

# Pacific Journal of Mathematics

**NOTE ON SOME SPECTRAL INEQUALITIES OF C. R.  
PUTNAM**

STERLING K. BERBERIAN

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 OF C. R. PUTNAM

S. K. BERBERIAN

It is shown that if  $A$  is any operator in Hilbert space and  $\lambda = re^{i\theta}$  is in the approximate point spectrum of  $A$ , then

$$\min A^*A \leq (\max J_\theta)^2$$

and

$$|r - \max J_\theta| \leq [(\max J_\theta)^2 - \min A^*A]^{1/2},$$

where

$$J_\theta = (1/2)(Ae^{-i\theta} + A^*e^{i\theta}).$$

Several corollaries are deduced for arbitrary operators, generalizing results of C. R. Putnam on semi-normal operators.

We employ the notations in Putnam's paper [3]. In particular if  $A$  is any operator (bounded linear, in a Hilbert space) and  $\theta$  is a real number,  $J_\theta = \operatorname{Re}(Ae^{-i\theta}) = (1/2)(Ae^{-i\theta} + A^*e^{i\theta})$ . We write  $\sigma(A)$  and  $\pi(A)$  for the spectrum and approximate point spectrum of  $A$ , and  $(x|y)$  for the inner product of vectors.

The following result extracts the essentials of the proof of Theorem 1 in Putnam's paper:

**THEOREM.** *If  $A$  is any operator and  $\lambda \in \pi(A)$ ,  $\lambda = re^{i\theta}$  ( $r \geq 0$ ), then*

$$(1) \quad \max J_\theta \geq r \geq (\min A^*A)^{1/2},$$

$$(2) \quad \max J_\theta - r \leq [(\max J_\theta)^2 - \min A^*A]^{1/2}.$$

*Proof.* Let  $x_n$  be a sequence of unit vectors with  $(A - \lambda I)x_n \rightarrow 0$ . Clearly  $(Ax_n|x_n) \rightarrow \lambda$ ,  $(x_n|Ax_n) \rightarrow \bar{\lambda}$ ; it follows that  $(J_\theta x_n|x_n) \rightarrow r$  and therefore  $\max J_\theta \geq r$ . Since  $\|Ax_n\|$  is bounded,

$$0 = \lim ((A - \lambda I)x_n|Ax_n) = \lim \{(A^*Ax_n|x_n) - \lambda(x_n|Ax_n)\},$$

thus  $(A^*Ax_n|x_n) \rightarrow \lambda\bar{\lambda} = r^2$  and therefore  $\min A^*A \leq r^2$ . Thus (1) is proved. Since  $(A - \lambda I)^*(A - \lambda I) = A^*A - 2rJ_\theta + r^2I$ , one has

$$\|(A - \lambda I)x_n\|^2 = (A^*Ax_n|x_n) - 2r(J_\theta x_n|x_n) + r^2,$$

hence

$$\begin{aligned} \min A^*A &\leq (A^*Ax_n|x_n) = \|(A - \lambda I)x_n\|^2 + 2r(J_\theta x_n|x_n) - r^2 \\ &\leq \|(A - \lambda I)x_n\|^2 + 2r \max J_\theta - r^2; \end{aligned}$$

letting  $n \rightarrow \infty$ ,

$$\min A^*A \leq 2r \max J_\theta - r^2 .$$

Thus  $\min A^*A \leq (\max J_\theta)^2 - (\max J_\theta - r)^2$ , which proves (2).

Incidentally, if  $\lambda = 0 \in \pi(A)$  then obviously  $\min A^*A = 0$  and the theorem yields no information other than  $\max J_\theta \geq 0$  for all  $\theta$ .

If the dependence of  $J_\theta$  on  $A$  is indicated by writing  $J_\theta = J_\theta(A)$ , evidently  $J_{-\theta}(A^*) = J_\theta(A)$ . One has  $\pi(A^*) \subset \sigma(A^*) = (\sigma(A))^*$ , thus  $(\pi(A^*))^* \subset \sigma(A)$ ; if  $\lambda = re^{i\theta} \in (\pi(A^*))^*$  then  $re^{-i\theta} \in \pi(A^*)$  and application of the theorem to  $A^*$  yields the following:

**COROLLARY 1.** *If  $A$  is any operator and  $\lambda \in (\pi(A^*))^*$ ,  $\lambda = re^{i\theta}$ , then*

$$(3) \quad \max J_\theta \geq r \geq (\min AA^*)^{1/2} ,$$

$$(4) \quad \max J_\theta - r \leq [(\max J_\theta)^2 - \min AA^*]^{1/2} .$$

If  $A$  is hyponormal ( $AA^* \leq A^*A$ ) then  $\pi(A^*) = \sigma(A^*) = (\sigma(A))^*$  [cf. 1, p. 1175] and Corollary 1 yields:

