

# Pacific Journal of Mathematics

**MEASURES ON COUNTABLE PRODUCT SPACES**

EDWIN O. ELLIOTT

## MEASURES ON COUNTABLE PRODUCT SPACES

E. O. ELLIOTT

A regular conditional measure  $\nu$  on a space  $Y$  relative to an outer measure  $\mu$  on a space  $X$  is defined as a function on  $X \times \mathcal{R}$  such that (1) for each  $x \in X$ ,  $\nu(x, \cdot)$  is an outer measure on  $Y$  and  $\mathcal{R}$  is the family of subsets of  $Y$  which are (Carathéodory) measurable under each of the measures  $\nu(x, \cdot)$ ,  $x \in X$ , and (2) for each  $\beta \in \mathcal{R}$  the function  $\nu(\cdot, \beta)$  on  $X$  is  $\mu$  integrable) i.e.,  $\int \nu(x, \beta) \mu dx \leq \infty$ .

Letting  $g$  be the function on the subsets of  $Z = X \times Y$  defined by

$$g(\beta) = \iint I_{\beta}(x, y) \nu(x, \cdot) dy \mu dx,$$

defining a covering family  $\mathfrak{S}$  to consist of those rectangles  $A \times B$  where  $A$  is  $\mu$  measurable,  $B \in \mathcal{R}$  and  $g(A \times B) < \infty$  or those sets  $N$  such that  $g(N) = 0$ , we obtain the outer measure  $\phi = (\mu \circ \nu)$  on  $Z$  generated by (the content)  $g$  and covering family  $\mathfrak{S}$ .

A system of regular conditional measures is a sequence begun by a measure  $\nu_0$  on a space  $X_1$  and followed by regular conditional measures  $\nu_i$  (relative to  $\mu_i$  on spaces  $X_{i+1}$  ( $i=1, 2, \dots$ )) where  $\mu_1 = \nu_0$  and  $\mu_{i+1} = (\mu_i \circ \nu_i)$  for  $i = 1, 2, \dots$ . Set  $X = \prod_i X_i$ , and for  $x \in X$  write  $x^i$  for the point  $(x_1, x_2, \dots, x_i)$  which is the projection of  $x$  onto the space  $X^i = \prod_{j=1}^i X_j$  and similarly write  $S^i = \prod_{j=1}^i S_j$  whenever the sets  $S_j$  are subsets of  $X_j$  ( $j = 1, \dots, i$ ).

For such a system of regular conditional measures a generalization of Tulcea's extension theorem for regular conditional probabilities holds, a Fubini-like theorem for integrable functions is obtained and finally, for topological spaces, a condition is given for the extension of inner regularity and almost Lindelöfness properties.

We let  $\mathcal{R}_1$  be the family of  $\nu_0$  measurable sets and let  $\mathcal{R}_i$  be the family of subsets of  $X_i$  which are measurable under each of the measures  $\nu_{i-1}(x^{i-1}, \cdot)$ ,  $x^{i-1} \in X^{i-1}$ , and let  $\mathfrak{S}_i$  be the family of subsets  $\gamma$  of  $X^i$  such that  $\mu_i(\gamma) = 0$  or  $\gamma = \alpha \times \beta$  where  $\alpha$  is  $\mu_{i-1}$  measurable and  $\beta \in \mathcal{R}_i$  and  $\mu_i(\gamma) < \infty$ . Thus  $\mathfrak{S}_i$  is the covering family which generates  $\mu_i$ .

Now, writing  $X_i^* = \prod_{j=i+1}^{\infty} X_j$  we define

$$\begin{aligned} \mathcal{R}^* &= \left\{ S: S = \prod_i \beta_i \text{ for some } \beta \text{ s.t. } \beta_i \in \mathcal{R}_i \text{ for each } i \right\} \\ \mathcal{R}^{**} &= \{ \beta: \text{for some } i, \beta = \alpha \times X_i^* \text{ where } \alpha \subset X^i \text{ and } \mu_i(\alpha) = 0 \}, \\ \mathcal{R} &= \mathcal{R}^* \cup \mathcal{R}^{**}, \end{aligned}$$

$g$  to be the function on  $\mathcal{R}$  which is zero on  $\mathcal{R}^{**}$  and given by

$$g(\beta) = \lim_i \mu_i(\beta^i)$$

on  $\mathcal{R}^*$ .

For  $\beta \in \mathcal{R}^*$  and  $x \in X$ , let

$$\rho_i(x, \beta) = \nu_0(\beta_1) \prod_{j=1}^{i-1} \nu_j(x^j, \beta_{j+1}),$$

and

$$\rho(x, \beta) = \lim_i \rho_i(x, \beta).$$

Let  $\mathcal{R}' = \{\beta \in \mathcal{R}^* : g(\beta) < \infty, \rho_i(x, \beta) \text{ is uniformly bounded on } \beta, \text{ and } \rho(x, \beta) \text{ exists for all } x \in X\}$  and  $\mathcal{R}' = \mathcal{R}' \cup \mathcal{R}^{**}$  and use  $\mathcal{R}'$  and  $g$  to generate a measure  $\varphi$  on  $X$ .

