A CLASSIFICATION OF CENTERS

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The purpose of this paper is to classify centers according to isomorphisms. We define three types of isomorphism, and for two of these types give necessary and sufficient conditions for two centers to be isomorphic. We also give necessary and sufficient conditions for the third type of isomorphism to be equivalent to one of the other two.

These isomorphisms are discussed in a more general situation by Taro Ura [7]. This paper was motivated by discussions with Taro Ura and Otomar Hájek.

In our investigation we construct a section which generates a neighborhood of the center by using a theorem from the theory of fibre bundles. This section may be constructed directly, using the existence of a transversal through each noncritical point of the dynamical system. Much insight, which is otherwise lost, into the structure of a center is obtained from the fibre bundle approach.

The concept of a transversal is essential in our investigation. The basic material on transversal theory in planar dynamical systems is found in [3].

Throughout this paper $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{R}^2$ will denote the nonnegative reals, the reals, and the plane respectively.

Let $(X, \pi)$ be a dynamical system on $X$, i.e., $X$ is a topological space and $\pi$ is a mapping of $X \times \mathbb{R}$ onto $X$ satisfying the following axioms (where $x \pi t = \pi(x, t)$ for $(x, t) \in X \times \mathbb{R}$):

1. Identity Axiom: $x \pi 0 = x$ for $x \in X$
2. Homomorphism Axiom: $(x \pi t) \pi s = x \pi (t + s)$ for $x \in X$ and $t, s \in \mathbb{R}$
3. Continuity Axiom: $\pi$ is continuous on $X \times \mathbb{R}$

Then for $x \in X$, $x \pi R$ is called the trajectory through $x$ and is denoted by $C(x)$. If $C(x) = \{x\}$, $x$ is called a critical point. If there exists $t \in \mathbb{R}$, $t \neq 0$, such that $x \pi t = x$, $C(x)$ is called periodic. If $C(x)$ is periodic and $x$ is not a critical point, $C(x)$, is called a cycle.

1. Definition and properties of a center. In the following $(\mathbb{R}^2, \pi)$ will denote a dynamical system on $\mathbb{R}^2$ and $P$ the set of noncritical periodic points of $(\mathbb{R}^2, \pi)$. Let $T: P \to \mathbb{R}$ be the mapping which associates with each point $x \in P$ its fundamental period $T(x)$. For the proof of the following result see [3, VII, 4.15].

PROPOSITION 1.1. $T$ is a continuous mapping of $P$ into $\mathbb{R}$.
**Definition 1.2.** A critical point $p$ of $(R^2, \pi)$ is called a center if and only if there exists a neighborhood $U$ of $p$ such that $C(x)$ is a cycle for every $x \in U - \{p\}$.

**Proposition 1.3.** Let $p$ be a center in $(R^2, \pi)$. Then $\{p\}$ is both positively and negatively stable.

**Proof.** Let $U$ be a neighborhood of $p$ as described in Definition 1.2. We will show that $D^+(p) = \{p\}$, where $D^+(p)$ denotes the positive prolongation of $p$ (see [1, 1.4.1]). This will prove (by [1, 2.6.6]) that $\{p\}$ is positively stable. Let $M$ be the component of $D^+(p)$ which contains $p$. By [1, 2.3.5], if $D^+(p)$ is compact, then it has exactly one component and if $D^+(p)$ is not compact, then none of its components is compact. We now have two cases:

$$M \cap (U - \{p\}) = \emptyset \quad \text{or} \quad M \cap (U - \{p\}) \neq \emptyset .$$

If $M \cap (U - \{p\}) = \emptyset$, then $M = \{p\}$ and $D^+(p) = M = \{p\}$. If $M \cap (U - \{p\}) \neq \emptyset$, let $y \in M \cap (U - \{p\})$.

Then there exist sequences $\{x_i\}_{i=1}^\infty$ in $U - \{p\}$ and $\{t_i\}_{i=1}^\infty$ in $R^+$, with $x_i \to p$ and $x_i \pi t_i \to y$. Since $x_i \in P$ for every $i$, we may assume $t_i \in [0, T(x_i))$. Since $C(y)$ is a cycle, $t_i \in [0, 2T(y))$ for all $i$ sufficiently large by the continuity of $T$. Let $\{t_n\}_{n=1}^\infty$ be a convergent subsequence of $\{t_i\}_{i=1}^\infty$ with limit $t_0$. Then

$$y \leftarrow x_i \pi t_n \to p \pi t_0 = p .$$

This contradicts our assumption that $y \in M - \{p\}$. Thus $D^+_p = \{p\}$ and $\{p\}$ is positively stable. Similarly $\{p\}$ is negatively stable.

**Definition 1.4.** A cycle $C(x)$ of $(R^2, \pi)$ decomposes $R^2$ into two components, one bounded and the other unbounded. $\text{int} C(x)$ and $\text{ext} C(x)$ will denote the bounded and unbounded components, respectively, of $R^2 - C(x)$.

**Proposition 1.5.** Let $C(x)$ be a cycle in $(R^2, \pi)$. Then $\text{int} C(x)$ and $\text{ext} C(x)$ are invariant.

**Proof.** The components of an invariant set are invariant.

In [3, VII, 4.8] it is proved that

**Proposition 1.6.** If $C(x)$ is a cycle in $(R^2, \pi)$, then $\text{int} C(x)$ contains a critical point.
**PROPOSITION 1.7.** Let p be a center in \((\mathbb{R}^2, \pi)\) and U be a neighborhood as described in Definition 1.2. Then there exists \(x \in U\) such that

(i) \(\text{int } C(x) \subset U\),
(ii) \(p \in \text{int } C(x)\), and
(iii) \(p \in \text{int } C(y)\) for every \(y \in \text{int } C(x) - \{p\}\).

