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TWISTED COHOMOLOGY AND ENUMERATION OF VECTOR BUNDLES

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In the present paper we give a technique for completely enumerating real 4-plane bundles over a 4-dimensional space, real 5-plane bundles over a 5-dimensional space, and real 6plane bundles over a 6-dimensional space. We give a complete table of real and complex vector bundles over real projective space P_k , for $k \leq 5$. Some interesting results are:

(0.1.1.) Over P_5 , there are four oriented 4-plane bundles which could be the normal bundle to an immersion of P^5 in R^9 , i.e., have stable class 2h + 2, where h is the canonical line bundle. Of these, two have a unique complex structure.

(0.1.2.) Over P_5 there is an oriented 4-plane bundle which we call C, which has stable class 6h - 2, which has two distinct complex structures. D, the conjugate of C, i.e., reversed orientation, has no complex structure.

(0.1.3) Over P_5 , there are no 4-plane bundles of stable class 5h-1 or 7h-3.

0.2. In reading the tables (4.5.2) and (4.6), remember that if ξ : $P_k \rightarrow BO(n)$ or $\xi: P_k \rightarrow BU(n)$ is a locally oriented (i.e., oriented over base-point) real or complex vector bundle, and if

$$a \in H^k(P_k; \pi_k(BO(n), \xi))$$

(local coefficients if ξ unoriented) or $a \in H^k(P_k; \pi_k(BU(n)))$, then $\xi + a$ is a vector bundle obtained by cutting out a disk in the top cell of P_k and joining a sphere with some vector bundle on it.

0.3. Since some of the homotopy groups of BO(n) are acted upon nontrivially by $Z_2 \cong \pi_1(BO(n))$ for *n* even, we study cohomology with local coefficients in § 3.

1.2. From here on, we assume that all spaces are connected C. W.-complexes with base-point, all maps are b.p.p. (base-point-preserving) and that all homotopies are b.p.p.

For any space Y, we choose a Postnikov system for Y, that is: for each integer $n \ge 0$, a space $(Y)_n$ and a map $P_n: Y \to (Y)_n$ which induces an isomorphism in homotopy through dimension n, where all homotopy groups of $(Y)_n$ are zero above n; for each $n \ge 1$ a fibration $p_n: (Y)_n \to (Y)_{n-1}$ such that $p_n P_n = P_{n-1}$. The fiber of each p_n is then an Eilenberg-MacLane space of type $(\pi_n(Y), n)$. If X is a space of finite dimension m, then [X; Y], the set of homotopy classes of maps from X to Y, is in one-to-one correspondence with $[X; (Y)_m]$.

DEFINITION (1.2.1). For any integer $n \ge 1$, let $G_n(Y)$ be the sheaf over $(Y)_1$ whose stalk over every y is defined to be $\pi_n(p^{-1}y)$, which is isomorphic to $\pi_n(Y)$ (where $p = p_2 \cdots p_n: (Y)_n \to (Y)_1$) if $n \ge 2$; $\pi_1((Y)_1, y)$ if n = 1. If X is any space and $f: X \to (Y)_1$ is a map, let $\pi_n(Y, f)$ be the sheaf $f^{-1}G_n(Y)$ over X. This sheaf depends only on the homotopy class of f. If $g: X \to (Y)_m$ is a map for any integer $m \ge 1$, or if $h: X \to Y$ is a map, let $\pi_n(Y, g)$ denote $\pi_n(Y, p_2 \cdots p_m g)$ and let $\pi_n(Y, n)$ denote $\pi_n(Y, P_1h)$.

DEFINITION (1.2.2). If f and g are maps from X to $(Y)_n$ for any $n \ge 2$, which agree on A, and if $F: X \times I \to (Y)_{n-1}$ is a homotopy of $p_n f$ with $p_n g$ which holds A fixed, let $\delta^n(f, g; F) \in H^n(X, A; \pi_n(Y, f))$ be the obstruction to lifting F to a homotopy of f with g which holds A fixed.

REMARK (1.2.3). If $g: X \to (Y)_n$ is another map which agrees with f on A, and if G is a homotopy of $p_n g$ with $p_n h$ which holds A fixed, then $\delta^n(f, g; F) + \delta^n(g, h; G) = \delta^n(f, h; F + G)$, where, for each $(x, t) \in X \times I$,

$$(F+G)(x,t) = egin{cases} F(x,2t) & ext{if} & 0 \leq t \leq rac{1}{2} \ G(x,2t-1) & ext{if} & rac{1}{2} \leq t \leq 1 \ . \end{cases}$$

DEFINITION (1.2.4). Let X be a space, let $A \subset X$ be any subcomplex (possible empty), let $f: X \to (Y)_n$ be a map for some integer $n \ge 2$, and let a be an element of $H^n(X, A; \pi_n(Y, f))$. We define f + a to be that map from X to $(Y)_n$, unique up to fiber homotopy with A held fixed, such that $p_n(f + a) = p_n f$ and $\partial^n(f, f + a) = a$, where C is the constant homotopy.

REMARK (1.2.5). If b is any other element of $H^n(X, A; \pi_n(Y, f))$, then f + (a + b) = (f + a) + b.

REMARK (1.2.6). If $g: (X', A') \rightarrow (X, A)$ is a map, where (X'A') is any other C. W. pair, then $(f + a)g = gf + g^*a$.

MAIN THEOREM (1.2.7). For any $a \in H^n(X, A; \pi_n(Y, f)), f + a$ is homotophic to f, rel A, if and only if $\delta^n(f, f; F) = a$ for some homotopy F of $p_n f$ with itself which holds A fixed.

Proof. Let C be the constant homotopy of $p_n f$ with itself. On the one hand, if F is any homotopy of $p_n f$ with itself which holds

A fixed, let $a = \delta^n(f, f; F)$. Then $\delta^n(f + a, f; F) = \delta^n(f + a, f; C) + \delta^n(f, f; F) = -a + a = 0$. Thus F may be lifted to a homotopy of f + a with f. On the other hand, if G is a homotopy of f + a with f, then $\delta^n(f, f; p_n G) = \delta^n(f, f + a; C) + \delta^n(f + a, f; p_n G) = a + 0 = a$.

DEFINITION (1.2.8). Let L_f be the subgroup of $H^n(X, A; \pi_n(Y, f))$ consisting of all a such that f + a is homotopic to f rel A. Then the set of all homotopy (rel A) classes of liftings of $p_n f$ to $(Y)_n$ which agree with f on A is in a one-to-one correspondence with the quotient group $H^n(X, A; \pi_n(Y, f))/L_f$; each coset $a + L_f$ corresponds to f + a. If $g: X \to Y$ is a map such that $p_n g = f$, let $L_g^n = L_f$. If $h: X \to (Y)_m$ is a map such that $p_{n+1} \cdots p_m h = f$, for $m \ge n$, let $L_h^n = L_f$.

REMARK (1.2.9). If $a \in H^n(X, A; \pi_n(Y, f))$, then $L_{f+a} = L_f$.

Proof. Let F be any homotopy of $p_n f = p_n(f+a)$ with itself, and let C be the constant homotopy. Then $\delta^n(f+a, f+a; F) = \delta^n(f+a, f; C) + \delta^n(f, f; F) + \delta^n(f, f+a; C) = -a + \delta^n(f, f; F) + a = \delta^n(f, f; F)$.

1.3. In order to calculate L_f in specific cases, such as X a projective space, A = base-point, and Y = BO(m) for some m, we use a spectral sequence which has the following properties:

(1.3.1) ${}^{f}E_{2}^{p,q} = E_{2}^{p,q} = H^{p}(X, A; \pi_{q}(Y, f)) \text{ if } 2 \leq q \leq n, 1 \leq p \leq q+1.$

(1.3.2) $E_2^{p,q} = 0$ for all other values of p and q.

(1.3.3) $d_r: E_r^{p,q} \to E_r^{p+r,q+r-1}$ for all $r \ge 2$.

