MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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This paper applies a certain method of iteration, of the mean value type introduced by W. R. Mann, to obtain two theorems on the approximation of a fixed point of a mapping of a Banach space into itself which is nonexpansive (i.e., a mapping which satisfies $||Tx - Ty|| \leq ||x - y||$ for each $x$ and $y$).

The first theorem obtains convergence of the iterates to a fixed point of a nonexpansive mapping which maps a compact convex subset of a rotund Banach space into itself.

The second theorem obtains convergence to a fixed point provided that the Banach space is uniformly convex and the iterating transformation is nonexpansive, maps a closed bounded convex subset of the space into itself, and satisfies a certain restriction on the distance between any point and its image.

We note that a rotation $T$ about zero of the closed unit disc in the complex plane satisfies the conditions of Theorems 1 and 2, but the usual sequence $\{T^n x\}$ of iterates of $x$ does not converge unless $x$ is zero.

DEFINITIONS. If $Y$ is a Banach space, $T$ is a mapping from $Y$ into itself, and $x \in Y$, then $M(x, T)$ is the sequence $\{v_n\}$ defined by $v_1 = x$ and $v_{n+1} = (1/2)(v_n + Tv_n)$.

Following Wilansky [3, pp. 107–111], we say that a Banach space $Y$ is rotund provided that if $w \in Y$, $y \in Y$, $w \neq y$, and $||w|| = ||y|| \leq 1$, then $(1/2)||w + y|| < 1$.

THEOREM 1. Let $Y$ be a rotund Banach space, $E$ be a compact convex subset of $Y$, and $T$ be a nonexpansive mapping which maps $E$ into itself. If $x \in E$ then $M(x, T)$ converges to a fixed point of $T$.

Proof. If, for some $n$, $v_n = Tv_n$, then clearly $M(x, T)$ converges to $v_n$.

Hence suppose that $v_n \neq Tv_n$, for each $n$. Let $z$ be a fixed point of $T$. Then $||v_n - z||$ is decreasing, for since $Y$ is rotund and

$$||Tv_n - z|| = ||Tv_n - Tz|| \leq ||v_n - z||,$$

we have that

$$||v_{n+1} - z|| = \left|\left| \frac{1}{2}(v_n + Tv_n) - z \right|\right| < ||v_n - z||.$$
Suppose that \( \lim_n |v_n - z| = b > 0 \). Let \( y \) be a cluster value of \( \{v_n\} \). Then clearly \( b = |y - z| \).

Suppose first that \( y = Ty \). Then for each \( n \),
\[
|Tv_n - y| = |Tv_n - Ty| \leq |v_n - y|.
\]
Since we have assumed that \( v_n \neq Tv_n \) for each \( n \), we have by the rotundity of \( Y \) that
\[
|v_{n+1} - y| = \left| \frac{1}{2} (v_n + Tv_n) - y \right| < |v_n - y|.
\]
Thus \( \{ |v_n - y| \} \) is decreasing, and since \( y \) is a cluster value of \( \{v_n\} \), \( M(x, T) \) converges to \( y \).

Now suppose that \( y \neq Ty \). Let \( d \) denote \( b - |(1/2)(y + Ty) - z| \). Then \( d > 0 \), since \( Y \) is rotund, for
\[
|Ty - z| = |Ty - Tz| \leq |y - z| = b.
\]
Let \( n \) be such that \( |y - v_n| < d \). Then since \( T \) is nonexpansive,
\[
\left| \frac{1}{2} (y + Ty) - v_{n+1} \right| = \left| \frac{1}{2} (y + Ty) - \frac{1}{2} (v_n + Tv_n) \right|
\leq \frac{1}{2} |y - v_n| + \frac{1}{2} |Ty - Tv_n|
\leq |y - v_n| < d.
\]
Hence
\[
|v_{n+1} - z| \leq \left| v_{n+1} - \frac{1}{2} (y + Ty) \right| + \left| \frac{1}{2} (y + Ty) - z \right|
< d + (b - d) = b,
\]
a contradiction. Therefore \( b = \lim_n |v_n - z| = 0 \), so that \( M(x, T) \) converges to \( z \).

F.E. Browder [1] has shown that each nonexpansive mapping which maps a closed bounded convex subset \( E \) of a uniformly convex Banach space into itself has a fixed point in \( E \).