**Proof.** Let \(V\) be a disc neighborhood of \(p\) contained in \(U\). Since \(p\) is positively stable there exists \(x \in V - \{p\}\) such that \(C(x) = C^+(x) \subset V\). Then \(\text{int } C(x) \subset V\) because \(V\) is simply connected. \(\text{int } C(x)\) contains a critical point by Proposition 1.6. This critical point must be \(p\) because \(p\) is the unique critical point in \(U\). Similarly, \(p \in \text{int } C(y)\) for every \(y \in \text{int } C(x) - \{p\}\).

Thus we may reformulate Definition 1.2 as:

**DEFINITION 1.2'.** A critical point \(p\) of \((\mathbb{R}^2, \pi)\) is called a center if and only if there exists a cycle \(C(x)\) such that \(p \in \text{int } C(x)\) and \(\text{int } C(x) - \{p\}\) consists of cycles. We choose a fixed \(C(x_0)\) satisfying this condition and henceforth denote \(\text{int } C(x_0)\) by \(U\).

**PROPOSITION 1.8.** If \(x \in U\), then \(C(x)\) is both positively and negatively stable. Also \(C(x_0)\) is stable relative to \(U\).

**Proof.** See [3, VIII, 3.3].

**PROPOSITION 1.9.** Let \(S\) be a transversal contained in \(U\). Then \(C(x) \cap S = \{x\}\) for every \(x \in S\).

**Proof.** Since \(S\) is a transversal and \(p\) is critical, \(p \in S\); thus \(x \in S \subset U - \{p\}\) implies \(C(x)\) is a cycle. A cycle intersects a transversal at a unique point, [3, VII, 4.4].

**PROPOSITION 1.10.** Let \(p\) be a center and \(U\) be a neighborhood of \(p\) as described in Definition 1.2'. If \(C_1\) and \(C_2\) are distinct cycles in \(U\), then \(C_1 \subset \text{int } C_2\) or \(C_2 \subset \text{int } C_1\).

**Proof.** By Proposition 1.7 \(p \in \text{int } C_1\) and \(p \in \text{int } C_2\). Thus

\[
\text{int } C_1 \cap \text{int } C_2 \neq \emptyset.
\]

Thus \(\text{int } C_1 \subset \text{int } C_2\) or \(\text{int } C_1 \cap \text{ext } C_2 \neq \emptyset\). In the first case \(\overline{\text{int } C_1} \subset \overline{\text{int } C_2}\). Therefore \(C_1 \subset \text{int } C_2\) or \(C_1 \cap C_2 \neq \emptyset\). The latter is impossible because \(C_1\) and \(C_2\) are distinct trajectories. In the second case, \(\partial(\text{int } C_2) \cap \text{int } C_1 \neq \emptyset\). Therefore \(C_2 \cap \text{int } C_1 \neq \emptyset\) and \(C_2 \subset \text{int } C_1\) since
int $C_i$ is invariant.

**COROLLARY 1.11.** If $C_1$ and $C_2$ are distinct cycles in $U$ such that $C_1 \subset \text{ext } C_2$, then $C_2 \subset \text{int } C_1$.

**Proof.** By Proposition 1.10, $C_2 \subset \text{int } C_1$ or $C_1 \subset \text{int } C_2$. $C_1$ cannot be contained in both $\text{int } C_2$ and $\text{ext } C_2$. Therefore $C_2 \subset \text{int } C_1$.

2. Bundles and cross-sections.

**DEFINITION 2.1.** Let $(R^2, \pi)$ be a dynamical system on $R^2$ and let $x, y \in R^2$. We define a relation $\sim$ on $R^2$ by letting $x \sim y$ if and only if $x \in C(y)$.

Evidently $\sim$ is an equivalence relation. The topology on $R^2/\sim$ will be the quotient topology.

**PROPOSITION 2.2.** Let $e$ be the natural mapping of $R^2$ onto $R^2/\sim$. Then $e$ is an open mapping.

**Proof.** $e$ is open if and only if $e^{-1}eG$ is open for every open set $G \subset R^2$. Now, $e^{-1}eG = G \pi R^1 = \bigcup_{t \in R^1} G \pi t$, and $G \pi t$ is open for every $t \in R^1$ since $\pi: R^2 \approx R^2$. Hence $G \pi R^1$ is open and $e$ is an open mapping.

**PROPOSITION 2.3.** If $V$ is an invariant subset of $R^2$, then $e(V)$ is homeomorphic to $V/(\sim \cap V \times V)$.

**Proof.** Since $e$ is an open mapping, the result follows from § I, 3.5 of [2].

We shall now write $e(V)$ as $V/\sim$ where it is understood that $\sim$ is restricted to $V \times V$.

**PROPOSITION 2.4.** $e \mid U$ is a closed mapping of $U$ onto $U/\sim$.

**Proof.** $e \mid U$ is closed if and only if $e^{-1}eF = F \pi R^1$ is closed in $U$ for every set $F$ which is closed in $U$. Let $x \in F \pi R^1 \cap U$. Then there exist sequences $\{x_i\}_{i=1}^\infty$ in $F$ and $\{t_i\}_{i=1}^\infty$ in $R^1$ such that $x_i \pi t_i \rightarrow x$. Thus $C(x_i) \rightarrow C(x)$. Let $y \in U - \text{int } C(x)$. Then $C(x) \subset \text{int } C(y)$ by Corollary 1.11 and $\text{int } C(y)$ is a compact neighborhood of $C(x)$. Thus $x_i \in \text{int } C(y)$ for $i$ sufficiently large. Let $\{x_i\}_{i=1}^\infty$ be a convergent subsequence of $\{x_i\}_{i=1}^\infty$ with limit $z$. Then $z \in F \cap C(x)$ since $F$ is closed and $C(x_i) \rightarrow C(x)$. Thus $x \in C(z) \subset F \pi R^1$ and $F \pi R^1$ is closed.