(1.3.4) $E_{\infty}^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, which, by (1.2.7) and (1.2.8) can be put into one-to-one correspondence with the set of rel A homotopy classes of maps $X \to (Y)_n$ whose projection to $(Y)_{n-1}$ is rel A homotopic to $p_n f$.

Basically, what is happening is as follows (where, for any space Z and any map $g: A \to Z$, the set of rel A homotopy classes of maps $X \to Z$ which agree with g on A is denoted "[X; Z; g]"); consider the function:

$$[X; (Y)_n: f \mid A] \xrightarrow{(p_n)_{\ddagger}} [X; (Y)_{n-1}: p_n f \mid A] .$$

Now $(p_n)_{\sharp}$ is just a function of sets, but $(p_n)_{\sharp}^{-1}(p_n f)$ is an Abelian group with 0 the homotopy class of f itself. This group, $E_{\infty}^{n,n}$ of our spectral sequence, depends on the choice of f.

We define our spectral sequence via an exact couple:



where $E_2^{p,q}$ is as defined in (1.3.1) and (1.3.2), where i_2, j_2 , and k_2 have bi-degrees (-1, -1), (2, 1), and (0, 0) respectively; and where (for all $t \leq n, M_t$ = space of maps from X to $(Y)_t$ which agree with $p_t^n f$ on A, compact-open topology):

 $(1.3.5) \quad D_{2}^{p,q} = \pi_{q-p}(M_{q}, p_{q}^{n}f) \text{ if } 0 \leq q \leq n, \text{ and } p \leq q.$

(1.3.6) $D_2^{p,q} = 0$ if q < p or q < 0.

(1.3.7) $D_2^{p,q} = D_2^{p-1,q-1}$ if q > n.

Note that $D_2^{p,q}$ is only a group if q = p + 1 and only a set if q = p. This will not affect our computation, however.

We proceed to define the homomorphisms i_2 , j_2 and k_2 .

(1.3.8) If q > n, let i_2 be the identity. If $q \leq n$, let $i_2 = (p_q)_*$.

(1.3.9) If $p \leq q$ and $0 \leq q < n$, any $x \in D_2^{p,q}$ represents a map $g: X \times I^{q-p} \to (Y)_q$, where $g(x, v) = p_q^n f(x)$ for all $(x, v) \in X \times \partial I^{q-p} \cup A \times I^{q-p}$. Let $j_2(x) = (s^{q-p})^{-1}\gamma^{q+2}(g)$, where $s^{q-p}: H^{p+2}(X, A; \pi_{q+1}(Y, f)) \to H^{q+2}(X \times I^{q-p}, X \times \partial I^{q-p} \cup A \times I^{q-p}; \pi_{q+1}(Y, g))$ is the (q-p)-fold suspension and $\gamma^{q+2}(g)$ is the obstruction to finding a lifting $h: X \times I^{q-p} \to (Y)_{q+1}$ of g such that $h(x, v) = p_{q+1}^n f(x)$ for all $(x, v) \in X \times \partial I^{q+p} \cup A \times I^{q-p}$. (If p > q or q < 0 or $q \geq n, j_2: D_2^{p,q} \to E_2^{p+2,q+1}$ is obviously the zero map, since $E_2^{p+2,q+1} = 0$.) This obstruction is zero if and only if g can be lifted; it follows immediately that:

(1.3.10) The sequence $D_2^{p+1,q+1} \xrightarrow{i_2} D_2^{p,q} \xrightarrow{j_2} E^{p+2,q+1}$ is exact.

Furthermore, since every homotopy, rel A, of $p_n f$ with itself represents a loop in M_{n-1} :

(1.3.11) L_f is the image of $j_2: D_2^{n-2,n-1} \rightarrow E_2^{n,n}$. For any $2 \leq q \leq n$, $1 \leq p \leq q$, and any $a \in E_2^{p,q}$, let

$$b=s^{q-p}a\in H^q(X imes I^{q-p},\,X imes \partial I^{q-p}\cup A imes I^{q-p};\pi_q(Y,C))$$
 ,

where $C(x, v) = p_q^n f(x)$ for every $(x, v) \in X \times I^{q-p}$. Let $k_2(a) \in D_2^{p,q}$ be that element represented by the map C + b (cf. 1.2.2). It follows from (1.2.3) that k_2 is a homomorphism if p < q; if p = q then $D_2^{p,q}$ is only a set anyway. (For other values of p and q, $k_2 = 0$.) Since $p_q(C+b) = p_q C$, and C represents $0 \in D_2^{p,q}$:

(1.3.12) Im $k_2 \subset \text{Ker } i_2$.

If, on the other hand, a map $g: X \times I^{q-p} \to (Y)_q$ such that g = Con $X \times \partial I^{q-p} \cup A \times I^{q-p}$ is a representative of a given $a \in \text{Ker } i_2$, then $p_q g$ is homotopic, rel $X \times \partial I^{q-p} \cup A \times I$, to $p_q C$ via a homotopy F, then $a = k_2((s^{q-p})^{-1} \partial^q(C, g; F))$. Thus:

(1.3.13) Ker $i_2 \subset \text{Im } k_2$.

Somewhat more difficult to show is:

(1.3.14) Ker $k_2 = \text{Im } j_2$ if $p \leq q$.

Proof. Let $2 \leq q \leq n, 1 \leq p \leq q$. Let $g(x, v) = p_q^n f(x) \in (Y)_q$ for all $(x, v) \in X \times I^{q-p}$; g represents $0 \in D_2^{p,q}$. Let $b \in E_2^{p,q}$. Then $b \in \operatorname{Ker} k_2$

if and only if $s^{q-p}b \in L_q$ (cf. 1.2.7). If $b = j_2 a$, then a represents F, a homotopy, rel $X \times \partial I^{q-p} \cup A \times I^{q-p}$ of $p_q q$ with itself, and $s^{q-p}b = \delta^q(g, g; F) \in L_q$. If, on the other hand, $s^{q-p}b \in L_q$, then $s^{q-p}b = \delta^q(g, g; F)$ for some homotopy F, rel $X \times \partial I^{q-p} \cup A \times I^{q-p}$, of $p_q g$ with itself; let $a = [F] \in D^{p-2,q-1}$, and $j_2 a = b$.

1.4. Since only finitely many of the E_2 terms are nonzero, we obtain E_{∞} after a finite number of steps. We also have, by straightforward algebra, an exact sequence

$$0 \longrightarrow E_{\infty} \xrightarrow{k_{\infty}} D_{\infty} \xrightarrow{i_{\infty}} D_{\infty} \longrightarrow 0 .$$

Consider now the commutative diagram with exact columns:

A typical element of $D_2^{n-2,n-1}$ is a rel $X \times \partial I \cup A \times I$ homotopy class of homotopies of $p_n f$ with itself; if F is such a homotopy, $j_2[F] = \delta^n(f, f; F)$, by (1.3.9). If $x \in H^n(X, A; \pi_n(Y, f)), k_2x = f + x$, by (1.3.11). Thus $\text{Im } j_2 = L_f$, and $E_{\infty}^{n,n} = H^n(X, A; \pi_n(Y, f))/L_f$, the set of rel A homotopy classes of liftings of $p_n f$.

1.5. If $g: (X', A') \rightarrow (X, A)$ is a map, g induces a map of spectral sequences.

(1.5.1) $g^*: {}^{f}E_r^{p,q} \to {}^{fg}E_r^{p,q}$ for all p, q, r. If $h: Y \to Z$ is a map, where Z is any other space, h determines a map $h_m: (Y)_m \to (Z)_m$ for each $m \ge 0$ [1]. Then $h_{\sharp}: \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ induces a sheaf homomorphism from $G_n(Y)$ to $(h_1)^{-1}G_n(Z)$ which in turn induces a homomorphism.

(1.5.2) $h_*: H^*(X, A; \pi_m(Y, f)) \to H^*(X, A; \pi_m(Z, hf))$ for all $m \ge 0$ and a map of spectral sequences

(1.5.3) $h_*: {}^{r}E_r^{p,q} \rightarrow {}^{hf}E_r^{p,q}$ for all p, q, r.

