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A NOTE ON CERTAIN DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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In this paper an exact solution is obtained for the dual series equations

$$(1) \quad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + n + 1)} L_n^{(\alpha)}(x) = f(x), \quad 0 \leq x < y,$$

$$(2) \quad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + \beta + n)} L_n^{(\sigma)}(x) = g(x), \quad y < x < \infty,$$

where $\alpha + \beta + 1 > \beta > 1 - m$, $\sigma + 1 > \alpha + \beta > 0$, m is a positive integer,

$$L_n^{(\alpha)}(x) = \binom{\alpha + n}{n} {}_1F_1[-n; \alpha + 1; x],$$

is the Laguerre polynomial and $f(x)$ and $g(x)$ are prescribed functions.

The method used is a generalization of the multiplying factor technique employed by Lowndes [4] to solve a special case of the above equations when

$$\sigma = \alpha, A_n = \Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n)C_n, \alpha + \beta > 0 \quad \text{and} \quad 1 > \beta > 0.$$

In another paper by the present author [5] equations (1) and (2) have been solved by considering separately the equations when (i) $g(x) \equiv 0$, (ii) $f(x) \equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation. Indeed it is easy to verify that the solution obtained earlier [5] is in complete agreement with the one given in this paper.

2. The following results will be required in the analysis.

(i) The orthogonality relation for Laguerre polynomials given by [3, p. 292 (2)] and [3, p. 293 (3)]:

$$(3) \quad \int_0^{\infty} e^{-x} x^{\alpha} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{mn}, \quad \alpha > -1,$$

where δ_{mn} is the Kronecker delta.

(ii) The formula (27), p. 190 of [2] in the form:

$$(4) \quad \frac{d^m}{dx^m} \{x^{\alpha+m} L_n^{(\alpha+m)}(x)\} = \frac{\Gamma(\alpha + m + n + 1)}{\Gamma(\alpha + n + 1)} x^{\alpha} L_n^{(\alpha)}(x).$$

(iii) The following forms of the known integrals [2, p. 191 (30)] and [3, p. 405 (20)]:

$$(5) \quad \int_0^\xi x^\alpha (\xi - x)^{\beta-1} L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha + n + 1)\Gamma(\beta)}{\Gamma(\alpha + \beta + n + 1)} \xi^{\alpha+\beta} L_n^{(\alpha+\beta)}(\xi),$$

where $\alpha > -1, \beta > 0$, and

$$(6) \quad \int_\xi^\infty e^{-x} (x - \xi)^{\beta-1} L_n^{(\alpha)}(x) dx = \Gamma(\beta) e^{-\xi} L_n^{(\alpha-\beta)}(\xi),$$

where $\alpha + 1 > \beta > 0$.

3. Solution of the equations. Multiplying equation (1) by $x^\alpha (\xi - x)^{\beta+m-2}$, where m is a positive integer, equation (2) by $e^{-x} (x - \xi)^{\sigma-\alpha-\beta}$, and integrating with respect to x over $(0, \xi)$, (ξ, ∞) respectively we find, on using (5) and (6), that

$$(7) \quad \begin{aligned} & \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha + \beta + m + n)} L_n^{(\alpha+\beta+m-1)}(\xi) \\ &= \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta + m - 1)} \int_0^\xi x^\alpha (\xi - x)^{\beta+m-2} f(x) dx, \end{aligned}$$

where $0 < \xi < y, \alpha > -1, \beta + m > 1$, and

$$(8) \quad \begin{aligned} & \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha + \beta + n)} L_n^{(\alpha+\beta-1)}(\xi) \\ &= \frac{e^\xi}{\Gamma(\sigma - \alpha - \beta + 1)} \int_\xi^\infty e^{-x} (x - \xi)^{\sigma-\alpha-\beta} g(x) dx, \end{aligned}$$

where $y < \xi < \infty, \sigma + 1 > \alpha + \beta > 0$.

If we now multiply equation (7) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ and use the formula (4) we see that it becomes

$$(9) \quad \begin{aligned} & \sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha + \beta + n)} L_n^{(\alpha+\beta-1)}(\xi) \\ &= \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta + m - 1)} \frac{d^m}{d\xi^m} \int_0^\xi x^\alpha (\xi - x)^{\beta+m-2} f(x) dx, \end{aligned}$$

where $0 < \xi < y, \alpha > -1$, and $\beta + m > 1$.

The left-hand sides of equations (8) and (9) are now identical and an application of the orthogonality relation (3) yields the solution of equations (1) and (2) in the form

$$(10) \quad \begin{aligned} A_n &= \frac{n!}{\Gamma(\beta + m - 1)} \int_0^y e^{-\xi} L_n^{(\alpha+\beta-1)}(\xi) F(\xi) d\xi \\ &+ \frac{n!}{\Gamma(\sigma - \alpha - \beta + 1)} \int_y^\infty \xi^{\alpha+\beta-1} L_n^{(\alpha+\beta-1)}(\xi) G(\xi) d\xi, \\ n &= 0, 1, 2, 3, \dots, \end{aligned}$$

where

$$(11) \quad F(\xi) = \frac{d^m}{d\xi^m} \int_0^\xi x^\alpha (\xi - x)^{\beta+m-2} f(x) dx$$

and

$$(12) \quad G(\xi) = \int_\xi^\infty e^{-x} (x - \xi)^{\sigma-\alpha-\beta} g(x) dx ,$$

provided that $\alpha + \beta + 1 > 1 - m$ and $\sigma + 1 > \alpha + \beta > 0$, m being a positive integer.

When $\sigma = \alpha$, $A_n = \Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n)C_n$, the above equations provide the solution to Lowndes' equations for

$$\alpha + \beta > 0, 1 > \beta > 1 - m ,$$

and when $m = 1$ the results are in complete agreement (see [4], p. 124). Note also that the dual equations considered recently by Askey [1, p. 683, Th. 3] are essentially the same as Lowndes' equations.

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