NOTE ON SOME SPECTRAL INEQUALITIES OF C. R. PUTNAM

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It is shown that if $A$ is any operator in Hilbert space and $\lambda = re^{i\theta}$ is in the approximate point spectrum of $A$, then

$$\min A^*A \leq (\max J_\theta)^2$$

and

$$|r - \max J_\theta| \leq [(\max J_\theta)^2 - \min A^*A]^{1/2},$$

where

$$J_\theta = (1/2) (A e^{-i\theta} + A^* e^{i\theta}).$$

Several corollaries are deduced for arbitrary operators, generalizing results of C. R. Putnam on semi-normal operators.

We employ the notations in Putnam's paper [3]. In particular if $A$ is any operator (bounded linear, in a Hilbert space) and $\theta$ is a real number, $J_\theta = \text{Re} (A e^{-i\theta}) = (1/2) (A e^{-i\theta} + A^* e^{i\theta})$. We write $\sigma(A)$ and $\pi(A)$ for the spectrum and approximate point spectrum of $A$, and $(x|y)$ for the inner product of vectors.

The following result extracts the essentials of the proof of Theorem 1 in Putnam's paper:

**Theorem.** If $A$ is any operator and $\lambda \in \pi(A)$, $\lambda = re^{i\theta}$ ($r \geq 0$), then

1. $\max J_\theta \geq r \geq (\min A^*A)^{1/2},$
2. $\max J_\theta - r \leq [(\max J_\theta)^2 - \min A^*A]^{1/2}.$

**Proof.** Let $x_n$ be a sequence of unit vectors with $(A - \lambda I)x_n \to 0$. Clearly $(Ax_n | x_n) \to \lambda$, $(x_n | Ax_n) \to \overline{\lambda}$; it follows that $(J_\theta x_n | x_n) \to r$ and therefore $\max J_\theta \geq r$. Since $||Ax_n||$ is bounded,

$$0 = \lim ((A - \lambda I)x_n | Ax_n) = \lim \{(A^*Ax_n | x_n) - \lambda(x_n | Ax_n)\},$$

thus $(A^*Ax_n | x_n) \to \lambda\overline{\lambda} = r^2$ and therefore $\min A^*A \leq r^2$. Thus (1) is proved. Since $(A - \lambda I)^*(A - \lambda I) = A^*A - 2rJ_\theta + r^2I$, one has

$$||(A - \lambda I)x_n||^2 = (A^*Ax_n | x_n) - 2r(J_\theta x_n | x_n) + r^2,$$

hence

$$\min A^*A \leq (A^*Ax_n | x_n) = ||(A - \lambda I)x_n||^2 + 2r(J_\theta x_n | x_n) - r^2 \leq ||(A - \lambda I)x_n||^2 + 2r \max J_\theta - r^2;$$
letting \( n \to \infty \),
\[
\min A^*A \leq 2r \max J_\theta - r^2.
\]
Thus \( \min A^*A \leq (\max J_\theta)^2 - (\max J_\theta - r)^2 \), which proves (2).

Incidentally, if \( \lambda = 0 \in \pi(A) \) then obviously \( \min A^*A = 0 \) and the theorem yields no information other than \( \max J_\theta \geq 0 \) for all \( \theta \).

If the dependence of \( J_\theta \) on \( A \) is indicated by writing \( J_\theta = J_\theta(A) \), evidently \( J_{-\theta}(A^*) = J_\theta(A) \). One has \( \pi(A^*) \subset \sigma(A^*) = (\sigma(A))^* \), thus \( (\pi(A^*))^* \subset \sigma(A) \); if \( \lambda = re^{i\theta} \in (\pi(A^*))^* \) then \( re^{-i\theta} \in \pi(A^*) \) and application of the theorem to \( A^* \) yields the following:

**Corollary 1.** If \( A \) is any operator and \( \lambda \in (\pi(A^*))^* \), \( \lambda = re^{i\theta} \), then

\[
\text{(3)} \quad \max J_\theta \geq r \geq (\min AA^*)^{1/2},
\]
\[
\text{(4)} \quad \max J_\theta - r \leq [(\max J_\theta)^2 - \min AA^*]^{1/2}.
\]

If \( A \) is hyponormal \( (AA^* \leq A^*A) \) then \( \pi(A^*) = \sigma(A^*) = (\sigma(A))^* \) [cf. 1, p. 1175] and Corollary 1 yields:

**Corollary 2.** If \( A \) is hyponormal then (3) and (4) hold for every \( \lambda \in \sigma(A) \), \( \lambda = re^{i\theta} \).

