INTRINSIC EXTENSIONS OF RINGS

JOHN JOSEPH HUTCHINSON
Faith posed the problem of characterizing the left intrinsic extensions of left quotient semisimple (simple) rings. In this paper a characterization is given for the left strongly intrinsic extensions of left quotient semisimple rings.

Section 1 consists of several definitions and known preliminary results. In §2 we define essential subdirect sums and develop several of their elementary properties. The results of §2 enable us to state and prove the main characterization theorem which appears in §3. In the last section it is shown that in the class of left quotient semisimple rings, the left strongly intrinsic extensions are exactly the left intrinsic extensions.

1. Preliminaries. Let $R$ and $S$ be nonzero associative rings (not necessarily with identities or commutative) where $S \subseteq R$. $S$ is left quotient simple, left quotient semisimple, a left Ore domain if $S$ has a left classical (and maximal) quotient ring which is respectively simple Artinian, semisimple Artinian, a division ring. The left classical quotient ring of $S$ will be denoted $\bar{S}$, and left quotient semisimple (left quotient simple) will be written lqss (lqs). $R$ is a left intrinsic extension of $S$ if every nonzero left ideal of $R$ has nonzero intersection with $S$. A left $S$-module $M$ (denoted $S M$) is an essential extension of a submodule $N$ if every nonzero submodule of $M$ has nonzero intersection with $N$ (we also say $N$ is essential in $M$). $R$ is a left essential extension of $S$ if $S R$ is an essential extension of $S$. It is clear that every left essential extension of $S$ is left intrinsic, but the converse is not always true (for instance when $R$ is a proper field extension of a field $S$). A left ideal $A$ of $S$ is closed if $S$ contains no proper left essential extensions of $A$ (as left $S$-modules). The symbol $L(S)$ will denote the set of closed left ideals of $S$. $R$ is a left strongly intrinsic extension of $S$ if $R$ is a left intrinsic extension of $S$, and for all $A \in L(S)$ there exists a left ideal $B$ of $R$ such that $B \cap S = A$. In any left $S$-module $M$, we denote by $Z(SM)$ the set of elements in $M$ whose annihilator in $S$ is an essential left ideal. Clearly $Z(SM)$ is a submodule of $M$.

**Theorem 1.1.** If $Z(S) = 0$, then $S S$ has a (unique up to isomorphism) maximal essential extension $Q$ (called the maximal quotient ring of $S$) which has a ring structure compatible with the module structure; and $Q$ is a regular, left self-injective ring such that
$Z(\mathcal{Q}) = 0$. Moreover $L(S)$ and $L(Q)$ are lattices, and $L(Q) \cong L(S)$ under contraction.

**Proof.** Follows from [1, Theorem 1, p 69] and [3, Corollary 2.6] and their proofs.

The following two lemmas appear in [2].

**Lemma 1.2.** If $R$ is a left strongly intrinsic extension of $S$, then the following are equivalent: (i) $Z(sS) = 0$, (ii) $Z(sR) = 0$, (iii) $Z(rR) = 0$.

**Lemma 1.3.** If $Z(sS) = 0$ and $R$ is a left strongly intrinsic extension of $S$, then $L(R) \cong L(S)$ under contraction.

2. Essential subdirect sums. If $R$ is a subdirect sum of rings $\{R_\alpha | \alpha \in A\}$ and $S = \sum_{\alpha \in A} R_\alpha$ is the complete direct sum of the $R_\alpha$, then the subdirect sum is essential if $R$ (identifying $R$ and its canonical isomorphic image in $S$) is an essential left $R$-submodule of $S$.

