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MEASURES ON COUNTABLE PRODUCT SPACES

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A regular conditional measure ν on a space Y relative to an outer measure μ on a space X is defined as a function on $X \times \mathcal{R}$ such that (1) for each $x \in X$, $\nu(x, \cdot)$ is an outer measure on Y and \mathcal{R} is the family of subsets of Y which are (Carathéodory) measurable under each of the measures $\nu(x, \cdot)$, $x \in X$, and (2) for each $\beta \in \mathcal{R}$ the function $\nu(\cdot, \beta)$ on X is μ integrable i.e., $\int \nu(x, \beta) \mu dx \leq \infty$.

Letting g be the function on the subsets of $Z = X \times Y$ defined by

$$g(\beta) = \iint I_{\beta}(x, y) \nu(x, \cdot) dy \mu dx ,$$

defining a covering family \mathfrak{S} to consist of those rectangles $A \times B$ where A is μ measurable, $B \in \mathcal{R}$ and $g(A \times B) < \infty$ or those sets N such that $g(N) = 0$, we obtain the outer measure $\phi = (\mu \circ \nu)$ on Z generated by (the content) g and covering family \mathfrak{S} .

A system of regular conditional measures is a sequence begun by a measure ν_0 on a space X_1 and followed by regular conditional measures ν_i (relative to μ_i) on spaces X_{i+1} ($i=1, 2, \dots$) where $\mu_1 = \nu_0$ and $\mu_{i+1} = (\mu_i \circ \nu_i)$ for $i = 1, 2, \dots$. Set $X = \prod_i X_i$, and for $x \in X$ write x^i for the point (x_1, x_2, \dots, x_i) which is the projection of x onto the space $X^i = \prod_{j=1}^i X_j$ and similarly write $S^i = \prod_{j=1}^i S_j$ whenever the sets S_j are subsets of X_j ($j = 1, \dots, i$).

For such a system of regular conditional measures a generalization of Tulcea's extension theorem for regular conditional probabilities holds, a Fubini-like theorem for integrable functions is obtained and finally, for topological spaces, a condition is given for the extension of inner regularity and almost Lindelöfness properties.

We let \mathcal{R}_1 be the family of ν_0 measurable sets and let \mathcal{R}_i be the family of subsets of X_i which are measurable under each of the measures $\nu_{i-1}(x^{i-1}, \cdot)$, $x^{i-1} \in X^{i-1}$, and let \mathfrak{S}_i be the family of subsets γ of X^i such that $\mu_i(\gamma) = 0$ or $\gamma = \alpha \times \beta$ where α is μ_{i-1} measurable and $\beta \in \mathcal{R}_i$ and $\mu_i(\gamma) < \infty$. Thus \mathfrak{S}_i is the covering family which generates μ_i .

Now, writing $X_i^* = \prod_{j=i+1}^{\infty} X_j$ we define

$$\begin{aligned} \mathcal{R}^* &= \left\{ S : S = \prod_i \beta_i \text{ for some } \beta \text{ s.t. } \beta_i \in \mathcal{R}_i \text{ for each } i \right\} \\ \mathcal{R}^{**} &= \{ \beta : \text{for some } i, \beta = \alpha \times X_i^* \text{ where } \alpha \subset X^i \text{ and } \mu_i(\alpha) = 0 \} , \\ \mathcal{R} &= \mathcal{R}^* \cup \mathcal{R}^{**} , \end{aligned}$$

g to be the function on \mathcal{R} which is zero on \mathcal{R}^{**} and given by

$$g(\beta) = \lim_i \mu_i(\beta^i)$$

on \mathcal{R}^* .

For $\beta \in \mathcal{R}^*$ and $x \in X$, let

$$\rho_i(x, \beta) = \nu_0(\beta_1) \prod_{j=1}^{i-1} \nu_j(x^j, \beta_{j+1}),$$

and

$$\rho(x, \beta) = \lim_i \rho_i(x, \beta).$$

Let $\mathcal{R}' = \{\beta \in \mathcal{R}^* : g(\beta) < \infty, \rho_i(x, \beta) \text{ is uniformly bounded on } \beta, \text{ and } \rho(x, \beta) \text{ exists for all } x \in \beta\}$ and $\mathcal{R}' = \mathcal{R}' \cup \mathcal{R}^{**}$ and use \mathcal{R}' and g to generate a measure φ on X .

Our first objectives are to prove that φ and g agree on the covering family \mathcal{R}' and that members of \mathcal{R} are φ measurable. To do this we need and state a generalization of Tulcea's extension theorem for regular conditional probabilities. The final objective is to show that the product topology on X is inner regular and almost Lindelöf [1] whenever the component spaces are provided the spaces are of finite measure and the conditional measures are continuous [1]. The proof of this parallels that given for general product measures [2].