Our first objectives are to prove that  $\varphi$  and  $g$  agree on the covering family  $\mathcal{R}'$  and that members of  $\mathcal{R}$  are  $\varphi$  measurable. To do this we need and state a generalization of Tulcea's extension theorem for regular conditional probabilities. The final objective is to show that the product topology on  $X$  is inner regular and almost Lindelöf [1] whenever the component spaces are provided the spaces are of finite measure and the conditional measures are continuous [1]. The proof of this parallels that given for general product measures [2].

1. A generalization of Tulcea's extension theorem. Let a regular conditional measure system  $\nu'_i$  be given as above and assume that  $\nu'_0(X_1) = 1$  and  $\nu'_i(x^i, X_{i+1}) = 1$  for each  $i$  and  $x^i \in X^i$ , i.e.,  $\nu'_i$  is a system of regular conditional probabilities. Define the measures  $\mu'_i$  as above with  $\mu'_1 = \nu'_0$  and  $\mu'_{i+1} = (\mu'_i \circ \nu'_i)$  and let  $\mathcal{K}$  be the family of subsets of  $X$  which are cylinders in  $X$  over sets which are  $\mu'_i$  measurable for some  $i$ .

Now let  $\Psi$  be the measure on  $X$  generated by the covering family  $\mathcal{K}$  and the content  $h$  defined by

$$h(\beta) = \mu'_i(\alpha)$$

where  $\alpha \subset X^i$  and  $\beta = \alpha \times X_{i+1}^* \in \mathcal{K}$ .

The measure  $\Psi$  differs from the conventional Tulcea extension of the conditional probabilities  $\nu'_i$  in that in going from  $\mu'_i$  to  $\mu'_{i+1}$  the sets  $\alpha \subset X^{i+1}$  for which

$$\iint I_\alpha(x^{i+1}) \nu'_i(x^i, \cdot) dx_{i+1} \mu'_i dx^i = 0$$

are assigned measure zero whereas, in the conventional extension they may not even be measurable. The conventional method of proof [3]

for Tulcea's extension theorem, however, may be carried through for this new extension with essentially no changes. Therefore we give without proof the following.

**THEOREM 1.1.** *The members of  $\mathcal{H}$  are  $\Psi$  measurable and  $\Psi(\beta) = h(\beta)$  for each  $\beta \in \mathcal{H}$ .*

**2. Agreement and measurability.** Consider a member  $S$  of  $\mathcal{R}^{**}$  and let  $\nu'_0(\cdot) = \nu_0(\cdot)/\nu_0(S_1)$  to get a normalized measure on  $S_1$ , and let  $\nu'_i(x^i, \cdot) = \nu(x^i, \cdot)/\nu_i(x^i, S_{i+1})$  to get a regular conditional probability on  $S_{i+1}$ . Extending this system of probabilities as in §1 yields a (probability) measure  $\Psi_S$  on the space  $S$ . Let the measures  $\mu'_i$  on  $S^i$  be associated with the  $\nu'_i$  as in §1.

If  $\beta \in \mathcal{R}^{**}$  and  $\beta \subset S$  then  $\mu_i(\beta^i)$  is given by an  $i$  fold integral in

$$\begin{aligned} \mu_i(\beta^i) &= \int (i) \int I_{\beta^i}(x^i) \nu_{i-1}(x^{i-1}, \cdot) dx_i \cdots \nu_1(x^1, \cdot) dx_2 \nu_0 dx_1 \\ &= \int (i) \int I_{\beta^i}(x^i) \rho_i(x, S) \nu'_{i-1}(x^{i-1}, \cdot) dx_i \cdots \nu'_1(x^1, \cdot) dx_2 \nu'_0 dx_1 \\ &= \int I_{\beta^i}(x^i) \rho_i(x, S) \mu'_i dx^i \\ &= \int I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx . \end{aligned}$$

Thus, employing Lebesgue's theorem, we have

$$\begin{aligned} g(\beta) &= \lim_i \mu_i(\beta^i) \\ &= \lim_i \int I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx \\ &= \int \lim_i I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx \\ &= \int I_\beta(x) \rho(x, S) \Psi_S dx . \end{aligned}$$