Clearly an essential subdirect sum of nonzero rings is irredundant [5], and in the case of a finite number of factors, is essentially irredundant in the sense of [1, p 114]. It is an easily verified property of subdirect sums that if $B_i(i \in D)$ are disjoint subsets of $A$ such that $A = \bigcup_{i \in D} B_i$ and $R_{B_i} = \{a \in \sum_{\alpha \in B_i} R_\alpha | \text{for some } b \in R \text{ if } a(\alpha) = b(\alpha) \text{ for all } \alpha \in B_i\}$, then for each $i \in D$, the $R_{B_i}$ are rings which are subdirect sums in a natural way of the rings $\{R_\alpha | \alpha \in B_i\}$, and $R$ is a subdirect sum in a natural way of the rings $\{R_{B_i} | i \in D\}$. If, in addition, each $R_\alpha$ is a subdirect sum of rings $\{T_{\alpha\gamma} | \gamma \in A_\alpha\}$, then $R$ is a subdirect sum in a natural way of the rings $\{T_{\alpha\gamma} | \gamma \in A_\alpha \text{ whenever } \alpha \in A\}$. Each of these constructed subdirect sums will be referred to as the induced subdirect sum, and whenever we say "the subdirect sum" we are referring either to the original fixed subdirect sum or one of its various induced subdirect sums. Loosely speaking, we may think of the preceding remarks as saying that subdirect sums satisfy a generalized associative law. The results in this section will show that finite essential subdirect sums also have this nice property (finite irredundant subdirect sums do not).

**Lemma 2.1.** Let $R$ be a subdirect sum of nonzero rings $R_1, \cdots, R_n$. The subdirect sum is essential if and only if $R \cap R_i$ is an essential left $R$-submodule of $R_i$ for $i = 1, 2, \cdots, n$.

**Proof.** If the subdirect sum is essential and $W_i$ is a nonzero $R$-submodule of $R_i$, then $W_i$ is also a nonzero left $R$-submodule of $\bigoplus_{i \in I} R_i$, and so $W_i \cap R \neq 0$. Since $W_i \cap (R \cap R_i) = W_i \cap R$ the
result follows. Conversely suppose \( R \cap R_i \) is an essential \( R \)-submodule of \( R_i \) for each \( i \). If \( n = 1 \) the result is trivial, so suppose \( n = 2 \). Let \( 0 \neq x_1 + x_2 \in R \oplus R_2(x_i \in R_i) \) and assume \( x_1 \neq 0 \). Since \( R \cap R_i \) is essential in \( R_i \), there exists an integer \( n \) and an \( r \in R \) such that \( 0 \neq rx_i + nx_i = (r + n)x_i \in R \cap R_i \). If \( (r + n)x_2 = 0 \), then \( 0 \neq (r + n)(x_1 + x_2) \in R \). If \( (r + n)x_2 \neq 0 \), then since \( R \cap R_i \) is essential in \( R_i \) there exists an integer \( m \) and an \( s \in R \) such that \( 0 \neq (s + m)(r + n)x_2 \in R \cap R_i \). Then clearly \( 0 \neq (s + m)(r + n)(x_1 + x_2) \in R \), and \( R \) is essential in \( R_i \oplus R_2 \). The result now follows by a simple induction.

**Proposition 2.2.** Let \( Q \) be a ring which is an essential subdirect sum of nonzero rings \( Q_1, \ldots, Q_n \). If each \( Q_i \) is an essential subdirect sum of nonzero rings \( Q_{i,1}, \ldots, Q_{i,k_i} \), then the induced subdirect sum of \( Q \) (of the \( Q_{i,j} \)) is essential. Also, if \( n_1, n_2, \ldots, n_{k+1} \) are integers such that \( 1 = n_1 < n_2 < \cdots < n_{k+1} = n + 1 \), and \( Q'_i(i = 1, 2, \cdots, k) \) are the induced subdirect sums of \( Q_{n_1}, \ldots, Q_{n_{i+1}-1} \), then \( Q \) is the essential subdirect sum of \( Q', \cdots, Q'_k \) and each \( Q'_i \) is the essential subdirect sum of \( Q_{n_1}, \cdots, Q_{n_{i+1}-1} \).

**Proof.** The result follows by a fairly straightforward application of Lemma 2.1.

The following theorem is a modification of a theorem of Levy, [5, Theorem 6.1].

**Theorem 2.3.** \( R \) is lqss if and only if it is an essential subdirect sum of a finite number of lqs rings \( R_1, \ldots, R_n \) for some \( n \). In this case, we have \( R = \bigoplus_{i=1}^n R_i \).

