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Faith posed the problem of characterizing the left intrinsic extensions of left quotient semisimple (simple) rings. In this paper a characterization is given for the left strongly intrinsic extensions of left quotient semisimple rings.

Section 1 consists of several definitions and known preliminary results. In §2 we define essential subdirect sums and develop several of their elementary properties. The results of §2 enable us to state and prove the main characterization theorem which appears in §3. In the last section it is shown that in the class of left quotient semisimple rings, the left strongly intrinsic extensions are exactly the left intrinsic extensions.

1. Preliminaries. Let R and S be nonzero associative rings (not necessarily with identities or commutative) where $S \subseteqq R$. S is *left quotient simple, left quotient semisimple,* a *left Ore domain* if S has a left classical (and maximal) quotient ring which is respectively simple Artinian, semisimple Artinian, a division ring. The left classical quotient ring of S will be denoted $\bar{S}$, and left quotient semisimple (left quotient simple) will be written lqss (lqs). R is a *left intrinsic extension* of S if every nonzero left ideal of R has nonzero intersection with S. A left S-module M (denoted ${}_SM$) is an *essential extension* of a submodule N if every nonzero submodule of M has nonzero intersection with N (we also say N is essential in M). R is a *left essential extension* of S if ${}_SR$ is an essential extension of ${}_SS$. It is clear that every left essential extension of S is left intrinsic, but the converse is not always true (for instance when R is a proper field extension of a field S). A left ideal A of S is *closed* if S contains no proper left essential extensions of A (as left S-modules). The symbol $L(S)$ will denote the set of closed left ideals of S. R is a *left strongly intrinsic extension* of S if R is a left intrinsic extension of S, and for all $A \in L(S)$ there exists a left ideal B of R such that $B \cap S = A$. In any left S-module M, we denote by $Z({}_SM)$ the set of elements in M whose annihilator in S is an essential left ideal. Clearly $Z({}_SM)$ is a submodule of M.

THEOREM 1.1. *If $Z({}_SS) = 0$, then ${}_SS$ has a (unique up to isomorphism) maximal essential extension Q (called the maximal quotient ring of S) which has a ring structure compatible with the module structure; and Q is a regular, left self-injective ring such that*

$Z({}_QQ) = 0$. *Moreover* $L(S)$ *and* $L(Q)$ *are lattices, and* $L(Q) \cong L(S)$ *under contraction.*

Proof. Follows from [1, Theorem 1, p 69] and [3, Corollary 2.6] and their proofs.

The following two lemmas appear in [2].

LEMMA 1.2. *If R is a left strongly intrinsic extension of S, then the following are equivalent*: (i) $Z({}_SS) = 0$, (ii) $Z({}_SR) = 0$, (iii) $Z({}_RR) = 0$.

LEMMA 1.3. *If* $Z({}_SS) = 0$ *and R is a left strongly intrinsic extension of S, then* $L(R) \cong L(S)$ *under contraction.*

2. **Essential subdirect sums.** If R is a subdirect sum of rings $\{R_\alpha \mid \alpha \in A\}$ and $S = \sum_{\alpha \in A}^c R_\alpha$ is the complete direct sum of the R_α, then the subdirect sum is essential if R (identifying R and its canonical isomorphic image in S) is an essential left R-submodule of S.

Clearly an essential subdirect sum of nonzero rings is irredundant [5], and in the case of a finite number of factors, is essentially irredundant in the sense of [1, p 114]. It is an easily verified property of subdirect sums that if $B_i (i \in D)$ are disjoint subsets of A such that $A = \bigcup_{i \in D} B_i$ and $R_{B_i} = \{a \in \sum_{\alpha \in B_i}^c R_\alpha \mid$ for some $b \in R$ $a(\alpha) = b(\alpha)$ for all $\alpha \in B_i\}$, then for each $i \in D$, the R_{B_i} are rings which are subdirect sums in a natural way of the rings $\{R_\alpha \mid \alpha \in B_i\}$, and R is a subdirect sum in a natural way of the rings $\{R_{B_i} \mid i \in D\}$. If, in addition, each R_α is a subdirect sum of rings $\{T_{\alpha\gamma} \mid \gamma \in A_\alpha\}$, then R is a subdirect sum in a natural way of the rings $\{T_{\alpha\gamma} \mid \gamma \in A_\alpha$ whenever $\alpha \in A\}$. Each of these constructed subdirect sums will be referred to as the *induced subdirect sum*, and whenever we say "the subdirect sum" we are referring either to the original fixed subdirect sum or one of its various induced subdirect sums. Loosely speaking, we may think of the preceding remarks as saying that subdirect sums satisfy a generalized associative law. The results in this section will show that finite essential subdirect sums also have this nice property (finite irredundant subdirect sums do not).