2. Nonbase-point-preserving homotopies.

2.1. Using the techniques of $\S1$, we can compute all b.p.p.

homotopy classes of maps from a finite-dimensional space X to a space Y. What if we want to know, instead, all free homotopy classes of maps?

2.2. Let $f: X \to Y$ be any b.p.p. map, and let $a \in \pi_1(Y, y_0)$. By the homotopy extension property, we can find a free homotopy F: $X \times I \to Y$ of f such that $F | \{x_0\} \times I$ represents a. Let $f^a(x) = F(x, 1)$ for any $x \in X$; f^a is unique up to b.p.p. homotopy, and $f^{ab}(f^a)^b$ for any other $b \in \pi_1(Y, y_0)$.

THEOREM (2.2.1). If f and g are any b.p.p. maps from X to Y, then f is freely homotopic to g if and only if f^a is b.p.p. homotopic to g for some $a \in \pi_1(Y, y_0)$.

Proof. If f^a is b.p.p. homotopic to g, then f is obviously freely homotopic to g since f is freely homotopic to f^a . If, on the other hand, $F: X \times I \to Y$ is a free homotopy of f with g, let a be that element of $\pi_1(Y, y_0)$ represented by the loop $F | \{x_0\} \times I$. Then $f^a = g$ (up to b.p.p. homotopy).

THEOREM (2.2.2). If $n \ge 2, f: X \rightarrow (Y)_n$ is a map,

 $a\in H^{\scriptscriptstyle n}(X,\,x_{\scriptscriptstyle 0};\,\pi_{\scriptscriptstyle n}(\,Y,\,f\,))$,

and $b \in \pi_1(Y, y_0)$, then $(f + a)^b = f^b + 1^b_*(a)$, where 1^b_* is the homomorphism induced by the map 1^b (cf. 1.5.2), where 1 is the identity map on $(Y)_n$.

Proof. The theorem follows from naturality of obstruction theory.

3. Sheaves of local coefficients.

3.1. The homotopy groups of BO(n) are sometimes acted on nontrivially by π_1 . We must therefore study twisted sheaves.

DEFINITION (3.1.1). A twisted group is an ordered pair (G, T), Gan Abelian group, $T: G \to G$ an automorphism of order 2. If X is a space, a (G, T)-sheaf over X is a fiber bundle over X with fiber G and structural group Z_2 , action determined by T. Let $G^T[u]$ be the (G, T)-sheaf over P_{∞} obtained by identifying (x, g) with (Tx, Tg) for all $(x, g) \in S^{\infty} \times G$, where $T: S^{\infty} \to S^{\infty}$ is the antipodal map.

DEFINITION (3.1.2). If $a \in H^1(X, x_0; Z_2)$ and $f: (X, x_0) \to (P_{\infty}, *)$ is a map where $f^*u = a$ (u = fundamental class of P_{∞}), let $G^{T}[a] = f^{-1}G^{T}[u]$. We call a the twisting class of $G^{T}[a]$. PROPOSITION (3.1.3). $G^{T}[u]$ is universal in the sense of Steenrod [6], that is, if G is a (G, T)-sheaf over a space X, $G \cong G^{T}[a]$ for some unique $a \in H^{1}(X, x_{0}; Z_{2})$.

Proof. $P_{\infty} = BZ_2$.

REMARK (3.1.4). If $F: X \times I \to P_{\infty}$ is a *free* homotopy of f with itself, where $f^*u = a$, then F induces an automorphism of $G^{r}[a]$; 1 or T depending on whether $F | \{x_0\} \times I$ is a trivial loop in P_{∞} or not.

3.2. If X is a space, $B \subset A \subset X$ are closed, and S is a sheaf over X, we have a long exact sequence:

$$\cdots \longrightarrow H^{n}(X, A; S) \longrightarrow H^{n}(X, B; S) \longrightarrow H^{n}(A, B; S)$$
$$\xrightarrow{\delta} H^{n+1}(X, A; S) \longrightarrow \cdots$$

PROPOSITION (3.2.1). If S is a sheaf over a space X, and $A \subset X$ is closed, we may find an isomorphism

$$s: H^*(X, A; S) \longrightarrow H^*(X \times I, X \times \partial I \cup A \times I; S \times I)$$
,

called the suspension, of degree 1, where $S \times I = p^{-1}S$; $p: X \times I \rightarrow X$ being the projection.

Proof. Let S' be that subsheaf of S such that S' | A = 0 and S' | (X - A) = S | (X - A). According to Bredon [1],

$$H^*(X, A; S) = H^*(X; S')$$

and

$$H^*(X imes I, X imes \partial I \cup A imes I; S imes I) = H^*(X imes I, X imes \partial I; S' imes I).$$

Now $H^*(X \times I, X \times \{t\}; S') = 0$ for any $t \in I$ [1], and by the long exact sequence of $(X \times I, X \times \partial I, X \times \{1\})$ and excision we have an isomorphism $H^*(X \times \{0\}; S' \times I) \xrightarrow{\cong} H^*(X \times I, X \times \partial I; S' \times I)$ of degree 1; the left group is isomorphic to $H^*(X; S')$.

3.3. Let X be a space, $A \subset X$ closed. If $\alpha: S \to S'$ is a homomorphism of sheaves over X, we get a homomorphism $\alpha_*: H^*(X, A; S) \to H^*(X, A; S')$. If S and S' are sheaves over X and

$$E \colon 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

is an extension of S' by S, then E determines a long exact sequence

$$\cdots \longrightarrow H^{n}(X, A; S) \xrightarrow{i_{*}} H^{n}(X, A; S'') \xrightarrow{p_{*}} H^{n}(X, A; S'') \xrightarrow{\delta^{E}} H^{n+1}(X, A; S) \longrightarrow \cdots$$

where δ^{E} is called the Bockstein of *E*.

PROPOSITION (3.3.1). If S and S' are sheaves over X and if

$$E: 0 \longrightarrow S \xrightarrow{i} S'' \xrightarrow{p} S' \longrightarrow 0$$

and

$$F: 0 \longrightarrow S \xrightarrow{j} U \xrightarrow{q} S' \longrightarrow 0$$

are elements of $\operatorname{Ext}(S', S)$, then $\delta^{E+F} = \delta^E + \delta^F$.

Proof. We use the Baer sum construction to find

$$E + F: 0 \longrightarrow S \longrightarrow V \longrightarrow S' \longrightarrow 0;$$

our result follows from the commutative diagram, where each row is exact:

3.4. As Abelian groups $\operatorname{Ext} (Z_2, Z_2) \cong Z_2$; the nonzero extension is Z_4 . Fix a space X; we study Ext of sheaves over X.

PROPOSITION 3.4.1. As sheaves over X,

 $\mathrm{Ext}\,(Z_2,\,Z_2)\cong Z_2\,+\,H^{\scriptscriptstyle 1}(X,\,x_0;\,Z_2)$.

For any $a \in H^1(X, x_0; Z_2)$, (0, a) corresponds to the extension

$$E_a^{_0} : 0 \longrightarrow Z_2 \overset{i_1}{\longrightarrow} (Z_2 + Z_2)^{_T} [a] \overset{p_2}{\longrightarrow} Z_2 \longrightarrow 0$$
 ,

where T(x, y) = (x + y, y), $i_1(x) = (x, 0)$, and $p_2(x, y) = y$; (1, a) corresponds to

$$E^{\scriptscriptstyle 1}_a {:} 0 {\, \longrightarrow \,} Z_2 {\, \stackrel{m}{\longrightarrow}\,} Z^{\scriptscriptstyle T}_4 [a] {\, \stackrel{e}{\longrightarrow}\,} Z_2 {\, \longrightarrow \,} 0$$
 ,

where T(x) = -x for all $x \in Z_4$, m(1) = 2, and e(1) = 1.

