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POLYHEDRON INEQUALITY AND STRICT CONVEXITY

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This paper considers convexity of functions defined on the "Grassmann cone" of simple r -vectors. It is proved that the strict polyhedron inequality does not imply strict convexity.

H. Busemann, in conjunction with others, (see [3]), has considered the problem of giving a suitable definition of the convexity of functions defined on nonconvex sets. An examination of various methods of defining convexity on the "Grassmann cone" (see [1]) is found in [2]. The most important open problems (see [3]) are whether weak convexity implies the area minimizing property (also called the polyhedron inequality) and whether the latter implies convexity. A modest result in this direction is proved below, namely, the strict area minimizing property does not imply strict convexity.

2. Basic definitions. Let a continuous function \mathcal{F} be defined on the Grassmann cone G_r^n of the simple r -vectors R in the linear space V_r^n of all r -vectors \tilde{R} (over the reals). Let \mathcal{F} be positive homogeneous, i.e., $\mathcal{F}(\lambda R) = \lambda \mathcal{F}(R)$ for $\lambda \geq 0$. To a Borel set F in an oriented r -flat \mathcal{R}^+ in the n -dimensional affine space A^n , we associate a simple r -vector as follows: $R = 0$ if F has r -dimensional measure 0, and otherwise $R = v_1 \wedge v_2 \wedge \cdots \wedge v_r$, is parallel to \mathcal{R}^+ and the measure of the parallelepiped spanned by v_1, v_2, \dots, v_r equals the measure of F . (Note a set of measure 0 and equality of measures in parallel r -flats are affine concepts and hence welldefined.) We denote below by \mathcal{R} an r -flat parallel to an r -vector R passing through the origin.

DEFINITION 1. We say that \mathcal{F} has the strict area minimizing property (*SFMA*) if: Whenever R_0, R_1, \dots, R_p are associated to r -dimensional faces of an r -dimensional oriented closed polyhedron P we have $\mathcal{F}(-R_0) < \sum \mathcal{F}(R_i)$, with $i = 1$ to p , unless $R_i = \lambda_i R_0$, $\lambda_i \geq 0$ for all $i = 1$ to p (called the strict Polyhedron Inequality).

DEFINITION 2. \mathcal{F} is said to be strictly weakly convex (*SWC*) if: Whenever R, R_1 and R_2 are simple, $R = R_1 + R_2$, R_1 is not a scalar multiple of R_2 , we have $\mathcal{F}(R) < \mathcal{F}(R_1) + \mathcal{F}(R_2)$.

DEFINITION 3. \mathcal{F} is said to be convex (*C*) if there exists a convex extension of \mathcal{F} to V_r^n .

DEFINITION 4. \mathcal{F} is said to be strictly convex (*SC*) if \mathcal{F} is *C* and if there is at least one convex extension F of \mathcal{F} to V_r^n which satisfies the following property: Whenever $\tilde{R} = \Sigma \tilde{R}_i$ with $\tilde{R}, \tilde{R}_i \in V_r^n$, \tilde{R} is not a scalar multiple of all \tilde{R}_i , then $F(\tilde{R}) < \Sigma F(\tilde{R}_i)$.

In terms of these definitions we wish to prove below that: if \mathcal{F} is *SWC* and *C* then it has the *SFMA* and that if \mathcal{F} is *SWC* and *C* it still need not be *SC*. This implies that the property *SFMA* is weaker than the property *SC*.

3. Some algebraic facts. We collect below some algebraic facts which are either known or are relatively easy to prove.

(a) Let R_1 and R_2 be simple vectors. Then $R_1 + R_2$ is simple if and only if \mathcal{R}_1 and \mathcal{R}_2 intersect in a flat of dimension $\geq r - 1$.

(b) Identify r -vectors with points representing them in V_r^n considered as an affine space. If a line in V_r^n contains three points corresponding to simple vectors, then the entire line consists of simple vectors. Put differently, if R_1 and R_2 are simple and $R_i + R_2$ is not simple, then the line joining R_1 and R_2 in V_r^n does not contain any simple vector other than R_1 and R_2 .

Suppose next that R_1, R_2 and R_3 are simple but that $R_i + R_j$ is nonsimple for all $i, j = 1$ to 3 when $i \neq j$. Then we have the following:

(c) The set $\{R_1, R_2, R_3\}$ is a linearly independent set of vectors.

(d) The plane π containing $\Delta R_1 R_2 R_3$ does not contain any line of simple vectors.

(e) The flat Ω spanned by the origin, R_1, R_2 and R_3 does not contain a 2-plane of simple vectors.

(f) If a line l lies in Ω and does not pass through the origin, then l cannot be a line of simple vectors, i.e., l cannot contain three distinct points corresponding to simple vectors.

4. An example. Busemann and Straus [2] give the following concrete example which we use here to illustrate the above algebraic facts. Let the vectors e_1, e_2, e_3, e_4 form a base for the four dimensional affine space A^4 . Denote by e_{ij} the 2-vectors $e_i \wedge e_j$. Let Ω denote the flat spanned by the origin, e_{12}, e_{34} and $(e_1 + e_3) \wedge (e_2 + e_4)$ in V_2^4 . We denote the vectors spanning Ω by z, R_1, R_2 and R_3 respectively. Then $R_i + R_j$ is nonsimple for all $i, j = 1$ to 3 when $i \neq j$. Thus any line l in Ω which does not pass through the origin cannot contain three distinct points representing simple vectors.

5. *SWC* with *C* is stronger than the *SFMA*.