LEMMA 2.1. *Let R be a subdirect sum of nonzero rings $R_1, \cdots, R_n$. The subdirect sum is essential if and only if $R \cap R_i$ is an essential left R-submodule of R_i for $i = 1, 2, \cdots, n$.*

Proof. If the subdirect sum is essential and W_1 is a nonzero R-submodule of R_1, then W_1 is also a nonzero left R-submodule of $\bigoplus \sum_{i=1}^n R_i$, and so $W_1 \cap R \neq 0$. Since $W_1 \cap (R \cap R_1) = W_1 \cap R$ the result follows. Conversely suppose $R \cap R_i$ is an essential R-submodule of R_i for each i. If $n = 1$ the result is trivial, so suppose $n = 2$. Let $0 \neq x_1 + x_2 \in R_1 \oplus R_2 (x_i \in R_i)$ and assume $x_1 \neq 0$. Since $R \cap R_1$ is essential in R_1, there exists an integer n and an $r \in R$ such that $0 \neq rx_1 + nx_1 = (r + n)x_1 \in R \cap R_1$. If $(r + n)x_2 = 0$, then $0 \neq (r+n)(x_1 + x_2) \in R$. If $(r + n)x_2 \neq 0$, then since $R \cap R_2$ is essential in R_2 there exists an integer m and an $s \in R$ such that $0 \neq (s + m)(r + n)x_2 \in R \cap R_2$. Then clearly $0 \neq (s + m)(r + n)(x_1 + x_2) \in R$, and R is essential in $R_1 \oplus R_2$. The result now follows by a simple induction.

PROPOSITION 2.2. *Let Q be a ring which is an essential subdirect sum of nonzero rings $Q_1, \cdots, Q_n$. If each Q_i is an essential subdirect sum of nonzero rings $Q_{i,1}, \cdots, Q_{i,k_i}$, then the induced subdirect sum of Q (of the $Q_{i,j}$) is essential. Also, if $n_1, n_2, \cdots, n_{k+1}$ are integers such that $1 = n_1 < n_2 < \cdots < n_{k+1} = n + 1$, and $Q'_i (i = 1, 2, \cdots, k)$ are the induced subdirect sums of $Q_{n_i}, \cdots, Q_{n_{i+1}-1}$, then Q is the essential subdirect sum of $Q'_1, \cdots, Q'_k$ and each Q'_i is the essential subdirect sum of $Q_{n_i}, \cdots, Q_{n_{i+1}-1}$.*

Proof. The result follows by a fairly straightforward application of Lemma 2.1.

The following theorem is a modification of a theorem of Levy, [5, Theorem 6.1].

THROREM 2.3. *R is lqss if and only if it is an essential subdirect sum of a finite number of lqs rings $R_1, \cdots, R_n$ for some n. In this case, we have $\bar{R} = \bigoplus \sum_{i=1}^{n} \bar{R}_i$.*

Proof. If R is lqss, then by [5, Theorem 6.1], R is an irredundant subdirect sum of lqs rings $R_1, \cdots, R_n$ for some n, and $R \subseteqq R_1 \oplus \cdots \oplus R_n \subseteqq \bar{R}_1 \oplus \cdots \oplus \bar{R}_n \subseteqq \bar{R}$. Since $\bar{R}$ is an essential extension of R, it follows that R is essential in $R_1 \oplus \cdots \oplus R_n$, and R is then an essential subdirect sum of $R_1, \cdots, R_n$. Conversely, if R is a finite essential (and so irredundant) subdirect sum of lqs rings $R_1, \cdots, R_n$, then R is lqss by [5, Theorem 6.1].