The following material on bundles is to be found in [6].
DEFINITION 2.5. A bundle $\beta$ is a collection as follows:

1. A space $B$ called the **bundle space**, 
2. a space $X$ called the **base space**, 
3. a map $p : B \to X$ of $B$ onto $X$ called the **projection**, 
4. a space $Y$ called the **fibre**, 
5. an effective topological transformation group $G$ of $Y$ (i.e., $g \cdot y = y$ for all $y \in G$ implies $g$ is the identity) called the **group of the bundle**, 
6. a family $\{V_j\}$ of open sets covering $X$ indexed by a set $J$, the $V_j$'s are called the **coordinate neighborhoods**, and 
7. for each $j$ in $J$, a homeomorphism $\varphi_j : V_j \times Y \to p^{-1}(V_j)$ called the coordinate function.

The coordinate functions are required to satisfy the following conditions:

8. $p \varphi_j(x, y) = x$ for $x \in V_j$, $y \in Y$
9. if the map $\varphi_{j,i} : Y \to p^{-1}(x)$ is defined by setting $\varphi_{j,i}(y) = \varphi_j(x, y)$ then for each pair $i, j$ in $J$, and each $x \in V_i \cap V_j$, the homeomorphism $\varphi_{j,i}^{-1} \varphi_{i,j} : Y \to Y$ coincides with the operation of an element of $G$ and 
10. for each pair $i, j$ in $J$, the map $g_{ji} : V_j \cap V_i \to G$ defined by $g_{ji}(x) = \varphi_{j,i}^{-1} \varphi_{i,j}$ is continuous.

Let $U - \{p\}$ be the bundle space, $U - \{p\}/\sim$ be the base space, the canonical mapping $e$ of $U - \{p\}$ onto $U - \{p\}/\sim$ be the projection. Then $S^1$ (the one-sphere) is the fibre; as the group take $S^1$ (with complex multiplication). $U - \{p\}$ can be covered by a countable family $\{U_j\}_{j=1}^{\infty}$ of open invariant sets which are generated by arc transversals $\{T_j\}_{j=1}^{\infty}$ minus their end points: if $a_j$ and $b_j$ are the end-points of $T_j$, then $U_j = (T_j - \{a_j \cup \{b_j\}\}) \pi R^1$. If we set $V_j = U_j / \sim$, then $\{V_j\}_{j=1}^{\infty}$ is an open covering of $U - \{p\}/\sim$. For any $(C(x), \xi) \in V_j \times S^2$ define

$$\varphi_j((C(x), \xi)) = (C(x) \cap T_j) \pi \lambda \xi T(x)$$

where $\xi = \exp [i \lambda \xi 2 \pi]$ and $\lambda \xi \in [0, 1)$. It is easily verified that the above satisfies (1) through (8). We will verify that it also satisfies (9) and (10) and is hence a bundle. Let $\delta = \delta(x) \in S^1$ be such that

$$(C(x) \cap T_j) \pi (\lambda \xi T(x)) = C(x) \cap T_j$$

It can be shown that $\delta(\cdot)$ is continuous. Then
\[ \varphi_{i,x}^{-1}(\xi) = \varphi_{i,x}((C(x) \cap T_i) \pi \lambda \xi T(x)) = \varphi_{i,x}((C(x) \cap T_i) \pi (\lambda_i - \lambda) T(x)) = \xi \delta^{-1}. \]

Thus \( \varphi_{i,x}^{-1} \varphi_{i,x} \) coincides with multiplication by \( \delta^{-1} \) and is continuous since \( \delta \) is a continuous function.

**Proposition 2.6.** \( U - \{ p \}/\sim \) is homeomorphic with \( (0, 1) \).

*Proof.* First \( U - \{ p \}/\sim \) is connected and locally connected since \( U - \{ p \} \) is such. Second, \( U - \{ p \}/\sim \) is a regular \( T \) space since \( e | U \) is a closed mapping. Since the topology of \( U - \{ p \} \) has a countable base and \( e \) is an open mapping, the topology of \( U - \{ p \}/\sim \) has a countable base. By Urysohn's metrization theorem \( U - \{ p \}/\sim \) is metrizable. It is known that if a metric space \( X \) is separable, connected, and locally connected, and such that on removing any point \( y \) of \( X \) the remaining set \( X - \{ y \} \) consists of exactly two components, then it is the homeomorphic image of \( (0, 1) \), \([8]\). Take any \( C(x) \in U - \{ p \}/\sim \). Then \( (U - \{ p \}) - C(x) \) consists of two components \( C_1 \) and \( C_2 \). (Indeed, \( C(x) \) is a Jordan curve in \( U \approx \mathbb{R}^2 \).) For \( i = 1, 2 \), \( e(C_i) \) is both open and closed and \( e \) is both open and closed, and \( e \) \( | U \) is both open and closed. \( (U - \{ p \})/\sim - C(x) = ((U - \{ p \}) - C(x))/\sim = e((U - \{ p \}) - C(x) = e(C_1 \cup C_2) = e(C_1) \cup e(C_2). \) Thus \( (U - \{ p \})/\sim - C(x) \) has exactly two components. Hence \( U - \{ p \}/\sim \) is homeomorphic with \( (0, 1) \).

**Definition 2.7.** A space \( Y \) will be called solid with respect to a space \( X \), if for every closed subset \( A \) of \( X \) and mapping \( f: A \rightarrow Y \), there exists a mapping \( f': X \rightarrow Y \) such that \( f' | A = f \).

