

# Pacific Journal of Mathematics

**A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS**

ERNEST LESTER STITZINGER

## A NONIMBEDDING THEOREM OF ASSOCIATIVE ALGEBRAS

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**Let  $A$  and  $B$  be associative algebras and define the Frattini subalgebra of  $A$ ,  $\phi(A)$ , to be the intersection of all maximal subalgebras of  $A$  if maximal subalgebras of  $A$  exist and as  $A$  otherwise. Conditions on  $B$  will be found such that  $B$  cannot be an ideal of  $A$  contained in  $\phi(A)$ .**

Hobby in [2] has shown that a nonabelian group  $G$  cannot be the Frattini subgroup of any  $p$ -group if the center of  $G$  is cyclic. Chao in [1] has shown that a nonabelian Lie algebra  $L$  can not be the Frattini subalgebra of any nilpotent Lie algebra if the center of  $L$  is one dimensional. In this note, we find a similiar result in the theory of associative algebras. However, in this case, it is not necessary to place any restrictions on the containing algebra.

Let  $A$  be an associative algebra over a field  $F$  and let  $B$  be an ideal of  $A$ . If  $x \in A$ , then  $x$  induces an endomorphism of the additive group of  $B$  by  $L_x(b) = xb$  for all  $b \in B$ . Let  $E(B, A)$  be the collection of all endomorphisms of this type. Then  $E(B, A)$  is a subspace of the vector space of all linear transformations from  $B$  into  $B$  and is an associative algebra under the compositions  $L_x + L_y = L_{x+y}$ ,  $\alpha L_x = L_{\alpha x}$  and  $L_x L_y = L_{xy}$  for all  $x, y \in A$  and all  $\alpha \in F$ . Clearly  $E(B, B)$  is an ideal of  $E(B, A)$ . If  $C$  is an ideal of  $A$  contained in  $B$ , then let  $E(B, A, C) = \{E \in E(B, A); E(c) = 0 \text{ for all } c \in C\}$ . Then  $E(B, A, C)$  is an ideal of  $E(B, A)$  and  $E(B, A)/E(B, A, C)$  is isomorphic to  $E(C, A)$ . Note that the mapping from  $A$  onto  $E(B, A)$  which assigns to each  $a \in A$  the element  $L_a$  is an algebra homomorphism. We define the right annihilating series of  $B$  inductively. Let  $r_1(B) = \{c \in B; bc = 0 \text{ for all } b \in B\}$  and let  $r_j(B)$  be the ideal of  $B$  such that  $r_j(B)/r_{j-1}(B) = r_1(B/r_{j-1}(B))$  for  $j > 1$ . Since  $B$  is an ideal in  $A$ ,  $r_i(B)$  is an ideal in  $A$  for all  $i$ .

The following lemma is immediate.

**LEMMA.** *If  $A$  and  $A'$  are associative algebras and  $\pi$  is a homomorphism from  $A$  onto  $A'$ , then  $\pi(\phi(A)) \subseteq \phi(\pi(A))$ . Furthermore, if the kernel of  $\pi$  is contained in  $\phi(A)$ , then  $\pi(\phi(A)) = \phi(\pi(A))$ .*

**THEOREM.** *Let  $B$  be an associative algebra such that  $\dim r_1(B) = 1$  and  $\dim r_2(B) = k$  where  $1 < k < \infty$ . Then  $B$  cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

*Proof.* Suppose that to the contrary  $B$  is an ideal contained in the Frattini subalgebra of the associative algebra  $A$ . Then

$$E(B, B) \subseteq \phi(E(B, A)).$$

For if  $T$  is the mapping from  $A$  onto  $E(B, A)$  defined by  $T(a) = L_a$  for all  $a \in A$ , then, by the lemma,

$$E(B, B) = T(B) \subseteq T(\phi(A)) \subseteq \phi(T(A)) = \phi(E(B, A)).$$

Let  $z_1, \dots, z_k$  be a basis for  $r_2(B)$  such that  $z_k$  is a basis  $r_1(B)$ . For notational convenience, let  $r_i = r_i(B)$  for all  $i$ . Let  $\pi$  be the natural homomorphism from  $E(B, A)$  onto  $E(r_2, A)$ . Since

$$\begin{aligned} E(B, B) + E(B, A, r_2)/E(B, A, r_2) &\simeq E(B, B)/E(B, A, r_2) \cap E(B, B) \\ &= E(B, B)/E(B, B, r_2) \simeq E(r_2, B) \end{aligned}$$

it follows that

$$E(r_2, B) \simeq \pi(E(B, B)) \subseteq \pi(\phi(E(B, A))) \subseteq \phi(E(r_2, A)).$$

