

Pacific Journal of Mathematics

**A SUBCOLLECTION OF ALGEBRAS IN A COLLECTION OF
BANACH SPACES**

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Let $D(p, r)$ with $1 \leq p < \infty$ and $-\infty < r < +\infty$ denote the Banach space consisting of certain analytic functions $f(z)$ defined in the unit disk. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a member of $D(p, r)$ if and only if

$$\sum_{n=0}^{\infty} (n+1)^r |a_n|^p < \infty.$$

We define the norm of f in $D(p, r)$ by

$$\|f\|_{p,r} = \left(\sum_{n=0}^{\infty} (n+1)^r |a_n|^p \right)^{1/p}.$$

By the product of two functions f and g in $D(p, r)$ we shall mean their product as functions, i.e., $[f \cdot g](z) = f(z)g(z)$. The purpose of this paper is to discover which of the spaces $D(p, r)$ are algebras.

THEOREM 1. *If $D(p, r)$ is an algebra, then there exists a real $c > 0$ with $\|fg\| \leq c \|f\| \|g\|$ for every $f, g \in D(p, r)$.*

Proof. Let h be a fixed element of $D(p, r)$. It suffices to show the map $f \rightarrow hf$ is a bounded linear transformation from $D(p, r)$ to itself. The proof is based on the closed graph theorem [2, p. 306]. Suppose h is a multiplier from $D(p_1, r_1)$ to $D(p_2, r_2)$ and suppose

- (i) $f_n \rightarrow f$ in $D(p_1, r_1)$ and
- (ii) $hf_n \rightarrow g$ in $D(p_2, r_2)$.

Then $f_n(z) \rightarrow f(z)$ for each z in the unit disk and so $h(z)f_n(z) \rightarrow h(z)f(z)$. On the other hand by (ii), $h(z)f_n(z) \rightarrow g(z)$ for each z in the unit disk. Hence $g = hf$, and so by the closed graph theorem multiplication by h is a continuous linear transformation. It follows from this [2, p. 183] that $D(p, r)$ is equivalent to a Banach algebra, and from this the theorem follows immediately.

COROLLARY 1. *If $D(p, r)$ is an algebra and $c > 0$ as above, then $|f(z)| \leq c \|f\| \forall f \in D(p, r)$ and $|z| < 1$.*

Proof. For each f in $D(p, r)$ let T_f denote the multiplication operator from $D(p, r)$ to itself determined by f , i.e., $T_f(g) = fg$. Then for z_0 satisfying $|z_0| < 1$ the map $T_f \rightarrow f(z_0)$ is a multiplicative linear functional on the Banach algebra of multiplication operators

$$T_f, f \in D(p, r)$$

with the usual norm. Hence

$$|f(z_0)| \leq \|T_f\| = \sup_{\|g\|=1} \|fg\| \leq c \|f\|, g \in D(p, r).$$

THEOREM 2. *If $p = 1$, then $D(p, r)$ is no algebra for $r < 0$. And if $1 < p < \infty$, then $D(p, r)$ is no algebra for $r \leq p - 1$.*

Proof. The function $f(z) = \sum_{n=0}^{\infty} [1/(n + 1)]z^n$ is an unbounded function on $|z| < 1$ but lies in $D(1, r)$ if $r \leq 0$. And similarly the function $f(z) = \sum_{n=0}^{\infty} 1/[(n + 1) \log(n + 1)]z^n$ is an unbounded function on $|z| < 1$ in $D(p, r)$ if $p > 1$ and $r \leq p - 1$. Therefore by Corollary 1 the spaces are not algebras.

THEOREM 3. *If $p = 1$, then $D(p, r)$ is an algebra for $r \geq 0$, and if $1 < p < \infty$ then $D(p, r)$ is an algebra for $r > p - 1$.*

Proof. (i) Suppose first $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ lie in $D(1, r)$ with $r \geq 0$. We will show $fg \in D(1, r)$

$$\begin{aligned} \|fg\| &= \sum_{n=0}^{\infty} (n + 1)^r \left| \sum_{k=0}^n a_k b_{n-k} \right| \leq \sum_{n=0}^{\infty} (n + 1)^r \sum_{k=0}^n |a_k| |b_{n-k}| \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (n + 1)^r |a_k| |b_{n-k}| \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j + k + 1)^r |a_k| |b_j| \quad \text{where } j = n - k \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j + k + 1)^r / [(k + 1)^r (j + 1)^r] (k + 1)^r |a_k| (j + 1)^r |b_j| \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} [(j + k + 1)/(jk + j + k + 1)]^r (k + 1)^r |a_k| (j + 1)^r |b_j| \\ &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k + 1)^r |a_k| (j + 1)^r |b_j| \\ &= \|f\| \|g\|. \end{aligned}$$

(ii) Now suppose $r > p - 1$, and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be two elements of $D(p, r)$. We will show there is a constant K such that $\|fg\| \leq K \|f\| \|g\|$. Define q by the equation $1/p + 1/q = 1$.

$$\begin{aligned} \|fg\|^p &= \sum_{n=0}^{\infty} (n + 1)^r \left| \sum_{k=0}^n a_k b_{n-k} \right|^p \leq \sum_{n=0}^{\infty} (n + 1)^r \\ &\quad \left\{ \sum_{k=0}^n 1/[(k + 1)^{r/p} (n - k + 1)^{r/p}] (k + 1)^{r/p} |a_k| (n - k + 1)^{r/p} |b_{n-k}| \right\}^p. \end{aligned}$$

Applying Holder's inequality we get

$$\begin{aligned} & \|fg\|^p \\ & \leq \sum_{n=0}^{\infty} (n+1)^r \left\{ \left(\sum_{k=0}^n [1/\{(k+1)^{r/p}(n-k+1)^{r/p}\}]^q \right)^{1/q} \right. \\ & \qquad \left. \left(\sum_{k=0}^n (k+1)^r |a_k|^p (n-k+1)^r |b_{n-k}|^p \right)^{1/p} \right\}^p \\ & = \sum_{n=0}^{\infty} [C_n] \sum_{k=0}^n (k+1)^r |a_k|^p (n-k+1)^r |b_{n-k}|^p \\ & \leq \sup_n [C_n] \|f\|^p \|g\|^p \end{aligned}$$

where

$$C_n = (n+1)^r \left(\sum_{k=0}^n \{1/[k+1)^{r/p}(n-k+1)^{r/p}\}]^q \right)^{p/q} .$$

We complete the proof of the theorem by showing

$$\begin{aligned} & \sup_n [C_n] < \infty . \\ C_n & = (n+1)^r \left(\sum_{k=0}^n \{1/[(k+1)^{r/p}(n-k+1)^{r/p}]\}^q \right)^{p/q} \\ & = (n+1)^r \left(\sum_{k=0}^n 1/(n+2)^{rq/p} \{1/(k+1) + 1/(n-k+1)\}^{rq/p} \right)^{p/q} \\ & = [(n+1)/(n+2)]^r \left[\sum_{k=0}^n \{1/(k+1) + 1/(n-k+1)\}^{rq/p} \right]^{p/q} \\ & \leq \left[\sum_{k=0}^n \{2/(k+1)\}^{rq/p} \right]^{p/q} \\ & \leq 2^r \left[\sum_{k=0}^{\infty} 1/(k+1)^{rq/p} \right]^{p/q} \end{aligned}$$

since

$$rq/p = r/(p-1) > 1 .$$

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