POLYHEDRON INEQUALITY AND STRICT CONVEXITY

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This paper considers convexity of functions defined on the
"Grassmann cone" of simple $r$-vectors. It is proved that the
strict polyhedron inequality does not imply strict convexity.

H. Busemann, in conjunction with others, (see [3]), has considered
the problem of giving a suitable definition of the convexity of func-
tions defined on nonconvex sets. An examination of various methods
of defining convexity on the "Grassmann cone" (see [1]) is found in
[2]. The most important open problems (see [3]) are whether weak
convexity implies the area minimizing property (also called the poly-
hedron inequality) and whether the latter implies convexity. A modest
result in this direction is proved below, namely, the strict area min-
imizing property does not imply strict convexity.

2. Basic definitions. Let a continuous function $\mathcal{F}$ be defined
on the Grassmann cone $G^r_*$ of the simple $r$-vectors $R$ in the linear
space $V^r_*$ of all $r$-vectors $\tilde{R}$ (over the reals). Let $\mathcal{F}$ be positive
homogeneous, i.e., $\mathcal{F}(\lambda R) = \lambda \mathcal{F}(R)$ for $\lambda \geq 0$. To a Borel set $F$ in
an oriented $r$-flat $\mathcal{R}^+$ in the $n$-dimensional affine space $A^n$, we as-
sociate a simple $r$-vector as follows: $R = 0$ if $F$ has $r$-dimensional
measure 0, and otherwise $R = v_1 \wedge v_2 \wedge \cdots \wedge v_r$, is parallel to $\mathcal{R}^+$
and the measure of the parallelepiped spanned by $v_1, v_2, \ldots, v_r$ equals
the measure of $F$. (Note a set of measure 0 and equality of measures
in parallel $r$-flats are affine concepts and hence welldefined.) We de-
note below by $\mathcal{R}$ an $r$-flat parallel to an $r$-vector $R$ passing through the
origin.

**DEFINITION 1.** We say that $\mathcal{F}$ has the strict area minimizing
property (SFMA) if: Whenever $R_0, R_1, \ldots, R_p$ are associated to $r$-
dimensional faces of an $r$-dimensional oriented closed polyhedron $P$
we have $\mathcal{F}(-R_0) < \Sigma \mathcal{F}(R_i)$, with $i = 1$ to $p$, unless $R_i = \lambda_i R_0, \lambda_i \geq 0$
for all $i = 1$ to $p$ (called the strict Polyhedron Inequality).

**DEFINITION 2.** $\mathcal{F}$ is said to be strictly weakly convex (SWC)
if: Whenever $R, R_1$ and $R_2$ are simple, $R = R_1 + R_2, R_1$ is not a
scalar multiple of $R_2$, we have $\mathcal{F}(R) < \mathcal{F}(R_1) + \mathcal{F}(R_2)$.

**DEFINITION 3.** $\mathcal{F}$ is said to be convex (C) if there exists a con-
vex extension of $\mathcal{F}$ to $V^r_*$.
DEFINITION 4. \( \mathcal{F} \) is said to be strictly convex (SC) if \( \mathcal{F} \) is C and if there is at least one convex extension \( F \) of \( \mathcal{F} \) to \( V^*_r \) which satisfies the following property: Whenever \( \bar{R} = \sum R_i \) with \( \bar{R}, R_i \in V^*_r \), \( \bar{R} \) is not a scalar multiple of all \( R_i \), then \( F(\bar{R}) < \sum F(R_i) \).

In terms of these definitions we wish to prove below that: if \( \mathcal{F} \) is SWC and C then it has the SFMA and that if \( \mathcal{F} \) is SWC and C it still need not be SC. This implies that the property SFMA is weaker than the property SC.

3. Some algebraic facts. We collect below some algebraic facts which are either known or are relatively easy to prove.

(a) Let \( R_i \) and \( R_2 \) be simple vectors. Then \( R_1 + R_2 \) is simple if and only if \( R_1 \) and \( R_2 \) intersect in a flat of dimension \( \geq r - 1 \).

