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ON UNIVERSAL TREE-LIKE CONTINUA

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R. M. Schori has conjectured that if T is a tree, but not an arc, then there is no universal T -like continuum. We show that if G is a finite collection of trees and there is a universal G -like continuum, then each element of G is an arc. It then follows that if G is a finite collection of one-dimensional (connected) polyhedra, and there is a universal G -like continuum, then each element of G is an arc.

1. **Definitions.** By a *continuum* here we mean a compact connected metric space; by a *polyhedron*, a nondegenerate (finitely) triangulable continuum. In a metric space, the distance between two points, A and B , is denoted by $d(A, B)$, and a similar notation is used for distances between points and point sets. The closure of a point set K is denoted by \bar{K} .

The point P of the continuum M is a *junction point* of M if and only if $M - P$ has at least three components.

A *tree* is a polyhedron that contains no simple closed curve. The point P of the tree T is an *endpoint* of T if and only if P is a noncutpoint of T .

The continuum M is an *n -od* if and only if n is a positive integer greater than 2 and there is a point P such that M is the sum of n arcs, each two intersecting only at P , which is an endpoint of both of them. If PQ is one of the n arcs, then $PQ - P$ is called a *ray* of M .

If $\varepsilon > 0$, a transformation f from a metric space X onto a space Y is called an *ε -map* if and only if f is continuous and if P is a point of Y , then $f^{-1}(P)$ has diameter $< \varepsilon$. The space X is *Y -like* if and only if there is an ε -map from X onto Y for each $\varepsilon > 0$. If G is a collection of spaces, the metric space X is *G -like* if and only if for each $\varepsilon > 0$, there is an ε -map from X onto some element of G [1].

2. Lemmas.

LEMMA 1. *If P is a junction point of the subcontinuum M of the continuum U , then there is an open set R in U containing P such that if R' is an open subset of R containing P , then there is a positive number ε such that every ε -map f from U onto a tree, T , throws some point of R' onto a junction point of T .*

Proof. Since $M - P$ has at least three components, $M - P$ is the sum of three mutually separated point sets, K_1, K_2 , and K_3 . For

each $i \leq 3$, let P_i denote a point of K_i . Let R denote an open set in U that contains P but not P_1, P_2 , or P_3 , and suppose R' is any open subset of R that contains P . Let ε denote a positive number less than the distance between any two of the sets $K_i - K_i \cdot R'$ ($i \leq 3$), and also less than $d(P_i, K_j)$, for $i \leq 3, j \leq 3, i \neq j$.

Now, suppose f is an ε -map from U onto a tree T . Since, if $i \leq 3, \bar{K}_i$ is a continuum, $f(\bar{K}_i)$ contains an arc α_i from $f(P_i)$ to $f(P)$. If no two of these arcs intersect except at $f(P)$, then $f(P)$ is a junction point of T . If the arc α_1 intersects the arc α_2 in a point distinct from $f(P)$, let Q denote the first point of α_2 on α_1 from $f(P_1)$ to $f(P)$. Clearly, Q must also be the first point of α_1 on α_2 from $f(P_2)$ to $f(P)$. Hence the three arcs, $[f(P), Q]$ and $[Q, f(P_1)]$ on α_1 , and $[Q, f(P_2)]$ on α_2 , intersect only in the point Q , and Q is a junction point of T . Moreover, Q is a point of $f(R')$, since $f^{-1}(Q)$ intersects both K_1 and K_2 , but cannot intersect both $K_1 - K_1 \cdot R'$ and $K_2 - K_2 \cdot R'$.

A similar argument suffices in case some other pair of the arcs α_1, α_2 , and α_3 intersect in a point distinct from $f(P)$.

LEMMA 2. *If N is an n -od with junction point P , lying in a continuum U , there is a positive number ε such that if f is an ε -map from U onto a tree T with at most one junction point then (1) T is a j -od with junction point Q , and $j \geq n$, (2) each endpoint of N is thrown by f into some ray of T , but no two into the same ray, and (3) if E is an endpoint of N and $f(P)$ lies in the ray of T that contains $f(E)$, then $f(P)$ lies in the arc in T from Q to $f(E)$.*

Proof. By Lemma 1 there is an open set R in U containing P and a positive number ε' such that (1) \bar{R} contains no endpoint of N and (2) if f is an ε' -map from U onto a tree T_0 , then $f(R)$ contains a junction point of T_0 . Let P_1, \dots, P_n denote the endpoints of N and, for each $i \leq n$, let Z_i denote the ray of N that contains P_i . Let ε denote a positive number less than each of the numbers $\varepsilon', d(P_i, R)$, and $d(P_i, N - Z_i)$, for $i \leq n$, and suppose that f is an ε -map from U onto a tree T with at most one junction point.