1. A generalization of Tulcea's extension theorem. Let a regular conditional measure system ν'_i be given as above and assume that $\nu'_i(X_1) = 1$ and $\nu'_i(x^i, X_{i+1}) = 1$ for each i and $x^i \in X^i$, i.e., ν'_i is a system of regular conditional probabilities. Define the measures μ'_i as above with $\mu'_1 = \nu'_0$ and $\mu'_{i+1} = (\mu'_i \circ \nu'_i)$ and let \mathcal{K} be the family of subsets of X which are cylinders in X over sets which are μ'_i measurable for some i .

Now let Ψ be the measure on X generated by the covering family \mathcal{K} and the content h defined by

$$h(\beta) = \mu'_i(\alpha)$$

where $\alpha \subset X^i$ and $\beta = \alpha \times X_i^* \in \mathcal{K}$.

The measure Ψ differs from the conventional Tulcea extension of the conditional probabilities ν'_i in that in going from μ'_i to μ'_{i+1} the sets $\alpha \subset X^{i+1}$ for which

$$\iint I_\alpha(x^{i+1}) \nu'_i(x^i, \cdot) dx_{i+1} \mu'_i dx^i = 0$$

are assigned measure zero whereas, in the conventional extension they may not even be measurable. The conventional method of proof [3]

for Tulcea's extension theorem, however, may be carried through for this new extension with essentially no changes. Therefore we give without proof the following.

THEOREM 1.1. *The members of \mathcal{H} are Ψ measurable and $\Psi(\beta) = h(\beta)$ for each $\beta \in \mathcal{H}$.*

2. Agreement and measurability. Consider a member S of $\mathcal{R}^{*'}$ and let $\nu_0(\cdot) = \nu_0(\cdot)/\nu_0(S_1)$ to get a normalized measure on S_1 , and let $\nu'_i(x^i, \cdot) = \nu(x^i, \cdot)/\nu_i(x^i, S_{i+1})$ to get a regular conditional probability on S_{i+1} . Extending this system of probabilities as in §1 yields a (probability) measure Ψ_S on the space S . Let the measures μ'_i on S^i be associated with the ν'_i as in §1.

If $\beta \in \mathcal{R}^{*'}$ and $\beta \subset S$ then $\mu_i(\beta^i)$ is given by an i fold integral in

$$\begin{aligned} \mu_i(\beta^i) &= \int (i) \int I_{\beta^i}(x^i) \nu_{i-1}(x^{i-1}, \cdot) dx_i \cdots \nu_1(x^1, \cdot) dx_2 \nu_0 dx_1 \\ &= \int (i) \int I_{\beta^i}(x^i) \rho_i(x, S) \nu'_{i-1}(x^{i-1}, \cdot) dx_i \cdots \nu'_1(x^1, \cdot) dx_2 \nu'_0 dx_1 \\ &= \int I_{\beta^i}(x^i) \rho_i(x, S) \mu'_i dx^i \\ &= \int I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx . \end{aligned}$$

Thus, employing Lebesgue's theorem, we have

$$\begin{aligned} g(\beta) &= \lim_i \mu_i(\beta^i) \\ &= \lim_i \int I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx \\ &= \int \lim_i I_{\beta^i}(x^i) \rho_i(x, S) \Psi_S dx \\ &= \int I_\beta(x) \rho(x, S) \Psi_S dx . \end{aligned}$$

Suppose now that $\mathcal{G} \subset \mathcal{R}'$, \mathcal{G} is countable, and $S = \cup \mathcal{G}$, then the members of \mathcal{G} are Ψ_S measurable since the members of $\mathcal{G}_1 = \mathcal{G} \cap \mathcal{R}^{*'}$ are countable intersections of members of \mathcal{H} (i.e., cylinders over μ'_i measurable sets for some i) and members of $\mathcal{G}_2 = \mathcal{G} \cap \mathcal{R}^{**}$ have Ψ_S measure zero. Hence,

$$I_S(x) \leq \sum_{\beta \in \mathcal{G}} I_\beta(x)$$

and

$$I_S(x) \leq \sum_{\beta \in \mathcal{G}_1} I_\beta(x) \quad \text{a.e. } \Psi_S .$$

