right)^{p+q-1} (pF^{p-1}Hf^q + qF^p f^{q-1}h) \quad \text{a.e.} \tag{30}$$

with similar bounds if $0 < p < 1$ or $0 \leqq q < 1$. Thus, Lebesgue's dominated convergence theorem gives (27). A similar argument gives (28).

By assumption $J(f)/\|f\|_r^{p+q} = A(0)/B(0)^{(p+q)/r}$ is maximal and hence

$$\frac{d}{d\varepsilon}(A(\varepsilon)B(\varepsilon)^{-(p+q)/r})\Big|_{\varepsilon=0} = 0 .$$

Differentiating and using $A(0) = K^*$ and $B(0) = 1$, we obtain

$$A'(0) - K^*((p+q)/r)B'(0) = 0 \tag{30}$$

or if we write $\lambda^* = K^*(p+q)/r$, we obtain

$$\int_a^b (r\lambda^* f^{r-1}h - pF^{p-1}f^q H - qF^p f^{q-1}h)d\mu = 0 . \tag{31}$$

By Fubini's theorem we have

$$\begin{aligned} \int_a^b F^{p-1}f^q H &= \int_a^b F^{p-1}(x)f^q(x)\Big(\int_a^b k(x,t)h(t)d\mu(t)\Big)d\mu(x) \\ &= \int_a^b h(t)\Big(\int_a^b k(x,t)F^{p-1}(x)f^q(x)d\mu(x)\Big)d\mu(t) . \end{aligned} \tag{32}$$

Thus, if we write T' for the operator with kernel $k(t,x)$ we have from (31) and (32)

$$\begin{aligned} 0 &= \int_a^b h(x)\{r\lambda^* f^{r-1} - q(Tf)^p f^{q-1} - pT'((Tf)^{p-1}f^q)\}d\mu(x) \\ &= \int_a^b h(x)G(x)d\mu(x) . \end{aligned} \tag{33}$$

To obtain (19) set $h(x) = f(x)\operatorname{sgn} G(x)$ in (33) and use the fact that $f(x) \neq 0$ a.e.

To see that λ^* is the largest value of λ for which a solution to (19) is possible with $\|f\|_r = 1$, note that if (19) holds then (33) and hence (31) hold for any $|h| \leqq f$ with λ in place of λ^*. Thus, setting $h = f$ in (31) (with λ for λ^*), we obtain $r\lambda\|f\|_r^r - (p+q)J(f) = 0$, and thus $\lambda = (p+q)J(f)/r \leqq (p+q)K^*/r = \lambda^*$.

REMARK. Part (a) of Lemma 2 may be strengthened by allowing k to vanish on more extensive sets. However, the precise condition that is needed to insure $f \neq 0$ a.e. depends on the relationship of p, q and r. For example, if $q > 0$ and $p < r$, and if there are no sets E with $\mu(E) > 0$ and $\mu(CE) > 0$ such that k vanishes on $(E \times CE) \cup (CE \times E)$ then for f as in Lemma 2, one has $f \neq 0$ a.e.

Proof of Theorem 1. By Lemma 1, $\sup\{J(f): \|f\|_r \leqq 1\} = K^* < \infty$, and there is an $f \geqq 0$ with $\|f\|_r = 1$ and $J(f) = K^*$. Since $m(x) > 0$ and $w(x) > 0$ a.e., Lemma 2 applies and we have $f \neq 0$ a.e. in $[a, b]$, and f satisfies

$$\lambda r f^{r-1}(x)m(x) - qF(x)^p f(x)^{q-1}w(x) - p\int_x^b F^{p-1}f^q w = 0 \text{ a.e.} \tag{34}$$

where $F(x) = \int_a^x f(t)dt$.

We claim that by modifying f on a set of measure zero, we will have $f \in C^1(a, b)$, $f(x) \neq 0$ in (a, b) and f will satisfy (34) everywhere. To see this, rewrite (34) as

$$f^{r-1} - A(x)f^{q-1} = B(x) \text{ a.e.} \tag{35}$$

where $A(x) \geqq 0$, and $B(x) > 0$ for all $x \in (a, b)$.