(b) Identify \( r \)-vectors with points representing them in \( V^*_r \) considered as an affine space. If a line in \( V^*_r \) contains three points corresponding to simple vectors, then the entire line consists of simple vectors. Put differently, if \( R_i \) and \( R_2 \) are simple and \( R_i + R_2 \) is not simple, then the line joining \( R_i \) and \( R_2 \) in \( V^*_r \) does not contain any simple vector other than \( R_i \) and \( R_2 \).

Suppose next that \( R_i, R_2 \) and \( R_3 \) are simple but that \( R_i + R_j \) is nonsimple for all \( i, j = 1 \) to \( 3 \) when \( i \neq j \). Then we have the following:

(c) The set \( \{R_1, R_2, R_3\} \) is a linearly independent set of vectors.

(d) The plane \( \pi \) containing \( \Delta R_i R_2 R_3 \) does not contain any line of simple vectors.

(e) The flat \( \Omega \) spanned by the origin, \( R_i, R_2 \) and \( R_3 \) does not contain a 2-plane of simple vectors.

(f) If a line \( l \) lies in \( \Omega \) and does not pass through the origin, then \( l \) cannot be a line of simple vectors, i.e., \( l \) cannot contain three distinct points corresponding to simple vectors.

4. An example. Busemann and Straus [2] give the following concrete example which we use here to illustrate the above algebraic facts. Let the vectors \( e_1, e_2, e_3, e_4 \) form a base for the four dimensional affine space \( A^4 \). Denote by \( e_{ij} \) the 2-vectors \( e_i \wedge e_j \). Let \( \Omega \) denote the flat spanned by the origin, \( e_{12}, e_{34} \) and \( (e_1 + e_3) \wedge (e_2 + e_4) \) in \( V^*_4 \). We denote the vectors spanning \( \Omega \) by \( z, R_1, R_2 \) and \( R_3 \) respectively. Then \( R_i + R_j \) is nonsimple for all \( i, j = 1 \) to \( 3 \) when \( i \neq j \). Thus any line \( l \) in \( \Omega \) which does not pass through the origin cannot contain three distinct points representing simple vectors.

5. SWC with C is stronger than the SFMA.
**Lemma A.** If a function $\mathcal{F}$ is SWC and $C$ then it has the SFMA.

**Proof.** Let $R_0, R_1, R_2, \ldots, R_p$ be $r$-vectors corresponding to $r$-faces of an $r$-dimensional oriented closed polyhedron $P$. We need consider only the case when not all $R_i$ are scalar multiples of $R_0$, $i > 0$. In such a case, since $P$ is closed, some other faces which are not parallel to the face represented by $R_0$ intersect the face represented by $R_0$ in an $(r-1)$-dimensional set. Let $R_i$ be associated with one such face. Then from §3a the vector $R_0 + R_i$ is simple. Also since $P$ is closed we have $-(R_0 + R_i) = \sum_{i=1}^p R_i$. Thus $\sum_{i=2}^p R_i$ is also simple. But then the equation: $-R_0 = R_i + \sum_{i=2}^p R_i$ shows that

$$\mathcal{F}(-R_0) < \mathcal{F}(R_i) + \mathcal{F}\left(\sum_{i=2}^p R_i\right),$$

and, since $\mathcal{F}$ is convex, $\mathcal{F}(-R_0) < \sum_{i=1}^p \mathcal{F}(R_i)$ so that $\mathcal{F}$ has the SFMA.

6. Existence of functions which are SWC and $C$ but not SC.

**Lemma B.** There exist functions which are SWC and $C$ but not SC.

**Proof.** We actually construct an absolutely homogeneous function of this type. Take three simple unit vectors $R_1, R_2, R_3$ in $V^*_r$ such that $R_i + R_j$ is nonsimple for all $i, j = 1$ to $3$ with $i \neq j$. Choose unit vectors $S_1, S_2, \ldots, S_p$ in $V^*_r$ such that the set of vectors $\{R_i S_j\}$, $i = 1$ to $3$, $j = 1$ to $p$ where

$$p = \left(\frac{n}{r}\right) - 3,$$

is a base for $V^*_r$. Thus given $\tilde{R} \in V^*_r$ we find unique numbers $\{a_i, b_j\}$ such that $\tilde{R} = \sum a_i R_i + \sum b_j S_j$. We denote this last written equality by the notation $\tilde{R} = (a_i, b_j)$. Now define the function $\mathcal{F}$ in $V^*_r$ in the following manner:

If $\tilde{R} = (a_i, b_j)$ then $\mathcal{F}(\tilde{R}) = \sum_{i,j} (a_i^2 + b_j^2)^{1/2} + (\sum_{j} b_j^2)^{1/2}$ with $i = 1$ to $3$ and $j = 1$ to $p$.