Proof. Routine computation shows that $E_a^x + E_b^y = E_{a+b}^{x+y}$ for any $x, y \in \mathbb{Z}_2$ and $a, b \in H^1(X, x_0; \mathbb{Z}_2)$. On the other hand, suppose that

 $E: 0 \longrightarrow Z_2 \stackrel{i}{\longrightarrow} G \stackrel{p}{\longrightarrow} Z_2 \longrightarrow 0$

is some extension. Then the stalk of G at x_0 is Z_4 , in which case $G = Z_4^T[a]$ for some $a \in H^1(X, x_0; Z_2)$, or it is $Z_2 + Z_2$. In that case, we have an exact sequence of stalks at x_0 :

$$0 \longrightarrow Z_2 \stackrel{i_1}{\longrightarrow} Z_2 + Z_2 \stackrel{p_2}{\longrightarrow} Z_2 \longrightarrow 0$$
 .

Since G is locally isomorphic to $Z_2 + Z_2$, it is a fiber bundle with fiber $Z_2 + Z_2$ and structural group Aut $(Z_2 + Z_2)$. But the only nontrivial automorphism which commutes with $i_1: Z_2 \rightarrow Z_2 + Z_2$ and $p_2: Z_2 + Z_2 \rightarrow Z_2$ is T given above. So the structural group of G may be reduced to Z_2 ; $G = (Z_2 + Z_2)^T[a]$ for some $a \in H^1(X, x_0; Z_2)$. This gives us the isomorphism.

We have the following commutative diagram with both rows exact, for any $a \in H^{1}(X, x_{0}; \mathbb{Z}_{2})$:

$$\begin{array}{cccc} 0 \longrightarrow Z^{T}[a] \xrightarrow{2} Z^{T}[a] \xrightarrow{\Pi} Z_{2} \longrightarrow 0 \\ & & & \downarrow^{\Pi} & & \downarrow^{\Pi} & 1 \\ 0 \longrightarrow Z_{2} & \xrightarrow{m} Z_{4}^{T}[a] \xrightarrow{e} Z_{2} \longrightarrow 0 \end{array}$$

DEFINITION (3.4.2). Let $\beta^{T}[a]$ (or simply β^{T} , when *a* is understood) denote the Bockstein of the top row of the above diagram, and let $(S_{a}^{1})^{T}[a]$ (or $(S_{a}^{1})^{T}$) denote the Bockstein of the bottom row.

Remark (3.4.3). $\Pi_*\beta^T = (S_q^1)^T$.

PROPOSITION (3.4.4). For any $n \ge 0$ and any $x \in H^n(X, A; Z_2)$, $(S_q^1)^T x = S_q^1 x + x \cup a$.

Proof. Samelson [5].

PROPOSITION (3.4.5). For any $n \ge 0$ and any $x \in H^n(X, A; Z_2)$ $\delta(x) = x \cup a$, where δ is the Bockstein of $E_a^0: 0 \to Z_2 \to (Z_2 + Z_2)^T[a] \to Z_2 \to 0$.

Proof. The result follows immediately from (3.3.1), (3.4.1), and (3.4.4).

3.5. Let T(n,m) = (m-n,m) for any $(n,m) \in \mathbb{Z} + \mathbb{Z}$. If S and S' are sheaves over a space X, and if $\mu: S \otimes S' \to S''$ is a sheaf homomorphism, then we have a cup product defined from

 $H^*(X, A; S) \otimes H^*(X, B; S')$

to $H^*(X, A \cup B; S'')$ for any closed $A \subset X$ and $B \subset X$. We have thus

cup products generated by the following relations:

$$egin{aligned} Z^{ au}[a]\otimes Z^{ au}[b]&=Z^{ au}[a+b], Z_2\otimes (Z_2+Z_2)^{ au}[a]\ &=(Z_2+Z_2)^{ au}[a], Z\otimes (Z+Z)^{ au}[a]\ &=(Z+Z)^{ au}[a], Z^{ au}[a]\otimes (Z+Z)^{ au}[a]=(Z+Z)^{ au}[a]\ & ext{(where}\quad n\otimes (p,q)=(np,2np-nq)), Z_4^{ au}[a]\otimes Z_4^{ au}[b]=Z_4^{ au}[a+b], \end{aligned}$$

and many others.

Let
$$(X, A)$$
 be a C. W.-pair. Let $a \in H^1(X, x_0; Z_2)$ and
 $lpha = eta^{ au}[a](1) \in H^1(X; Z^{ au}[a])$.

We have the following commutative diagram; where

$$i_1x = (x, 0), T(x, y) = (y - x, y), j_1x = (x, 2x),$$

and $q_2(x, y) = y - 2x$.

PROPOSITION (3.5.1). The Bockstein homomorphisms δ_1 and δ_2 are both cup products with α .

Proof. By (3.4.3) and (3.4.4) we may compute that

$$H^1(P_{\infty};Z^{T}[u])\cong Z_2$$

and is generated by $\bar{u} = \beta^{T}(1)$.

Let $x \in H^n(X, A; Z)$. If n = 0, then the universal example is $X = P_{\infty}, A = \emptyset, x = 1$. Then $\alpha = \overline{u}$. Now $H^0(P_{\infty}; Z^T) = 0$, so $(j_1)_*$: $H^0(P_{\infty}; Z) \leftarrow H^0(P_{\infty}; (Z + Z)^T)$ is an isomorphism, and $p_2 j_1 = 2$. Thus $1 \notin \text{Im } (p_2)_*$, so $\delta_1(1) = \overline{u}$. If $n \ge 1$, the universal example is $X = K(Z, n) \times P_{\infty}, A = * \times P_{\infty}, x = v_n \times 1$. Then $\alpha = p^*\overline{u}$, where $p: X \to P_{\infty}$ is projection onto the second factor. Now routine computations using (3.4.3) and (3.4.4) show that $H^{n+1}(X, A; Z^T) \cong Z_2$ and is generated by $(v_n \times 1) \cup p^*\overline{u}$, which is mapped onto $\Pi_* v_n \times u$ under $\Pi_*: H^*(; Z^T) \to H^*(; Z_2)$. The result follows from (3.4.5).

Let $x \in H^n(X, A; Z^T)$. If n = 0, x = 0. If n = 1, the universal example is $X = K(Z^T, n), A = P_{\infty}$, and $x = v_n^T$, where $K(Z^T, n)$ is obtained as follows:¹ Let K(Z, n) be a topogical group, let $T(g, y) = (g^{-1}, Ty)$ for all $g \in K(Z, n)$ and $y \in S^{\infty}$. Let

¹ Personal communication from C. T. C. Wall.

$$\mathit{K}(Z^{\scriptscriptstyle T},\,n) = \mathit{K}(Z,\,n) imes S^{\infty}/T$$
 .

We have inclusion and projection

$$P_{\infty} \xrightarrow{i} K(Z^{T}, n) \xrightarrow{p} P_{\infty}$$

where i[y] = [*, y] and $p[g, y] = [y]; P_{\infty}$ may thus be considered to be a subset of $K(Z^{T}, n)$, and its cohomology group is a direct summand¹. Then $v_{n}^{T} \in H^{n}(K(Z^{T}, n), P_{\infty}; Z^{T}[u])$ is the fundamental class.

$$H^n(X, A; Z_2) \cong Z_2$$

is generated by $\Pi_* v_n^{\scriptscriptstyle T}$; $H^{n+1}(X, A; Z_2) \cong Z_2$ generated by $\Pi_* v_n^{\scriptscriptstyle T} \cup u$. Thus, by (3.4.3) and (3.4.4), $H^{n+1}(X, A; Z) \cong Z_2$ generated by $v_n^{\scriptscriptstyle T} \cup \bar{u}$, and the result follows from (3.4.5).