LEMMA A. *If a function \mathcal{F} is SWC and C then it has the SFMA.*

Proof. Let $R_0, R_1, R_2, \dots, R_p$ be r -vectors corresponding to r -faces of an r -dimensional oriented closed polyhedron P . We need consider only the case when not all R_i are scalar multiples of R_0 , $i > 0$. In such a case, since P is closed, some other faces which are not parallel to the face represented by R_0 intersect the face represented by R_0 in an $(r - 1)$ -dimensional set. Let R_1 be associated with one such face. Then from § 3a the vector $R_0 + R_1$ is simple. Also since P is closed we have $-(R_0 + R_1) = \sum_{i=2}^p R_i$. Thus $\sum_{i=2}^p R_i$ is also simple. But then the equation: $-R_0 = R_1 + \sum_{i=2}^p R_i$ shows that

$$\mathcal{F}(-R_0) < \mathcal{F}(R_1) + \mathcal{F}\left(\sum_{i=2}^p R_i\right),$$

and, since \mathcal{F} is convex, $\mathcal{F}(-R_0) < \sum_{i=1}^p \mathcal{F}(R_i)$ so that \mathcal{F} has the SFMA.

6. Existence of functions which are SWC and C but not SC.

LEMMA B. *There exist functions which are SWC and C but not SC.*

Proof. We actually construct an absolutely homogeneous function of this type. Take three simple unit vectors R_1, R_2, R_3 in V_r^n such that $R_i + R_j$ is nonsimple for all $i, j = 1$ to 3 with $i \neq j$. Choose unit vectors $\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_p$ in V_r^n such that the set of vectors $\{R_i \tilde{S}_j\}$, $i = 1$ to 3, $j = 1$ to p where

$$p = \binom{n}{r} - 3,$$

is a base for V_r^n . Thus given $\tilde{R} \in V_r^n$ we find unique numbers $\{a_i, b_j\}$ such that $\tilde{R} = \sum a_i R_i + \sum b_j \tilde{S}_j$. We denote this last written equality by the notation $\tilde{R} = (a_i, b_j)$. Now define the function \mathcal{F} in V_r^n in the following manner:

If $\tilde{R} = (a_i, b_j)$ then $\mathcal{F}(\tilde{R}) = \sum_{i,j} (a_i^2 + b_j^2)^{1/2} + (\sum_j b_j^2)^{1/2}$ with $i = 1$ to 3 and $j = 1$ to p .

We verify that \mathcal{F} has the required property.

(i) \mathcal{F} is clearly absolutely homogeneous ($\mathcal{F}(\lambda R) = |\lambda| \mathcal{F}(R)$) and a convex function on V_r^n , hence convex on G_r^n .

(ii) We next show \mathcal{F} is SWC. Let $R = (a_i, b_j)$ and

$$R' = (a'_i, b'_j)$$

be two simple r -vectors such that $R + R'$ is also simple. Assume

further that $\mathcal{F}(R + R') = \mathcal{F}(R) + \mathcal{F}(R')$. We prove \mathcal{R} is parallel to \mathcal{R}' . Assume that \mathcal{R} is not parallel to \mathcal{R}' . Then the line l in V_r^n joining R to R' is a line of simple vectors and l does not pass through the origin. Therefore from the algebraic facts, l does not lie in the flat Ω spanned by the origin, R_1, R_2 and R_3 . Therefore either a b_j or a b'_j is different from zero. Without loss of generality assume that $b_1 \neq 0$. We make the simple observation that when numbers α_i and β_i are such that $\alpha_i \leq \beta_i$ and $\Sigma \alpha_i = \Sigma \beta_i$ then each $\alpha_k = \beta_k$. From this and $\mathcal{F}(R + R') = \mathcal{F}(R) + \mathcal{F}(R')$ we have the following equalities:

$$(E) \quad (\Sigma b_j^2)^{1/2} + (\Sigma b'_j{}^2)^{1/2} = (\Sigma (b_j + b'_j)^2)^{1/2}.$$

For all (i, j) ,

$$(E_{ij}) \quad (a_i^2 + b_j^2)^{1/2} + (a_i'^2 + b_j'^2)^{1/2} = ((a_i + a_i')^2 + (b_j + b'_j)^2)^{1/2}.$$

From the equality (E) we see that there exists a number μ such that

$$(F) \quad (b'_1, b'_2, \dots, b'_p) = \mu(b_1, b_2, \dots, b_p).$$

Also from the equalities (E_{i1}) we have numbers μ_i such that

$$(F_i) \quad (a'_i, b'_i) = \mu_i(a_i, b_i).$$

But combining (F_i) with (F) and remembering that $b_1 \neq 0$ we have $\mu = b'_1/b_1 = \mu_i$. This shows $(a'_i, b'_i) = \mu(a_i, b_i)$ which would mean that \mathcal{R} and \mathcal{R}' are parallel. This proves \mathcal{F} is SWC.

(iii) However, \mathcal{F} is not SC. This can be proved as follows: Take any simple vector R which is linearly dependent on R_1, R_2, R_3 say $R = a_1R_1 + a_2R_2 + a_3R_3$ with $a_i \neq 0, i = 1$ to 3. Then we have $\mathcal{F}(R) = |a_1| + |a_2| + |a_3| = \mathcal{F}(a_1R_1) + \mathcal{F}(a_2R_2) + \mathcal{F}(a_3R_3)$, which violates strict inequality even on G_r^n . Consequently it is impossible to extend \mathcal{F} to a strictly convex function on V_r^n . We note here that in the example of § 4 all vectors $a_1R_1 + a_2R_2 + (-a_1a_2/a_1 + a_2)R_3$ are simple. This completes the proof of Lemma B.

7. THEOREM. *The strict area minimizing property does not imply strict convexity.*

Proof. By Lemma A we have the SFMA implied by SWC and C. But by Lemma B, SWC and C do not imply SC. Hence, the SFMA does not imply SC. Briefly SFMA \leq SWC + C < SC.

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