Consider the equation $\zeta^{r-1} - \xi\zeta^{q-1} = \eta$. For $\eta > 0$, $\xi \geqq 0$ this has a unique positive solution $\zeta = \varphi(\xi, \eta)$ which can be extended to be C^∞ on an open region containing the set $\{(\xi, \eta): \xi \geqq 0, \eta \geqq 0, \xi + \eta > 0\}$. To see this, consider the function $\psi(\zeta) = \zeta^{r-1} - \xi\zeta^{q-1}$ for fixed ξ, r and q. First suppose $q \geqq 1$, and $\xi > 0$, then $\psi'(\zeta)$ has a single positive zero $\zeta_0 = \zeta_0(\xi)$, and ψ decreases from $\psi(0) = 0$ to $\psi(\zeta_0) < 0$ and is strictly increasing on $[\zeta_0, \infty)$ to $+\infty$. Thus $\psi(\zeta) = \eta$ has a unique solution for $\eta > 0$ which we denote $\varphi(\xi, \eta)$. We define $\varphi(\xi, \eta)$ for $\xi > 0$ and $0 \geqq \eta > \psi(\zeta_0(\xi))$ to be that solution of $\eta = \psi(\zeta)$ with $\zeta > \zeta_0$. If $q \geqq 1$ and $\xi \leqq 0$, then ψ is strictly increasing from $\psi(0) = 0$, hence $\psi(\zeta) = \eta$ has a unique solution for $\eta \geqq 0$. Thus, for $q \geqq 1$, $\psi(\xi, \eta)$ is defined on an open set containing $Q = \{(\xi, \eta): \xi \geqq 0, \eta \geqq 0, \zeta + \eta > 0\}$, and since $\psi'(\varphi(\xi, \eta)) > 0$, the implicit function theorem shows that $\varphi \in C^\infty$. To show that $\varphi(\xi, \eta) \to \varphi(0, 0) = 0$ as $(\xi, \eta) \to (0, 0)$ in Q, we note that if $0 \leqq \xi \leqq \delta$, $0 \leqq \eta \leqq \delta$ and $\zeta_1 = \alpha\delta^{1/(r-1)}$ with $\alpha = 2^{1/(r-q)}$, then

$$\psi(\zeta_1) \geqq \alpha^{r-1}\delta - \delta\alpha^{q-1}\delta^{(q-1)/(r-1)} \geqq \alpha^{q-1}(\alpha^{r-q} - 1)\delta \geqq \delta \text{ (if } \delta \leqq 1) .$$

Thus $\varphi(\xi, \eta) \leqq \alpha\delta^{1/(r-1)}$ for $0 \leqq \xi \leqq \delta$, $0 \leqq \eta \leqq \delta$ proving the assertion.

If $0 \leqq q < 1$ and $\xi > 0$, then ψ is strictly increasing from $-\infty$ to ∞ on $(0, \infty)$ so $\psi(\zeta) = \eta$ has a unique solution for all η. If $0 \leqq q < 1$ and $\xi < 0$, then $\psi(\zeta) \to \infty$ as $\zeta \to 0+$ or $\zeta \to \infty$, and ψ has a minimum at a point ζ_0 where $\psi(\zeta_0) = \gamma|\xi|^{(r-1)/(r-q)}$ and $\gamma > 0$. If $\xi = 0$, $\psi(\xi) = \eta$ has a unique solution for $\eta \geqq 0$. Again we have $\varphi \in C^\infty$ on an open set containing Q and that $\varphi(\xi, \eta) \to 0$ as $(\xi, \eta) \to (0, 0)$ in Q.

Now, from (35), by modifying f on a null set, we have

$$f(x) = \varphi(A(x), B(x)) \quad \text{for all } x \in (a, b) . \tag{36}$$

If $w, m \in C^1$ then A, B are absolutely continuous so (36) shows that f

is absolutely continuous. But then $F \in C^1$ so in fact, $A, B \in C^1$ and (36) shows that $f \in C^1$. That $f(x) \neq 0$ for $x \in (a, b)$ follows immediately from (36).