(3.5.2). We summarize the results of (3.4.5) and (3.5.1) in the following commutative diagram with all rows exact:

3.6. Applying the results of 3.4 and 3.5, we compute the cohomology of real projective space P_k , for $k \ge 1$:

$$(3.6.1) \qquad H^n(P_k; Z_z) \cong \begin{cases} Z_z, \text{ generated by } u^n, \text{ if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

$$(3.6.2) \qquad H^n(P_k; Z) \cong \begin{cases} Z_z, \text{ generated by } \bar{u}^n, \text{ if } n \\ \text{even, } 0 < n \leq k \\ Z, \text{ generated by } 1, \text{ if } n = 0 \\ 0, \text{ if } n \text{ odd, } 0 < n < k \\ Z, \text{ generated by } t(P_k), \text{ the top class, if } n = k \text{ odd } \\ 0 & \text{if } n > k \end{cases}$$

$$(Z_z, \text{ generated by } \bar{u}^n, \text{ if } n \text{ odd } \end{cases}$$

$$(3.6.3) \qquad H^n(P_k;Z^{\scriptscriptstyle T}[u]) \cong \begin{cases} Z_2, \text{ generated by } u^*, \text{ If } n \text{ odd}, \\ 0 < n \leq k \\ 0, \text{ if } n \text{ even}, \ 0 < n < k \\ Z, \text{ generated by } t(P_k), \text{ the top } \\ \text{class, if } n = k \text{ even} \\ 0, \text{ if } n > k \text{ .} \end{cases}$$

$$(3.6.4) \qquad H^n P_k, \, {}^*; \, Z^{\scriptscriptstyle T}[u]) \cong \begin{cases} 0, & \text{if } n = 0 \\ Z, & \text{generated by } \bar{u}, \, \text{if } n = 1 \\ H^n(P_k; \, Z^{\scriptscriptstyle T}[u]) & \text{if } n > 1 \end{cases}$$

$$(3.6.5) H^n(P_k; Z_2 + Z_2) \cong H^n(P_k; Z_2) \bigoplus H^n(P_k; Z_2) \ .$$

$$(3.6.6) Hn(Pk; Z + Z) \cong Hn(Pk; Z) \oplus Hn(Pk; Z) .$$

$$(3.6.7) \quad H^{n}(P_{k}; (Z+Z)^{T}[u]) \cong \begin{cases} Z, \text{ generated by } (j_{1})_{*}1, \\ \text{if } n = 0 \\ 0, \text{ if } 0 < n < k \\ Z, \text{ generated by } \frac{1}{2}(i_{1})_{*}t(P_{k}) = \\ (q_{2})_{*}^{-1}t(P_{k}) \text{ if } n = k \text{ is even} \\ Z, \text{ generated by } \frac{1}{2}(j_{1})_{*}t(P_{k}) = \\ (p_{2})_{*}^{-1}t(P_{k}) \text{ if } n = k \text{ is odd} \\ 0, \text{ if } n > k \end{cases}$$
$$(3.6.8) \quad H^{n}(P_{k}; (Z_{2} + Z_{2})^{T}[u]) \cong \begin{cases} Z_{2}, \text{ generated by } (i_{1})_{*}1 \\ \text{ if } n = 0 \\ 0, \text{ if } 0 < n < k \\ Z_{2}, \text{ generated by } (p_{2})_{*}^{-1}u^{k} \\ (= \Pi_{*}\frac{1}{2}(i_{1})_{*}t(P_{k})) \text{ if } k \\ \text{ even, } = \Pi_{*}\frac{1}{2}(j_{1})_{*}t(P_{k}) \text{ if } k \\ \text{ odd) if } n = k \\ 0, \text{ if } n > k . \end{cases}$$

4. Evaluation of the differentials.

4.1. We need two remarks.

(4.1.1) If Y_1 and Y_2 are spaces, and $h: Y_1 \to Y_2$ is a map, h induces a map $(Y_1)_{n-1} \to (Y_2)_{n-1}$ and a sheaf homomorphism $\tilde{h}: \pi_n(Y_1, 1) \to \pi_n(Y_2, h)$. If k_1^{n+1} and k_2^{n+1} are the n^{th} k-invariants of Y_1 and Y_2 respectively, $\tilde{h}_* k_1^{n+1} = h^* k_1^{n+2} \in H^{n+1}((Y_1)_{n-1}; \pi_n(Y_2, h))$.

(4.1.2) Let X and Y be spaces, $2 \leq m < n$ integers such that $\pi_k(Y) = 0$ for all m < k < n, and $f: X \to (Y)_n$ a map. If the k-invariant k^{n+1} of Y is based on the relation $\theta(1, k^{m+1}) = 0$, where θ is a map cohomology operation and $1: (Y)_{m-1} \to (Y)_{m-1}$ is the identity map, then; for any

$$x \in H^{m-1}(X; \pi_m(Y, f)), \, d_r(x) = s^{-2} heta(p^n_{m-1}fP, s^2 x), \, r = n - m + 1 \; ,$$

where $P: X \times S^2 \rightarrow X$ is projection,

$$s^{\mathfrak{2}}$$
: $H^{st}(X, x_{\scriptscriptstyle 0})
ightarrow H^{st+2}(X imes S^{\mathfrak{2}}, X imes {st} \cup x_{\scriptscriptstyle 0} imes S^{\mathfrak{2}})$

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is suspension and $p_{m-1}^n = p_m \cdots p_n : (Y)_n \rightarrow (Y)_{m-1}$.

Proof. Let $(S^1, *)$ be a circle, which we think of as the unit interval with end-points identified. Let $C: X \times S^1 \to (Y)_m$ be the constant homotopy of $p_m^n f$ with itself. Now $p_m(C + sx) = p_m C$, where C + sx is as defined in (1.2.2) and $d_r(x) = \delta^n(f, f; C + sx)$ by (1.3). Finally, $s\delta^n(f, f; C + sx) = (C + sx)^*k^{n+1} = s^{-1}\theta(p_{m-1}^n fP, s^2x)$.

4.2. Kervaire [3, p. 162] gives us the following table of homotopy groups:

	BO(1)	BO(2)	BO(3)	<i>BO</i> (4)	BO(5)	BO(6)	BO(n)	for $7 \leq n \leq \infty$
π_1	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	
π_2	0	Z	Z_2	Z_2	Z_2	Z_2	Z_2	
π_3	0	0	0	0	0	0	0	
π_4	0	0	Z	Z + Z	Z	Z	Z	
π_5	0	0	Z_2	$Z_2 + Z_2$	Z_2	0	0	
π_6	0	0	Z_2	$Z_2 + Z_2$	Z_2	Z	0.	

Now $\pi_1(BO(n)) = Z_2$ acts on $\pi_k(BO(n))$ for all $n \ge 1$, $k \ge 1$; this action is trivial if $\pi_k(BO(n))$ is stable, that is, k < n; because BO is simple. For n even, Z_2 acts nontrivially on $\pi_n(BO(n))$, because the first relative k-invariant of $BO(n) \to BO$ is

$$k^{n+1} = \beta^T[w_1]w_n \in H^{n+1}(BO; Z^T[w_1])$$
.

(Because Π_*k^{n+1} , the reduction mod 2, must be w_{n+1}). Z_2 acts trivially on $\pi_4(BO(3))$ because if acts trivially on $\pi_4(BO)$ and the map $Z \cong$ $\pi_4(BO(3)) \to \pi_4(BO) \cong Z$ is just multiplication by 2. Since Z_2 can only act trivially on Z_2 , we need only now examine the action on $\pi_4(BO(4))$ for k = 4, 5, 6.

PROPOSITION (4.2.1). We may choose generators x and y of $\pi_4(BO(4))$ such that T(x) = -x, T(y) = x + y, and the maps

$$i_4^3$$
: $\pi_4(BO(3)) \longrightarrow \pi_4(BO(4))$ and i_4^4 : $\pi_4(BO(4)) \longrightarrow \pi_4(BO(5))$

have the properties $i_{4}^{3}(1) = x + 2y$, $i_{4}^{4}(x) = 0$ and $i_{4}^{4}(y) = 1$.

