

# Pacific Journal of Mathematics

**CONDITIONS FOR A MAPPING TO HAVE THE SLICING  
STRUCTURE PROPERTY**

GERALD S. UNGAR

## CONDITIONS FOR A MAPPING TO HAVE THE SLICING STRUCTURE PROPERTY

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Let  $p: E \rightarrow B$  be a fibering in the sense of Serre. As is well known the fibering need not be a fibering in any stronger sense. However it is expected that if certain conditions are placed on  $E, p$  or  $B$  then  $p$  might be a fibration in a stronger sense. This paper gives such conditions.

The main result of this paper is:

**THEOREM 1.** Let  $p$  be an  $n$ -regular perfect map from a complete metric space  $(E, d)$  onto a locally equiconnected space  $B$ . If  $\dim E \times B \leq n$  then  $p$  has the slicing structure property (in particular  $p$  is a Hurewicz fibration).

The following definitions will be needed.

**DEFINITION 1.** A space  $X$  is *locally equiconnected* if for each point  $x$ , there exists a neighborhood  $U_x$  of  $x$  and a map

$$N: U_x \times U_x \times I \rightarrow X$$

satisfying  $N(a, b, 0) = a$ ,  $N(a, b, 1) = b$ , and  $N(a, a, t) = a$ .

**DEFINITION 2.** A map  $p$  from  $E$  to  $B$  is  $n$ -regular if it is open and satisfies the following property: given any  $x$  in  $E$  and any neighborhood  $U$  of  $x$  there exists a neighborhood  $V$  of  $x$  such that if  $f: S^m \rightarrow V \cap p^{-1}(y)$  for some  $y \in B$  ( $m \leq n$ ) then there exists

$$F: B^{m+1} \rightarrow U \cap p^{-1}(y)$$

which is an extension of  $f$ .

**DEFINITION 3.** A family  $\mathcal{S}$  of sets of  $Y$  is *equi- $LC^n$*  if for every  $y \in S \in \mathcal{S}$  and every neighborhood  $U$  of  $y$  in  $Y$  there exists a neighborhood  $V$  of  $y$  such that for every  $S \in \mathcal{S}$ , every continuous image of an  $m$ -sphere ( $m \leq n$ ) in  $S \cap V$  is contractible in  $S \cap U$ .

*Note 1.* If  $p: E \rightarrow B$  is  $n$ -regular then the collection  $\{p^{-1}(b) \mid b \in B\}$  is *equi- $LC^n$* .

**DEFINITION 4.** A family  $\mathcal{S}$  of sets of a metric space  $(Y, d)$  is *uniformly equi- $LC^n$*  with respect to  $d$  if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $f: S^m \rightarrow S \cap N(x, \delta)$  ( $m \leq n$  and  $S \in \mathcal{S}$ ) then there exists  $F: B^{m+1} \rightarrow S \cap N(x, \varepsilon)$  which is an extension of  $f$ .

**DEFINITION 5.** A map  $p: E \rightarrow B$  has the *covering homotopy pro-*

*perty for a class of spaces* if given any space  $X$  in the class and maps  $F: X \times I \rightarrow B$  and  $g: X \rightarrow E$  such that  $F(x, 0) = pg(x)$  then there exists a map:  $G: X \times I \rightarrow E$  such that  $pG = F$  and  $G(x, 0) = g(x)$ .

**DEFINITION 6.** A map  $p: E \rightarrow B$  is a *Serre fibration* if  $p$  has the covering homotopy property for the class of polyhedra. It is a *Hurewicz fibration* if it has the covering homotopy property for all spaces.

**DEFINITION 7.** A map  $p: E \rightarrow B$  has the *slicing structure property* (SSP) if for each point  $b \in B$  there exists a neighborhood  $U_b$  of  $b$  and a map  $\psi_b: p^{-1}(U_b) \times U_b \rightarrow p^{-1}(U_b)$  such that (1)  $\psi_b(e, p(e)) = e$  and (2)  $p\psi_b = \pi_2$  (the projection onto  $U_b$ ).

**DEFINITION 8.** A function  $\varphi: X \rightarrow 2^Y$  ( $Y$  metric) is continuous if given  $\epsilon > 0$ ; every  $x_0 \in x$  has a neighborhood  $U$  such that for every  $x \in U$ ,  $\varphi(x_0) \subset N_\epsilon(\varphi(x))$  and  $\varphi(x) \subset N_\epsilon(\varphi(x_0))$ .