If such a mapping satisfies one additional requirement, we may approximate one of its fixed points using \( M(x, T) \):

**Theorem 2.** Let \( Y \) be a uniformly convex Banach space, \( E \) be a closed bounded convex subset of \( Y \), and let \( T \) be a nonexpansive mapping which maps \( E \) into itself. Let \( F \) denote the set of fixed point of \( T \) in \( E \), and suppose that there is a number \( c \) in \( (0, 1) \) such that if \( x \in E \), then
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\[ \| x - Tx \| \geq cd(x, F) , \]

where \( d(x, F) \) denotes \( \sup_{x \in F} \| x - z \| \).

If \( x \in E \) then \( M(x, T) \) converges to a fixed point of \( T \).

**Proof.** The theorem is trivial if \( x \in F \). Suppose that \( x \in E - F \) and that \( M(x, T) \) does not converge to a member of \( F \). Then \( v_n \notin F \) for each \( n \). Since \( Y \) is uniformly convex, we have as in the proof of Theorem 1 that if \( z \in F \) then \( \| v_n - z \| \) is decreasing.

Suppose that \( b = \lim_n d(v_n, F) > 0 \). Since \( Y \) is uniformly convex, there is an \( r \) in \( (0, 2b) \) such that, for \( w, y, \) and \( z \) in \( Y \), the relations

\[ \| w - z \| \leq \| y - z \| \leq 2b \quad \text{and} \quad \| w - y \| \geq cb \]

imply that

\[ \| \frac{1}{2} (w + y) - z \| \leq \| y - z \| - r . \]

There is a positive integer \( n \) and a member \( z \) of \( F \) such that

\[ \| v_n - z \| < b + \frac{r}{2} , \]

so that since

\[ \| T v_n - z \| = \| T v_n - T z \| \leq \| v_n - z \| < 2b \]

and

\[ \| T v_n - v_n \| \geq cd(v_n, F) \geq cb , \]

we have that

\[ \| v_{n+1} - z \| = \| \frac{1}{2} (v_n + T v_n) - z \| \leq \| v_n - z \| - r < b + \frac{r}{2} - r < b , \]

an contradiction. Hence \( \lim_n d(v_n, F) = 0 \).

We now need the following:

**Lemma.** If \( s > 0, z \in F, \) and \( r > 0 \) such that for some \( n, v_n \) is in the open sphere \( S(z, r) \) with center \( z \) and radius \( r \), then there exist \( t \) in \( (0, s) \), \( w \) in \( F \), and an \( m \) such that the closed sphere \( \bar{S}(w, t) \) lies in \( S(z, r) \), and for each \( p, v_{m+p} \in S(w, t) \).

**Proof.** Recall that \( \{ \| v_p - z \| \} \) is decreasing and that we are supposing that \( \{ v_p \} \) does not converge to \( z \). Let \( a = \lim_p \| v_p - z \| . \)
Then $0 < a < r$. Let $t = (1/3) \min \{r - a, s\}$.

Since $\lim_p \|v_p - z\| = a$, $\lim_p d(v_p, F) = 0$, and $v_p \in F$ for each $p$, there exist $w$ in $F$ and an $m$ such that $\|v_m - z\| < a + t$ and $\|v_m - w\| < t$.

Since $w \in F$, $\|v_{m+p} - w\|$ decreases as $p$ increases, so that $v_{m+p} \in S(w, t)$ for each $p$. Also, if $y \in \bar{S}(w, t)$, then $y \in S(z, r)$, for

$$
\|y - z\| \leq \|y - w\| + \|w - v_m\| + \|v_m - z\|
< t + t + (a + t)
\leq 3 \left( \frac{r - a}{3} \right) + a = r .
$$

The lemma guarantees the existence of a sequence $\{z_i\}$ in $F$, a sequence $\{t_i\}$ of positive numbers with limit 0, and a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that for each $i$ and each $p$,

$$S(z_{i+1}, t_{i+1}) \text{ lies in } S(z_i, t_i)$$
and

$$v_{n_i+p} \in S(z_{i}, t_i) .$$

By the Cantor Intersection Theorem, $\bigcap_{i=1}^{\infty} S(z_i, t_i)$ contains exactly one point, say $w$. Clearly $\{z_i\}$ converges to $w$ and $w \in F$. Further, $\{\|v_n - w\|\}$ is decreasing and $\{v_{n_i}\}$ converges to $w$, so that $\{v_n\}$ converges to $w$. Thus we have contradicted our assumption that $M(x, T)$ does not converge to a member of $F$.

REFERENCES


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