Since f is also an ε' -map from U onto T , $f(R)$ contains a junction point Q of T . Hence T is, for some positive integer j , a j -od. Now, if $i \leq n, d(P_i, R) > \varepsilon$ and Q is in $f(R)$, so $f(P_i) \neq Q$, and $f(P_i)$ lies in a ray of T .

Suppose i and k are two integers such that $f(P_i)$ and $f(P_k)$ lie in the same ray of T . The arc in T from $f(P_i)$ to $f(P_k)$ must contain $f(P)$, for otherwise either $f(Z_i)$ contains $f(P_k)$ or $f(Z_k)$ contains $f(P_i)$, neither of which is possible, since $d(P_i, N - Z_i) > \varepsilon$ and $d(P_k, N - Z_k) > \varepsilon$. But then if $m \leq n$ and $i \neq m \neq k$, either (1) $f(P_m)$ lies in $f(Z_i + Z_k)$ or (2) $f(P_i + P_k)$ intersects $f(Z_m)$, neither of which is possible. So the

images of different endpoints of N lie in different rays of T , and $j \geq n$.

Finally, suppose $i \leq n$ and $f(P)$ lies in the ray W of T that contains $f(P_i)$, but $f(P)$ is not on the arc in T from Q to $f(P_i)$. Then $f(P_i)$ is on the arc in T from Q to $f(P)$. So, if $k \leq n$, and $k \neq i$, then since $f(P_k)$ is not in W , $f(Z_k)$ contains $f(P_i)$. But $d(P_i, N - Z_i) > \varepsilon$.

LEMMA 3. *Suppose (1) $I_1; I_2$; and I_3 are the intervals in the plane with endpoints $(-1, 1), (-1, -1); (-1, 0), (1, 0)$; and $(1, 1), (1, -1)$, respectively, and (2) $H = I_1 + I_2 + I_3$. Then if T is any tree with at least two junction points, and $\varepsilon > 0$, there is an ε -map from H onto T .*

Proof. Let A and B denote the points $(-1, 0)$ and $(1, 0)$, respectively. Since T has two junction points, T contains an arc α whose endpoints, X and Y , are junction points of T , but no other point of α is a junction point of T . Let E denote the sum of all the components of $T - X$ that do not contain $\alpha - X$. Then E contains two mutually exclusive arcs β_1 and β_2 such that if $i \leq 2$, then β_i contains no junction point of T , and one endpoint of β_i is an endpoint of T . If $i \leq 2$, let Q_i denote the endpoint of β_i that is not an endpoint of T . Then $[E - (\beta_1 + \beta_2)] + X + Q_1 + Q_2$ is a tree.

Now, suppose $\varepsilon > 0$. Let $C_1; D$; and C_2 denote the subintervals of I_1 with endpoints $(-1, 1), (-1, \varepsilon/2); (-1, \varepsilon/2), (-1, -\varepsilon/2)$; and $(-1, -\varepsilon/2), (-1, -1)$, respectively. There is a continuous transformation g_1 from I_1 onto $E + X$ such that (1) if $i \leq 2$, $g_1|C_i$ is a homeomorphism from C_i onto β_i , (2) $f(A) = X$, and (3) $f(D) = [E - (\beta_1 + \beta_2)] + X + Q_1 + Q_2$. Clearly, g_1 is an ε -map. Similarly, there is an ε -map from I_3 onto $[T - (E + \alpha)] + B$ which may be combined with a homeomorphism from I_2 onto α to obtain an ε -map from H onto T .

3. Theorems.

THEOREM 1. *If k is a positive integer and G is a collection each element of which is a tree with not more than k junction points, but some element of G has two junction points, then there is no universal G -like continuum.*

Proof. Suppose U is a universal G -like continuum. Then by Lemma 3, the continuum H defined in Lemma 3 is G -like, and so U contains a continuum H' homeomorphic to H . Let T denote an element of G such that no element of G has more junction points than T , and let j denote the number of junction points of T . Let T_0 denote the continuum obtained from T by replacing, with a pseudo-arc, each arc in T which is maximal with respect to the property that each interior

point of it is of order 2, in such a way that T_0 is T -like, and hence G -like. Again, U contains a continuum T' homeomorphic to T_0 .