**Proof.** If \( R \) is lqss, then by [5, Theorem 6.1], \( R \) is an irredundant subdirect sum of lqs rings \( R_1, \ldots, R_n \) for some \( n \), and \( R \subseteq R_1 \oplus \cdots \oplus R_n \subseteq R_1 \oplus \cdots \oplus R_n \subseteq R \). Since \( R \) is an essential extension of \( R \), it follows that \( R \) is essential in \( R_1 \oplus \cdots \oplus R_n \), and \( R \) is then an essential subdirect sum of \( R_1, \ldots, R_n \). Conversely, if \( R \) is a finite essential (and so irredundant) subdirect sum of lqs rings \( R_1, \ldots, R_n \), then \( R \) is lqss by [5, Theorem 6.1].

**Theorem 2.4.** If \( R \) is an essential subdirect sum of prime rings \( R_1, \ldots, R_n \) \( (S_1, \ldots, S_n) \), \( S \subseteq R \), and each \( R_i \) is a left intrinsic extension of the corresponding \( S_i \), then \( R \) is a left intrinsic extension of \( S \).

**Proof.** If \( 0 \neq x \in R \), then \( x = x_1 + \cdots + x_n(x_i \in R_i) \). We may
assume \( x_1 \neq 0 \), and so \( 0 \neq R_1 x_1 = R_1 x \) since \( R_1 \) is prime. Thus \( R_1 x \) is a nonzero left ideal of \( R_1 \), so \( R_1 x \cap S_i \neq 0 \). \( R_1 x \cap S_i \) is a nonzero left \( S \)-submodule of \( S_i \oplus \cdots \oplus S_n \), so \( S \cap R_1 x \cap S_i \neq 0 \). Since \( S_i \) is prime, \((S \cap R_1 x \cap S_i) = 0 \). Hence there exists \( s, s' \in S \cap R_1 x \cap S_i \) such that \( 0 \neq ss' \in S \). If \( s' = r_1x \), then there exists \( r \in R \) such that \( r = r_1 + \cdots + r_n \). Hence \( srx = s'rx = ss' \in Rx \cap S \), so \( R \) is a left intrinsic extension of \( S \).

3. The main theorem.

**Lemma 3.1.** If rings \( R \) and \( S \) are direct sums of division rings \( R_1, \cdots, R_n \) and \( S_1, \cdots, S_m \) respectively, and \( R \) is a left intrinsic extension of \( S \) such that their identities coincide, then \( m = n \) and \( R_i \cap S = S_i \) for a suitable arrangement of the \( R_i \).

*Proof.* Since \( R_i \) is a nonzero ideal of \( R_i \), it follows that \( R_i \cap S \) is a nonzero ideal of \( S \). The ideals of \( S \) are direct sums of some of the \( S_i \), so \( R_i \cap S = \bigoplus S_i \) for some rearrangement of the \( S_i \). But \( k = 1 \), otherwise \( R_i \) has nonzero zero divisors. Similarly, each \( R_i \) contains exactly one \( S_i \), so \( n \leq m \) and \( R_i \cap S = S_i \) for \( i = 1, 2, \cdots, n \). Each \( S_i \) is a multiplicative subgroup of \( R_i \), so their identities coincide. Equating identities leads to a contradiction if \( n < m \), so we must have \( n = m \).

**Proposition 3.2.** If \( S \) is a left Ore domain, then \( R \) is a left intrinsic extension of \( S \) if and only if \( R \) is a left strongly intrinsic extension of \( S \).

*Proof.* Let \( R \) be a left intrinsic extension of \( S \). By Theorem 1.1, \( L(S) \cong L(\tilde{S}) = \{0, \tilde{S}\} \), so \( L(S) = \{0, S\} \). The zero ideal of \( R \) and \( R \) itself contract to the elements of \( L(S) \), so \( R \) is a left strongly intrinsic extension of \( S \). Note that by Lemma 1.3, \( L(R) = \{0, R\} \).

**Proposition 3.3.** If \( R \) is a left intrinsic extension of a left Ore domain \( S \), then \( R \) is a left Ore domain.