Now, defining $y = F$ and differentiating (34) once gives (P) (i). The conditions (ii) and (iii) are apparent from (34). The problem (P) thus has solutions for $\lambda = K^*(p + q)/r$. To identify the largest eigenvalue of (P) as $K^*(p + q)/r$, we note that a solution of (P) gives a solution of (34) and by Lemma 2 the largest eigenvalue of (34) is $K^*(p + q)/r$.

The inequality (5) and the statement concerning equality are now obvious.

REMARK. If $m(x) > 0$ and $w(x) > 0$ for all $x \in [a, b]$, and if $q > 0$, then $A(x) > 0$ unless $x = a$ and $B(x) > 0$ unless $x = b$. Hence equation (36) shows that $f(x) > 0$ for all $x \in [a, b]$; and $f \in C^1[a, b]$. We also note that if $\lim_{x \to b} A(x)$ is finite and $\lim_{x \to a} B(x)$ is finite then $f(a) < \infty$ and $f(b) < \infty$. This will be used in § 3.

3. Some inequalities on a finite interval. As an application of Theorem 1, we obtain the best constants in case (a, b) is a finite interval and $m(x) \equiv w(x) \equiv 1$. We immediately consider

$$\int_0^1 |y|^p |y'|^q dx \leqq K(p, q, r)\left\{\int_0^1 |y'|^r dx\right\}^{(p+q)/r} \tag{37}$$

where y is absolutely continuous and $y(0) = 0$.

Some special cases of (37) are known. The case $q = 0, p = r = 2k$ (k a positive integer) is inequality 256 of [8], which was derived there by classical variational methods using the Weierstrass sufficient condition. This case was handled by elementary methods in [3]. Opial's inequality is the case $p = q = 1, r = 2$. If $q = 1, r = p + 1$, the best constant can be obtained by Hölder's inequality (see [5], for example). The case $r = p + q$ was considered in [6] but the best constant was found only when $q = 1$ or $r = 1$.

Note that if $q > r$, there is no inequality of the form (37), since for $y(x) = 1 - (1 - x)^{1-r}$, $q^{-1} < \gamma < r^{-1}$, the left member of (37) is infinite while $\| y' \|_r < \infty$. The case $p = 0$ is simply Hölder's inequality with $K(0, q, r) = 1$.

THEOREM 2. *For $r \geqq 1, p > 0, 0 \leqq q \leqq r$, the inequality* (37) *is valid with a finite constant $K(p, q, r)$. The best such constant is given by the following expressions*

(a) *if $p > 0, r > 1, 0 \leqq q < r$, then*

$$K(p, q, r) = \frac{(r - q)p^p}{(r - 1)(p + q)} \beta^{p+q-r} I(p, q, r)^{-p} \tag{38}$$

where

$$\beta = \left\{\frac{p(r-1)+(r-q)}{(r-1)(p+q)}\right\}^{1/r}$$

and

$$I(p,q,r) = \int_0^1 \left\{1 + \frac{r(q-1)}{r-q}\,t\right\}^{-(q+p+rp)/rp} \{1+(q-1)t\}t^{1/p-1}dt\ .$$

(b) *If* $r = 1$, *then*

$$K(p,q,1) = \begin{cases} q^q(p+q)^{-q}, & q > 0 \\ 1 & , q = 0\ . \end{cases}$$

(c) *If* $q = r$, *then*

$$K(p,r,r) = \frac{rp^p}{p+r}\left(\frac{p}{p+r}\right)^{p/r} B\left(\frac{1}{r}+1, \frac{1}{p}\right)^{-p}\ . \tag{39}$$

If $r = 1, q = 0$, *there is strict inequality for all* $y \not\equiv 0$ *while in all other cases there is equality only for multiples of a single function* $y(p, q, r, x)$ *which is in* $C^\infty(0,1)$, *and is concave if* $0 \leqq q < 1$, *convex if* $q > 1$, *linear if* $q = 1$.