Consequently,

$$\int I_s(x)\rho(x, S)\Psi_s dx \leq \sum_{\beta \in \mathcal{C}_1} \int I_\beta(x)\rho(x, S)\Psi_s dx + 0$$

and

$$\begin{aligned} g(S) &\leq \sum_{\beta \in \mathcal{C}_1} g(\beta) + 0 \\ &\leq \sum_{\beta \in \mathcal{C}} g(\beta) , \end{aligned}$$

and we conclude $g(S) = \varphi(S)$ proving the

THEOREM 2.1. *If $S \in \mathcal{R}'$ then $\varphi(S) = g(S)$. Let*

$$\mathcal{M} = \{A: A = X^{i-1} \times \beta_i \times X_i^* \text{ for some } i \text{ and } \beta_i \in \mathcal{R}_i\}$$

and note that if $A \in \mathcal{M}$ and $S \in \mathcal{R}'$ then

$$S \cap A \in \mathcal{R}' \text{ and } S - A \in \mathcal{R}'$$

and

$$\varphi(S) = \varphi(S \cap A) + \varphi(S - A) .$$

We consequently learn that members of \mathcal{M} are φ measurable since \mathcal{R}' is the covering family for φ . Countable intersections of members of \mathcal{M} are hence measurable proving the next

THEOREM 2.2. *If $\beta \in \mathcal{R}$ then β is φ measurable.*

For $x^i \in X^i$ let $\xi_i(\cdot) = \nu_i(x^i, \cdot)$, write $x^i y^j$ for the point $(x_1, \dots, x_i, y_1, \dots, y_j)$ and let $\xi_j(y^j, \cdot) = \nu_i(x^i y^j, \cdot)$, $j = 1, 2, \dots$. The regular conditional measure system ξ_j then determines a measure $\lambda_i(x^i, \cdot)$ on X_i^* . For $\beta \subset X$ let us agree that $\beta_x^i = \{y: (x_1, \dots, x_i, y_1, y_2, \dots) \in \beta\}$. Then we may state the

THEOREM 2.3. *If β is φ measurable then*

$$\varphi(\beta) = \int \lambda_i(x^i, \beta_x^i) \mu_i dx^i = \iint I_\beta(x^i y) \lambda_i(x^i, \cdot) dy \mu_i dx^i$$

and λ_i is a regular conditional measure associated with μ_i .

From [1, 1.6] we obtain the Fubini-like

THEOREM 2.4. *If f is φ integrable¹ then*

¹ $-\infty \leq \int f(z)\varphi dz \leq \infty$ and $\{z: f(z) \neq 0\}$ is σ -finite under φ .

$$\int f(z)\varphi dz = \iint f(x^i, y)\lambda_i(x^i, \cdot)dy \mu_i dx^i .$$

3. **Topological measures.** To review the topological notions in [1] let us suppose that T is a topological space with \mathcal{S} being its family of open sets, and let θ be a measure on T for which the open sets are measurable. Then \mathcal{S} is almost Lindelöf (a.L.) provided each covering of T by open sets contains a countable subfamily which covers almost all of T , and \mathcal{S} is inner regular (i.r.) provided each open set can be approximated in measure by closed subsets of finite measure, i.e., for each $\beta \in \mathcal{S}$,

$$\theta(\beta) = \text{Sup}_{\gamma \text{ closed}} \theta(\gamma) < \infty .$$

Now let us assume that each of the spaces X_i is endowed with a topology \mathcal{S}_i and that \mathcal{S}^i is the product of the topologies \mathcal{S}_j , $1 \leq j \leq i$. Then the sequence \mathcal{S}_i will be called a.L. and i.r. provided \mathcal{S}_1 is a.L. and i.r. relative to ν_0 and \mathcal{S}_i is a.L. and i.r. relative to $\nu_{i-1}(x^{i-1}, \cdot)$ for each $x^{i-1} \in X^{i-1}$, and the sequence ν_i will be called continuous provided that for each $i = 1, 2, \dots$, the function $\nu_i(\cdot, \beta)$ is finite and \mathcal{S}^i continuous for each set β which is measurable under all measures $\nu_i(x^i, \cdot)$ where $x^i \in X^i$.

From [1, 2.3] and mathematical induction we obtain the

THEOREM 3.1. *If \mathcal{S}_i is a.L. and i.r., ν_i is continuous and $\mu_i(X^i) < \infty$ for each i , then \mathcal{S}^i is a.L. and i.r. relative to μ_i for each i .*

Let \mathcal{S} be the product topology on X obtained from the \mathcal{S}_i . Then we have the

THEOREM 3.2. *If \mathcal{S}_i is a.L. and i.r., ν_i is continuous, $\mu_i(X^i) < \infty$ for each i , and $\varphi(X) < \infty$ then \mathcal{S} is a.L. and i.r. relative to φ .*

Proof. Suppose $A \in \mathcal{S}$, then for some countable family \mathcal{G} such that each $\alpha \in \mathcal{G}$ is a cylinder $\alpha' \times \alpha''$ where $\alpha' \in \mathcal{S}^{i(\alpha)}$ and $\alpha'' = X_{i(\alpha)}^*$ we have $A = \cup \mathcal{G}$. Since α' above is $\mu_{i(\alpha)}$ measurable, α is φ measurable and consequently A is φ measurable. Since $\varphi(X) < \infty$ and each set α' can be $\mu_{i(\alpha)}$ approximated by a closed subset as closely as desired, it follows that each $\alpha \in \mathcal{G}$ can be φ approximated as closely as desired by the (closed) cylinders over those closed subsets. Since $\varphi(A) < \infty$ a finite subfamily \mathcal{G}' of \mathcal{G} can be chosen so that $\varphi(\cup \mathcal{G}')$ is as close to $\varphi(A)$ as desired. Hence A may be φ approximated as closely as desired by closed subsets (which are the union of the closed cylinders

associated with \mathcal{G}'). Thus \mathcal{F} is i.r. relative to φ .