THEOREM 2.4. *If R (S) is an essential subdirect sum of prime rings $R_1, \cdots, R_n$ $(S_1, \cdots, S_n)$, $S \subseteqq R$, and each R_i is a left intrinsic extension of the corresponding S_i, then R is a left intrinsic extension of S.*

Proof. If $0 \neq x \in R$, then $x = x_1 + \cdots + x_n (x_i \in R_i)$. We may assume $x_1 \neq 0$, and so $0 \neq R_1 x_1 = R_1 x$ since R_1 is prime. Thus $R_1 x$ is a nonzero left ideal of R_1, so $R_1 x \cap S_1 \neq 0$. $R_1 x \cap S_1$ is a nonzero left S-submodule of $S_1 \oplus \cdots \oplus S_n$, so $S \cap R_1 x \cap S_1 \neq 0$. Since S_1 is prime, $(S \cap R_1 x \cap S_1)^2 \neq 0$. Hence there exists $s, s' \in S \cap R_1 x \cap S_1$ such that $0 \neq ss' \in S$. If $s' = r_1 x$, then there exists $r \in R$ such that $r = r_1 + \cdots + r_n$. Hence $srx = sr_1 x = ss' \in Rx \cap S$, so R is a left intrinsic extension of S.

3. The main theorem.

LEMMA 3.1. *If rings R and S are direct sums of division rings $R_1, \cdots, R_n$ and $S_1, \cdots, S_m$ respectively, and R is a left intrinsic extension of S such that their identities coincide, then $m = n$ and $R_1 \cap S = S_i$ for a suitable arrangement of the R_i.*

Proof. Since R_1 is a nonzero ideal of R, it follows that $R_1 \cap S$ is a nonzero ideal of S. The ideals of S are direct sums of some of the S_i, so $R_1 \cap S = \bigoplus \sum_{i=1}^{k} S_i$ for some rearrangement of the S_i. But $k = 1$, otherwise R_1 has nonzero zero divisors. Similarly, each R_i contains exactly one S_i, so $n \leqq m$ and $R_i \cap S = S_i$ for $i = 1, 2, \cdots, n$. Each S_i^* $(i = 1, 2, \cdots, n)$ is a multiplicative subgroup of R_i^*, so their identities coincide. Equating identities leads to a contradiction if $n < m$, so we must have $n = m$.

PROPOSITION 3.2. *If S is a left Ore domain, then R is a left intrinsic extension of S if and only if R is a left strongly intrinsic extension of S.*

Proof. Let R be a left intrinsic extension of S. By Theorem 1.1, $L(S) \cong L(\bar{S}) = \{0, \bar{S}\}$, so $L(S) = \{0, S\}$. The zero ideal of R and R itself contract to the elements of $L(S)$, so R is a left strongly intrinsic extension of S. Note that by Lemma 1.3, $L(R) = \{0, R\}$.

PROPOSITION 3.3. *If R is a left intrinsic extension of a left Ore domain S, then R is a left Ore domain.*

Proof. By Proposition 3.2, R is a left strongly intrinsic extension of S, and $\{0, R\} = L(R)$ which clearly satisfies the maximum condition. By Lemma 1.2, $Z({}_R R) = Z({}_S S) = 0$. If A is a nilpotent left ideal of R, then $A \cap S$ is a nilpotent left ideal of S. Thus $A \cap S = 0$, and $A = 0$. Hence R is semiprime, and by [3, Theorem 4.4], R is lqss. Thus $\{0, R\} = L(R) \cong L(\bar{R}) = \{0, \bar{R}\}$. By [1, Proposition 5, p. 71], $L(\bar{R})$ consists of the annihilator left ideals of $\bar{R}$. Since $\bar{R}$ has an identity, $\bar{R}$ is a domain. It follows that R is a left Ore domain.

