Pacific Journal of Mathematics

ON GENERAL Z.P.I.-RINGS

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Vol. 30, No. 3

November 1969

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A commutative ring in which each ideal can be expressed as a finite product of prime ideals is called a general Z.P.I.ring (for Zerlegungsatz in Primideale). A general Z.P.I.-ring in which each proper ideal can be uniquely expressed as a finite product of prime ideals is called a Z.P.I.-ring. Such rings occupy a central position in multiplicative ideal theory. In case R is a domain with identity, it is clear that R is a Dedekind domain¹ and the ideal theory of R is well known. If R is a domain without identity, the following result of Gilmer gives a somewhat less known characterization of R: If D is an integral domain without identity in which each ideal is a finite product of prime ideals, then each nonzero ideal of D is principal and is a power of D; the converse also holds. Also somewhat less known is the characterization of a general Z.P.I.-ring with identity as a finite direct sum of Dedekind domains and special primary rings.²

This paper considers the one remaining case: R is a general Z.P.I.-ring with zero divisors and without identity. A characterization of such rings is given in Theorem 2. This result is already contained in a more obscure form in a paper by S. Mori. The main contribution here is in the directness of the approach as contrasted to that of Mori.

In order to prove Theorem 2 we need to establish two basic properties of a general Z.P.I.-ring R: R is Noetherian and primary ideals of R are prime powers. Having established these two properties of R, the following result of Butts and Gilmer in [3], which we label as (BG), is applicable and easily yields our characterization of general Z.P.I.-rings without identity.

(BG), [3; Ths. 13 and 14]: If R is a commutative ring such that $R \neq R^2$ and such that every ideal in R is an intersection of a finite number of prime power ideals, then $R = F_1 \bigoplus \cdots \bigoplus F_k \bigoplus T$ where each F_i is a field and T is a nonzero ring without identity in which every nonzero ideal is a power of T.

It is important to note that we do not use Butts and Gilmer's

¹ M. Sono [14] and E. Noether [13] were among the first to consider Dedekind domains. For a historical development of the theory of Dedekind domains see [4; pp. 31-32].

² S. Mori in [11] considered both general Z.P.I.-rings with identity and Z.P.I.rings without identity which contain no proper zero divisors, but Mori's results in these cases are not as sharp as those of Asano and Gilmer.

paper [3] to prove that a general Z.P.I.-ring is Noetherian, while Butts and Gilmer do use this result from Mori's paper [11; Th. 7]. Theorem 2 gives a finite direct sum characterization of a general Z.P.I.-ring whereas Theorems 3 and 4 and Corollary 2 give characterizations of a general Z.P.I.-ring in terms of ideal-theoretic conditions.

Since we are only concerned with commutative rings, "ring" will always mean "commutative ring". The notation and terminology is that of [16] with two exceptions: \subseteq denotes containment and \subset denotes proper containment, and we do not assume that a Noetherian ring contains an identity. If A is an ideal of a ring R, we say that A is a *prper ideal* of R if $(0) \subset A \subset R$ and that A is a *genuine ideal* of R if $A \subset R$.

2. Structure theorem of a general Z.P.I.-ring. In this section section we prove directly that a general Z.P.I.-ring is Noetherian by proving that each of its prime ideals is finitely generated. We then use result (BG) to prove the structure theorem of a general Z.P.I.ring.

DEFINITION. Let R be a ring. If there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of n+1 prime ideals of R where $P_n \subset R$, but no such chain of n+2 prime ideals, then we say that R has dimension n and we write dim R = n.

LEMMA 1. If R is a general Z.P.I.-ring, R contains only finitely many minimal prime ideals and dim $R \leq 1$.

Proof. If R contains no proper prime ideal, then the lemma is clearly true. Therefore, we assume R contains a proper prime ideal P and we show that R contains a minimal prime ideal. If P is not a minimal prime of R, there exists a prime ideal P_1 such that $P_1 \subset P \subset R$. It follows that R/P_1 is a domain containing a proper prime ideal in which each ideal can be represented as the product of finitely many prime ideals. This implies that R/P_1 is a Dedekind domain [6]. Therefore, P_1 is a minimal prime of R. This also shows that dim $R \leq 1$.

Since R is a general Z.P.I.-ring, there exist prime ideals Q_1, \dots, Q_n in R and positive integers e_1, \dots, e_n such that $(0) = Q_i^{e_1} \dots Q_n^{e_n}$. If M is a minimal prime ideal of R, $(0) = Q_i^{e_1} \dots Q_n^{e_n} \subseteq M$ which implies that $Q_i \subseteq M$ for some *i*. Hence, $M = Q_i$ and it follows that the collection $\{Q_1, \dots, Q_n\}$ contains all the minimal prime ideals of R. Therefore, R contains only finitely many minimal prime ideals.

