Pacific Journal of Mathematics

# LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS PRESERVING A FIXED RANK

DRAGOMIR Z. DJOKOVIC

Vol. 30, No. 2 October 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 2, 1969

# LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS PRESERVING A FIXED RANK

D. Ž. DJOKOVIĆ

**In this paper $T$ is a linear transformation from a tensor product $X \otimes Y$ into $U \otimes V$, where $X, Y, U, V$ are vector spaces over an infinite field $F$. The main result gives a characterization of surjective transformations $T$ for which there is a positive integer $k$ ($k < \dim U$, $k < \dim V$) such that whenever $z \in X \otimes Y$ has rank $k$ then also $Tz \in U \otimes V$ has rank $k$. It is shown that $T = A \otimes B$ or $T = S \circ (C \otimes D)$ where $A, B, C, D$ are appropriate linear isomorphisms and $S$ is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.**

Let $F$ be an infinite field and $X$, $Y$, $U$, $V$ vector spaces over $F$. We denote by $T$ a linear transformation of the tensor product $X \otimes Y$ into $U \otimes V$. The rank of a tensor $z \in X \otimes Y$ is denoted by $\rho(z)$. By definition $\rho(o) = 0$. The subspace of $X$ spaned by the vectors $x_1, \cdots, x_n \in X$ will be denoted by $< x_1, \cdots, x_n >$.

LEMMA 1. *Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. Then $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$ for all $z$.*

*Proof.* If this is not true then for some $z \in X \otimes Y, z \neq 0$, we have $\rho(z) < k$ and $\rho(Tz) > k$. There exists $t \in X \otimes Y$ such that $\rho(t) + \rho(z) = k$ and moreover $\rho(z + \lambda t) = k$ for all $\lambda \neq 0, \lambda \in F$. Let

$$Tz = \sum_{i=1}^{m} u_i \otimes v_i \, , \qquad m = \rho(Tz) \, .$$

Since $u_i \in U$ are linearly independent and also $v_i \in V$ we can consider them as contained in a basis of $U$ and $V$, respectively. The matrix of coordinates of $Tz$ has the form

$$\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$$

where $I_m$ is the identity $m \times m$ matrix. Let

$$\begin{pmatrix} A_m & B \\ C & D \end{pmatrix}$$

be the matrix of coordinates of $Tt$. Then the minor $| I_m + \lambda A_m |$ of the matrix of $T(z + \lambda t)$ has the form

$$1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots .$$

Since $F$ is infinite we can choose $\lambda \neq 0$ so that $|I_m + \lambda A_m| \neq 0$. For this value of $\lambda$ we have

$$\rho(z + \lambda t) = k , \qquad \rho(T(z + \lambda t)) \geqq m > k$$

which contradicts our assumption. This proves the lemma.

LEMMA 2. *Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply $\rho(Tz) \leqq k$. If $T$ is surjective and $k <$ dim $U$, $k <$ dim $V$ then $\rho(z) \geqq \rho(Tz)$ for all $z$.*

*Proof.* Assume that for some $z$ we have $\rho(z) < \rho(Tz)$. Clearly, we can assume in addition that $\rho(z) = 1$. Therefore $k > 1$. By assumption $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$. Let $s \leqq k$ be the maximal integer such that there exists $z \in X \otimes Y$ satisfying $\rho(z) < s$ and $\rho(Tz) = s$. Let

$$Tz = \sum_{i=1}^{s} u_i \otimes v_i .$$

We can choose $u_{s+1} \in U, v_{s+1} \in V$ such that $u_{s+1} \notin \langle u_1, \cdots, u_s \rangle$ and $v_{s+1} \notin \langle v_1, \cdots, v_s \rangle$. Since $u_i \in U$ are linearly independent and $v_i \in V$ also linearly independent we can assume that these vectors are contained in a basis of $U$ and $V$, respectively. Since $T$ is surjective there exists $t \in X \otimes Y$ such that $\rho(t) = 1$ and the $(s + 1, s + 1)$-coordinate $a_{s+1, s+1}$ of $Tt$ is nonzero. The minor of order $s + 1$ in the upper left corner of the matrix of $T(z + \lambda t)$ has the form

$$a_{s+1,s+1}\lambda + \alpha_2\lambda^2 + \cdots .$$

Since $a_{s+1,s+1} \neq 0$ we can choose $\lambda \neq 0$ so that the minor is nonzero. For this value of $\lambda$ we have

$$\rho(z + \lambda t) \leqq \rho(z) + 1 \leqq s \leqq k ,$$
$$\rho(T(z + \lambda t)) \geqq s + 1 .$$

If $s = k$ this contradicts our assumption. If $s < k$ this contradicts the maximality of $s$. Hence, Lemma 2 is proved.

LEMMA 3. *Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If $T$ is surjective and $k <$ dim $U$, $k <$ dim $V$ then $\rho(z) = \rho(Tz)$ for each $z \in X \otimes Y$ satisfying $\rho(z) \leqq k$.*

*Proof.* The assertion is trivial if $\rho(z) = 0$ or $k$. Let $0 < \rho(z) < k$. Choose $t \in X \otimes Y$ such that

$$\rho(z + t) = \rho(z) + \rho(t) = k .$$

Using this and Lemmas 1 and 2 we deduce

$$\begin{aligned} &\rho(T(z + t)) = \rho(Tz + Tt) = k , \\ &\rho(Tz) + \rho(Tt) \geqq k , \\ &\rho(Tz) + \rho(t) \geqq k , \\ &\rho(Tz) \geqq \rho(z) . \end{aligned}$$

Since by Lemma 2, $\rho(Tz) \leqq \rho(z)$ we are ready.

The following Theorem is an immediate consequence of Lemma 3 and Theorem 3.4 of [3]:

THEOREM 1. *Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. If $T$ is surjective and $k <$ dim $U$, $k <$ dim $V$ then*

$$T = A \otimes B , \tag{1}$$

*or*

$$T = S \circ (C \otimes D) , \tag{2}$$

*where*

$$\begin{aligned} A : X \to U , \qquad & B : Y \to V , \\ C : X \to V , \qquad & D : Y \to U , \end{aligned}$$

*are bijective linear transformations and $S$ is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.*

This theorem gives a partial answer to a conjecture of Marcus and Moyls [2].

From Lemma 2 and Theorem 3.4 of [3] we get the following variant:

THEOREM 2. *Let $k$ be a positive integer such that $z \in X \otimes Y$ and $\rho(z) \leqq k$ imply that $\rho(Tz) \leqq k$. If $T$ is bijective and $k <$ dim $U$, $k <$ dim $V$ then* (1) *or* (2) *holds.*

When $X = Y = U = V$, dim $X = n$, $k = n - 1$ we get a result of Dieudonné [1].

## REFERENCES

1. J. Dieudonné, *Sur une généralisation du groupe orthogonale à quatre variables*, Archiv der Math. **1** (1948), 282-287.

2. M. Marcus and B. N. Moyls, *Transformations on tensor product spaces,* Pacific J. Math. **9** (1959), 1215-1221.
3. R. Westwick, *Transformations on tensor spaces,* Pacific J. Math. **23** (1967), 613-620.

Received August 21, 1968. This work was supported in part by N. R. C. Grant A-5285.

UNIVERSITY OF WATERLOO

# PACIFIC JOURNAL OF MATHEMATICS





Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

# Pacific Journal of Mathematics

**Vol. 30, No. 2 October, 1969**