Proof. We know that i_4^i is onto. Choose x to be a generator of Ker i_{4i}^i , and pick a such that $i_{4i}^i a = 1$. Now $2a - i_{4i}^3(1) \in \text{Ker } i_{4i}^i$, since $i_{4i}^i i_{4i}^2 = 2$. So $2a - i_{4i}^3(1)$ is a multiple of x. It can't be an even multiple, because then $i_{4i}^3(1)$ would be divisible by 2, and $i_{3i}^3\pi_4(BO(3))$ is a direct summand of $\pi_4(BO(4))$. So for some k, $2a - i_{4i}^3(1) = (2k - 1)x$. Let y = a - kx; then $i_{4i}^3(1) = x + 2y$, $i_{4i}^4(x) = 0$, and $i_{4i}^4(y) = 1$. Now $T(x) \in \text{Ker } i_{4i}^4$, so T(x) must be -x. T(x + 2y) = x + 2y so $T(y) = \frac{1}{2}(x + 2y - Tx) = x + y$. We are done.

We represent $\pi_4(BO(4))$ as ordered pairs of integers, where (p, q) represents px + qy.

PROPOSITION (4.2.2). $\pi_5(BO(4))$ and $\pi_6(BO(4))$ may be represented as ordered pairs of elements of Z_2 , such that $i_5^3(x) = i_6^3(x) = (x, 0)$, $i_5^4(x, y) = i_6^4(x, y) = y$, and T(x, y) = (x + y, y) for all $x, y \in Z_2$.

Proof. $\pi_5(BO(n))$ and $\pi_6(BO(n))$ are the images, under η and η^2 respectively, of $\pi_4(BO(n))$, for n = 3, 4, or 5. Apply (4.2.1).

REMARK (4.2.3). There are two possible choices of x in (4.2.1) we retroactively make that choice such that the image of $\pi_5(BU(2)) \cong Z_2$, under the classifying map of the reallification $BU(2) \rightarrow BO(4)$, is generated by $(0, 1) \in \pi_5(BO(4))$.

4.3. We need to describe k-invariants for BO(n).

(4.3.1) For all n, k^3 of BO(n) is zero, since the projection

$$P_1: BO(n) \longrightarrow (BO(n))_1 = K(Z_2, 1) = BO(1)$$

has a lifting, namely, the map induced by the inclusion of O(1) in O(n). Also $k^4 = 0$, since $\pi_3(BO(n)) = 0$.

(4.3.2) For $BO(3), k^5 = \pm \beta_4 \mathfrak{P} w_2$, where β_4 is the Bockstein of $Z \rightarrow Z_4$ and $\mathfrak{P}: H^2(; \mathbb{Z}_2) \rightarrow H^4(; \mathbb{Z}_4)$ is the Pontrjagin square [2], and k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.3) For BO(5), $k^5 = 2\beta_4 \Re w_2 = \beta w_2^2$ (see [4]), and $k^6 = w_6$, based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$.

(4.3.4) Using (4.3.2), (4.3.3), we get that for BO(4), $k^5 = \iota \beta_4 \mathfrak{P} w_2$, where $\iota: H^*(; Z) \to H^*(; (Z + Z)^T)$ is $(j_1)_*$ as described in (3.5.2), and k^5 is of order 4 and generates $H^5((BO(4))_4; (Z + Z)^T[w_1])$. Also, k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5$, where

$$S_q^2: H^*(; (Z_2 + Z_2)^T[a]) \longrightarrow H^{*+2}(; (Z_2 + Z_2)^T[a])$$

is that unique operation which is ordinary S_q^2 on each factor when a = 0, and $w_2 \cup$ is as described in (3.5).

(4.3.5) For $BO(6), k^5 = 2\beta_4 \mathfrak{P} w_2 = \beta w_2^2$, and $k^7 = \beta^T [w_1] w_6$, based on the relation $\beta^T (S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5) = 0$.

4.4. Using (4.1.1) and (4.1.2) we can now evaluate some differentials $d_r = d_r^f$ for a map $f: X \to (Y)_k$.

(4.4.1) If Y = BO(1) or $BO(2), d_r = 0$.

(4.4.2) If Y = BO(3) and k < 4, $d_r = 0$. If k = 4, $d_2 = 0$: by (4.1.2), $d_3(x) = \beta(x^3 + x \cup f^*w_2) \in H^4(X; Z)$ for all $x \in H^1(X; Z_2)$. This was also known to Dold and Whitney [2]. If

$$k=5,\, d_{\scriptscriptstyle 2}(x)=S_{\scriptscriptstyle q}^{\scriptscriptstyle 2}\varPi_{\,*}x+f^{*}w_{\scriptscriptstyle 2}\cup\varPi_{\,*}x\in H^{\scriptscriptstyle 5}(X;\,Z_{\scriptscriptstyle 2})$$
 ,

for all $x \in H^{3}(X; Z)$ by (4.1.2); $d_{3} = 0$, and d_{4} requires special computation.

(4.4.3) If Y = BO(4) and k < 4, $d_r = 0$. If k = 4, $d_2 = 0$; and by (4.1.2),

$$d_3(x) = \iota eta(x^3 + x \cup f^*w_2) \in H^4(X; (Z + Z)^T[f^*w_1])$$

for all $x \in H^1(X; \mathbb{Z}_2)$; if

$$k = 5, d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; (Z_2 + Z_2)^T [f^* w_1])$$

for all $x \in H^{3}(X; (Z + Z)^{T}[f^{*}w_{1}])$ by (4.1.2), $d_{3} = 0$, and d_{4} must be computed specially.

(4.4.4) If Y = BO(5) and k < 5, $d_r = 0$. If

$$k = 5, d_2(x) = S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x \in H^5(X; Z_2)$$

for all $x \in H^{3}(X; \mathbb{Z})$, $d_{3} = 0$, and

$$egin{aligned} d_4(x) &= x^5 + f^* w_1 \cup x^4 + f^* w_2 \cup x^3 + f^* w_3 \cup x^2 \ &+ f^* w_4 \cup x + \operatorname{Im} d_2 \! \in \! E_4^{5,5} = H^5(X; Z_2) / \mathrm{Im} \, d_2 \end{aligned}$$

for all $x \in H^1(X; \mathbb{Z}_2)$.

Proof. We have a map $S: \Sigma K(Z, 1) - BSO$, such that $S^* w_{i+1} = su^i$ for all $i \ge 1$, where u is the fundamental class. Now $(BO(5))_4 = (BO)_4$ has the same homotopy as BO up through dimension 7, so we identify $H^k((BO(5))_4)$ with $H^k(BO)$ for $0 \le k \le 7$. Let $h: \Sigma K(Z_2, 1) - (BO(5))_4$ be given by the commutative diagram:

 $(BO(5))_4$ has an *H*-space structure $\mu: (BO(5))_4 \times (BO(5))_4 \rightarrow (BO(5))_4$ and $\mu^* w_6 = \sum_{i=0}^6 w_i \times w_{6-i}$. Let QX be the space obtained from $X \times S^1$ by collapsing $x_0 \times S^1$; let $J: QX \rightarrow \Sigma X$ be the map which collapses $X \times *$, and let $p_1: QX \rightarrow X$ be projection onto the first factor. For any $x \in (H^*X)$, let $qx = p_1^*x$ and let $Qx = J^*sx$, both in $H^*(QX)$. We showed in [4, 5.1] that $qa \cup qb = q(a \cup b), qa \cup Qb = Q(a \cup b)$, and $Qa \cup Qb = 0$ for all $a, b \in H^*(X)$. Let $C: X \rightarrow K(Z_2, 1)$ be a classifying map for a given $x \in H^1(X; Z_2)$, and let $F: QX \rightarrow (BO(5))_4$ be a map, which represents a homotopy of p_5f with itself, defined by composing the following maps:

$$QX \xrightarrow{\Delta} QX \times QX \xrightarrow{J \times p_1} \Sigma X \times X \xrightarrow{\Sigma C \times p_5 f} \Sigma K(Z_2, 1) \times (BO(5))_4$$
$$\xrightarrow{h \times 1} (BO(5))_4 \times (BO(5))_4 \xrightarrow{\mu} (BO(5))_4 .$$

By (1.3), $d_4(x)$ contains $\delta^5(f, f; F)$. Now routine computation shows that $f^*w_6 = Q(x^5 + x^4f^*w_1 + x^3f^*w_2 + x^2f^*w_3 + xf^*w_4)$, and the result follows from [4, 5.2].