**Proposition 2.8.** \( S^1 \) is solid with respect to \( U - \{ p \}/\sim \).

*Proof.* It suffices, by Proposition 2.6, to show that \( S^1 \) is solid with respect to \( (0, 1) \). We will only indicate the proof. Let \( I \) denote \( (0, 1) \) and \( A \) a closed subset of \( I \). The components of \( I - A \) are open intervals and there are at most countably many of them. If \( A = I \) there is nothing to show. Let \( f: A \rightarrow S^1 \) be continuous, \( A \neq I \). Let \( V \) be a component of \( I - A \). Since \( A \neq I \), \( V \) must have an endpoint \( a \) contained in \( (0, 1) \). If \( a \) is the only end-point of \( V \) in \( (0, 1) \) define \( f^1: V \rightarrow S^1 \) by \( f^1(x) = f(a) \) for all \( x \in V \). If \( V \) has another end-point \( b \) contained in \( (0, 1) \), we have two cases: \( f(a) \neq f(b) \) or \( f(a) = f(b) \). If \( f(a) = f(b) \) define \( f^1: V \rightarrow S^1 \) by \( f^1(x) = f(a) \) for all \( x \in V \). If \( f(a) \neq f(b) \), then the points \( f(a) \) and \( f(b) \) are the end-points of two subarcs of \( S^1 \). Let \( S_i \) be the one of shorter arc length, and if the two arcs are of equal length \( S_i \) is chosen to be either arc. Then there exists a homeo-
morphism \( f^* \) of \( \tilde{V} \) onto \( S \), such that \( f^*(a) = f(a) \) and \( f^*(b) = f(b) \). We repeat this construction for every component of \( I - A \) and let \( g \) denote the union of all such mappings. The continuity of \( g \) follows from the fact that in any compact subinterval of \( I \) there can be only a finite number of components of \( I - A \) whose end-points have \( f \) images which are diametrically opposite.

The following theorem from [6, 12.2] gives the existence of cross-sections to bundles \( p: B \rightarrow X \), i.e., a continuous mapping \( f: X \rightarrow B \) such that \( pf(x) = x \) for every \( x \in X \).

**THEOREM.** Let \( X \) be a normal space with the property that every covering of \( X \) by open sets is reducible to a countable covering. Let \( \beta \) be a bundle over \( X \) with fibre \( Y \) which is solid. Let \( f \) be a cross-section of \( \beta \) defined on a closed subset \( A \) of \( X \). Then \( f \) can be extended to a cross-section over all of \( X \). (Taking \( A = \emptyset \), it follows that \( \beta \) has a cross-section.)

It should be noted that in the proof of this theorem it is not necessary that \( Y \) be solid, but only that \( Y \) be solid with respect to \( X \), i.e., that any continuous mapping \( f: A \rightarrow Y \), \( A \) closed in \( X \), be continuously extendable to a mapping \( f': X \rightarrow Y \). Hence

**PROPOSITION 2.9.** There exists a continuous map \( f: U - \{p\}/\sim \rightarrow U - \{p\} \) such that \( ef(C(x)) = C(x) \) for every \( C(x) \in U - \{p\}/\sim \).

**COROLLARY 2.10.** Let \( f \) be as in Proposition 2.9 and \( S = f(U - \{p\}/\sim) \); then \( S \) is homeomorphic with \((0, 1)\).

**Proof.** This is a consequence of the fact that if \( \alpha: X \rightarrow Y \) has a cross-section \( \beta: Y \rightarrow X \), then \( Y \) is homeomorphic with \( \beta(Y) \).

**COROLLARY 2.11.** \( C(x) \cap S = \{x\} \) for each \( x \in S \) and \( S \pi R^1 = U - \{p\} \).

**PROPOSITION 2.12.** Let \( h: (0, 1) \rightarrow S \) be a homeomorphism. Then either \( \lim_{t \rightarrow 1} h(t) = p \) or \( \lim_{t \rightarrow 0} h(t) = p \).

**Proof.** Let \( x \in S \) and \( a \in (0, 1) \) be such that \( h(a) = x \). Then either \( h((0, a)) \subset \text{int} C(x) \) or \( h((a, 1)) \subset \text{int} C(x) \) since \( S \pi R^1 = U - \{p\} \) and \( S \cap C(x) = \{x\} \). Since this is true for every \( x \in S \) we must have

\[
\bar{S} - (S \cup C(x_0) \cup \{p\}) = \emptyset.
\]

Thus if \( h((0, a)) \subset \text{int} C(x) \), then \( \lim_{t \rightarrow 0} h(t) = p \) since \( \overline{\text{int} C(x)} \) is compact. Similarly if \( h((a, 1)) \subset \text{int} C(x) \), then \( \lim_{t \rightarrow 1} h(t) = p \).
Corollary 2.13. \( S \cup \{p\} \) is homeomorphic with \([0, 1)\).

Remark 2.14. Let \( x \in S \) and \( S_i \) be the subarc of \( S \cup \{p\} \) with end-points \( x \) and \( p \). In what follows we will assume \( x = x_0 \) and \( S = S_i \).

3. Type-N-isomorphisms. The classification of dynamical systems in terms of the following types of isomorphisms is due to Ura [7].

Let \((X_1, \pi_1)\) and \((X_2, \pi_2)\) be two dynamical systems. An isomorphism of \((X_1, \pi_1)\) onto \((X_2, \pi_2)\) is a pair of mappings \((h, \varphi)\) which satisfies one of the sets of conditions which follow. An isomorphism which satisfies the condition of Type \( N \) will be called a type-\( N \)-isomorphism.