We verify that $\mathcal{F}$ has the required property.

(i) $\mathcal{F}$ is clearly absolutely homogeneous ($\mathcal{F}(\lambda R) = |\lambda| \mathcal{F}(R)$) and a convex function on $V^*_r$, hence convex on $G^*_r$.

(ii) We next show $\mathcal{F}$ is SWC. Let $R = (a_i, b_j)$ and

$$R' = (a'_i, b'_j)$$

be two simple $r$-vectors such that $R + R'$ is also simple. Assume
further that $\mathcal{F}(R + R') = \mathcal{F}(R) + \mathcal{F}(R')$. We prove $R$ is parallel to $R'$. Assume that $R$ is not parallel to $R'$. Then the line $l$ in $V^*$ joining $R$ to $R'$ is a line of simple vectors and $l$ does not pass through the origin. Therefore from the algebraic facts, $l$ does not lie in the flat $\Omega$ spanned by the origin, $R_1, R_2, R_3$. Therefore either $b_j$ or $b'_j$ is different from zero. Without loss of generality assume that $b_j \neq 0$. We make the simple observation that when numbers $\alpha_i$ and $\beta_i$ are such that $\alpha_i \leq \beta_i$ and $\Sigma \alpha_i = \Sigma \beta_i$ then each $\alpha_k = \beta_k$. From this and $\mathcal{F}(R + R') = \mathcal{F}(R) + \mathcal{F}(R')$ we have the following equalities:

\[(E) \quad (\Sigma b^2_i)^{1/2} + (\Sigma b^2'_j)^{1/2} = (\Sigma (b_j + b'_j)^2)^{1/2}.\]

For all $(i, j)$,

\[(E_{ij}) \quad (a^2_i + b^2_j)^{1/2} + (a^2'_i + b^2'_j)^{1/2} = ((a_i + a'_i)^2 + (b_j + b'_j)^2)^{1/2}.\]

From the equality $(E)$ we see that there exists a number $\mu$ such that

\[(F) \quad (b'_i, b'_2, \ldots, b'_p) = \mu(b_1, b_2, \ldots, b_p).\]

Also from the equalities $(E_{ij})$ we have numbers $\mu_i$ such that

\[(F_i) \quad (a'_i, b'_i) = \mu_i(a_i, b_i).\]

But combining $(F_i)$ with $(F)$ and remembering that $b_i \neq 0$ we have $\mu = b'_i/b_i = \mu_i$. This shows $(a'_i, b'_i) = \mu_i(a_i, b_i)$ which would mean that $R$ and $R'$ are parallel. This proves $\mathcal{F}$ is SWC.

(iii) However, $\mathcal{F}$ is not SC. This can be proved as follows: Take any simple vector $R$ which is linearly dependent on $R_1, R_2, R_3$, say $R = a_1R_1 + a_2R_2 + a_3R_3$ with $a_i \neq 0$, $i = 1$ to 3. Then we have

\[\mathcal{F}(R) = |a_1| + |a_2| + |a_3| = \mathcal{F}(a_1R_1) + \mathcal{F}(a_2R_2) + \mathcal{F}(a_3R_3),\]

which violates strict inequality even on $G^*$. Consequently it is impossible to extend $\mathcal{F}$ to a strictly convex function on $V^*$. We note here that in the example of § 4 all vectors $a_1R_1 + a_2R_2 + (-a_i/a_1 + a_3)R_3$ are simple. This completes the proof of Lemma B.

7. Theorem. The strict area minimizing property does not imply strict convexity.

Proof. By Lemma A we have the SFMA implied by SWC and C. But by Lemma B, SWC and C do not imply SC. Hence, the SFMA does not imply SC. Briefly SFMA $\subseteq$ SWC + C $<$ SC.

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