Suppose that one of the junction points of H' is not also a junction point of T' . Then U contains at least $j + 1$ points P_1, P_2, \dots, P_{j+1} each of which is a junction point of a subcontinuum of U . By successive applications of Lemma 1, there is a positive number ε and a sequence R_1, R_2, \dots, R_{j+1} of open sets in U such that (1) $d(R_i, R_n) > \varepsilon$, for $i \leq j + 1, n < j + 1$, and $i \neq n$, and (2) if f is an ε -map from U onto a tree, T , then if $i \leq j + 1, f(R_i)$ contains a junction point, J_i , of T . Note that the points J_1, J_2, \dots, J_{j+1} must all be distinct; hence T must have at least $j + 1$ junction points. But since U is G -like, U can be ε -mapped onto some tree in G , and no tree in G has $j + 1$ junction points. Thus we have a contradiction, and both junction points, A and B , of H' are also junction points of T' .

So U contains both an arc from A to B , and a continuum (T') that contains A and B , but no arc from A to B . Since U is treelike, and so hereditarily unicoherent, this is impossible.

Thus, there is no universal G -like continuum.

THEOREM 2. *If G is a finite collection each element of which is a tree, and there is a universal G -like continuum, then each element of G is an arc.*

Proof. Suppose some element of G is not an arc, but U is a universal G -like continuum. If some element of G has two junction points, then Theorem 1 is contradicted. Thus each element of G is an arc or, for some n , an n -od. Let n denote the greatest positive integer j such that G contains a j -od. Then U contains (1) an n -od N , and (2) a continuum H which is the sum of n pseudo-arcs, all joined at only one point. By arguments used in the proof of Theorem 1, the junction point, P , of N is also the junction point of H .

Let (1) ε_1 denote a positive integer for the subcontinuum N of U as in Lemma 2, (2) ε_2 and R denote a positive number and an open set in U , respectively, such that R contains P , and if E is an endpoint of N , then $d(E, R) > \varepsilon$, and (3) C denote the component of $U \cdot R$ that contains P .

\bar{C} is a subset of N , for suppose A is a point of \bar{C} not in N . Let ε denote a positive number less than $\varepsilon_1, \varepsilon_2$, and $d(A, N)$. Since U is G -like, there is an ε -map f from U onto an element T of G . Since $\varepsilon < \varepsilon_1$ we have, using Lemma 2, that (1) T is an n -od with junction point Q , (2) each ray of T contains the image of one, and only one, endpoint of N , and (3) there is an endpoint E of N such that $f(P)$ lies in the arc in T from Q to $f(E)$. Since $d(A, N) > \varepsilon$, $f(A)$ does not intersect $f(N)$, so there is an endpoint E' of N such that $f(E')$

lies in the arc in T from Q to $f(A)$. Since \bar{C} is a continuum that contains A and a point of $f^{-1}(Q)$, $f(\bar{C})$ contains $f(A)$ and Q , and so $f(\bar{C})$ contains $f(E')$. But since $d(E', C) > \varepsilon$, this is impossible.

Thus \bar{C} is a subset of N . Since the component C' of $H \cdot R$ that contains P is a subset of C , \bar{C}' contains an arc. But H itself contains no arc, and we have a contradiction.

THEOREM 3. *If G is a finite collection each element of which is a one-dimensional polyhedron, and there is a universal G -like continuum, then each element of G is an arc.*

Proof. If some element of G contains a simple closed curve, then by a theorem of M.C. McCord [2, Th. 4, p. 72], there is no universal G -like continuum. So each element of G is a tree, and by Theorem 2, each element of G is an arc.

We note that if each element of G is an arc, there is a universal G -like continuum [3].

REFERENCES

1. S. Mardešić and J. Segal, ε -mappings onto polyhedra, *Trans. Amer. Math. Soc.* **109** (1963), 146-164.
2. M. C. McCord, *Universal \mathcal{S} -like compacta*, *Michigan Math. J.* **13** (1966), 71-85.
3. R. M. Schori, *A universal snake-like continuum*, *Proc. Amer. Math. Soc.* **16** (1965), 1313-1316.
4. ———, *Universal spaces*, *Proceedings of the Second Prague Topological Symposium*, 1966, 320-322.

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