LEMMA 2. If R is a general Z.P.I.-ring containing a genuine

prime ideal, then each minimal prime ideal of R is finitely generated.

*Proof.*³ Let P be a minimal prime ideal of R and let $\{P_1, \dots, P_n\}$ be the collection of minimal primes of R distinct from P. If P = (0), the proof is clear. If $(0) \subset P$, we show that P is finitely generated by an inductive argument; that is, we show how to select a finite number of elements in P which generate P. We divide the proof into three cases.

Case 1. $P = P^2$. Since $P = P^2 \subseteq RP \subseteq P$, P = RP. Now,

$$P \not\subseteq \bigcup_{i=1}^n P_i$$

since $P \not\subseteq P_i$ for $1 \leq i \leq n$ so let $\mathbf{x}_1 \in P \setminus (\bigcup_{i=1}^n P_i)$. Thus, there exist prime ideals M_1, \dots, M_s , positive integers e_0, e_1, \dots, e_s , and a non-negative integer e_{s+1} such that

$$(x_1) = P^{e_0}M_1^{e_1}\cdots M_s^{e_s}R^{e_{s+1}} = PM_1^{e_1}\cdots M_s^{e_s}R^{e_{s+1}} = PM_1^{e_1}\cdots M_s^{e_s}$$

since P = RP. Let $\delta = \sum_{i=1}^{s} e_i$. If $P = (x_1)$, we are done. If $(x_1) \subset P$, then by choice of x_1 each M_i is a maximal prime ideal of R. Then [2; Proposition 2, p. 70] implies that $P \nsubseteq \{(x_1) \cup (\bigcup_{i=1}^{n} P_i)\}$. If $x_2 \in P \setminus \{(x_1) \cup (\bigcup_{i=1}^{n} P_i)\}$, then

$$(x_2) = PM_1^{f_1} \cdots M_s^{f_s} R^{f_{s+1}} Q_1^{g_1} \cdots Q_t^{g_t} = PM_1^{f_1} \cdots M_s^{f_s} Q_1^{g_1} \cdots Q_t^{g_t}$$

where Q_j is a maximal prime ideal of R for $1 \leq j \leq t$, $f_i \in \omega_0$ for $1 \leq i \leq s + 1$, and $g_j \in w$ for $1 \leq j \leq t$. Since $(x_2) \not\equiv (x_1)$, we have that $e_{i_0} > f_{i_0}$ for some $i_0, 1 \leq i_0 \leq s$. Therefore,

$$egin{aligned} & (x_1, x_2) = PM_1^{e_1} \cdots M_s^{e_s} + PM_1^{f_1} \cdots M_s^{f_s}Q_1^{q_1} \cdots Q_t^{q_t} \ & = PM_1^{m_1} \cdots M_s^{m_s}(M_1^{e_1-m_1} \cdots M_s^{e_s-m_s} + M_1^{f_1-m_1} \cdots M_s^{f_s-m_s}Q_1^{q_1} \cdots Q_t^{q_t}) \end{aligned}$$

where $m_i = \min \{e_i, f_i\}$ for $1 \leq i \leq s$. By the definition of m_i , if $e_i - m_i \neq 0$, then $f_i - m_i = 0$, and if $f_i - m_i \neq 0$, then $e_i - m_i = 0$. Let $A = M_1^{e_1 - m_1} \cdots M_s^{e_s - m_s}$ and let $B = M_1^{f_1 - m_1} \cdots M_s^{f_s - m_s} Q_1^{g_1} \cdots Q_t^{g_t}$, we show that A + B is contained in no maximal prime ideal of R. Note that $e_{i_0} - m_{i_0} \neq 0$ since $e_{i_0} > f_{i_0}$. If M is a maximal prime ideal of R. Note that $e_i = M_k \equiv 0$ and $M_k \equiv M$. Since M_k is a maximal prime ideal of R, it follows that $M = M_k$. Now, $e_k - m_k \neq 0$ implies that $f_k - m_k = 0$ which shows that $B \not\equiv M_k = M$. Thus, if M is a maximal prime ideal of R, it follows that $B \not\equiv M_k = M$. Thus, if M is a maximal prime ideal of R + B is containing A, M does not contain B. It follows that A + B is contained in no maximal prime ideal of R. Therefore, there exists a positive integer λ such that $A + B = R^{\lambda}$ and $(x_1, x_2) = PM_1^{m_1} \cdots M_s^{m_s}(A + B) = PM_1^{m_1} \cdots M_s^{m_s}R^{\lambda} = PM_1^{m_1} \cdots M_s^{m_s}$. By our choice of m_i , we have $e_i \geq m_i$ for $1 \leq i \leq s$. But $e_{i_0} < f_{i_0} = m_{i_0}$ implies that

³ The proof of Lemma 2 was suggested to the author by Professor Gilmer.