THEOREM 3.4. *If S is lqss, then R is a left strongly intrinsic extension of S if and only if $S \subseteqq R$ and one of the following is true:*

(i) $\bar{S} = \bar{R}$ *is semisimple Artinian,*

(ii) S *(and R) is an essential subdirect sum of left Ore domains* $S'_1, \cdots, S'_n$ ($R'_1, \cdots, R'_n$) *where R'_i is a left intrinsic extension of the corresponding S'_i,*

(iii) S *(and R) is an essential subdirect sum of nonzero rings S_1 and S_2 (R_1 and R_2) where $S_i \subseteqq R_i$ for $i = 1, 2$ and such that* (i) *holds for S_2 and R_2 and* (ii) *holds for S_1 and R_1.*

Proof. By [3, Theorem 4.4], $L(S)$ satisfies the maximum condition, so by Lemma 1.3, so does $L(R)$. By Lemma 1.2, $Z({}_RR) = Z({}_SS) = 0$; and as in Proposition 3.3, R is semiprime. Thus by [3, Theorem 4.4], R is lqss. By Theorem 1.1, $\bar{R}$ is a regular, semisimple, left self-injective ring. The lattice of principal left ideals of $\bar{R}$ is complete by [6, Theorem 1], so by [6, Corollary to Theorem 4], $\bar{R}$ can be decomposed into the direct sum of two ideal Q_1 and Q_2 in such a way that Q_1 is strongly regular and Q_2 does not contain any nonzero strongly regular ideals. By [2, Theorem 2.5], there is a subring T of Q_1 with the properties that:

(a) T contains every idempotent of Q_1,

(b) T is a strongly regular self-injective ring,

(c) $\bar{S} = T \oplus Q_2$.

Since $\bar{S}$ ($\bar{R}$) is semisimple Artinian, $\bar{S} = \bigoplus \sum_{i=1}^{m} F_i$ ($\bar{R} = \bigoplus \sum_{i=1}^{m'} D_i$) where each F_i (D_i) is simple Artinian. Since $\bar{S} = T \oplus Q_2$ ($\bar{R} = Q_1 \oplus Q_2$), we have $T = \bigoplus \sum_{i=1}^{n} F_i$ ($Q_1 = \bigoplus \sum_{i=1}^{n'} D_i$) where $0 \leqq n \leqq m$ ($0 \leqq n' \leqq m'$), and the F_i (D_i) are suitably arranged. Since strongly regular rings have no nonzero nilpotent elements, it follows that $F_1, \cdots, F_n, D_1, \cdots, D_{n'}$, are division rings (if $n \neq 0 \neq n'$). It is clear that Q_1 is a left intrinsic extension of T (so $T = 0$ if and only if $Q_1 = 0$). By property (a) the identities of T and Q_1 coincide, so by Lemma 3.1, $n = n'$ and $D_i \cap T = F_i$ for $i = 1, 2, \cdots, n$.

Let e_1 be the identity of Q_1 (and T) and e_2 the identity of Q_2. If $T \neq 0$, let $d_1, \cdots, d_n$ be the identities of $D_1, \cdots, D_n$ (and of $F_1, \cdots, F_n$). Let $R_i = Re_i$ and $S_i = Se_i$ for $i = 1, 2$. Clearly $S_i \subseteqq R_i \subseteqq Q_i$; $Q_i = 0$ if and only if $R_i = 0$ if and only if $S_i = 0$; and S (R) is a subdirect sum of S_1 and S_2 (R_1 and R_2).