If there exists a type-\( N \)-isomorphism of \((X_1, \pi_1)\) onto \((X_2, \pi_2)\), then we say that \((X_1, \pi_1)\) and \((X_2, \pi_2)\) are type-\( N \)-isomorphic.

**Type 1.** (Topological isomorphisms.)

1. \( h \) is a homeomorphism of \( X_1 \) onto \( X_2 \).
2. \( \varphi \) is a homeomorphic group-isomorphism of the real additive group \( R^i \) onto itself, i.e., \( \varphi(t) = ct \) for some nonzero constant \( c \).
3. (Homomorphism condition) \( h(x\pi_t) = h(x)\pi_x\varphi(t) \) for all \( x \in X_1 \) and \( t \in R^i \).

**Type 2.**

1. \( h \) is a homeomorphism of \( X_1 \) onto \( X_2 \).
2. \( \varphi \) is a continuous mapping of \( X_1 \times R^i \) onto \( R^i \) such that for every fixed \( x \in X_1 \), \( \varphi(x, \cdot) \) is a homeomorphic group-isomorphism of the real additive group \( R^i \) onto itself such that \( \varphi(x, 0) = 0 \), i.e., there exists a continuous mapping \( \varphi: X_1 \to R^i \) such that \( \varphi(x, t) = \varphi(x)t \) for all \( x \in X_1 \) and \( t \in R^i \).
3. (Homomorphism condition) \( h(x\pi_t) = h(x)\pi_x\varphi(x, t) \) for all \( x \in X_1 \) and \( t \in R^i \).

**Type 2'.** (Phase-map with reparameterization [4].)

1. \( h \) is a homeomorphism of \( X_1 \) onto \( X_2 \).
2. \( \varphi \) is a continuous mapping of \( X_1 \times R^i \) onto \( R^i \) such that for every fixed \( x \in X_1 \), \( \varphi(x, \cdot) \) is a homeomorphism of \( R^i \) onto \( R^i \) such that \( \varphi(x, 0) = 0 \).
3. (Homomorphism condition) \( h(x\pi_t) = h(x)\pi_x\varphi(x, t) \) for all \( x \in X_1 \) and \( t \in R^i \).

Remark. Type 1 \( \subset \) Type 2 \( \subset \) Type 2'.

Under certain restrictions we will show that isomorphisms of types 2 and 2' are equivalent for centers, and for \( i = 1, 2 \) give necessary and sufficient conditions for two centers to be type-\( N \)-isomorphic. The proof of the following assertion is in [7].
PROPOSITION 3.1. "type-N-isomorphic" is an equivalence relation on the family of all dynamical systems.

4. Classification of centers. We will now classify centers in terms of type-N-isomorphisms. Let \((\mathbb{R}^2, \pi_0)\) be the dynamical system defined by
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x .
\end{align*}
\]
The phase portrait consists of a single critical point—the origin—and cycles of fundamental period \(2\pi\) which are concentric circles about the origin. Let \(x \in \mathbb{R}^2\) and \(t \in \mathbb{R}^1\); then \(x\pi_0 t = xe^{it}\). Let
\[
U_0 = \{x \in \mathbb{R}^2: |x| \leq 1\}
\]
be as before.

PROPOSITION 4.1. \((U_0-\{0\}, \pi_0)\) and \((U-\{p\}, \pi)\) are type-2-isomorphic.

Proof. Let \(S\) be an arc such that \(S\pi\mathbb{R}^1 = U\) and let \(f: [0, 1] \to S\) be a homeomorphism such that \(f(0) = p\). If \(x \in U_0 - \{0\}\), there exists a unique \(t_x \in [0, 2\pi]\) such that \(x\pi_0 t_x = |x|\). Define \(h: U_0 \to U\) as follows:
\[
h(x) = \begin{cases} 
(f(|x|) - \frac{t_x}{2\pi} T(f(|x|))) & \text{if } x \in U_0 - \{0\} \\
p & \text{if } x = 0
\end{cases}
\]
h is easily verified to be continuous. Let \(x, y \in U_0 - \{0\}\) be such that \(h(x) = h(y)\). Then
\[
f(|x| - \frac{t_x}{2\pi} T(f(|x|))) = f(|y| - \frac{t_y}{2\pi} T(f(|y|)) .
\]
Thus \(f(|x|)\) and \(f(|y|)\) are on the same trajectory and both are elements of \(S\). Hence \(f(|x|) = f(|y|)\) and \(|x| = |y|\) since \(f\) is a homeomorphism. Next, \(t_x, t_y \in [0, 2\pi]\) implies \(t_x = t_y\). Thus \(x = |x| e^{itx} = |y| e^{ity} = y\); this shows that \(h\) is one-to-one.

If \(y \in U - \{p\}\) there exists a \(\tau_y \in [0, T(y)]\) such that \(y\pi\tau_y \in S\). Then \(h^{-1}(y) = f^{-1}(y\pi\tau_y)\exp[-2\pi i \tau_y/T(y)]\) and \(h\) is onto. Since each continuous, one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism, \(h\) is a homeomorphism of \(U_0\) onto \(U\).

Now let \(x \in U_0 - \{0\}\) and \(t \in \mathbb{R}^1\). Then \(x\pi_0 t_x = |x| = (x\pi_0 t)\pi_0 t_{x\pi_0 t}\) implies \(t_x = t + t_{x\pi_0 t} + 2n\pi\) for some integer \(n\).
\[
h(x\pi_0 t) = f(|x\pi_0 t| - \frac{t_{x\pi_0 t}}{2\pi} T(f(|x\pi_0 t|))
\]
\begin{align*}
&= f(|x|)\pi - \frac{t_x - t - 2n\pi}{2\pi} T(f(|x|)) \\
&= f(|x|)\pi - \frac{t_x - t}{2\pi} T(f(|x|)) \\
&= h(x)\pi \frac{t}{2\pi} T(f(|x|)).
\end{align*}

Since \( h(x) \) and \( f(|x|) \) are on the same trajectory, we have \( T(h(x)) = T(f(|x|)) \). Thus

\[ h(x\pi o t) = h(x)\pi - t - T(h(x)). \]

Set \( \phi(x, t) = (t/2\pi)T(h(x)) \) for all \( x \in U_0 - \{0\} \) and for all \( t \in R' \). Evidently \( (h \mid U_0 - \{0\}, \phi) \) satisfies the conditions of type 2.