Another way of fulfilling (3) and (4) is via the relation
\[
\partial \sigma(A) \subset \pi(A) \cap (\pi(A^*))^*.
\]
If \( \lambda = re^{i\theta} \in \partial \sigma(A) \), the boundary of \( \sigma(A) \), then \( \lambda \in \pi(A) \) [cf. 2, p. 39] hence (1) and (2) hold by the theorem. Moreover, \( \lambda \in (\partial \sigma(A))^* = \partial(\sigma(A))^* = \partial \sigma(A^*) \subset \pi(A^*) \), i.e., \( \lambda \in (\pi(A^*))^* \) and so (3) and (4) hold by Corollary 1. Thus:

**Corollary 3.** If \( A \) is any operator and \( \lambda = re^{i\theta} \) is a boundary point of \( \sigma(A) \), then (1), (2), (3), (4) hold.

Corollary 3 is stated in [3, Th. 1; 4, p. 44, Th. 3.3.1] assuming \( AA^* \geq A^*A \) (i.e., \( A^* \) hyponormal).

It follows readily from Corollary 3, as in [3], that the spectrum of a nonunitary isometry is the entire closed unit disc. The proof is similar to, and simpler than, the proof of the following corollary, which extends a result in [3, Corollary 2; 4, p. 44, Corollary 1] (the formulation there is inaccurate):
COROLLARY 4. If $A$ is an operator such that $\min A^*A > 0$ and $0 \in \sigma(A)$, then, for each real $\theta$, $\sigma(A)$ contains the segment

$$S_\theta = \{se^{i\theta}: 0 \leq s \leq R_\theta\},$$

where

$$R_\theta = \max J_\theta - [(\max J_\theta)^2 - \min A^*A]^{1/2} > 0.$$

Moreover, $\min_\theta R_\theta > 0$, thus $\sigma(A)$ contains the disc $\{\lambda: |\lambda| \leq \min_\theta R_\theta\}$.

Proof. The condition $\min A^*A > 0$ means that $0 \in \pi(A)$ and therefore $0 \in \partial\sigma(A)$, thus $0$ is an interior point of $\sigma(A)$. (Incidentally, $\pi(A) \neq \sigma(A)$, so $A$ is nonnormal; indeed, $A^*$ cannot be hyponormal.)

Fix $\theta$ and let $L$ be the ray from $0$ at angle $\theta$. If $\lambda = re^{i\theta}$ is a boundary point of $\sigma(A)$ on $L$, then (Corollary 3) by (1) one has $(\max J_\theta)^2 \leq \min A^*A > 0$; since $\max J_\theta$ is nonnegative (indeed $\geq r$) it follows that $R_\theta > 0$. Moreover, by (2) one has $|\lambda| = r \geq R_\theta$.

To show that $S_\theta \subset \sigma(A)$, suppose $\mu = se^{i\theta}, 0 < s \leq R_\theta$. For any $s_1, 0 \leq s_1 < s$, the segment $\{te^{i\theta}: s_1 \leq t \leq s\}$ must contain a point of $\sigma(A)$ since otherwise some internal point $\lambda$ of $S_\theta$ would belong to $\partial\sigma(A)$, contrary to the preceding paragraph; thus $\mu$ is adherent to, and therefore in, $\sigma(A)$.

Finally, since $J_\theta$ and therefore $R_\theta$ is a continuous function of $\theta$ ($0 \leq \theta \leq 2\pi$, $0$ and $2\pi$ identified) one has $\min_\theta R_\theta > 0$.

In view of the symmetry in Corollary 3, the proof of Corollary 4 also shows: If $\min AA^* > 0$ and $0 \in \sigma(A)$, then, for each real $\theta$, $\sigma(A)$ contains the segment $\{se^{i\theta}: 0 \leq s \leq R'_\theta\}$, where

$$R'_\theta = \max J_\theta - [(\max J_\theta)^2 - \min AA^*]^{1/2} > 0;$$

if, in addition, $A^*$ is hyponormal, then $R'_\theta \geq R_\theta$, which strengthens the conclusion of Corollary 4 [cf. 3, Corollary 2].

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