We claim that if $Q_1 \neq 0$, then $\bar{R}_1 = Q_1$ and $\bar{S}_1 = T$; and if $Q_2 \neq 0$, then $\bar{S}_2 = \bar{R}_2 = Q_2$. Suppose $Q_1 \neq 0$ and r is regular in R. Clearly $re_1 \in R_1$ is regular in R_1, so R_1 has regular elements. If r_1 is any regular element in R_1 and $q_1 r_1 = 0$ where $q_1 \in Q_1$, then $q_1 = c^{-1}b$ ($c, b \in R$), so $0 = br_1 = (be_1)r_1$. Since r_1 is regular in R_1, it follows that $be_1 = 0$, and so $q_1 = q_1e_1 = c^{-1}be_1 = 0$. Hence r_1 is not a zero divisor in Q_1, so by [1, Corollary 4, p. 70], r_1 is invertible in Q_1. If q_1 is given, then $q_1 = d^{-1}b$ $(d, b \in R)$ and $q_1 = q_1e_1 = (d^{-1}b)e_1 = (de_1)^{-1}(be_1)$. Hence $Q_1 = \bar{R}_1$, and exactly the same argument gives that $T = \bar{S}_1$ and (if $Q_2 \neq 0$) that $Q_2 = \bar{R}_2 = \bar{S}_2$.

Since $S \subseteqq S_1 \oplus S_2 \subseteqq T \oplus Q_2 = \bar{S}$ and $R \subseteqq R_1 \oplus R_2 \subseteqq Q_1 \oplus Q_2 = \bar{R}$, it follows that S (R) is an essential subdirect sum of S_1 and S_2 (R_1 and R_2).

If $R_1 = 0$, then $S_1 = 0$; so $S = S_2 \subseteqq R_2 = R$ and $\bar{S} = \bar{S}_2 = Q_2 = \bar{R}_2 = \bar{R}$. This is condition (i).

If $R_1 \neq 0$, then $e_1 = d_1 + \cdots + d_n$, $S_1 = Se_1 \subseteqq Sd_1 + \cdots + Sd_n$, and $R_1 = Re_1 \subseteqq Rd_1 + \cdots + Rd_n$. If $S_i' = S_1d_i = Sd_i$ $(R_i' = R_1d_i = Rd_i)$ for $i = 1, 2, \cdots n$, it follows as before that S_1 (R_1) is a subdirect sum of $S_1', \cdots, S_n'$ $(R_1', \cdots, R_n')$. In exactly the same way as we proved that $\bar{R}_1 = Q_1$, we get that $\bar{S}_i' = F_i$ and $\bar{R}_i' = D_i$ for $i = 1, 2, \cdots, n$. Also, as before, the subdirect sums are essential.

We next show that R_1' is a left intrinsic extension of S_1' (and similarly for $R_2', \cdots, R_n'$). Let $0 \neq x = rd_1 \in Rd_1 = R_1'$ $(r \in R)$. Then $R_1'x = Rx \neq 0$, so $Rx \cap R \neq 0$ (since $Rx \subseteqq \bar{R}$). Since R is a left intrinsic extension of S, we have $Rx \cap R \cap S = Rx \cap S \neq 0$. Thus if $0 \neq s = r'x \in Rx \cap S$ $(r' \in R)$, we have $0 \neq s = sd_1 \in Sd_1$. Hence $0 \neq s \in (Rd_1)x \cap Sd_1 = R_1'x \cap S_1'$, and R_1' is a left intrinsic extension of S_1'.

If $R_2 = 0$, then $S_2 = 0$; so $S = S_1$ and $R = R_1$ which gives condition (ii). If R_1 and R_2 are not zero, then condition (iii) is satisfied.

Conversely, suppose condition (i) is true. Hence $S \subseteqq R \subseteqq \bar{S}$, and $\bar{R}$ exists. Thus by [3, Corollary 2.6], $L(S) \cong L(\bar{S}) = L(\bar{R}) \cong L(R)$ under contraction, so R is a left strongly intrinsic extension of S.

In condition (ii), we have by Theorem 2.3 that $\bar{S} = \oplus \sum_{i=1}^{n} \bar{S}_i'$, and $\bar{R} = \oplus \sum_{i=1}^{n} \bar{R}_i'$, where $\bar{S}_i'$, and $\bar{R}_i'$, are division rings. Clearly $L(\bar{S}) \cong L(\bar{R})$ under contraction, and since $L(S) \cong L(\bar{S})$ and $L(R) \cong L(\bar{R})$, it follows that $L(S) \cong L(R)$ under contraction. By Theorem 2.4, R is a left intrinsic extension of S, so R is a left strongly intrinsic extension of S.