**Proposition 4.2.** The following three conditions are equivalent:

(i) \((U_0, \pi_0)\) and \((U, \pi)\) are type-2-isomorphic.

(ii) \((U_0, \pi_0)\) and \((U, \pi)\) are type-2'-isomorphic.

(iii) \( \lim s_{y_p} T(y) \) exists, is finite, and nonzero.

**Proof.** We shall show that (iii) \(\Rightarrow\) (i) and (ii) \(\Rightarrow\) (iii). Assume \( \lim s_{y_p} T(y) \) exists and equals \( \lambda \), \( 0 < \lambda \in R' \). Let \( h \) and \( \phi \) be as in the proof of Proposition 4.1 and define \( \tilde{\phi} : U_0 \times R' \to R' \) as follows:

\[
\tilde{\phi}(x, t) = \begin{cases} 
\phi(x, t) & \text{if } x \in U_0 - \{0\} \text{ and } t \in R' \\
\frac{t\lambda}{2\pi} & \text{if } x = 0 \text{ and } t \in R'.
\end{cases}
\]

Evidently \( \tilde{\phi} \) is a continuous extension of \( \phi \) to \( U_0 \times R' \) and \( (h, \tilde{\phi}) \) satisfies the conditions of type 2.

Now assume \( (h, \phi) \) is a type-2'-isomorphism of \((U_o, \pi_o)\) onto \((U, \pi)\). \( h(0) = h(0 \pi t) = h(0)\pi \phi(0, t) \) for every \( t \in R' \). Thus \( h(0) \) is critical and must equal \( p \). Since \( h \) is a homeomorphism, \( h(x) = p \) if and only if \( x = 0 \). Let \( x \in U_0 - \{0\} \). Then \( h(x) = h(x\pi, 2\pi) = h(x)\pi \phi(x, 2\pi) \) and \( h(x) \neq h(x\pi o t) \) for \( 0 < t < 2\pi \) imply that \( |\phi(x, 2\pi)| \) is the fundamental period of \( h(x) \), i.e., \( |\phi(x, 2\pi)| = T(h(x)) \) for all \( x \in U_0 - \{0\} \). By the continuity of \( \phi(\cdot, 2\pi) \), we have that \( \lim s_{y_p} T(h(x)) \) exists and is finite. \( \phi(x, \cdot) \) a homeomorphism such that \( \phi(x, 0) = 0 \) implies \( \lim s_{y_p} T(h(x)) \neq 0 \). Since \( h \) is a homeomorphism, \( \lim s_{y_p} T(y) \) exists, is finite, and nonzero. This completes the proof.

Let \((R^2, \pi_i)\) and \((R^2, \pi_2)\) be two dynamical systems with centers \( p_i \) and \( p_2 \) respectively. For \( i = 1, 2 \), let \( U_i \) be a neighborhood of \( p_i \) as described in Remark 2.14, \( S_i \) be the arc which generates \( U_i \), and \( T \) be the mapping which associates with \( x \in U_i - \{p_i\} \) its fundamental
THEOREM 4.3. \((U_1 - \{p_1\}, \pi_1)\) and \((U_2 - \{p_2\}, \pi_2)\) are type-2-isomorphic.

Proof. This is an immediate consequence of Propositions 3.1 and 4.1.

PROPOSITION 4.4. If \(f: S_1 \to S_2\) is a homeomorphism, then there exists a type-2-isomorphism \((h, \varphi)\) of \((U_1 - \{p_1\}, \pi_1)\) onto \((U_2 - \{p_2\}, \pi_2)\) such that \(h|S_1 = f\) and \(\varphi(x, t) = tT_2(h(x))/T_1(x)\) for all \(x \in U_1 - \{p_1\}\) and for all \(t \in R^1\).

Proof. Analogous to that of Proposition 4.1.

DEFINITION 4.5. Let \((X_1, \pi_1)\) and \((X_2, \pi_2)\) be dynamical systems. A homeomorphism \(h\) of \(X_1\) onto \(X_2\) is said to be trajectory preserving if and only if \(h(C_i(x)) = C_i(h(x))\) for every \(x \in X_1\).

PROPOSITION 4.6. \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are type-2-isomorphic if and only if there exists a trajectory preserving homeomorphism \(h: U_1 \to U_2\) such that \(\lim_{t \to \lambda} T_2(h(y))/T_1(y)\) exists, is finite, and nonzero.