For special cases of (a), (28) reduces to a simpler form. First, if $r = p + q$, we have

$$K(p,q,p+q) = q(p+q)^{p-1}\{pL(p,q)+q\}^{-p},\ q \neq 0 \tag{40}$$

where

$$L(p,q) = \int_0^1 \frac{ds}{1-ks^p},\ k = \frac{(p+q)(q-1)}{(p+q-1)q}\ .$$

In particular,

$$K(1,q,1+q) = \left\{1 + \frac{2q}{q^2-1}\log q\right\}^{-1},\ q \neq 0, 1\ .$$

If $q = 0$, and $r > 1$, we have

$$K(p,0,r) = A(p,r)^p$$
$$A(p,r) = (r')^{1/p}p^{1/r'}(p+r')^{1/r-1/p}B(1/p, 1/r')^{-1}\ ,$$

where $r' = r/(r-1)$. Note that $A(p,r)$ is the norm of the mapping $T: L^r \to L^p$ where here $Tf(x) = \int_0^x f(t)dt$. By (4), if $\|f\|_r = 1$ we have $J(f) \leqq \|T\|^p$, where $\|T\|$ is the norm of T as a mapping from $L^r_m \to L^s_m$, $(s = pr/(r-q))$, and so we always have $K(p,q,r) \leqq A(s,r)^p$.

We also note that in (38), if $q \neq 1$, one may make the replacement

$$(r-q)I(p,q,r)^{-p} = r^{p+1}(q-1)I_1(p,q,r)^{-p}\ , \tag{41}$$

where

$$I_1(p,q,r) = \pm (r-q)\int_0^{\pm T} t^{1/p}(1 \mp t)^{\gamma-1}dt + r\int_0^{\pm T} t^{(1/p)-1}(1 \mp t)^{\gamma}dt\ ,$$

where $T_1 = 1 - [(r-q)/q(r-1)]$, $\gamma = (p+q-r)/rp$, and the upper sign is used with $q > 1$, the lower sign with $q < 1$.

Proof. In case (a), Theorem 1 applies since certainly $|||\, T_1 \,||| < \infty$. We seek solutions of the problem (P). We first observe that by the remark at the end of § 2, we have $y \in C^2[a,b]$ and $0 < y'(0) < \infty$, $0 < y'(1) < \infty$ except in case $q = 0$ when we have $y'(1) = 0$. To see this note that the functions A and B which appear in (35) are here just

$$A(x) = q(\lambda r)^{-1}y(x)^p,\ B(x) = p(\lambda r)^{-1}\int_x^1 y(t)^{p-1}y'(t)^q dt\ .$$

But $y(1) = \int_0^1 y'(t)dt \leqq \| y' \|_r < \infty$, so $A(1) < \infty$, and

$$\int_0^1 y^{p-1}y'^q \leqq \left\{\int_0^1 y^{r(p-1)/(r-q)}\right\}^{(r-q)/r} \| y' \|_r^q$$

which shows that $B(0) < \infty$.

Notice that equation (i) of (P) has the integrating factor y' from which we obtain

$$(r-1)\lambda y'^r - (q-1)y^p y'^q = \alpha\lambda\ , \tag{42}$$

where α is a constant which is evaluated by using $\| y' \|_r = 1$ and $J(y') = r\lambda/(p+q)$. Thus we have

$$\alpha = (r-1) - \frac{(q-1)r}{p+q} = \frac{pr-p-q+r}{p+q} > 0\ . \tag{43}$$

Solving (42) for $y = \lambda^{1/p}G(y')$ and differentiating leads to a variables separable equation for y', and if we write $z = y'$ we have, for $q \neq 1$

$$dx = \frac{\lambda^{1/p}}{p}\left\{\frac{(r-1)z^{r-q} - \alpha z^{-q}}{q-1}\right\}^{(1/p)-1} \times \left\{\frac{r-1)(r-q)z^{r-q-2} + \alpha q z^{-q-2}}{q-1}\right\}dz\ . \tag{44}$$

To obtain boundary conditions, we use (42) and (ii) and thus, since $z(0) \neq 0$ and $z(1) \neq 0$ for $q \neq 0$, we obtain

$$z(0)^r = \alpha/(r-1) \quad \text{and} \quad z(1)^r = \alpha q/(r-q)\,. \tag{45}$$

We now integrate (44) from $x = 0$ to $x = 1$ using (45) and make the change of variables (for $q \neq 1$)

$$t = \frac{r-q}{r\alpha} \cdot \frac{(r-1)z^r - \alpha}{q-1}$$

which leads to equation (38).