Suppose now that  $\mathcal{G} \subset \mathcal{R}'$ ,  $\mathcal{G}$  is countable, and  $S = \cup \mathcal{G}$ , then the members of  $\mathcal{G}$  are  $\Psi_S$  measurable since the members of  $\mathcal{G}_1 = \mathcal{G} \cap \mathcal{R}^{**}$  are countable intersections of members of  $\mathcal{H}$  (i.e., cylinders over  $\mu'_i$  measurable sets for some  $i$ ) and members of  $\mathcal{G}_2 = \mathcal{G} \cap \mathcal{R}^{**}$  have  $\Psi_S$  measure zero. Hence,

$$I_S(x) \leq \sum_{\beta \in \mathcal{G}} I_\beta(x)$$

and

$$I_S(x) \leq \sum_{\beta \in \mathcal{G}_1} I_\beta(x) \quad \text{a.e. } \Psi_S .$$

Consequently,

$$\int I_S(x)\rho(x, S)\Psi_S dx \leq \sum_{\beta \in \mathcal{F}_1} \int I_\beta(x)\rho(x, S)\Psi_S dx + 0$$

and

$$\begin{aligned} g(S) &\leq \sum_{\beta \in \mathcal{F}_1} g(\beta) + 0 \\ &\leq \sum_{\beta \in \mathcal{F}} g(\beta), \end{aligned}$$

and we conclude  $g(S) = \varphi(S)$  proving the

**THEOREM 2.1.** *If  $S \in \mathcal{R}'$  then  $\varphi(S) = g(S)$ . Let*

$$\mathcal{M} = \{A: A = X^{i-1} \times \beta_i \times X_i^* \text{ for some } i \text{ and } \beta_i \in \mathcal{R}_i\}$$

and note that if  $A \in \mathcal{M}$  and  $S \in \mathcal{R}'$  then

$$S \cap A \in \mathcal{R}' \text{ and } S - A \in \mathcal{R}'$$

and

$$\varphi(S) = \varphi(S \cap A) + \varphi(S - A).$$

We consequently learn that members of  $\mathcal{M}$  are  $\varphi$  measurable since  $\mathcal{R}'$  is the covering family for  $\varphi$ . Countable intersections of members of  $\mathcal{M}$  are hence measurable proving the next

**THEOREM 2.2.** *If  $\beta \in \mathcal{R}$  then  $\beta$  is  $\varphi$  measurable.*

For  $x^i \in X^i$  let  $\xi_0(\cdot) = \nu_i(x^i, \cdot)$ , write  $x^i y^j$  for the point  $(x_1, \dots, x_i, y_1, \dots, y_j)$  and let  $\xi_j(y^j, \cdot) = \nu_i(x^i y^j, \cdot)$ ,  $j = 1, 2, \dots$ . The regular conditional measure system  $\xi_j$  then determines a measure  $\lambda_i(x^i, \cdot)$  on  $X_i^*$ . For  $\beta \subset X$  let us agree that  $\beta_x i = \{y: (x_1, \dots, x_i, y_1, y_2, \dots) \in \beta\}$ . Then we may state the

**THEOREM 2.3.** *If  $\beta$  is  $\varphi$  measurable then*

$$\varphi(\beta) = \int \lambda_i(x^i, \beta_x i) \mu_i dx^i = \iint I_\beta(x^i y) \lambda_i(x^i, \cdot) dy \mu_i dx^i$$

and  $\lambda_i$  is a regular conditional measure associated with  $\mu_i$ .

From [1, 1.6] we obtain the Fubini-like

**THEOREM 2.4.** *If  $f$  is  $\varphi$  integrable<sup>1</sup> then*

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<sup>1</sup>  $-\infty \leq \int f(z) \varphi dz \leq \infty$  and  $\{z: f(z) \neq 0\}$  is  $\sigma$ -finite under  $\varphi$ .

$$\int f(z)\varphi dz = \iint f(x^i, y)\lambda_i(x^i, \cdot)dy \mu_i dx^i .$$

3. **Topological measures.** To review the topological notions in [1] let us suppose that  $T$  is a topological space with  $\mathcal{S}$  being its family of open sets, and let  $\theta$  be a measure on  $T$  for which the open sets are measurable. Then  $\mathcal{S}$  is almost Lindelöf (a.L.) provided each covering of  $T$  by open sets contains a countable subfamily which covers almost all of  $T$ , and  $\mathcal{S}$  is inner regular (i.r.) provided each open set can be approximated in measure by closed subsets of finite measure, i.e., for each  $\beta \in \mathcal{S}$ ,

$$\theta(\beta) = \text{Sup}_{\gamma \text{ closed } \subset \beta} \theta(\gamma) < \infty .$$

Now let us assume that each of the spaces  $X_i$  is endowed with a topology  $\mathcal{S}_i$  and that  $\mathcal{S}^i$  is the product of the topologies  $\mathcal{S}_j$ ,  $1 \leq j \leq i$ . Then the sequence  $\mathcal{S}_i$  will be called a.L. and i.r. provided  $\mathcal{S}_1$  is a.L. and i.r. relative to  $\nu_0$  and  $\mathcal{S}_i$  is a.L. and i.r. relative to  $\nu_{i-1}(x^{i-1}, \cdot)$  for each  $x^{i-1} \in X^{i-1}$ , and the sequence  $\nu_i$  will be called continuous provided that for each  $i = 1, 2, \dots$ , the function  $\nu_i(\cdot, \beta)$  is finite and  $\mathcal{S}^i$  continuous for each set  $\beta$  which is measurable under all measures  $\nu_i(x^i, \cdot)$  where  $x^i \in X^i$ .