We now show that  $E(r_2, B) \not\subseteq \phi(E(r_2, A))$  by showing that  $E(r_2, B)$  is complemented in  $E(r_2, A)$ . For  $i = 1, \dots, k-1$ , define linear transformations  $e_i$  from  $r_2$  onto  $r_1$  by

$$e_i(z_j) = \begin{cases} \delta_{ij}z_k & \text{for } j = 1, \dots, k-1 \\ 0 & \text{for } j = k \end{cases}$$

where  $\delta_{ij}$  is the Kronecker delta. Let  $S = ((e_1, \dots, e_{k-1}))$ . We claim that  $S = E(r_2, B)$ . Since  $r_1 = ((z_k))$  and  $B \cdot r_2 \subseteq r_1$ ,  $E(r_2, B) \subseteq S$ . To show that  $S = E(r_2, B)$ , we shall show that  $\dim E(r_2, B) = k-1 = \dim S$ . For each  $x \in B$ ,  $L_x$  induces a linear transformation from  $r_2$  into  $r_1 \simeq F$ , where  $F$  is the ground field. Therefore, we may consider each  $L_x$ ,  $x \in B$  as a linear functional on  $r_2$ . That is,  $E(r_2, B) \subseteq (r_2)^*$  where  $(r_2)^*$  is the dual space of  $r_2$ . Consequently,  $\dim E(r_2, B) = \dim r_2 - \dim r_2^B$  where  $r_2^B = \{z \in r_2; L_x(z) = 0 \text{ for all } x \in B\}$ . Clearly  $r_2^B = r_1$ . Then, since  $\dim r_2 = k$  and  $\dim r_1 = 1$ ,  $\dim E(r_2, B) = k-1$  and  $S = E(r_2, B)$ .

We now show that  $S$  is complemented in  $E(r_2, A)$ . Let

$$M = \{E \in E(r_2, A); E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij}z_j, \lambda_{ij} \in F, i = 1, \dots, k-1$$

and  $E(z_k) = \lambda_k z_k, \lambda_k \in F\}$ .  $M$  is clearly a subalgebra of  $E(r_2, A)$  and  $M \cap S = 0$ . We claim that  $M + S = E(r_2, A)$ . Let  $E \in E(r_2, A)$ . Then  $E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij}z_j + \lambda_{ik}z_k$  for  $i = 1, \dots, k-1$  and  $E(z_k) = \lambda_k z_k$ . However  $E = E - \sum_{i=1}^{k-1} \lambda_{ik}e_i + \sum_{i=1}^{k-1} \lambda_{ik}e_i$  where  $E - \sum_{i=1}^{k-1} \lambda_{ik}e_i \in M$  and  $\sum_{i=1}^{k-1} \lambda_{ik}e_i \in S$ . Therefore  $M + S = E(r_2, A)$ . We claim that  $M \neq 0$ . If  $M = 0$ , then  $E(r_2, A) = E(r_2, B)$  which contradicts

$$E(r_2, B) \subseteq \phi(E(r_2, A)) \subset E(r_2 A) .$$

Consequently,  $S$  is complemented in  $E(r_2, A)$ , contradicting  $S \subseteq \phi(E(r_2, A))$ . This contradiction establishes the result.

**COROLLARY.** *Let  $B$  be a finite dimensional nontrivial nilpotent associative algebra with  $\dim r_1(B) = 1$ . Then  $B$  cannot be an ideal contained in the Frattini subalgebra of any associative algebra.*

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2. C. Hobby, *The Frattini subgroup of a  $p$ -group*, Pacific J. Math. **10** (1960), 209-212.

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