For $q = 1$, we note that $I(p, 1, r) = p$, $\beta = 1$, and so (38) gives $K(p, 1, r) = (p+1)^{-1}$ which is the correct result by our earlier remarks. In the equation $y = \lambda^{1/p} G(y')$, G is increasing if $q > 1$ and decreasing if $q < 1$. Thus, since y is increasing, we must have y' increasing if $q > 1$ and decreasing if $q < 1$. The solution to problem (P) with $\lambda = \lambda^*$ can be seen to be unique in the following way. We know that a solution of (P) must satisfy $y = \lambda^{1/p} G(y')$ and thus also $y' = \lambda^{1/p} G'(y')y''$, and hence y' satisfies

$$\lambda^{1/p} \int_{y'(0)}^{y'(x)} G'(z)\,\frac{dz}{z} = x\,. \tag{46}$$

But, for $q \neq 1$, $G'(z)$ does not change sign on the interval from $y'(0)$ to $y'(1)$ so (46) has a unique solution for $y'(x)$, and hence (P) has a unique solution when $\lambda = \lambda^*$.

To obtain the alternate expression for $(r-q)I(p, q, r)^{-p}$ given in (41), we make the change of variable $t = 1 - \alpha(r-1)^{-1}z^{-r}$ in (44).

To obtain the formula (40), we make the following change of variables in (38)

$$1 + \frac{r(q-1)}{r-q}t = \left(1 - \frac{r(q-1)}{q(r-1)}s^p\right)^{-1} = R(s)^{-1}\,.$$

Then $t = (p/q(r-1))s^p R(s)^{-1}$, and $t = 0, 1$ correspond to $s = 0, 1$ and one has

$$\begin{aligned} I(p, q, r) &= \text{const}\left\{\int_0^1 R(s)^{(q/rp)+(1/r)+1-(1/p)-1}(pR(s)^{-1} + q)ds\right\} \\ &= \text{const}\,\{pL(p, q) + q\}, \text{ since } r = p + q\,. \end{aligned} \tag{47}$$

The formula (40) can be obtained in a more direct way by making the substitution $u = (q/r\lambda)^{1/p}(y/y')$ in equation (i), where we assume $q \neq 0$ so $y'(x) > 0$ for $x \in [a, b]$. Then the conditions (ii) give $u(0) = 0$, $u(1) = 1$, and equation (i) reduces to

$$(r-1)\left\{1 - \frac{r(q-1)}{q(r-1)}u^p\right\} = p\left(\frac{r\lambda}{q}\right)^{1/p} u' \qquad (r = p + q)\,. \tag{48}$$

Separating variables and integrating gives (40).

For case (b), let $z(x) = \int_0^x |y'|dt$, and then if $q > 0$

$$\int_0^1 |y|^p |y'|^q dx \leqq \int_0^1 (z^{p/q}z')^q dx$$
$$\leqq \left\{\int_0^1 dx\right\}^{1/q}\left\{\int_0^1 z^{p/q}z'dx\right\}^q = \left(\frac{q}{p+q}\right)^q z^{p+q},$$

using Hölder's inequality with exponents $1/q > 1$ and $1/(1-q)$. Equality holds only if $z^{p/q}z'$ is constant, and $y = z$ which means $y(x) = cx^{q/(p+q)}$. For $q = 0$, we have

$$\int_0^1 |y|^p dx \leqq \int_0^1 |z|^p dx \leqq z(1)^p = \int_0^1 |z'(x)|dx .$$

Equality holds only if $y = z$, and $z(x) = z(1)$ for all x, so $y(x) = z(x) \equiv 0$.