*Proof.* By Proposition 3.2, \( R \) is a left strongly intrinsic extension of \( S \), and \( \{0, R\} = L(R) \) which clearly satisfies the maximum condition. By Lemma 1.2, \( Z(R) = Z(S) = 0 \). If \( A \) is a nilpotent left ideal of \( R \), then \( A \cap S \) is a nilpotent left ideal of \( S \). Thus \( A \cap S = 0 \), and \( A = 0 \). Hence \( R \) is semiprime, and by [3, Theorem 4.4], \( R \) is lqss. Thus \( \{0, R\} = L(R) \cong L(\tilde{R}) = \{0, \tilde{R}\} \). By [1, Proposition 5, p. 71], \( L(\tilde{R}) \) consists of the annihilator left ideals of \( \tilde{R} \). Since \( \tilde{R} \) has an identity, \( \tilde{R} \) is a domain. It follows that \( R \) is a left Ore domain.
THEOREM 3.4. If $S$ is lqss, then $R$ is a left strongly intrinsic extension of $S$ if and only if $S\subseteq R$ and one of the following is true:

(i) $\bar{S} = \bar{R}$ is semisimple Artinian,

(ii) $S$ (and $R$) is an essential subdirect sum of left Ore domains $S_1, \cdots, S_n$ ($R_1, \cdots, R_n$) where $R_i$ is a left intrinsic extension of the corresponding $S_i$,

(iii) $S$ (and $R$) is an essential subdirect sum of nonzero rings $S_1$ and $S_2$ ($R_1$ and $R_2$) where $S_i \subseteq R_i$ for $i = 1, 2$ and such that (i) holds for $S_2$ and $R_2$ and (ii) holds for $S_1$ and $R_1$.

Proof. By [3, Theorem 4.4], $L(S)$ satisfies the maximum condition, so by Lemma 1.3, so does $L(R)$. By Lemma 1.2, $Z(R) = Z(S) = 0$; and as in Proposition 3.3, $R$ is semiprime. Thus by [3, Theorem 4.4], $R$ is lqss. By Theorem 1.1, $\bar{R}$ is a regular, semisimple, left self-injective ring. The lattice of principal left ideals of $\bar{R}$ is complete by [6, Theorem 1], so by [6, Corollary to Theorem 4], $\bar{R}$ can be decomposed into the direct sum of two ideal $\bar{Q}_1$ and $\bar{Q}_2$ in such a way that $\bar{Q}_1$ is strongly regular and $\bar{Q}_2$ does not contain any nonzero strongly regular ideals. By [2, Theorem 2.5], there is a subring $T$ of $Q_1$ with the properties that:

(a) $T$ contains every idempotent of $Q_i$,

(b) $T$ is a strongly regular self-injective ring,

(c) $\bar{S} = T \oplus \bar{Q}_2$.

Since $\bar{S} (\bar{R})$ is semisimple Artinian, $\bar{S} = \bigoplus F_i \oplus (\bar{R} = \bigoplus D_i)$ where each $F_i (D_i)$ is simple Artinian. Since $\bar{S} = T \oplus \bar{Q}_2$, we have $T = \bigoplus F_i (Q_i = \bigoplus n' \leq n \leq m)$, and the $F_i (D_i)$ are suitably arranged. Since strongly regular rings have no nonzero nilpotent elements, it follows that $F_1, \cdots, F_n, D_1, \cdots, D_n$, are division rings (if $n \neq 0 \neq n'$). It is clear that $Q_i$ is a left intrinsic extension of $T$ (so $T = 0$ if and only if $Q_i = 0$). By property (a) the identities of $T$ and $Q_i$ coincide, so by Lemma 3.1, $n = n'$ and $D_i \cap T = F_i$ for $i = 1, 2, \cdots, n$.

Let $e_i$ be the identity of $Q_i$ (and $T$) and $e_2$ the identity of $Q_2$. If $T \neq 0$, let $d_1, \cdots, d_n$ be the identities of $D_1, \cdots, D_n$ (and of $F_1, \cdots, F_n$). Let $R_i = Re_i$ and $S_i = Se_i$ for $i = 1, 2$. Clearly $S_i \subseteq R_i \subseteq Q_i$; $Q_i = 0$ if and only if $R_i = 0$ if and only if $S_i = 0$; and $S (R)$ is a subdirect sum of $S_i$ and $S_i (R_i$ and $R_i$).