 $\delta - 1 \geq \sum_{i=1}^{s} m_i \geq 0.$

Assume that we have chosen, as described above, x_1, x_2, \dots, x_u in P such that $(x_1, \dots, x_u) = PM_1^{v_1} \dots M_s^{v_s}$ and $\delta - (u-1) \ge \sum_{i=1}^s v_i \ge 0$. Then by the above method, either $P = (x_1, \dots, x_u)$ or there exists an $x_{u+1} \in P \setminus \{(x_1, \dots, x_u) \cup (\bigcup_{i=1}^s P_i)\}$ such that

$$(x_1, \cdots, x_u, x_{u+1}) = PM_1^{v_1} \cdots M_s^{v_s'}$$

where $v'_i \in \omega_0$ and $\delta - (u + 1 - 1) \ge \sum_{i=1}^s v'_i \ge 0$. Since $\sum_{i=1}^s e_i$ is a finite positive number, there exists a positive integer q and $x_1, \dots, x_q \in P$ such that $P = (x_1, \dots, x_q)$; that is, P is a finitely generated ideal of R.

Case 2. $P^2 \subset P$ and P = RP. Now, $P \not\subseteq \{P^2 \cup (\bigcup_{i=1}^n P_i)\}$ by [2; Proposition 2, p. 70] so let $x_1 \in P \setminus \{P^2 \cup (\bigcup_{i=1}^n P_i)\}$. Then there exist prime ideals M_1, \dots, M_s of R, $e_1, \dots, e_s \in \omega$, and $e_{s+1} \in \omega_0$ such that $(x_1) = PM_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} = PM_1^{e_1} \cdots M_s^{e_s}$ since P = RP. If $P = (x_1)$ we are done. If $(x_1) \subset P$, then we can choose an

$$x_2 \in P \setminus \{(x_1) \cup P^2 \cup (\bigcup_{i=1}^n P_i)\}$$

by [2; Proposition 2, p. 70]. We now consider (x_1, x_2) and the remainder of the proof of Case 2 is the same as the proof of Case 1. Thus, P is a finitely generated ideal of R.

Case 3. $P^2 \subset P$ and $RP \subset P$. Let $x \in P \setminus RP$. Then there exist prime ideals M_1, \dots, M_s of R and $e_1, \dots, e_{s+1} \in \omega_0$ such that $(x) = PM_1^{e_1} \cdots M_s^{e_s} R^{e_{s+1}} \not\subseteq RP$. Thus, $e_i = 0$ for $1 \leq i \leq s+1$; that is, P = (x).

LEMMA 3. Each prime ideal of a general Z.P.I.-ring is finitely generated.

Proof. Let R be a general Z.P.I.-ring.

Case 1. R contains no proper prime ideal. If $R = R^2$, let $r \in R \setminus \{0\}$. Since R is a general Z.P.I.-ring, there exists a positive integer n such that $(r) = R^n = R$. If $R^2 \subset R$, let $r \in R \setminus R^2$. Then (r) = R.

Case 2. R contains a proper prime ideal. Let M be a nonzero prime ideal of R. If M is a minimal prime ideal of R, M is finitely generated by Lemma 2. If M is not a minimal prime ideal of R, the proof of Lemma 1 implies that there exists a minimal prime ideal P of R such that $P \subset M$. Thus, R/P is Noetherian which implies that M/P is a finitely generated ideal of R/P. Since P is a finitely generated ideal of R, it follows that M is a finitely generated ideal of R.

Thus, each prime ideal of R is finitely generated.

THEOREM 1. A general Z.P.I.-ring is Noetherian.

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Proof. Let A be an ideal of R, a general Z.P.I.-ring. Then there exist prime ideals P_1, \dots, P_n of R and positive integers e_1, \dots, e_n such that $A = P_1^{e_1} \cdots P_n^{e_n}$. Since each P_i is finitely generated by Lemma 3, it follows that A is finitely generated. Thus, R is Noetherian.