From [1, 2.3] and mathematical induction we obtain the

**THEOREM 3.1.** *If  $\mathcal{S}_i$  is a.L. and i.r.,  $\nu_i$  is continuous and  $\mu_i(X^i) < \infty$  for each  $i$ , then  $\mathcal{S}^i$  is a.L. and i.r. relative to  $\mu_i$  for each  $i$ .*

Let  $\mathcal{S}$  be the product topology on  $X$  obtained from the  $\mathcal{S}_i$ . Then we have the

**THEOREM 3.2.** *If  $\mathcal{S}_i$  is a.L. and i.r.,  $\nu_i$  is continuous,  $\mu_i(X^i) < \infty$  for each  $i$ , and  $\varphi(X) < \infty$  then  $\mathcal{S}$  is a.L. and i.r. relative to  $\varphi$ .*

*Proof.* Suppose  $A \in \mathcal{S}$ , then for some countable family  $\mathcal{G}$  such that each  $\alpha \in \mathcal{G}$  is a cylinder  $\alpha' \times \alpha''$  where  $\alpha' \in \mathcal{S}^{i(\alpha)}$  and  $\alpha'' = X_{i(\alpha)}^*$  we have  $A = \cup \mathcal{G}$ . Since  $\alpha'$  above is  $\mu_{i(\alpha)}$  measurable,  $\alpha$  is  $\varphi$  measurable and consequently  $A$  is  $\varphi$  measurable. Since  $\varphi(X) < \infty$  and each set  $\alpha'$  can be  $\mu_{i(\alpha)}$  approximated by a closed subset as closely as desired, it follows that each  $\alpha \in \mathcal{G}$  can be  $\varphi$  approximated as closely as desired by the (closed) cylinders over those closed subsets. Since  $\varphi(A) < \infty$  a finite subfamily  $\mathcal{G}'$  of  $\mathcal{G}$  can be chosen so that  $\varphi(\cup \mathcal{G}')$  is as close to  $\varphi(A)$  as desired. Hence  $A$  may be  $\varphi$  approximated as closely as desired by closed subsets (which are the union of the closed cylinders

associated with  $\mathcal{G}'$ ). Thus  $\mathcal{T}$  is i.r. relative to  $\varphi$ .

To see that  $\mathcal{T}$  is a.L., let  $\mathcal{H}$  be an open covering of  $X$  and let  $\mathcal{H}_i$  be the family of open sets in  $X^i$  such that each cylinder in  $X$  over one of these open sets is a subset of some member of  $\mathcal{H}$ . Thus, letting

$$\mathcal{C}_i = \{\beta: \beta = \alpha \times X_i^* \text{ for some } \alpha \in \mathcal{H}_i\}$$

we see that members of  $\mathcal{C}_i$  belong to the base for the topology  $\mathcal{T}$  and that  $X = \bigcup \mathcal{H} = \bigcup_i \bigcup \mathcal{C}_i$ . Using the fact that  $\mathcal{T}^i$  is both i.r. and a.L. we can select a countable subfamily  $\mathcal{H}'_i$  of  $\mathcal{H}_i$  for which  $\mu_i(\bigcup \mathcal{H}_i - \bigcup \mathcal{H}'_i) = 0$ . Now, letting

$$\mathcal{C}'_i = \{\beta: \beta = \alpha \times X_i^* \text{ for some } \alpha \in \mathcal{H}'_i\}$$

we have  $\phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{C}'_i) = 0$  and taking  $\mathcal{B}_i$  to be such a countable subfamily of  $\mathcal{H}$  that each member of  $\mathcal{C}'_i$  is a subset of some member of  $\mathcal{B}_i$ , we obtain further that  $\phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) = 0$ .

Finally, let  $\mathcal{B} = \bigcup_i \mathcal{B}_i$  and conclude,

$$\begin{aligned} X - \bigcup \mathcal{B} &= \bigcup_i \bigcup \mathcal{C}_i - \bigcup_i \bigcup \mathcal{B}_i \\ &\subset \bigcup_i (\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) \end{aligned}$$

and

$$\phi(X - \bigcup \mathcal{B}) \leq \sum_i \phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) = 0.$$

Noting that  $\mathcal{B}$  is a countable subfamily of  $\mathcal{H}$  completes the proof.

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