Proof. Let \(h\) be a trajectory preserving homeomorphism of \(U_1\) onto \(U_2\) such that \(\lim_{t \to \lambda} T_2(h(y))/T_1(y)\) exists, is finite, and nonzero. Then \(h(x\pi_iR^1) = h(x)\pi_2R^1,\) and, for all \(x \in S_i, h(x)\pi_2R^1 \cap h(S_i) = \{h(x)\}\) since \(x\pi_iR^1 \cap S_i = \{x\}\). \(h|S_i\) is a homeomorphism of \(S_i\) onto \(h(S_i)\). By Proposition 4.4 there exists a homeomorphism \(g\) of \(U_1\) onto \(U_2\) such that \((g|U_1 - \{p_1\}, \varphi)\) is a type-2-isomorphism of \((U_1 - \{p_1\}, \pi_1)\) onto \((U_2 - \{p_2\}, \pi_2)\). Moreover \(g|S_1 = h|S_1\) and \(\varphi(x, t) = tT_2(g(x))/T_1(x)\). Then \(\varphi(x, t) = tT_2(h(x))/T_1(x)\) for all \(x \in S - \{p\}\) since \(g|S_1 = h|S_1\) and \(\lim_{t \to \lambda} tT_2(h(x))/T_1(x) = \lambda t\) for some nonzero \(\lambda\) by our assumption on \(h\). Define \(\bar{\varphi}: U_1 \times R^1 \to R^1\) as follows:

\[
\bar{\varphi}(x, t) = \begin{cases} 
\varphi(x, t) & \text{if } x \in U_1 - \{p_1\} \text{ and } t \in R^1 \\
\lambda t & \text{if } x = p_1 \text{ and } t \in R^1 
\end{cases}
\]

\(\bar{\varphi}\) is evidently a continuous extension of \(\varphi\) and \((g, \bar{\varphi})\) a type-2-isomorphism of \((U_1, \pi_1)\) onto \((U_2, \pi_2)\).

Now assume that \((h, \varphi)\) is a type-2-isomorphism of \((U_1, \pi_1)\) onto \((U_2, \pi_2)\). Then \(\varphi_z(\cdot)\) a homeomorphic group isomorphism of \(R^1\) onto itself such that \(\varphi_z(0) = 0\); thus there exists a continuous function \(f: U_1 \to R^1\) such that \(\varphi_z(t) = f(x)t\) for all \(x \in U_1\) and for all \(t \in R^1\). Indeed, \(f(x) = \varphi(x, 1).\) If \(x \in U_1 - \{p_1\},\) then
\( h(x) = h(x\pi, T_1(x)) = h(x)\pi \phi(x, T_1(x)) \)

and \( h(x) \neq h(x\pi, t) \) for \( 0 < t < T_1(x) \). Thus \( |\phi(x, T_1(x))| \) is the fundamental period of \( h(x) \). Thus \( |\phi_2(T_1(x))| = |f(x)| T_1(x) = |f(p_1)| T_1(h(x)) \). Therefore \( |f(x)| = T_1(h(x))/T_1(x) \) and \( \lim_{x\to p_1} T_1(h(x))/T_1(x) = \frac{f(p_1)}{T_1(h(x))} = 0 \) since \( f \) is continuous and \( \phi_{p_1} \) is a homeomorphic group isomorphism of \( \mathbb{R}^t \) onto itself. This completes the proof.

**Corollary 4.7.** If both \( \lim_{x\to p_1} T_1(x) \) and \( \lim_{y\to p_2} T_2(y) \) exist, are finite, and nonzero, then \( (U_1, \pi_1) \) and \( (U_2, \pi_2) \) are type-2-isomorphic.

**Proof.** Since \( S_1 \) and \( S_2 \) are both homeomorphic to \([0, 1]\) by Remark 2.14), \( S_1 \) and \( S_2 \) are homeomorphic. By Proposition 4.4 there exists a trajectory preserving homeomorphism of \( U_1 \setminus \{p_1\} \) onto \( U_2 \setminus \{p_2\} \). This can be extended to a trajectory preserving homeomorphism \( h \) of \( U_1 \) onto \( U_2 \) by mapping \( p_1 \) onto \( p_2 \). Then \( \lim_{x\to p_1} T_1(h(x))/T_1(x) \) exists, is finite, and nonzero since both \( \lim_{x\to p_1} T_1(x) \) and \( \lim_{y\to p_2} T_2(y) \) are such. The result follows from Proposition 4.6.

By assumption \( U_1 \) and \( U_2 \) are neighborhoods of \( p_1 \) and \( p_2 \) respectively such that there exist \( x_1, x_2 \in \mathbb{R}^t \) with \( \text{int} C_i(x_i) = U_1 \) and \( \text{int} C_i(x_2) = U_2 \). Moreover \( x_i \) can be chosen so that \( S_i = S_1 \cup \{x_i\} \cup \{p_i\} \), \( i = 1, 2 \).

(See Remark 2.14.)

**Corollary 4.8.** If \( \lim_{x\to p_1} T_1(x) = \lim_{y\to p_2} T_2(y) \) (with values 0 and \( \infty \) as allowed), \( T_1(x_1) = T_2(x_2) \) and both \( T_1|S_1 \) and \( T_2|S_2 \) are one-to-one, \( (U_1, \pi_1) \) and \( (U_2, \pi_2) \) are type-2-isomorphic.

**Proof.** Since \( T_i(S_i) \) is connected, \( T_i(S_i) \) is an interval for \( i = 1, 2 \). Moreover \( T_i(S_i) = T_2(S_2) \) since

\[ T_1(x_i) = T_2(x_2) \text{ and } \lim_{x\to p_1} T_1(x) = \lim_{y\to p_2} T_2(y). \]

If \( V \) is a compact subset of \( S_i \), then \( T_i|V \) is a homeomorphism because a continuous, one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism. Since this is true for every compact subset \( V \) of \( S_i \), \( T_i \) is a homeomorphism, \( i = 1, 2 \). Define \( g: S_1 \to S_2 \) as follows:

\[
g(x) = \begin{cases} T_2^{-1} T_1(x) & \text{if } x \in S_1 \setminus \{p_1\} \\ p_2 & \text{if } x = p_1. \end{cases}
\]

Evidently \( g \) is a homeomorphism of \( S_1 \) onto \( S_2 \). By Proposition 4.4 \( g \) can be extended to a trajectory preserving homeomorphism \( h: U_1 \to U_2 \). Then
\[
\lim_{y \to p_1} \frac{T_2(h(y))}{T_1(y)} = \lim_{y \to p_1} \frac{T_2(h(C_1(y) \cap S_i))}{T_1(C_1(y) \cap S_i)} \\
= \lim_{y \to p_1} \frac{T_2(h(y))}{T_1(y)} = \lim_{y \to p_1} \frac{T_2(g(y))}{T_1(y)} \\
= \lim_{y \to p_1} \frac{T_2(T^{-1}T_1(y))}{T_1(y)} = 1.
\]

The result now follows by Proposition 4.6.