For case (c), we let $q \to r-$ in formula (38), using the equation (41) to evaluate $\lim (r-q)I(p, q, r)^{-p}$. This shows that the best constant is given by (39), because if $y' \in L^r$ and $q < r$, then

$$\int_0^1 |y|^p |y'|^q \to \int_0^1 |y|^p |y'|^r$$

by dominated convergence. To handle the case of equality we cannot apply Lemma 2 directly since the proof of Lemma 2(a) used $r > q$. However, if there is an f with $J(f) = K(p, r, r) = K^*$ then we know that $f \geqq 0$ a.e. Now referring to the proof of Lemma 2(a), since $r > 1$ we do have (24) which proves that if $E = \{x: f(x) = 0\}$, then $k(x, t) = 0$ a.e. on $CE \times E$. This means that

$$(CE \times E) \cap \{(x, t): 0 \leqq t \leqq x \leqq 1\}$$

is a set of measure zero. This implies that E differs from an interval $[c, 1]$ by a set of measure zero. To see this, let

$$c = \sup\{x \leqq 1: [0, x] \cap E \text{ is of measure zero}\},$$

and let $d = \inf\{x \geqq 0: [x, 1] \cap CE$ is of measure zero$\}$. Clearly $d \geqq c$. But, if $d > c$, and $e = (d + c)/2$, then $[c, e] \cap E$ and $[e, d] \cap CE$ have positive measure; but then $CE \times E$ intersects $\{(x, t): 0 \leqq t \leqq x \leqq 1\}$ in a set of positive measure which is absurd.

However if equality held for such an f, we would have (writing $f(x) = y'(x)$),

$$\int_0^c |y|^p |y'|^r dx = K^*\left\{\int_0^c |y'|^r dx\right\}^{(p+r)/r}. \tag{49}$$

Define $z(t) = y(ct)$ so $z'(t) = cy'(ct)$, and from (49) we obtain

$$\int_0^1 |z|^p |z'|^r dx = K^* c^{-p(r-1)/r} \left\{ \int_0^1 |z'|^r dx \right\}^{(p+r)/r} . \tag{50}$$

But, if $c < 1$, then $K^* c^{-p(r-1)/r} > K^*$ contradicting the maximality of K^*. Thus $c = 1$, so $f(x) = y'(x) > 0$ a.e. on $[0, 1]$.

Still proceeding on the assumption that there exists an f with $J(f) = K^*$ we have shown $f(x) > 0$ a.e. on $[0, 1]$, so the proof of Lemma 2(b) is valid and f satisfies

$$\lambda r f^{r-1}(x) - rF(x)^p f^{r-1}(x) - p\int_x^1 F^{p-1} f^r dt = 0 \text{ a.e.} \tag{51}$$

where $F(x) = y(x) = \int_0^x f(t)dt$, and $\lambda = (p + q)K^*/r$.

If x is any point where $f(x) > 0$ and (51) holds then (51) shows that $\lambda f^{r-1}(x) > F^p(x) f^{r-1}(x)$, so $F^p(x) < \lambda$ a.e. But F is strictly increasing so $F(x)^p < \lambda$ for $0 \leqq x < 1$. Now we can solve (51) for f and obtain

$$f(x) = \varphi(A(x), B(x)) \quad \text{for almost all} \quad x \in [0, 1) . \tag{52}$$

where $\varphi(\xi, \eta) = (\eta/(1 - \xi))^{1/(r-1)}$, $A(x) = \lambda^{-1} F(x)^p < 1$ for $0 \leqq x < 1$, and $B(x) = p(r\lambda)^{-1} \int_x^1 F^{p-1} f^r dt > 0$ for $0 \leqq x \leqq 1$. Now we proceed as in the proof of Theorem 1. If we modify f on a null set so that it satisfies (52) everywhere then we obtain $f \in C^1[0, 1)$, and $f(x) > 0$ for $0 \leqq x < 1$. Thus we see that if $J(f) = K^*$ and $y' = f$ then y must be a solution of the problem (P), with $\lambda = \lambda^* = (p + r)K^*/r$. But a solution of (P) must be a solution of (42) (with $q = r$) which is

$$\lambda y'^r - y^p y'^r = p\lambda/(p + r) . \tag{53}$$

However if (53) has a solution then it must also satisfy

$$\int_0^{y(x)} (\lambda - u^p)^{1/r} du = (p\lambda/(p + r))^{1/r} x . \tag{54}$$

