RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE

Surender Kumar Jain, Saad H. Mohamed and Surjeet Singh
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S. K. JAIN, S. H. MOHAMED AND SURJEET Singh

It is well known that if every right ideal of a ring \( R \) is injective, then \( R \) is semi simple Artinian. The object of this paper is to initiate the study of a class of rings in which each right ideal is quasi-injective. Such rings will be called \( q \)-rings. It is shown by an example that a \( q \)-ring need not be even semi prime. A number of important properties of \( q \)-rings are obtained.

Throughout this paper, unless otherwise stated, we assume that every ring has unity \( 1 \neq 0 \). If \( M \) is a right \( R \)-module, then \( \hat{M} \) will denote the injective hull of \( M \). For any positive integer \( n \), \( R_n \) will denote the ring of all \( n \times n \) matrices over the ring \( R \). \( R' \), \( J(R) \) and \( B(R) \) will denote the right singular ideal, the Jacobson radical and the prime radical respectively. A ring \( R \) is said to be a right duo ring if every right ideal of \( R \) is two-sided. Left duo rings are defined symmetrically. By a duo ring we mean a ring which is both right and left duo ring.

It is shown that \( R_n (n > 1) \) is a \( q \)-ring if and only if \( R \) is semi-simple Artinian. Some of the main results are: (i) a prime \( q \)-ring is simple Artinian, (ii) a semi-prime \( q \)-ring is a direct sum of two rings \( S \) and \( T \), where \( S \) is a complete direct sum of simple Artinian rings, and \( T \) is a semi-prime \( q \)-ring with zero socle, and (iii) a semi-prime \( q \)-ring is a direct sum of two rings \( A \) and \( B \), where \( A \) is a right self injective duo ring, and \( B \) is semi-simple artinian.

2. Let \( R \) be a right self injective ring. If \( B \) is any right ideal of \( R \), then \( \hat{B} = eR \) for some idempotent \( e \) of \( R \). Let \( K = \text{Hom}_R(\hat{B}, \hat{B}) \). Then \( K \cong eRe \). In fact every element in \( K \) can be realized by the left multiplication of some element of \( eRe \). By Johnson and Wong ([3], Theorem 1.1) \( B \) is a quasi injective as a right \( R \)-module if and only if \( KB = B \). Hence \( B \) is quasi injective if and only if \( B = KB = (eRe)B = (eR)(eB) = \hat{B}B \). Hence every two-sided ideal in a right self injective ring is quasi-injective. So, the following is immediate.

2.1. Every commutative self injective ring is a \( q \)-ring. 

Now, we give an example of a \( q \)-ring which is not semi-prime.

Example 2.2. Let \( Z \) be the ring of integers. Set \( R = Z/(4) \). It is trivial that \( R \) is a \( q \)-ring. But \( R \) is not semi-prime, since its only proper ideal is nilpotent.

73
In fact, \( \mathbb{Z}/(n) \) is a q-ring for every integer \( n > 1 \), since it is self injective (cf. Levy [5]). Also we remark that \( \mathbb{Z}/(n) \) has nonzero nilpotent ideals if \( n \) is not square free.

Next we prove

**Theorem 2.3.** The following are equivalent

1. \( R \) is a q-ring
2. \( R \) is right self injective, and every right ideal of \( R \) is of the form \( eI, e \) is an idempotent in \( R \), \( I \) is a two sided ideal in \( R \).
3. \( R \) is right self-injective, and every large right ideal of \( R \) is two sided.

**Proof.** Assume (1). Therefore \( R \) is right self injective. Let \( B \) be any right ideal of \( R \). Then \( B = eR \) for some idempotent \( e \). Since \( B \) is quasi injective \( B = BB = eRB = eI \), where \( I = RB \), the smallest two-sided ideal of \( R \) containing \( B \). Hence (1) implies (2).

Assume (2). Let \( A \) be a large right ideal of \( R \). Then \( A = eI, e^2 = e, I \) is a two sided ideal. Since \( A \cap (1 - e)R = 0, (1 - e)R = 0 \). This implies that \( e = 1 \). Hence \( A = I \), proving (3).

Now assume (3). Let \( B \) be a right ideal of \( R \). If \( K \) is a complement of \( B \), then \( B \oplus K \) is large in \( R \). By assumption \( B \oplus K \) is a two-sided ideal in \( R \), hence quasi-injective. This implies \( B \) is a quasi-injective, completing the proof.

**Theorem 2.4.** Let \( n > 1 \) be an integer. Then \( R_n \) is a q-ring if and only if \( R \) is semi-simple Artinian.

**Proof.** Suppose that \( R \) is not semi-simple Artinian. By Lambek ([4], Proposition 2, p. 61), there exists a large right ideal \( B \) of \( R \) such that \( B \neq R \). Let \( e_{ij}, 1 \leq i, j \leq n \) be the matrix units of \( R_n \) and let \( E = \{ \sum a_{ij}e_{ij}; a_{ij} \in B, 1 \leq j \leq n \text{ and } a_{ij} \in R, 1 \leq i, j \leq n \} \). It is clear that \( E \) is a right ideal in \( R_n \). But \( E \) is not two-sided, for \( e_{nn} \in E \) and \( e_{nn}e_{nn} = e_{nn} \notin E \). Now, we prove that \( E \) is a large right ideal in \( R_n \). Let \( 0 \neq x = \sum_{i,j=1}^n b_{ij}e_{ij} \). If \( b_{ij} = 0, 1 \leq j \leq n \), then \( x \in E \). So, let \( b_{ik} \neq 0 \) for some \( k \). Since \( B \) is large in \( R \), there exists \( a \in R \) such that \( 0 \neq b_{ik}a \in B \). Then,

\[
x(ae_{kk}) = (\sum_{i,j=1}^n b_{ij}e_{ij})(ae_{kk}) = \sum_{i=1}^n b_{ik}ae_{ik} \in E.
\]

Hence, \( 0 \neq x(ae_{kk}) \in E \). Therefore \( E \) is a large right ideal in \( R_n \), which is not two-sided, and by Theorem 2.3, \( R_n \) is not a q-ring. This proves "only if" part. Other part is obvious.