In condition (iii), $\bar{S}_2 = \bar{R}_2$ are semisimple Artinian, so S_2 and R_2 are lqss. Let $\bar{S}_2 = \bar{R}_2 = \oplus \sum_{i=1}^{m} F_i$, where F_i are simple Artinian rings with identities e_i $(i = 1, 2, \cdots, m)$. Let $S_{n+i}' = S_2e_i$ and $R_{n+i}' = R_2e_i$ for $i = 1, 2, \cdots, m$. By Theorem 2.3 and the proof of [5, Theorem 6.1], we have that S_2 (R_2) is an essential subdirect sum of the lqs rings S_i' (R_i') $(i = n + 1, \cdots, n + m)$, and $\bar{S}_i' = \bar{R}_i' = F_{i-n}$ for $i = n + 1, \cdots, n + m$. Since $S_i' \subseteqq R_i' \subseteqq \bar{S}_i'$, (for $i = n + 1, \cdots, n + m$), it follows that $\bar{R}_i'$ is a left intrinsic extension of S_i' for $i = 1, \cdots, n + m$. Thus by Proposition 2.2 and Theorem 2.4, R is a left intrinsic extension of S. Also $\bar{S} = \oplus \sum_{i=1}^{n+m} \bar{S}_i, = \bar{S}_1 \oplus \bar{S}_2$, and $\bar{R} = \oplus \sum_{i=1}^{n+m} \bar{R}_i, = \bar{R}_1 + \bar{R}_2$, and as in the proof of case (ii), $L(\bar{S}_1) \cong L(\bar{R}_1)$ under contraction. Since

$L(\bar{S}_2) = L(\bar{R}_2)$, it follows that $L(\bar{S}) \cong L(\bar{R})$ under contraction. Again $L(S) \cong L(R)$ under contraction, so R is a left strongly intrinsic extension of S.

COROLLARY 3.5. *R is a left strongly intrinsic extension of a lqs ring S if and only if either $S \subseteqq R \subseteqq \bar{S}$ or S and R are left Ore domains such that R is a left intrinsic extension of S.*

Proof. If $S \subseteqq R \subseteqq \bar{S}$, then R is a left strongly intrinsic extension of S by case (i). If R and S are left Ore domains such that R is a left intrinsic extension of S, then the result follows from Proposition 3.2.

Conversely, since $\bar{S} = T \oplus Q_2$, we have either $T = 0$ or $Q_2 = 0$. If $Q_2 = 0$, then $\bar{S} = T = F_1$ and $\bar{R} = Q_1 = D_1$, so R and S are left Ore domains and R is a left intrinsic extension of S. If $T = 0$; then $S_1 = R_1 = 0, S = S_2$, and $R = R_2$ which is case (i).

4. **Left intrinsic extensions.** In this section, it is shown that, in the case of lqss rings, every left intrinsic extension is left strongly intrinsic.

LEMMA 4.1. *If R is a left intrinsic extension of S, then $Z(_SS) \subseteqq Z(_SR) \subseteqq Z(_RR)$.*

Proof. The first containment is clear. If $x \in R$ and $x \notin Z(_RR)$, then the left annihilator in R of x (denoted $l_R(x)$) is not an essential left ideal of R. Thus there exists a nonzero left ideal A of R such that $l_R(x) \cap A = 0$. Thus $0 = l_R(x) \cap A \cap S = l_S(x) \cap (A \cap S)$, and $A \cap S \neq 0$. Hence $x \notin Z(_SR)$, and so $Z(_SR) \subseteqq Z(_RR)$.

