

Pacific Journal of Mathematics

**MEAN VALUE ITERATION OF NONEXPANSIVE MAPPINGS IN
A BANACH SPACE**

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This paper applies a certain method of iteration, of the mean value type introduced by W. R. Mann, to obtain two theorems on the approximation of a fixed point of a mapping of a Banach space into itself which is nonexpansive (i.e., a mapping which satisfies $\|Tx - Ty\| \leq \|x - y\|$ for each x and y).

The first theorem obtains convergence of the iterates to a fixed point of a nonexpansive mapping which maps a compact convex subset of a rotund Banach space into itself.

The second theorem obtains convergence to a fixed point provided that the Banach space is uniformly convex and the iterating transformation is nonexpansive, maps a closed bounded convex subset of the space into itself, and satisfies a certain restriction on the distance between any point and its image.

We note that a rotation T about zero of the closed unit disc in the complex plane satisfies the conditions of Theorems 1 and 2, but the usual sequence $\{T^n x\}$ of iterates of x does not converge unless x is zero.

DEFINITIONS. If Y is a Banach space, T is a mapping from Y into itself, and $x \in Y$, then $M(x, T)$ is the sequence $\{v_n\}$ defined by $v_1 = x$ and $v_{n+1} = (1/2)(v_n + Tv_n)$.

Following Wilansky [3, pp. 107-111], we say that a Banach space Y is *rotund* provided that if $w \in Y$, $y \in Y$, $w \neq y$, and $\|w\| = \|y\| \leq 1$, then $(1/2)\|w + y\| < 1$.

THEOREM 1. *Let Y be a rotund Banach space, E be a compact convex subset of Y , and T be a nonexpansive mapping which maps E into itself. If $x \in E$ then $M(x, T)$ converges to a fixed point of T .*

Proof. If, for some n , $v_n = Tv_n$, then clearly $M(x, T)$ converges to v_n .

Hence suppose that $v_n \neq Tv_n$, for each n . Let z be a fixed point of T . Then $\{\|v_n - z\|\}$ is decreasing, for since Y is rotund and

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leq \|v_n - z\|,$$

we have that

$$\|v_{n+1} - z\| = \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| < \|v_n - z\|.$$

Suppose that $\lim_n \|v_n - z\| = b > 0$. Let y be a cluster value of $\{v_n\}$. Then clearly $b = \|y - z\|$.

Suppose first that $y = Ty$. Then for each n ,

$$\|Tv_n - y\| = \|Tv_n - Ty\| \leq \|v_n - y\|.$$

Since we have assumed that $v_n \neq Tv_n$ for each n , we have by the rotundity of Y that

$$\|v_{n+1} - y\| = \left\| \frac{1}{2}(v_n + Tv_n) - y \right\| < \|v_n - y\|.$$

Thus $\{\|v_n - y\|\}$ is decreasing, and since y is a cluster value of $\{v_n\}$, $M(x, T)$ converges to y .

Now suppose that $y \neq Ty$. Let d denote $b - \|(1/2)(y + Ty) - z\|$. Then $d > 0$, since Y is rotund, for

$$\|Ty - z\| = \|Ty - Tz\| \leq \|y - z\| = b.$$

Let n be such that $\|y - v_n\| < d$. Then since T is nonexpansive,

$$\begin{aligned} \left\| \frac{1}{2}(y + Ty) - v_{n+1} \right\| &= \left\| \frac{1}{2}(y + Ty) - \frac{1}{2}(v_n + Tv_n) \right\| \\ &\leq \frac{1}{2}\|y - v_n\| + \frac{1}{2}\|Ty - Tv_n\| \\ &\leq \|y - v_n\| < d. \end{aligned}$$

Hence

$$\begin{aligned} \|v_{n+1} - z\| &\leq \left\| v_{n+1} - \frac{1}{2}(y + Ty) \right\| + \left\| \frac{1}{2}(y + Ty) - z \right\| \\ &< d + (b - d) = b, \end{aligned}$$

a contradiction. Therefore $b = \lim_n \|v_n - z\| = 0$, so that $M(x, T)$ converges to z .

F. E. Browder [1] has shown that each nonexpansive mapping which maps a closed bounded convex subset E of a uniformly convex Banach space into itself has a fixed point in E .

If such a mapping satisfies one additional requirement, we may approximate one of its fixed points using $M(x, T)$:

THEOREM 2. *Let Y be a uniformly convex Banach space, E be a closed bounded convex subset of Y , and let T be a nonexpansive mapping which maps E into itself. Let F denote the set of fixed point of T in E , and suppose that there is a number c in $(0, 1)$ such that if $x \in E$, then*

$$\|x - Tx\| \geq cd(x, F) ,$$

where $d(x, F)$ denotes $\sup_{z \in F} \|x - z\|$.

If $x \in E$ then $M(x, T)$ converges to a fixed point of T .

Proof. The theorem is trivial if $x \in F$. Suppose that $x \in E - F$ and that $M(x, T)$ does not converge to a member of F . Then $v_n \notin F$ for each n . Since Y is uniformly convex, we have as in the proof of Theorem 1 that if $z \in F$ then $\{\|v_n - z\|\}$ is decreasing.