To see that (54) has a unique solution for $0 \leqq x \leqq 1$, we note that

$$\int_0^{\lambda^{1/p}} (\lambda - u^p)^{1/r} du = p^{-1} \lambda^{1/p} B\left(\frac{1}{r} + 1, \frac{1}{p}\right) = (p\lambda/(p + r))^{1/r} , \tag{55}$$

using the formula for $K(p, r, r) = K^*$ and $\lambda = (p + r)K^*/r$. Since $\lambda - u^p > 0$ for $0 \leqq u < \lambda^{1/p}$, (54) has a unique solution $y = y(x)$ which is strictly increasing and has $y(0) = 0$, $y(1) = \lambda^{1/p}$. To complete the proof we must show that y in fact satisfies (i), (ii) and (iii) of (P). By the implicit function theorem $y \in C^2(a, b)$, and differentiating (54) twice shows that y satisfies (i). Clearly $y(0) = 0$. For the other part of (ii), we note that for $0 \leqq x < 1$, we have

$$(\lambda - y^p(x))^{1/r} y'(x) = (p\lambda/(p + r))^{1/r} ,$$

and since $y^p(x) \to \lambda$ as $x \to 1-$, we have $y'(x) \to \infty$ as $x \to 1-$. But this means that

$$(56) \quad (\lambda - y^p(x))y'(x)^{r-1} = (p\lambda/(p+r))y'(x)^{-1} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow 1 .$$

To verify that $\| y' \|_r = 1$, let us first introduce the function g by

$$(57) \qquad g(t) = (p\lambda)^{-1/r}(p+r)^{1/r} \int_0^t (\lambda - u^p)^{1/r} du ,$$

so $g(y(x)) = x$ for $x \in [0, 1]$ and hence $y(g(t)) = t$ for $t \in [0, \lambda^{1/p}]$. Now

$$(58) \qquad \begin{aligned} \int_0^1 y'(x)^r dx &= \frac{p\lambda}{p+r} \int_0^1 \frac{dx}{\lambda - y^p(x)} \\ &= \left(\frac{p\lambda}{p+r}\right)^{1-(1/r)} \int_0^{\lambda^{1/p}} (\lambda - t^p)^{(1/r)-1} dt , \end{aligned}$$

where we use the change of variable $x = g(t)$. Now using the formula for λ, we obtain

$$(59) \qquad \int_0^1 y'(x)^r dx = \frac{p}{p+r} B\left(\frac{1}{r}, \frac{1}{p}\right) / B\left(\frac{1}{r} + 1, \frac{1}{p}\right) = 1 .$$

REMARKS. (1) As was mentioned above, the method of this paper applies to inequalities of the form (1) with $n > 1$. In this case T becomes

$$w(x)^{1/p} m(x)^{-1/p} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt .$$

A discussion of the special case $p = q = 1, r = 2$ will be found in [4] where, for $m(x) \equiv w(x) \equiv 1$, $[a, b] = [0, 1]$, the best constant is shown to be asymptotic to $1/4n!$

(2) The method is equally applicable to inequalities in which the function y is restricted by other boundary conditions. For example, if $[a, b]$ is a finite interval we may treat

$$\int_a^b y^p y''^q w(x) dx \leqq K \left\{ \int_a^b y''^r m(x) dx \right\}^{(p+q)/r}$$
$$y(a) = y(b) = 0 .$$

In this case, if f is a given function in L_m^r, the boundary value problem $y'' = -f$, $y(a) = y(b) = 0$ has a solution $y(x) = \int_a^b G(x, t) f(t) dt$, where $G(x, t) \geqq 0$ a.e. Hence our lemmas apply.

(3) When Theorem 1 is specialized to the situation studied by Beesack in [1] ($q = 0, p = r$), the results are not as general as his. This is because we can effectively handle only those inequalities where

we can insure in advance that equality is possible. There is some compensation in the fact that the existence of solutions to the Euler-Lagrange equations (P) is a conclusion of our theorem rather than a hypothesis as in [1] and [5].

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