We claim that if $Q_1 \neq 0$, then $\bar{R}_i = Q_1$ and $\bar{S}_i = T$; and if $Q_2 \neq 0$, then $\bar{S}_i = \bar{R}_i = \bar{Q}_2$. Suppose $Q_1 \neq 0$ and $r$ is regular in $R$. Clearly $r e_i \in R_i$ is regular in $R_i$, so $R_i$ has regular elements. If $r_i$ is any regular element in $R_i$ and $q r_i = 0$ where $q_i \in Q_i$, then $q_i = c^{-1} b (c, b \in R)$,
so \( 0 = b r_i = (b e_i)r_i \). Since \( r_i \) is regular in \( R_i \), it follows that \( b e_i = 0 \), and so \( q_i = q_i e_i = c^{-1}b e_i = 0 \). Hence \( r_i \) is not a zero divisor in \( Q_i \), so by [1, Corollary 4, p. 70], \( r_i \) is invertible in \( Q_i \). If \( q_i \) is given, then \( q_i = d^{-1}b (d, b \in R) \) and \( q_i = q_i e_i = (d^{-1}b)e_i = (de_i)^{-1}(be_i) \). Hence \( Q_i = \tilde{R}_i \), and exactly the same argument gives that \( T = \tilde{S}_1 \) and (if \( Q_i \neq 0 \)) that \( Q_i = \tilde{R}_i = \tilde{S}_i \).

Since \( S \subseteq S_1 \oplus S_2 \subseteq T \oplus Q_i = \tilde{S} \) and \( R \subseteq R_1 \oplus R_2 \subseteq Q_1 \oplus Q_2 = \tilde{R} \), it follows that \( S(R) \) is an essential subdirect sum of \( S_1 \) and \( S_2 \) (\( R_1 \) and \( R_2 \)).

If \( R_1 = 0 \), then \( S_1 = 0 \); so \( S = S_2 \subseteq R_2 = R \) and \( S = \tilde{S}_2 = Q_2 = \tilde{R}_2 = \tilde{R} \). This is condition (i).

If \( R_1 \neq 0 \), then \( d_i = d_1 + \cdots + d_n \), \( S_i = S_i e_i \subseteq S d_i + \cdots + S d_n \), and \( R_i = R d_i \subseteq R d_1 + \cdots + R d_n \). If \( S_i = S_i d_i = S d_i (R_i = R d_i = R d_i) \) for \( i = 1, 2, \ldots, n \), it follows as before that \( S_i (R_i) \) is a subdirect sum of \( S_i', \ldots, S_n (R_i', \ldots, R_n') \). In exactly the same way as we proved that \( \tilde{R}_i = Q_i \), we get that \( \tilde{S}_i = F_i \) and \( \tilde{R}_i = D_i \) for \( i = 1, 2, \ldots, n \). Also, as before, the subdirect sums are essential.

We next show that \( R_i \) is a left intrinsic extension of \( S_i' \) (and similarly for \( R_1', \ldots, R_n' \)). Let \( 0 \neq x = r d_i \in R d_i = R_i' (r \in R) \). Then \( R_i' x = R x = 0 \), so \( R x \cap R = 0 \) (since \( R x \subseteq \tilde{R} \)). Since \( R \) is a left intrinsic extension of \( S \), we have \( R x \cap R \cap S = R x \cap S \neq 0 \). Thus if \( 0 \neq s = r' x \in R x \cap S \) \((r' \in R)\), we have \( 0 \neq s = s d_i \in S d_i \). Hence \( 0 \neq s = (R d_i) x \cap S d_i = R_i' x \cap S_i' \), and \( R_i' \) is a left intrinsic extension of \( S_i' \).

If \( R_2 = 0 \), then \( S_2 = 0 \); so \( S = S_1 \) and \( R = R_1 \) which gives condition (ii). If \( R_1 \) and \( R_2 \) are not zero, then condition (iii) is satisfied.

Conversely, suppose condition (ii) is true. Hence \( S \subseteq R \subseteq \tilde{S} \), and \( \tilde{R} \) exists. Thus by [3, Corollary 2.6], \( L(S) \cong L(\tilde{S}) = L(R) \cong L(R) \) under contraction, so \( R \) is a left strongly intrinsic extension of \( S \).