REMARK. Theorem 1 also follows from the fact that a ring R is Noetherian if and only if each prime ideal of R is finitely generated. [4; Th. 2].

RESULT 1. If Q is a P-primary ideal in a ring R such that Q can be represented as a finite product of prime ideals, then Q is a power of P.

Proof. By hypothesis there exist distinct prime ideals P_1, \dots, P_n and positive integers e_1, \dots, e_n such that $Q = P_1^{e_1} \dots P_n^{e_n}$. Since $Q = P_1^{e_1} \dots P_n^{e_n} \subseteq P, P_i \subset P$ for some *i*—say i = 1. Now, $P = \sqrt{Q} = P_1 \cap \dots \cap P_n$ which implies that $P \subseteq P_i$ for each *i*. Therefore, $P \subseteq P_1 \subseteq P$; that is, $P_1 = P$. We have that $Q = P^{e_1}P_2^{e_2} \dots P_n^{e_n}$ where $P \subset P_i$ for $2 \leq i \leq n$. Since

$$Q = P^{e_1}(P_2^{e_2}\cdots P_n^{e_n}) \subseteq Q$$

and $P_{2}^{e_{2}} \cdots P_{n}^{e_{n}} \not\subseteq P$, it follows that $P^{e_{1}} \subseteq Q$. Hence, $Q = P^{e_{1}}$.

DEFINITIONS. Let R be a ring. We say that R has property (α) , if each primary ideal of R is a power of its (prime) radical [3]. If each ideal of R is an intersection of a finite number of prime power ideals, we say that R has property (δ) [3]. Finally, we say that R satisfies property (\sharp) if R is a ring without identity such that each nonzero ideal of R is a power of R.

REMARK. If R is a ring satisfying property (#), it follows that either R is an integral domain in which $\{R^i\}_{i=1}^{\infty}$ is the collection of nonzero ideals of R or R is not an integral domain and $\{R, R^2, \dots, R^n = (0)\}$ is the collection of all ideals of R for some $n \in \omega$.

COROLLARY 1. A general Z.P.I.-ring has property (α).

Proof. This follows immediately from Result 1.

THEOREM 2. Structure theorem of a general Z.P.I.-ring. A ring R is a general Z.P.I.-ring if and only if R has the following structure:

(a) If $R = R^2$, then $R = R_1 \oplus \cdots \oplus R_n$ where R_i is either a Dedekind domain or a special P.I.R. for $1 \leq i \leq n$.

(b) If $R \neq R^2$, then either $R = F \bigoplus T$ or R = T where F is a field and T is a ring satisfying property (\sharp).

Proof. (\rightarrow) If R is a general Z.P.I.-ring, then R is Noetherian and has property (α). Hence, [3; Corollary 6] implies that (δ) holds in R. If $R = R^2$, then R contains an identity by [5; Corollary 2]. Therefore, [1; Th. 1] implies that part (a) holds. If $R \neq R^2$, then by (BG) $R = F_1 \oplus \cdots \oplus F_u \oplus T$ where each F_i is a field and T is a nonzero ring satisfying property (\sharp). Using a contrapositive argument, we show that $u \not\geq 2$.

Assume that $u \ge 2$. We show that R is not a general Z.P.I.ring. Since $u \ge 2$, it is clear that T is an ideal of R that is not prime. The prime ideals of R containing T are R and

$$P_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_u \oplus T$$

for $1 \leq i \leq u$ where $T \subset P_i$ for each *i*. Now

and $R^2 = F_1 \bigoplus \cdots \bigoplus F_u \bigoplus T^2$. Since $T^2 \subset T$, it follows that $T \nsubseteq P_i P_j$, $T \nsubseteq RP_i$, and $T \nsubseteq R^2$ for $1 \le i, j \le u$. Thus, T cannot be represented as a finite product of prime ideals of R; that is, R is not a general Z.P.I.-ring. Therefore, if R is a general Z.P.I.-ring, $u \ngeq 2$; that is, $R = F_1 \bigoplus T$ or R = T where F_1 is a field and T is a ring satisfying property (\ddagger).

 (\leftarrow) If R is a direct sum of finitely many Dedekind domains and special P.I.R.'s R is a general Z.P.I.-ring by [1; Th. 1]. If R = Twhere T is a ring satisfying property (#), then R is clearly a general Z.P.I.-ring. If $R = F \bigoplus T$ where F is a field and T is a ring satisfying property (#), then $\{F \bigoplus T^i, T^i, (0): i \in \omega\}$ is the collection of ideals of R. It follows that each ideal of R is a finite product of prime ideals. Therefore, if R satisfies either (a) or (b), R is a general Z.P.I.-ring.