**DEFINITION 9.** A selection for a function  $\varphi: X \rightarrow 2^Y$  is a map  $g: X \rightarrow Y$  such that  $g(x) \in \varphi(x)$ .

A mapping is a continuous function. All spaces will be Hausdorff. The  $n$ -dimensional sphere will be denoted by  $S^n$  and the ball which it bounds  $B^{n+1}$ . If  $f$  is a mapping  $\text{Gr}(f)$  will denote the graph of  $f$ .

The following theorem of Michael will be needed:

**THEOREM M.** *Let  $Z$  be paracompact, let  $X = Z \times I$  and let  $Y$  be a complete metric space with metric  $\rho$ . Let  $\mathcal{S} \subset 2^Y$  be uniformly equi- $LC^n$  with respect to  $\rho$  and let  $\varphi: X \rightarrow \mathcal{S}$  be continuous with respect to  $\rho$ . Let  $\dim Z \leq n$  and let  $A = (Z \times 0) \cup (C \times I)$  where  $C$  is closed in  $Z$ . Then every selection for  $\varphi|_A$  can be extended to a selection for  $\varphi$ .*

## 2. Proof of Theorem 1 and its consequences.

*Proof.* Let  $b_0 \in B$ . Since  $B$  is locally equiconnected at  $b_0$  there exists a neighborhood  $U$  of  $b_0$  and a map  $N_U: U \times U \times I \rightarrow B$  such that  $N_U(x, y, 0) = x$ ,  $N_U(x, y, 1) = y$ , and  $N_U(x, x, t) = x$ . Let  $P_U = p|_{p^{-1}(U)}$  and define  $g: \text{Gr}(p_U) \rightarrow p^{-1}(U)$  by  $g(e, p(e)) = e$ . Also define  $F: p^{-1}(U) \times B \rightarrow B$  by  $F(e, b) = b$  and

$$H: p^{-1}(U) \times U \times I \rightarrow p^{-1}(U) \times B$$

by  $H(e, b, t) = (e, N_U(p(e), b, t))$ . Note  $H(e, b, 0) = (e, N_U(p(e), b, 0)) = (e, p(e))$  and  $H(e, b, 1) = (e, N_U(p(e), b, 1)) = (e, b)$ . Further define

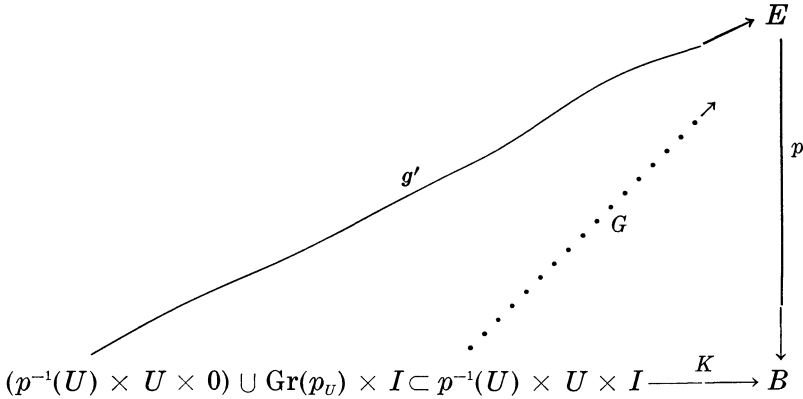
$$g': (p^{-1}(U) \times U \times 0) \cup (\text{Gr}(P_v) \times I) \rightarrow E$$

by  $g'(e, b, t) = e$  and  $K: p^{-1}(U) \times U \times I \rightarrow B$  by

$$K(e, b, t) = F(H(e, b, t))$$

and note that  $pg' = K|_{(p^{-1}(U) \times U \times 0) \cup \text{Gr}(P_v) \times I}$ .

Therefore we have the following commutative diagram.