**COROLLARY 2.** *If  $A$  is hyponormal then (3) and (4) hold for every  $\lambda \in \sigma(A)$ ,  $\lambda = re^{i\theta}$ .*

Another way of fulfilling (3) and (4) is via the relation

$$\partial\sigma(A) \subset \pi(A) \cap (\pi(A^*))^* .$$

If  $\lambda = re^{i\theta} \in \partial\sigma(A)$ , the boundary of  $\sigma(A)$ , then  $\lambda \in \pi(A)$  [cf. 2, p. 39] hence (1) and (2) hold by the theorem. Moreover,  $\bar{\lambda} \in (\partial\sigma(A))^* = \partial(\sigma(A))^* = \partial\sigma(A^*) \subset \pi(A^*)$ , i.e.,  $\lambda \in (\pi(A^*))^*$  and so (3) and (4) hold by Corollary 1. Thus:

**COROLLARY 3.** *If  $A$  is any operator and  $\lambda = re^{i\theta}$  is a boundary point of  $\sigma(A)$ , then (1), (2), (3), (4) hold.*

Corollary 3 is stated in [3, Th. 1; 4, p. 44, Th. 3.3.1] assuming  $AA^* \geq A^*A$  (i.e.,  $A^*$  hyponormal).

It follows readily from Corollary 3, as in [3], that the spectrum of a nonunitary isometry is the entire closed unit disc. The proof is similar to, and simpler than, the proof of the following corollary, which extends a result in [3, Corollary 2; 4, p. 44, Corollary 1] (the formulation there is inaccurate):

COROLLARY 4. *If  $A$  is an operator such that  $\min A^*A > 0$  and  $0 \in \sigma(A)$ , then, for each real  $\theta$ ,  $\sigma(A)$  contains the segment*

$$S_\theta = \{se^{i\theta} : 0 \leq s \leq R_\theta\},$$

where

$$R_\theta = \max J_\theta - [(\max J_\theta)^2 - \min A^*A]^{1/2} > 0.$$

Moreover,  $\min_\theta R_\theta > 0$ , thus  $\sigma(A)$  contains the disc  $\{\lambda : |\lambda| \leq \min_\theta R_\theta\}$ .

*Proof.* The condition  $\min A^*A > 0$  means that  $0 \notin \pi(A)$  and therefore  $0 \notin \partial\sigma(A)$ , thus  $0$  is an interior point of  $\sigma(A)$ . (Incidentally,  $\pi(A) \neq \sigma(A)$ , so  $A$  is nonnormal; indeed,  $A^*$  cannot be hyponormal.)

Fix  $\theta$  and let  $L$  be the ray from  $0$  at angle  $\theta$ . If  $\lambda = re^{i\theta}$  is a boundary point of  $\sigma(A)$  on  $L$ , then (Corollary 3) by (1) one has  $(\max J_\theta)^2 \geq \min A^*A > 0$ ; since  $\max J_\theta$  is nonnegative (indeed  $\geq r$ ) it follows that  $R_\theta > 0$ . Moreover, by (2) one has  $|\lambda| = r \geq R_\theta$ .

To show that  $S_\theta \subset \sigma(A)$ , suppose  $\mu = se^{i\theta}$ ,  $0 < s \leq R_\theta$ . For any  $s_1$ ,  $0 \leq s_1 < s$ , the segment  $\{te^{i\theta} : s_1 \leq t \leq s\}$  must contain a point of  $\sigma(A)$  since otherwise some internal point  $\lambda$  of  $S_\theta$  would belong to  $\partial\sigma(A)$ , contrary to the preceding paragraph; thus  $\mu$  is adherent to, and therefore in,  $\sigma(A)$ .

Finally, since  $J_\theta$  and therefore  $R_\theta$  is a continuous function of  $\theta$  ( $0 \leq \theta \leq 2\pi$ ,  $0$  and  $2\pi$  identified) one has  $\min_\theta R_\theta > 0$ .

In view of the symmetry in Corollary 3, the proof of Corollary 4 also shows: If  $\min AA^* > 0$  and  $0 \in \sigma(A)$ , then, for each real  $\theta$ ,  $\sigma(A)$  contains the segment  $\{se^{i\theta} : 0 \leq s \leq R'_\theta\}$ , where

$$R'_\theta = \max J_\theta - [(\max J_\theta)^2 - \min AA^*]^{1/2} > 0;$$

if, in addition,  $A^*$  is hyponormal, then  $R'_\theta \geq R_\theta$ , which strengthens the conclusion of Corollary 4 [cf. 3, Corollary 2].

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