3. Necessary and sufficient conditions on a general Z.I.P.ring. In this section we again use results of Butts and Gilmer in [3] to derive several necessary and sufficient conditions for a ring to be a general Z.P.I.-ring.

DEFINITION. Let A be an ideal of a ring R. We say that A

is simple if there exist no ideals properly between A and A^2 . To avoid conflicts with other definitions of a simple ring we say in case A = R that R satisfies property S.

LEMMA 4. Let A be an ideal of a Noetherian ring R. If $B = \bigcap_{i=1}^{\infty} A^i$, then AB = B.

Proof. See [15; L_1].

LEMMA 5. If A is a genuine ideal of a Noetherian domain D, then $\bigcap_{i=1}^{\infty} A^i = (0)$.

Proof. Let K be the quotient field of D and let $D^* = D[e]$ where e is the identity of K. Then D^* is Noetherian by [5; Th. 1], and since A is also an ideal of D^* , [16; Corollary 1, p. 216] shows that $\bigcap_{i=1}^{\infty} A^i = (0)$.

LEMMA 6. Let A be a simple ideal of a ring R. Then for each $i \in \omega$ there are no ideals properly between A^i and A^{i+1} . Further, the only ideals between A and A^n for $n \in \omega$ are A, A^2, \dots, A^n .

Proof. See [7; Lemma 3].

LEMMA 7. Let A be a proper simple ideal of a Noetherian ring R. If there exists a prime ideal P of R such that $(0) \subset P \subset A \subset R$, P is unique and $P = \bigcap_{i=1}^{\infty} A^i$. Also, if Q is a P-primary ideal of R, Q = P.

Proof. We first show by an inductive argument that $P \subset A^i$ for each $i \in \omega$. By hypothesis $P \subset A$. Assume that $P \subset A^k$ for some $k \in \omega$. Since A/P is a proper ideal of R/P, a Noetherian integral domain, $A^k/P \supset (A^k/P)(A/P) = (A^{k+1} + P)/P \supset P/P$ by [5; Corollary 1] which shows that $A^k \supset A^{k+1} + P \supseteq A^{k+1}$. Therefore, $A^{k+1} + P = A^{k+1}$. Since $A^{k+1} + P \supset P$, it follows that $P \subset A^{k+1}$. Thus, $P \subset A^i$ for each $i \in \omega$.

We now show that $P = \bigcap_{i=1}^{\infty} A^i$. Since A/P is a proper ideal of a Noetherian domain, $P/P = \bigcap_{i=1}^{\infty} (A/P)^i$ by Lemma 5. Also, since $\bigcap_{i=1}^{\infty} (A/P)^i = \bigcap_{i=1}^{\infty} ((A^i + P)/P) = \bigcap_{i=1}^{\infty} (A^i/P) = (\bigcap_{i=1}^{\infty} A^i)/P$, it follows that $P = \bigcap_{i=1}^{\infty} A^i$.

Finally, we show that if Q is a P-primary ideal of R, then Q = P. Lemma 4 shows that $P = A(\bigcap_{i=1}^{\infty} A^i) = AP$. There exists an $a \in A$ such that ap = p for each $p \in P$ by [5; Corollary 1]; that is, ap - p = 0 for each $p \in P$. If $x \in R \setminus A$, then $p(ax - x) = apx - px = 0 \in Q$ for each $p \in P$. Since $x \notin A$, $ax - x \notin A$ which shows that $ax - x \notin P$. Thus, $p \in Q$ for each $p \in P$ since $p(ax - x) \in Q$ for each

 $p \in P$, $ax - x \notin P$, and Q is a P-primary ideal of R. Thus, $P \subseteq Q$ which shows that Q = P.

THEOREM 3. Let R be a ring.

(A) If R contains an identity, then R is a general Z.P.I.-ring if and only if R satisfies the following two conditions:

(1) R is Noetherian.

(2) Each maximal ideal of R is simple.

(B) If R does not contain an identity and R contains a proper prime ideal, then R is a general Z.P.I.-ring if and only if R satisfies the following four conditions:

(1) R is Noetherian.

(2) R satisfies property S.

(3) Each maximal prime ideal of R is simple.

(4) $\bigcap_{i=1}^{\infty} R^i$ is a field.