Suppose that $b = \lim_n d(v_n, F) > 0$. Since Y is uniformly convex, there is an r in $(0, 2b)$ such that, for w, y , and z in Y , the relations

$$\|w - z\| \leq \|y - z\| \leq 2b \quad \text{and} \quad \|w - y\| \geq cb$$

imply that

$$\left\| \frac{1}{2}(w + y) - z \right\| \leq \|y - z\| - r .$$

There is a positive integer n and a member z of F such that

$$\|v_n - z\| < b + \frac{r}{2} ,$$

so that since

$$\|Tv_n - z\| = \|Tv_n - Tz\| \leq \|v_n - z\| < 2b$$

and

$$\|Tv_n - v_n\| \geq cd(v_n, F) \geq cb ,$$

we have that

$$\begin{aligned} \|v_{n+1} - z\| &= \left\| \frac{1}{2}(v_n + Tv_n) - z \right\| \\ &\leq \|v_n - z\| - r < b + \frac{r}{2} - r < b , \end{aligned}$$

an contradiction. Hence $\lim_n d(v_n, F) = 0$.

We now need the following:

LEMMA. *If $s > 0, z \in F$, and $r > 0$ such that for some n, v_n is in the open sphere $S(z, r)$ with center z and radius r , then there exist t in $(0, s)$, w in F , and an m such that the closed sphere $\bar{S}(w, t)$ lies in $S(z, r)$, and for each $p, v_{m+p} \in \bar{S}(w, t)$.*

Proof. Recall that $\{\|v_p - z\|\}$ is decreasing and that we are supposing that $\{v_p\}$ does not converge to z . Let $a = \lim_p \|v_p - z\|$.

Then $0 < a < r$. Let $t = (1/3) \min \{r - a, s\}$.

Since $\lim_p \|v_p - z\| = a$, $\lim_p d(v_p, F) = 0$, and $v_p \notin F$ for each p , there exist w in F and an m such that $\|v_m - z\| < a + t$ and $\|v_m - w\| < t$.

Since $w \in F$, $\|v_{m+p} - w\|$ decreases as p increases, so that $v_{m+p} \in S(w, t)$ for each p . Also, if $y \in \bar{S}(w, t)$, then $y \in S(z, r)$, for

$$\begin{aligned} \|y - z\| &\leq \|y - w\| + \|w - v_m\| + \|v_m - z\| \\ &< t + t + (a + t) \\ &\leq 3\left(\frac{r - a}{3}\right) + a = r. \end{aligned}$$

The lemma guarantees the existence of a sequence $\{z_i\}$ in F , a sequence $\{t_i\}$ of positive numbers with limit 0, and a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that for each i and each p ,

$$\bar{S}(z_{i+1}, t_{i+1}) \text{ lies in } S(z_i, t_i)$$

and

$$v_{n_i+p} \in S(z_i, t_i).$$

By the Cantor Intersection Theorem, $\bigcap_{i=1}^{\infty} S(z_i, t_i)$ contains exactly one point, say w . Clearly $\{z_i\}$ converges to w and $w \in F$. Further, $\{\|v_n - w\|\}$ is decreasing and $\{v_{n_i}\}$ converges to w , so that $\{v_n\}$ converges to w . Thus we have contradicted our assumption that $M(x, T)$ does not converge to a member of F .

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Willard Ellis Baxter, <i>Topological rings with property (Y)</i>	563
Sterling K. Berberian, <i>Note on some spectral inequalities of C. R. Putnam</i>	573
David Theodore Brown, <i>Galois theory for Banach algebras</i>	577
Dennis K. Burke and R. A. Stoltenberg, <i>A note on p-spaces and Moore spaces</i>	601
Rafael Van Severen Chacon and Stephen Allan McGrath, <i>Estimates of positive contractions</i>	609
Rene Felix Dennemeyer, <i>Conjugate surfaces for multiple integral problems in the calculus of variations</i>	621
Edwin O. Elliott, <i>Measures on countable product spaces</i>	639
John Moss Grover, <i>Covering groups of groups of Lie type</i>	645
Charles Lemuel Hagopian, <i>Concerning semi-local-connectedness and cutting in nonlocally connected continua</i>	657
Velmer B. Headley, <i>A monotonicity principle for eigenvalues</i>	663
John Joseph Hutchinson, <i>Intrinsic extensions of rings</i>	669
Harold H. Johnson, <i>Determination of hyperbolicity by partial prolongations</i>	679
Tilla Weinstein, <i>Holomorphic quadratic differentials on surfaces in E^3</i>	697
R. C. Lacher, <i>Cell-like mappings. I</i>	717
Roger McCann, <i>A classification of centers</i>	733
Curtis L. Outlaw, <i>Mean value iteration of nonexpansive mappings in a Banach space</i>	747
Allan C. Peterson, <i>Distribution of zeros of solutions of a fourth order differential equation</i>	751
Bhalchandra B. Phadke, <i>Polyhedron inequality and strict convexity</i>	765
Jack Wyndall Rogers Jr., <i>On universal tree-like continua</i>	771
Edgar Andrews Rutter, <i>Two characterizations of quasi-Frobenius rings</i>	777
G. Sankaranarayanan and C. Suyambulingom, <i>Some renewal theorems concerning a sequence of correlated random variables</i>	785
Joel E. Schneider, <i>A note on the theory of primes</i>	805
Richard Peter Stanley, <i>Zero square rings</i>	811
Edward D. Tymchatyn, <i>The 2-cell as a partially ordered space</i>	825
Craig A. Wood, <i>On general Z.P.I.-rings</i>	837