LEMMA 4.2. *Let S have a left classical quotient ring. If $Z(_RR) = 0$ and R is a left intrinsic extension of S, then every regular element of S is a regular element of R.*

Proof. Let s be a regular element of S, and $r \in R$. If $rs = 0$, then $r \in l_R(s)$. Clearly, $l_S(s) = l_R(s) \cap S = 0$, so $l_R(s) = 0$ and $r = 0$. If $sr = 0$, then $(Ss)r = 0$ and $r \in Z(_SR)$. By Lemma 4.1, $Z(_SR) \subseteqq Z(_RR) = 0$, so $r = 0$. Thus s is regular in R.

LEMMA 4.3. *Let S have a classical left quotient ring $\bar{S}$. If R is a left intrinsic extension of S where $Z(_RR) = 0$, then $\bar{S} \subseteqq Q$ where Q is the maximal left quotient ring of R.*

Proof. Let M be the injective hull of R as a left R-module. By [1, Theorem 1, p. 69], $Q = \mathrm{Hom}_R(M, M) \cong M$. If d is a regular elements of S, define the map $\bar{f}: Rd \to R$ by $\bar{f}(rd) = r$ for all $r \in R$. The map is well defined by Lemma 4.2, and by the injectivity of M, there exists $f \in \mathrm{Hom}_R(M, M)$ such that $f|_{Rd} = \bar{f}$. By [1, Theorem 1, p. 69], the canonical isomorphic image of d in Q is the unique $g \in \mathrm{Hom}_R(M, M)$ such that $g(r) = rd$ for all $r \in R$. If 1 denotes the identity of Q, it follows that $R \subseteqq \ker(1 - gf)$ and $Rd \subseteqq \ker(1 - gf)$. R is left essential in Q, and it is easy to verify that Rd is also essential in Q. By [1, Theorem 1, p. 44], $1 - gf$ and $1 - fg$ are in the Jacobson radical of the semisimple ring Q. Hence $gf = fg = 1$. By the canonical injection of R into Q, we can consider R to be a subring of Q, and so d has a two-sided inverse f (henceforth denoted d^{-1}) in Q. Hence every regular element of S has a two-sided inverse in Q. If $T = \{a^{-1}b \mid b \in S,\ a \text{ regular in } S\}$, then $T \subseteqq Q$ and T is a ring by Ore's condition for S, [see 4, p. 109]. Hence $\bar{S} = T \subset Q$.

LEMMA 4.4. *If S is a left self-injective ring and $Z({}_SS) = 0$, then every left intrinsic extension of S is a left strongly intrinsic extension of S.*

Proof. Let R be a left intrinsic extension of S. By [1, Theorem 1, p. 69], S is its own maximal left quotient ring and is a regular ring. If $A \in L(S)$, then by [1, Theorem 4, p. 70], $A = Se$ where $e^2 = e \in S$. Hence $Re \cap S = Se$, and R is a left strongly intrinsic extension of S.

THEOREM 4.5. *If R is an extension of a lqss ring S and $Z({}_RR) = 0$, then R is a left intrinsic extension of S if and only if R is a left strongly intrinsic extension of S.*

Proof. Let R be a left intrinsic extension of S and Q the maximal left quotient ring of R. By Lemma 4.3, $\bar{S} \subseteqq Q$, and clearly Q is a left intrinsic extension of S. Also $\bar{S}$ is left self-injective and $Z({}_S\bar{S}) = 0$, so by Lemma 4.4, Q is a left strongly intrinsic extension of the lqss ring $\bar{S}$. By Lemma 1.3, $L(Q) \cong L(\bar{S})$ under contraction, and since $L(Q) \cong L(R)$ and $L(S) \cong L(\bar{S})$ under contraction, it follows that $L(S) \cong L(R)$ under contraction. Hence R is a left strongly intrinsic extension of S.

THEOREM 4.6. *If R is a left intrinsic extension of a lqss ring S, then the following are equivalent:*

(i) $Z({}_RR) = 0$,

(ii) *R is a left strongly intrinsic extension of S,*

(iii) *R is lqss.*

Proof. (i) $\Rightarrow$ (ii) by Theorem 4.5. (ii) $\Rightarrow$ (iii) by the proof of Theorem 3.4. (iii) $\Rightarrow$ (i) follows from [3, Theorem 4.4].

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