In condition (ii), we have by Theorem 2.3 that \( \tilde{S} = \bigoplus \sum_{i=1}^{n} \tilde{S}_i \) and \( \tilde{R} = \bigoplus \sum_{i=1}^{m} \tilde{R}_i \), where \( \tilde{S}_i \) and \( \tilde{R}_i \) are division rings. Clearly \( L(S) \cong L(\tilde{R}) \) under contraction, and since \( L(S) \cong L(\tilde{S}) \) and \( L(R) \cong L(R) \), it follows that \( L(S) \cong L(R) \) under contraction. By Theorem 2.4, \( R \) is a left intrinsic extension of \( S \), so \( R \) is a left strongly intrinsic extension of \( S \).

In condition (iii), \( \tilde{S} = \tilde{R} \) are semisimple Artinian, so \( S_i \) and \( R_i \) are lqss. Let \( \tilde{S}_i = \tilde{R}_i = \bigoplus \sum_{i=1}^{m} F_i \), where \( F_i \) are simple ring with identities \( e_i \) \((i = 1, 2, \ldots, m) \). Let \( S'_{n+1} = S e_i \) and \( R_{n+1} = R e_i \) for \( i = 1, 2, \ldots, m \). By Theorem 2.3 and the proof of [5, Theorem 6.1], we have that \( S_i (R_i) \) is an essential subdirect sum of the lqss rings \( S_i' (R_i) \) \((i = 1 + 1, \ldots, n + m) \), and \( S_i = R_i = F_{n + n} \) for \( i = n + 1, \ldots, n + m \). Since \( S_i' \subseteq R_i' \subseteq S_i' \) \((i = n + 1, \ldots, n + m) \), it follows that \( \tilde{R}_i \) is a left intrinsic extension of \( S_i' \) \((i = 1, \ldots, n + m) \). Thus by Proposition 2.2 and Theorem 2.4, \( R \) is a left intrinsic extension of \( S \). Also \( \tilde{S} = \bigoplus \sum_{i=1}^{n} \tilde{S}_i \) and \( \tilde{R} = \bigoplus \sum_{i=1}^{n} \tilde{R}_i \), and as in the proof of case (ii), \( L(\tilde{S}) \cong L(\tilde{R}) \) under contraction. Since
L(S₂) = L(R₂), it follows that L(Ś) ≅ L(Ř) under contraction. Again
L(S) ≅ L(R) under contraction, so R is a left strongly intrinsic ex-
tension of S.

**Corollary 3.5.** R is a left strongly intrinsic extension of a
lqs ring S if and only if either S ⊆ R ⊆ Ś or S and R are left Ore
domains such that R is a left intrinsic extension of S.

**Proof.** If S ⊆ R ⊆ Ś, then R is a left strongly intrinsic extension
of S by case (i). If R and S are left Ore domains such that R is a
left intrinsic extension of S, then the result follows from Proposition 3.2.

Conversely, since Ś = T ⊕ Q₂, we have either T = 0 or Q₂ = 0. If
Q₂ = 0, thenŚ = T = F₁ and Ř = Q₁ = D₁, so R and S are left Ore
domains and R is a left intrinsic extension of S. If T = 0; then
S₁ = R₁ = 0, S = S₂, and R = R₂ which is case (i).

4. Left intrinsic extensions. In this section, it is shown that,
in the case of lqs rings, every left intrinsic extension is left strongly
intrinsic.

**Lemma 4.1.** If R is a left intrinsic extension of S, then
Z(𝒮S) ⊆ Z(𝒮R) ⊆ Z(RR).

**Proof.** The first containment is clear. If x ∈ R and x ∈ Z(𝒮R),
then the left annihilator in R of x (denoted lₓ(R)) is not an essential
left ideal of R. Thus there exists a nonzero left ideal A of R
such that lₓ(A) ∩ A = 0. Thus 0 = lₓ(A) ∩ A ∩ S = lₓ(A) ∩ (A ∩ S),
and A ∩ S ≠ 0. Hence s ∈ Z(𝒮R), and so Z(𝒮R) ⊆ Z(RR).

**Lemma 4.2.** Let S have a left classical quotient ring. If Z(RR) = 0
and R is a left intrinsic extension of S, then every regular element
of S is a regular element of R.