(4.4.5) If Y = BO(6) and k < 6, $d_r = 0$. If k = 6, $d_2 = 0$ and $d_3(x) = \beta^T (S_q^2 \Pi_* x + f^* w_2 \cup \Pi_* x) \in H^6(X; Z^T[f^* w_1])$ for all $x \in H^3(X; Z)$; $d_4 = 0$ and

$$egin{aligned} d_5(x) &= eta^{ au}(x^5 + x^4 f^* w_1 + x^3 f^* w_2 + x^2 f^* w_3 + x f^* w_4) \ &+ \operatorname{Im} d_2 \in E_5^{6,6} = H^6(X; Z^{ au}[f^* w_1]) / \operatorname{Im} d_2 \end{aligned}$$

for all $x \in H^1(X; \mathbb{Z}_2)$.

Proof. same as (4.4.4).

4.5. We are now ready to classify real vector bundles over P_k , for $k \leq 5$.

DEFINITION (4.5.1). A locally oriented real *n*-dimensional vector bundle over a space X shall be a b.p.p. homotopy class of maps from X to BO(n). If $f: X \to BO(n)$ represents a locally oriented v.b. ξ , let $\sim \xi$, or ξ conjugate, be that locally oriented v.b. given by a map $g: X \to BO(n)$ which is connected to f via a free homotopy which sends the base-point of X around a nontrivial loop of BO(n). Obviously $\sim \xi \cong \xi$, and conjugate classes of locally oriented vector bundles correspond to equivalence classes of vector bundles.

TABLE (4.5.2). For $k \ge 1$, let $h: P_k \to BO(1)$ be the canonical line bundle. Let " \oplus " denote Whitney sum. We give a complete list of all locally oriented real *n*-dimensional vector bundles over P_k , each *n* and *k*; all bundles are self-conjugate unless otherwise specified.

Let G denote $(q_1)^{-1}_*t(P_4) = \frac{1}{2}(i_1)_*t(P_4)$ which generates

$$H^{4}(P_{4}; (Z + Z)^{T}[u])$$
.

Also $(p_2^*)^{-1}u^5$ generates $H^5(P_5; (Z_2 + Z_2)^T[u])$. Locally oriented real *n*-dimensional vector bundles over P_k , for $n-1 \leq k \leq 5$:

Over P_1		Over P ₂					
1	2	1	2	3			
h	$h\oplus 1$	h	$T_p = (h \oplus 1) + pt(P_2)$, for all $p \in Z$;	$h\oplus 2$			
			stable class $h+1$ if p even,	$2h \oplus 1 = 3 + u^2$			
			$3h-1$ if p odd; $\sim T_p = T_{-p}$.	$3h=(h\oplus 2)+u^2$			
			$2h=2+ar{u^2}$				

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Over P_3			0	ver F	P ₄				
1	2	3	4	1	2	$3=3+\bar{u}^4$	$4=4+(\bar{u}^4,0)$	5	
h	$h \oplus 1$	$h \oplus 2$	$h \oplus 3$	h	$h \oplus 1$	$h{\oplus}2$	$2h{\oplus}2$	$h{\oplus}4$	
	2h	$2h\oplus 1$	$2h{\oplus}2$		2h	$(h\oplus 2) + \bar{u}^4$	$2h \oplus 2 + (\bar{u}^4, 0)$	$2h \oplus 3$	
		3h	$3h\oplus 1$			$2h \oplus 1$	$4h = 4 + (0, \bar{u}^4) = 4h + (\bar{u}^4, 0)$	$3h{\oplus}2$	
						$(2h\oplus 1)+\bar{u}^4$	$2h \oplus 2+(0, \overline{u}^4)$; stable	$4h \oplus 1$	
						$3h\!=\!3h\!+\!ar{u}^4$	class $6h-2$	5h	
						$2h{\oplus}2{+}(ar{u}^{_4},ar{u}^{_4}){=}$	$((2h\oplus 2)+(0, \bar{u}^4))\oplus 1;$		
							$\sim (2h \oplus 2 + (0, \bar{u}^4))$	stably $6h - 1$	
			$E_p = h \oplus 3 + pG$ for all	$F_1 \oplus 1$; stable class					
$p \in \mathcal{I}$		$p \in Z$; stable class	7h - 2						
h+3 if p even, $5h-1$									
							if p odd; $\sim E_p = E_{-p}$		
							$F_p = 3h \oplus 1 + pG$ for all		
						$p \in Z$; stable class			
		$3h{+}1$ if p even,							
							7h-3 if p odd;		
							$\sim F_p = F_{-p}$		

Over P	5			
1 2	3	4	$5 = 5 + u^5$	6
$h \mid h \oplus 1$	$3+u^{5}$	$4+(u^5,0)$	$h{\oplus}4$	$h{\oplus}5$
2h	$h\oplus 2$	$4+(0, u^5)$	$h \oplus 4 + u^5$	$2h{\oplus}4$
	$h \oplus 2 + u^5$	$4+(u^5, u^5) = \sim (4+(0, u^5))$	$2h{\oplus}3$	$3h \oplus 3$
	$A \!=\! A \!+\! u^{5};$	$h \oplus 3$	$2h \oplus 3 + u^{5}$	$4h{\oplus}2$
	$A P_4\!=\!h{\oplus}2\!+\!ar u^4$	$h \oplus 3 + (p_2^*)^{-1} u_5$	$3h{\oplus}2$	
	$2h{\oplus}1$	$2h\oplus 2$	$3h \oplus 2 + u^5$	$5h{\oplus}1$
	$2h \oplus 1 + u^5$	$2h \oplus 2 + (u^5, 0)$	$4h \oplus 1$	6h
	$B = B + u^{5};$	$2h \oplus 2 + (0, u^5)$	$4h \oplus 1 + u^5$	$C \oplus h \oplus 1$
	$B P_4=2h\oplus 1+ar u^4$	$2h \oplus 2 + (u^5, u^5) = \sim (2h \oplus 2 + (0, u^5))$	$5h\!=\!5h\!+\!u^{5}$	
	3h	$B \oplus 1 = B \oplus 1 + (u^5, 0)$	$C \oplus 1 = C \oplus 1 + u^5$	
	$3h\!+\!u^5$	$B \oplus 1 + (0, u^5) = B \oplus 1 + (u^5, u^5)$	$C \oplus h = C \oplus h + u^5$	
		$3h\oplus 1$		
		$3h \oplus 1 + (p_2^*)^{-1} u_5$		
		4h		
		$4h + (u^5, 0)$		
		$4h + (0, u^5)$		
		$4h + (u^5, u^5) = \sim (4h + (0, u^5))$		
		$C = C + (0, u^5); C P_4 = 2h \oplus 2 + (0, \bar{u}^4)$		
		$D=D+(0, u^5)=\sim C$		
		$C+(u^5, 0)=C+(u^5, u^5)$		
		$D+(u^{5}, 0)=$		
		\sim (C+(u ⁵ , 0))=D+(u ⁵ , u ⁵)		

	BU(1)	BU(2)	BU(n)	for $3 \leq n \leq \infty$
π_1	0	0	0	
π_2	Z	Z	Z	
π_3	0	0	0	
π_4	Z	Z	Z	
π_5	0	Z_2	0	

4.6. Similarly, we can classify all complex vector bundles over P_k , for $k \leq 5$. We give a table of homotopy groups:

The only nonzero k-invariant in this range is k^{ϵ} of BU(2), which is $\Pi_*(c_1c_2) + S_q^2 \Pi_*c_2$, where $c_i \in H^{2i}(BU(2); Z)$ are the Chern classes. We thus have:

REMARK (4.6.1). For any space X, all complex line bundles over X correspond to $H^2(X; Z)$.

REMARK (4.6.2). For any space X of dimension ≤ 5 , all complex *n*-bundles, for $n \geq 3$, over X correspond to KU(X), satisfying the exact sequence $0 \rightarrow H^4(X; Z) \rightarrow KU(X) \rightarrow H^2(X; Z) \rightarrow 0$.