**EXAMPLE 4.9.** If \( \lim_{x \to p_1} T_1(x) = \lim_{y \to p_2} T_2(y) = 0 \), it is not necessarily true that \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are type-2-isomorphic. Let \((U_0, \pi_0)\) be as before and define \(\pi_1\) and \(\pi_2\) as follows (\(f\) and \(g\) shall be chosen later):

\[
x\pi_1t = x\pi_0\frac{t}{f(x)} \quad \text{for all } x \in U_0 \text{ and for all } t \in R^1 \\
x\pi_2t = x\pi_0\frac{t}{g(x)} \quad \text{for all } x \in U_0 \text{ and for all } t \in R^1.
\]

If there exists a type-2-isomorphism \((h, \varphi)\) of \((U_0, \pi_0)\) onto \((U_0, \pi_0)\) then by Proposition 4.6 \(\lim_{y \to 0} T_2(h(y))/T_1(y)\) exists and is nonzero. Note that \(T_1(x) = f(x)\) and \(T_2(x) = g(x)\). Restricting our attention to \(S_1\) and \(S_2\), the problem may be reduced to the following:

Given continuous functions \(f, g: [0, 1] \to [0, 1]\) such that \(f(0) = g(0) = 0\) and \(f(x) > 0 < g(x)\) for \(x \in (0, 1]\). Does there exist a homeomorphism \(h: [0, 1] \to [0, 1]\) such that \(\lim_{x \to 0} f(h_i(x))/g(x)\) exists and is nonzero? It is not hard to see that there exist functions \(f\) and \(g\) satisfying our assumptions and such that \(\lim_{x \to 0} f(h_i(x))/g(x)\) does not exist for any homeomorphism \(h_i: [0, 1] \to [0, 1]\). Hence for these choices of \(f\) and \(g\), \((U_0, \pi_0)\) and \((U_0, \pi_0)\) are not type-2-isomorphic.

Similarly, if \(\lim_{x \to p_1} T_1(x) = \lim_{y \to p_2} T_2(y) = +\infty\), it is not necessarily true that \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are type-2-isomorphic.

**PROPOSITION 4.10.** \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are type-1-isomorphic if and only if there exists a trajectory preserving homeomorphism \(h\) of \(U_1\) onto \(U_2\) and a constant \(\lambda\) such that \(T_2(h(x)) = \lambda T_1(x)\) for all \(x \in U_1 - \{p\}\).

**Proof.** Assume \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are type-1-isomorphic. Then there exist a homeomorphism \(h: U_1 \to U_2\) and a nonzero constant \(\lambda\) such that \(h(x\pi_1t) = h(x)\pi_2t\) for all \(x \in U_1\) and for all \(t \in R\). Evidently \(h\) is trajectory preserving. Let \(x \in U_1 - \{p\}\). Then

\[
h(x) = h(x\pi_1T_1(x)) = h(x)\pi_2\lambda T_1(x)
\]
and \( h(x) \neq h(x)\pi_2 x \) for \( t \in (0, T_1(x)) \). Thus

\[
T_2(h(x)) = |\lambda T_1(x)| = |\lambda| T_1(x).
\]

Now let \( h \) be a trajectory preserving homeomorphism of \( U_1 \) onto \( U_2 \) and \( \lambda \) be a nonzero constant such that \( T_2(h(x)) = \lambda T_1(x) \) for every \( x \in U_1 - \{p\} \). Then \( h|S_i \) is a homeomorphism of \( S_i \) onto \( h(S_i) \). By Proposition 4.4 there exists a homeomorphism \( g: U_1 - \{p\} \rightarrow U_2 - \{p_2\} \) such that \( g|S_i = h|S_i \) and \( (g, \varphi) \) is a type-2-isomorphism of

\[
(U_1 - \{p_1\}, \pi_1) \text{ onto } (U_2 - \{p_2\}, \pi_2)
\]

where \( \varphi(x, t) = T_2(g(x))t/T_1(x) \) for all \( x \in U - \{p\} \) and \( t \in R^1 \). Then \( \varphi(x, t) = T_2(h(x))t/T_1(x) \) for all \( x \in S_i - \{p_i\} \). Thus \( \varphi(x, t) = \lambda t \) for all \( x \in S_i - \{p_i\} \). Define \( \bar{\varphi}: U_1 \times R^1 \rightarrow R^1 \) as follows:

\[
\bar{\varphi}(x, t) = \lambda t.
\]

Then it is easy to show \( g(x, t) = g(x)\pi_2 x + t \) for all \( x \in U_1 - \{p_1\} \) and for all \( t \in R^1 \). \( g \) can be extended to a homeomorphism \( \bar{g} \) of \( U_1 \) onto \( U_2 \) by mapping \( p_1 \) onto \( p_2 \). Then \( (\bar{g}, \bar{\varphi}) \) is a type-1-isomorphism of \( (U_1, \pi_1) \) onto \( (U_2, \pi_2) \).

**Corollary 4.11.** \( (U_1, \pi_1) \) and \( (U_2, \pi_2) \) are type-1-isomorphic if and only if \( T_1(\cdot) \) is constant on \( U_1 - \{p_1\} \).

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