Now Theorem *M* will be applied. Let  $Z = p^{-1}(U) \times U$ ,  $Y = E$ , and  $\varphi: Z \times I \rightarrow \mathcal{S} \subset 2^Y$  be defined by  $\varphi(z, t) = p^{-1}K(z, t)$  and let  $C = \text{Gr}(p_v)$ . Note  $Z$  is paracompact and  $\varphi$  is continuous since  $p$  is perfect. Since  $p$  is  $n$ -regular  $\{p^{-1}(b)\}$  in equi- $LC^n$  and by Proposition 2.1 [3] there exists a metric  $\sigma$  on  $E$  agreeing with the topology such that  $\sigma \geq d$  and  $\{p^{-1}(b)\}$  is uniformly equi- $LC^n$ . Since  $\sigma \geq d$ ,  $(E, \sigma)$  is a complete metric space. It should also be noted that  $\dim Z \leq n$  and that  $g'$  is a selection for  $\varphi|_{(Z \times 0) \cup (C \times I)}$ . Hence by Theorem *M*,  $g'$  could be extended to a selection  $G$  for  $\varphi$  (i.e.,

$$G: p^{-1}(U) \times U \times I \rightarrow E$$

in such a way that the above diagram will still be commutative with the addition of  $G$ ).

Define  $\varphi_v: p^{-1}(U) \times U \rightarrow p^{-1}(U)$  by  $\varphi_v(e, b) = G(e, b, 1)$ . Note if  $(e, b) \in p^{-1}(U) \times U$  then

$$G(e, b, 1) \in p^{-1}K(e, b, 1) = p^{-1}FH(e, b, 1) = p^{-1}F(e, N_v(p(e), b, 1)) \\ = p^{-1}F(e, b) = p^{-1}(b) \in p^{-1}(U).$$

Hence the range of  $\varphi_v$  is as stated. It is now easy to see that  $\varphi_v$  satisfies the conditions to be a slicing function. This completes the proof.

*Note 2.* The hypothesis that  $p$  be perfect was used only to show that  $\{p^{-1}(b) \mid b \in B\}$  is a continuous collection and that  $B$  is paracom-

fact. Hence if this could be shown some other way a stronger theorem will be obtained.

**COROLLARY 1.** *If  $p: E \rightarrow B$  is a Serre fibration and  $E$  and  $B$  are finite dimensional compact ANR's then  $p$  has the SSP.*

*Proof.* It is well known that ANR's are locally equiconnected.

It also follows from [2] that  $p$  is  $n$ -regular for all  $n$ . Hence the proof follows from Theorem 1.

Theorem 1 and Corollary 1 allow us to get the following generalizations of Raymond's results in [5].

**COROLLARY 2.** *Let  $p: E \rightarrow B$  be a Serre fibration of a connected compact metric finite dimensional ANR onto a compact metric finite dimensional ANR. Suppose that  $E$  is an  $n$ -gm over  $L$  (a field or the integers). Then:*

- (a) *each fiber  $F_b$  is a  $k$ -gm over  $L$*
- (b)  *$B$  is an  $(n-k)$ -gm over  $L$ .*

**COROLLARY 3.** *Let  $p: E \rightarrow B$  is a Serre fibration of a connected compact metric finite dimensional ANR onto a compact metric finite dimensional ANR base  $B$ . Suppose that  $E$  is a (generalized) manifold (over a principal ideal domain) and some fiber has a component of dimension  $\leq 2$ . Then  $p$  is locally trivial.*

Another theorem which follows from Michael's Theorem 1.2 [3] is the following:

**THEOREM 2.** *Let  $p: E \rightarrow B$  be an  $n$ -regular map from a complete metric space  $E$  onto a paracompact space  $B$ . Assume that*

$$\dim E \times B \leq n + 1$$

*and  $p^{-1}(b)$  is  $C^n$  for every  $b \in B$ . Then  $p$  has the SSP and the slicing structure could be chosen with only one slicing function.*

*Proof.* Define  $g: \text{Gr}(p) \rightarrow E$  by  $g(e, p(e)) = e$  and  $F: E \times B \rightarrow B$  by  $F(e, b) = e$ . The  $\varphi(e, b) = p^{-1}F(e, b)$  is a carrier and  $g$  is a selection for  $\varphi|_{\text{Gr}(p)}$ . Hence by Theorem 1.2 [3]  $g$  could be extended to a selection  $G$  for  $\varphi$ . It is easily seen that  $G$  is the desired slicing function.

*Note 3.* Theorem 2 has corollaries similar to those of Theorem 1 and the author leaves them to the reader to develop.

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