To see that \mathcal{F} is a.L., let \mathcal{H} be an open covering of X and let \mathcal{H}_i be the family of open sets in X^i such that each cylinder in X over one of these open sets is a subset of some member of \mathcal{H} . Thus, letting

$$\mathcal{C}_i = \{\beta: \beta = \alpha \times X_i^* \text{ for some } \alpha \in \mathcal{H}_i\}$$

we see that members of \mathcal{C}_i belong to the base for the topology \mathcal{F} and that $X = \bigcup \mathcal{H} = \bigcup_i \bigcup \mathcal{C}_i$. Using the fact that \mathcal{F}^i is both i.r. and a.L. we can select a countable subfamily \mathcal{H}'_i of \mathcal{H}_i for which $\mu_i(\bigcup \mathcal{H}_i - \bigcup \mathcal{H}'_i) = 0$. Now, letting

$$\mathcal{C}'_i = \{\beta: \beta = \alpha \times X_i^* \text{ for some } \alpha \in \mathcal{H}'_i\}$$

we have $\phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{C}'_i) = 0$ and taking \mathcal{B}_i to be such a countable subfamily of \mathcal{H} that each member of \mathcal{C}'_i is a subset of some member of \mathcal{B}_i , we obtain further that $\phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) = 0$.

Finally, let $\mathcal{B} = \bigcup_i \mathcal{B}_i$ and conclude,

$$\begin{aligned} X - \bigcup \mathcal{B} &= \bigcup_i \bigcup \mathcal{C}_i - \bigcup_i \bigcup \mathcal{B}_i \\ &\subset \bigcup_i (\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) \end{aligned}$$

and

$$\phi(X - \bigcup \mathcal{B}) \leq \sum_i \phi(\bigcup \mathcal{C}_i - \bigcup \mathcal{B}_i) = 0.$$

Noting that \mathcal{B} is a countable subfamily of \mathcal{H} completes the proof.

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Willard Ellis Baxter, <i>Topological rings with property (Y)</i>	563
Sterling K. Berberian, <i>Note on some spectral inequalities of C. R. Putnam</i>	573
David Theodore Brown, <i>Galois theory for Banach algebras</i>	577
Dennis K. Burke and R. A. Stoltenberg, <i>A note on p-spaces and Moore spaces</i>	601
Rafael Van Severen Chacon and Stephen Allan McGrath, <i>Estimates of positive contractions</i>	609
Rene Felix Dennemeyer, <i>Conjugate surfaces for multiple integral problems in the calculus of variations</i>	621
Edwin O. Elliott, <i>Measures on countable product spaces</i>	639
John Moss Grover, <i>Covering groups of groups of Lie type</i>	645
Charles Lemuel Hagopian, <i>Concerning semi-local-connectedness and cutting in nonlocally connected continua</i>	657
Velmer B. Headley, <i>A monotonicity principle for eigenvalues</i>	663
John Joseph Hutchinson, <i>Intrinsic extensions of rings</i>	669
Harold H. Johnson, <i>Determination of hyperbolicity by partial prolongations</i>	679
Tilla Weinstein, <i>Holomorphic quadratic differentials on surfaces in E^3</i>	697
R. C. Lacher, <i>Cell-like mappings. I</i>	717
Roger McCann, <i>A classification of centers</i>	733
Curtis L. Outlaw, <i>Mean value iteration of nonexpansive mappings in a Banach space</i>	747
Allan C. Peterson, <i>Distribution of zeros of solutions of a fourth order differential equation</i>	751
Bhalchandra B. Phadke, <i>Polyhedron inequality and strict convexity</i>	765
Jack Wyndall Rogers Jr., <i>On universal tree-like continua</i>	771
Edgar Andrews Rutter, <i>Two characterizations of quasi-Frobenius rings</i>	777
G. Sankaranarayanan and C. Suyambulingom, <i>Some renewal theorems concerning a sequence of correlated random variables</i>	785
Joel E. Schneider, <i>A note on the theory of primes</i>	805
Richard Peter Stanley, <i>Zero square rings</i>	811
Edward D. Tymchatyn, <i>The 2-cell as a partially ordered space</i>	825
Craig A. Wood, <i>On general Z.P.I.-rings</i>	837