LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS
PRESERVING A FIXED RANK

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In this paper $T$ is a linear transformation from a tensor product $X \otimes Y$ into $U \otimes V$, where $X, Y, U, V$ are vector spaces over an infinite field $F$. The main result gives a characterization of surjective transformations $T$ for which there is a positive integer $k$ ($k < \dim U, k < \dim V$) such that whenever $z \in X \otimes Y$ has rank $k$ then also $Tz \in U \otimes V$ has rank $k$. It is shown that $T = A \otimes B$ or $T = S \circ (C \otimes D)$ where $A, B, C, D$ are appropriate linear isomorphisms and $S$ is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

Let $F$ be an infinite field and $X, Y, U, V$ vector spaces over $F$. We denote by $T$ a linear transformation of the tensor product $X \otimes Y$ into $U \otimes V$. The rank of a tensor $z \in X \otimes Y$ is denoted by $\rho(z)$. By definition $\rho(o) = 0$. The subspace of $X$ spaned by the vectors $x_1, \cdots, x_n \in X$ will be denoted by $\langle x_1, \cdots, x_n \rangle$.

**Lemma 1.** Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. Then $\rho(z) \leq k$ implies that $\rho(Tz) \leq k$ for all $z$.

**Proof.** If this is not true then for some $z \in X \otimes Y, z \neq 0$, we have $\rho(z) < k$ and $\rho(Tz) > k$. There exists $t \in X \otimes Y$ such that $\rho(t) + \rho(z) = k$ and moreover $\rho(z + \lambda t) = k$ for all $\lambda \neq 0, \lambda \in F$. Let

$$Tz = \sum_{i=1}^{m} u_i \otimes v_i, \quad m = \rho(Tz).$$

Since $u_i \in U$ are linearly independent and also $v_i \in V$ we can consider them as contained in a basis of $U$ and $V$, respectively. The matrix of coordinates of $Tz$ has the form

$$\begin{pmatrix}
I_m & 0 \\
0 & 0
\end{pmatrix}$$

where $I_m$ is the identity $m \times m$ matrix. Let

$$\begin{pmatrix}
A_m & B \\
C & D
\end{pmatrix}$$

be the matrix of coordinates of $Tt$. Then the minor $|I_m + \lambda A_m|$ of the matrix of $T(z + \lambda t)$ has the form
\[ 1 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots . \]

Since \( F \) is infinite we can choose \( \lambda \neq 0 \) so that \( |I_m + \lambda A_m| \neq 0 \). For this value of \( \lambda \) we have

\[ \rho(z + \lambda t) = k, \quad \rho(T(z + \lambda t)) \geq m > k \]

which contradicts our assumption. This proves the lemma.

**Lemma 2.** Let \( k \) be a positive integer such that \( z \in X \otimes Y \) and \( \rho(z) \leq k \) imply \( \rho(Tz) \leq k \). If \( T \) is surjective and \( k < \dim U, k < \dim V \) then \( \rho(z) \geq \rho(Tz) \) for all \( z \).

**Proof.** Assume that for some \( z \) we have \( \rho(z) < \rho(Tz) \). Clearly, we can assume in addition that \( \rho(z) = 1 \). Therefore \( k > 1 \). By assumption \( \rho(z) \leq k \) implies that \( \rho(Tz) \leq k \). Let \( s \leq k \) be the maximal integer such that there exists \( z \in X \otimes Y \) satisfying \( \rho(z) < s \) and \( \rho(Tz) = s \).

Let

\[ Tz = \sum_{i=1}^s u_i \otimes v_i. \]

We can choose \( u_{s+1} \in U, v_{s+1} \in V \) such that \( u_{s+1} \in \langle u_1, \ldots, u_s \rangle \) and \( v_{s+1} \in \langle v_1, \ldots, v_s \rangle \). Since \( u_i \in U \) are linearly independent and \( v_i \in V \) also linearly independent we can assume that these vectors are contained in a basis of \( U \) and \( V \), respectively. Since \( T \) is surjective there exists \( t \in X \otimes Y \) such that \( \rho(t) = 1 \) and the \((s + 1, s + 1)\)-coordinate \( a_{s+1,s+1} \) of \( Tt \) is nonzero. The minor of order \( s + 1 \) in the upper left corner of the matrix of \( T(z + \lambda t) \) has the form

\[ a_{s+1,s+1} \lambda + \alpha_2 \lambda^2 + \cdots. \]

Since \( a_{s+1,s+1} \neq 0 \) we can choose \( \lambda \neq 0 \) so that the minor is nonzero. For this value of \( \lambda \) we have

\[ \rho(z + \lambda t) \leq \rho(z) + 1 \leq s \leq k, \]

\[ \rho(T(z + \lambda t)) \geq s + 1. \]

If \( s = k \) this contradicts our assumption. If \( s < k \) this contradicts the maximality of \( s \). Hence, Lemma 2 is proved.

**Lemma 3.** Let \( k \) be a positive integer such that \( z \in X \otimes Y \) and \( \rho(z) = k \) imply that \( \rho(Tz) = k \). If \( T \) is surjective and \( k < \dim U, k < \dim V \) then \( \rho(z) = \rho(Tz) \) for each \( z \in X \otimes Y \) satisfying \( \rho(z) \leq k \).

**Proof.** The assertion is trivial if \( \rho(z) = 0 \) or \( k \). Let \( 0 < \rho(z) < k \). Choose \( t \in X \otimes Y \) such that
\[ \rho(z + t) = \rho(z) + \rho(t) = k. \]

Using this and Lemmas 1 and 2 we deduce
\[
\begin{align*}
\rho(T(z + t)) &= \rho(Tz + Tt) = k, \\
\rho(Tz) + \rho(Tt) &\geq k, \\
\rho(Tz) + \rho(t) &\geq k, \\
\rho(Tz) &\geq \rho(z).
\end{align*}
\]

Since by Lemma 2, \( \rho(Tz) \leq \rho(z) \) we are ready.

The following Theorem is an immediate consequence of Lemma 3 and Theorem 3.4 of [3]:

**Theorem 1.** Let \( k \) be a positive integer such that \( z \in X \otimes Y \) and \( \rho(z) = k \) imply that \( \rho(Tz) = k \). If \( T \) is surjective and \( k < \dim U \), \( k < \dim V \) then
\[
\begin{align*}
(1) & \quad T = A \otimes B, \\
(2) & \quad T = S \circ (C \otimes D),
\end{align*}
\]
where
\[
A : X \to U, \quad B : Y \to V, \\
C : X \to V, \quad D : Y \to U,
\]
are bijective linear transformations and \( S \) is the canonical isomorphism of \( V \otimes U \) onto \( U \otimes V \).

This theorem gives a partial answer to a conjecture of Marcus and Moyls [2].

From Lemma 2 and Theorem 3.4 of [3] we get the following variant:

**Theorem 2.** Let \( k \) be a positive integer such that \( z \in X \otimes Y \) and \( \rho(z) \leq k \) imply that \( \rho(Tz) \leq k \). If \( T \) is bijective and \( k < \dim U \), \( k < \dim V \) then (1) or (2) holds.

When \( X = Y = U = V \), \( \dim X = n \), \( k = n - 1 \) we get a result of Dieudonné [1].

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