REMARK (4.6.3). If $f: X \to (BU(2))_5$ is a map, then $d_2(x) = \Pi_*(c_1x) + S_q^2 \Pi_* x \in H^5(X; Z_2)$

for all $x \in H^{3}(X; Z); d_{3} = 0; d_{4}(x) = \Pi_{*}(f^{*}c_{2} \cup x) + \operatorname{Im} d_{2}$ for all

 $x \in H^{\scriptscriptstyle 1}(X;Z)$.

Proof. Let $S: S^2 = \Sigma K(Z, 1) \rightarrow BU$ be the generator of $\pi_2(BU)$; then $S^*c_1 = \sigma$, the fundamental class of S^2 , and $S^*c_2 = 0$. The result follows just as in (4.4.4).

TABLE (4.6.4). We summarize complex *n*-bundles over P_k , $2n - 1 \leq k \leq 5$. The reallification is given in square brackets.

	Over P_2			Ove	r P ₃		
1 <i>H</i>	$[2] \\ [2h]$	$2 \\ H \oplus 1 = 2 +$	$\begin{bmatrix} 4 \\ u^2 & [2h \oplus 2] \end{bmatrix}$	1 <i>H</i>	$\begin{matrix} [4] \\ [2h] \end{matrix}$	$2 \\ H \oplus 1$	$\begin{matrix} [4] \\ [2h \oplus 2] \end{matrix}$
·	Over P_4					— <u>————————————————————————————————————</u>	
1 H	[2] [2h]	2 $H \bigoplus 1$ $2H = 2 + \bar{u}^4$ $H \bigoplus 1 + \bar{u}^4$	[4] [$2h \oplus 2$] [$4h$] [$2h \oplus 2 + (\tilde{u}^4, 0)$] Stable class 3H	H-1	$3 H \oplus 2$ $2H \oplus 1 =$ $3H = H \oplus$	$=3+ar{u}^4 \oplus 2+ar{u}^4$	[6] $[2h \oplus 4]$ $[4h \oplus 2]$ [6h]

Over P_5			
1 [2 H [2	2] [] [2h]	2 $2 + u^{5}$ $H \oplus 1$ $H \oplus 1 \oplus u^{5}$ $2H$ $2H + u^{5}$	$\begin{array}{c} [4] \\ [4+(0, u^5)] \\ [2h \oplus 2] \\ [2h \oplus 2+(0, u^5)] \\ [4h] \\ [4h+(0, u^5)] \end{array}$
		$egin{array}{c} C \ C + u^5 \end{array}$	[C] [C]

4.7. We give a few representative examples of evaluating those difficult differentials. Is $f: P_5 \rightarrow (BO_4)_5$ is a map representing a 4-plane bundle ξ , then $d_4^f(u)$ is defined if and only if

$$d_2^f(u) = (j_1)_*eta(u^3 + uf^*w_2) = 0 \in H^4(P_5; (Z+Z)^{ extsf{T}}[f^*w_1]) \;.$$

If $d_2(u) = 0$, then $d'_4(u) = 0$ if and only if there is a map $F: QP_5 \rightarrow (BO_4)_5$ which represents a homotopy of f with itself, such that $F^*w_2 = qf^*w_2 + Qu$, where QX is as given in [4; 5].

EXAMPLE (4.7.1). If $\xi = 4$ or 4h, then $f^*w_2 = 0$, so $d_2(u) = (\bar{u}^4, 0)$ and $d_4(u)$ is not defined. Thus $4, 4 + (u^5, 0), 4 + (0, u^5)$, and $4 + (u^5, u^5)$ are all distinct oriented vector bundles.

EXAMPLE (4.7.2). If $\xi = 2h \bigoplus 2$, then $f^*w_2 = u^2$, so $d_2(u) = 0$.

Let η_1 be that line bundle over QP_5 such that $w_1(\eta_1) = qu$; now 2-plane bundles over a space X with $w_1 = x$ are classified by $H^2(X; Z^r[x])$; let η_2 be that 2-plane bundle over QP_5 with $w_1(\eta_2) = qu$ classified by $Q\bar{u}$. Then $w_2(\eta_2) = Qu$. Let $c: QP_5 \rightarrow BO(4)$ be the classifying map of $\eta_1 \oplus \eta_2 \oplus 1$; $c^*w_2 = qu^2 + Qu$ and $(\eta_1 \oplus \eta_2 \oplus 1) | P_5 = 2h \oplus 2$. Thus F, the projection of c onto $(BO(4))_5$, and $d_4^r(u) = 0$.

EXAMPLE (4.7.3). If $\xi = C$, then $f^*w_2 = u^2$, so $d_2^f(u) = 0$, and $d_4^f(u)$ is defined. Now $p_5C = p_5(2h \oplus 2) + (0, \overline{u}^4)$,



and so $d_4(u) = 0$ if and only if we can lift the map

$$p_5F + q(0, \, ar{u}^4)$$
: $Qp_5 \longrightarrow ((BO(4))_4)$

to $(BO(4))_5$, where F is the map given in (4.7.2). Now the k-invariant k^6 is based on the relation $S_q^2 \Pi_* k^5 + w_2 \cup \Pi_* k^5 = 0$, and $(p_5 F)^* k^6 = 0$, so $(p_5 F + a)^* k^6 = S_q^2 \Pi_* a + (p_5 F)^* w_2 \cup \Pi_* a$ which, when $a = q(0, \bar{u}^4)$, equals $S_q^2 q(0, u^4) + (qu^2 + Qu) \cup q(0, u^4) = Q(0, u^5)$. So, by [4; 5.2], $d_4(u) = (0, u^5)$. Thus $C + (0, u^5) = C$, but $C + (u^5, 0)$ is different. We also have that there are two complex structures on C, because since C is the reallification of the complex bundle C, $C = C + (0, u^5)$ is the reallification of $C + u^5$.

4.8. We would like to know how vector bundles behave under tensor products. If L is any line bundle over any space, $L \otimes L = 1$. Furthermore:

REMARK (4.8.1). If η_1 and η_2 are locally oriented real *n*-plane bundles over a space X, which agree on X^{k-1} , and if ξ is a locally oriented real *m*-plane bundle over X, then $i_*\delta^k(\eta_1, \eta_2) = \delta^k(\eta_1 \oplus \xi, \eta_1 \oplus \xi)$ and $j_*\delta^k(\eta_1, \eta_2) = d^k(\eta_1 \otimes \xi, \eta_2 \otimes \xi)$, where $i: BO(n) \to BO(n + m)$ and $j: BO(n) \subset BO(nm)$ are the maps induced by the inclusion of O(n) in O(n + m) and O(nm). Similarly for complex vector bundles.

REMARK (4.8.2). If ξ is an oriented real vector bundle which has a complex structure, and if η is any other locally oriented real vector bundle, then $\xi \otimes \eta$ also has a complex structure.

Proof. Let $C(\eta)$ be the complexification of η , and let ξ' be a complex bundle whose reallification is ξ . Then we can see routinely that the reallification of $\xi' \otimes C(\eta)$ is $\xi \otimes \eta$.

With the above information, we can almost completely determine the action of " \oplus " and " \otimes " on all locally oriented real vector bundles over $P_k, k \leq 5$. For example,

$$egin{aligned} A\otimes h&=B, C\otimes h=C, 4\otimes h=4h, (4+(0,u^{\mathrm{5}}))\otimes h=4h+(0,u^{\mathrm{5}}),\ T_{p}\otimes h&=T_{p}, E_{p}\otimes h=F_{p}, (4h+(u^{\mathrm{5}},u^{\mathrm{5}}))\oplus 1=4h\oplus 1+u^{\mathrm{5}}\ . \end{aligned}$$

The only unsolved questions are whether $A \oplus h = B \oplus 1$; it is also possible that $A \oplus h = B \oplus 1 + (0, u^5)$; and whether $B \oplus 2$ equals $2h \oplus 3$ or $2h \oplus 3 + u^5$.

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