(C) If R does not contain an identity and R contains no proper prime ideal, then R is a general Z.P.I.-ring if and only if R satisfies the following two conditions:

(1) R is Noetherian.

(2) R satisfies property S.

Proof of (A). Part (A) follows immediately from [1; Th. 5].

Proof of (B). (\rightarrow) Assume that R is a general Z.P.I.-ring. Then R is Noetherian by Theorem 1. Since R contains a proper prime ideal, Theorem 2 shows that $R = F \oplus T$ where F is a field and T is a ring satisfying property (\sharp). Hence, R clearly satisfies property S. If T is a domain, then F and T are the maximal prime ideals of R. If T is not a domain, then T is the maximal prime ideal of R. It follows that each maximal prime ideal of R is simple. Finally, $\bigcap_{i=1}^{\infty} R^i = \bigcap_{i=1}^{\infty} (F \oplus T)^i = F$, a field.

 (\leftarrow) Assume that conditions (1), (2), (3), and (4) hold. Let Q be a *P*-primary ideal of *R*. If P = R or if *P* is a maximal prime ideal of *R*, there exists an integer *n* such that $P^n \subseteq Q$ since *R* is Noetherian. Hence Lemma 6 shows that there exists an integer *k* such that $Q = P^k$. If *P* is a proper nonmaximal prime ideal of *R*, there exists a maximal prime ideal *M* of *R* such that $P \subset M \subset R$, and it follows from Lemma 7 that Q = P. Thus, *R* is a Noetherian ring having property (α) which shows that (δ) holds in *R*. [3; Corollary 6]. Therefore, by (BG) $R = F_1 \bigoplus \cdots \bigoplus F_m \bigoplus T$ where each F_i is a field and *T* satisfies property (\sharp). Since *R* contains a proper prime ideal, $m \ge 1$; condition (4) implies that $m \ge 1$. Hence $R = F_1 \bigoplus T$ which implies that *R* is a general Z.P.I.-ring.

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Proof of (C). (\rightarrow) If R is a general Z.P.I.-ring containing no proper prime ideal, then R = T where T is a ring satisfying property (#), Hence, R is Noetherian and satisfies property S.

 (\leftarrow) Assume that conditions (1) and (2) hold. Since R is Noetherian and since R is the only nonzero prime ideal in R, R has property (α) . Thus, R is a general Z.P.I.-ring by an argument similar to that given in part (B) above.

LEMMA 8. A ring R has property (δ) if and only if R satisfies the following three conditions:

- (1) R is Noetherian.
- (2) R satisfies property S.
- (3) Each maximal prime ideal of R is simple.

Proof. (\rightarrow) Assume that R has property (δ) . If $R = R^2$, [3; Th. 11] implies that R is a general Z.P.I.-ring. Therefore, (1), (2), and (3) hold by Theorem 3. If $R \neq R^2$, then [3; Th. 12] implies that R is Noetherian. From (BG) we have that $R = F_1 \bigoplus \cdots \bigoplus F_m \bigoplus T$ where each F_i is a field and T satisfies property (\sharp). It follows from the representation of R, that (2) and (3) hold.

 (\leftarrow) We showed in the proof of Theorem 3 (B) that if (1), (2), and (3) hold in a ring R, then (δ) holds in R.

LEMMA 9. In a Noetherian ring R, property (α) is equivalent to the following two conditions:

- (2) R satisfies property S.
- (3) Each maximal prime ideal of R is simple.

Proof. This follows immediately from Lemma 8 and [3; Corollary 6].

THEOREM 4. If R is a ring with identity, R is a general Z.P.I.-ring if and only if R is Noetherian and (α) holds in R.

Proof. The necessity follows from Theorem 1 and Corollary 1 and the sufficiency follows from [3; Corollary 6 and Th. 11].

COROLLARY 2. Let R be a ring without identity.

(A) If R contains a proper prime ideal, then R is a general Z.P.I.-ring if and only if R satisfies the following three conditions:

- (1) R is Noetherian.
- (2') (a) holds in R.
- (4) $\bigcap_{i=1}^{\infty} R^i$ is a field.
- (B) If R contains no proper prime ideal, then R is a general

Z.P.I.-ring if and only if R satisfies the following two conditions: (1) R is Noetherian.

(2') (α) holds in R.

Proof. This follows immediately from Theorem 3 and Lemma 9.

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Received November 5, 1968. This paper is a portion of the author's doctoral dissertation, which was written under the direction of Professor Robert W. Gilmer Jr., at The Florida State University.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17. Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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