**Proof.** Let s be a regular element of S, and r ∈ R. If rs = 0,
then r ∈ lₛ(R). Clearly, lₛ(s) = lₛ(s) ∩ S = 0, so lₛ(s) = 0 and r = 0. If
sr = 0, then (Sr)₀ = 0 and r ∈ Z(RR). By Lemma 4.1, Z(𝒮R) ⊆ Z(RR) = 0,
so r = 0. Thus s is regular in R.

**Lemma 4.3.** Let S have a classical left quotient ring Š. If R
is a left intrinsic extension of S where Z(RR) = 0, then Š ⊆ Q where
Q is the maximal left quotient ring of R.

**Proof.** Let M be the injective hull of R as a left R-module.
By [1, Theorem 1, p. 69], Q = Hom_R(M, M) ≅ M. If d is a regular
elements of $S$, define the map $\bar{f}: Rd \rightarrow R$ by $\bar{f}(rd) = r$ for all $r \in R$. The map is well defined by Lemma 4.2, and by the injectivity of $M$, there exists $f \in \text{Hom}_R(M, M)$ such that $f \mid_{rd} = \bar{f}$. By [1, Theorem 1, p. 69], the canonical isomorphic image of $d$ in $Q$ is the unique $g \in \text{Hom}_R(M, M)$ such that $g(r) = rd$ for all $r \in R$. If 1 denotes the identity of $Q$, it follows that $R \subseteq \ker (1 - gf)$ and $Rd \subseteq \ker (1 - gf)$. $R$ is left essential in $Q$, and it is easy to verify that $Rd$ is also essential in $Q$. By [1, Theorem 1, p. 44], $1 - gf$ and $1 - fg$ are in the Jacobson radical of the semisimple ring $Q$. Hence $gf = fg = 1$. By the canonical injection of $J_\pi$ into $Q$, we can consider $J_\pi$ to be a subring of $Q$, and so $d$ has a two-sided inverse $f$ (henceforth denoted $d^{-1}$) in $Q$. Hence every regular element of $S$ has a two-sided inverse in $Q$. If $T = \{a^{-1}b \mid b \in S$, a regular in $S\}$, then $T \subseteq Q$ and $T$ is a ring by Ore’s condition for $S$, [see 4, p. 109]. Hence $\bar{S} = T \subseteq Q$.

**Lemma 4.4.** If $S$ is a left self-injective ring and $Z(S) = 0$, then every left intrinsic extension of $S$ is a left strongly intrinsic extension of $S$.

**Proof.** Let $R$ be a left intrinsic extension of $S$. By [1, Theorem 1, p. 69], $S$ is its own maximal left quotient ring and is a regular ring. If $A \in L(S)$, then by [1, Theorem 4, p. 70], $A = Se$ where $e^2 = e \in S$. Hence $Re \cap S = Se$, and $R$ is a left strongly intrinsic extension of $S$.

**Theorem 4.5.** If $R$ is an extension of a lqss ring $S$ and $Z(\pi R) = 0$, then $R$ is a left intrinsic extension of $S$ if and only if $R$ is a left strongly intrinsic extension of $S$.

**Proof.** Let $R$ be a left intrinsic extension of $S$ and $Q$ the maximal left quotient ring of $R$. By Lemma 4.3, $\bar{S} \subseteq Q$, and clearly $Q$ is a left intrinsic extension of $S$. Also $\bar{S}$ is left self-injective and $Z(\bar{S}) = 0$, so by Lemma 4.4, $Q$ is a left strongly intrinsic extension of the lqss ring $\bar{S}$. By Lemma 1.3, $L(Q) \cong L(\bar{S})$ under contraction, and since $L(Q) \cong L(R)$ and $L(S) \cong L(\bar{S})$ under contraction, it follows that $L(S) \cong L(R)$ under contraction. Hence $R$ is a left strongly intrinsic extension of $S$.

**Theorem 4.6.** If $R$ is a left intrinsic extension of a lqss ring $S$, then the following are equivalent:

(i) $Z(\pi R) = 0$,

(ii) $R$ is a left strongly intrinsic extension of $S$,

(iii) $R$ is lqss.
Proof. (i) $\Rightarrow$ (ii) by Theorem 4.5. (ii) $\Rightarrow$ (iii) by the proof of Theorem 3.4. (iii) $\Rightarrow$ (i) follows from [3, Theorem 4.4].

References


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