We are now ready to show the existence of right self injective rings which are not q-rings.
Example 2.5. Let \( R \) be a right self injective ring which is not semi-simple (we can take \( R = \mathbb{Z}/(4) \)). Let \( n > 1 \) be an integer. By Utumi ([6], Th. 8.3) \( R_n \) is right self injective. But \( R_n \) is not a q-ring, by the above theorem.

Next we prove

Theorem 2.6. A simple ring is a q-ring if and only if it is Artinian.

Proof. Let \( R \) be a simple q-ring. Let \( B \) be a large right ideal in \( R \). Then \( B \) is two-sided, and hence \( B = R \). This proves that \( R \) does not contain any proper large right ideal. Hence \( R \) is Artinian. The converse is trivial.

Now, we give an example of a right self injective simple ring which is not a q-ring.

Example 2.7. Let \( S \) be a noncommutative integral domain which is not a right Ōre domain (cf. Goldie [1]). Let \( R = S \). Then \( R \) is a right self injective simple regular ring which is not Artinian. By the above theorem \( R \) is not a q-ring.

Lemma 2.8. Let \( R \) be a q-ring. Then \( B(R) \) is essential in \( J(R) \) as a right \( R \)-module.

Proof. Since \( R \) is self injective, \( J(R) = R^t \), by Utumi ([6], Lemma 4.1). Let \( 0 \neq x \in J(R) \). There exist a large right ideal \( E \) of \( R \) such that \( xE = 0 \). Then \( xE \subseteq P \) for every prime ideal \( P \) of \( R \). Since \( R \) is a q-ring, \( E \) is two-sided. This implies that either \( x \in P \) or \( E \subseteq P \).

Let \( \{P_i\}_{i \in I} \) be the set of all prime ideals of \( R \) such that \( x \in P_i \) for every \( i \in I \), and \( \{P_j\}_{j \in J} \) be the set of all prime ideals of \( R \) such that \( x \in P_j \) for every \( j \in J \). Let \( X = \bigcap_{i \in I} P_i \), and \( Y = \bigcap_{j \in J} P_j \). \( X \neq 0 \), since \( 0 \neq x \in X \). On the other hand, \( E \subseteq P_j \) for every \( j \in J \). Thus \( E \subseteq Y \), which implies that \( Y \) is large in \( R \). Therefore \( B(R) = X \cap Y \neq 0 \). Moreover, there exists \( a \in R \) such that \( 0 \neq xa \in Y \). This implies that \( 0 \neq xa \in X \cap Y = B(R) \), completing the proof.

Hence, we have the following

Theorem 2.9. A q-ring is regular if and only if it is semi-prime.
Proof. The result follows by the above lemma, and Utumi ([6], Corollary 4.2).

**Theorem 2.10.** Let \( V \) be a vector space over a division ring \( D \), and let \( R = \text{Hom}_O(V, V) \). Then \( R \) is a \( q \)-ring if and only if \( V \) is of finite dimension over \( D \).

Proof. The “if” part is obvious. Conversely, suppose that \( V \) is of infinite dimension over \( D \). Let \( X = \{x_1, x_2, \cdots \} \) be a denumerable set of linearly independent elements of \( V \). \( X \) can be extended to a basis \( X \cup Y \) of \( V \). Let \( F \) be the ideal in \( R \) consisting of all elements of finite rank. Let \( \sigma \in R \) be defined by \( \sigma(x_{2i}) = x_{2i} \), \( \sigma(x_{2i-1}) = 0 \) for every \( i \), and \( \sigma(y) = 0 \) for every \( y \in Y \). Let \( E = \sigma R + F \). Then \( F \subset E \). Since \( F \) is a two-sided ideal in \( R \), \( F \) is large. Therefore \( E \) is a large right ideal in \( R \). We proceed to prove that \( E \) is not two-sided. Let \( \lambda_1, \lambda_2 \in R \) be defined by: \( \lambda_1(x_i) = x_{2i} \) for every \( i \), and \( \lambda_1(y) = 0 \) for every \( y \in Y \), \( \lambda_2(x_{2i}) = x_i \), \( \lambda_2(x_{2i-1}) = 0 \), for every \( i \), and \( \lambda_2(y) = 0 \) for every \( y \in Y \). Let \( \lambda = \lambda_2 \sigma \lambda_1 \). Then \( \lambda(x_i) = x_i \) for every \( i \). Hence \( X \subset \lambda(V) \). We assert that \( \lambda \not\in E \); for otherwise, let \( \lambda = \sigma r + f, r \in R, f \in F \). Then \( X \subset \lambda(V) = (\sigma r + f)(V) \subset \sigma(V) + f(V) \). But since \( f \) is of finite rank, there exists an integer \( n \) such that \( x_{2n-1} \not\in f(V) \). Also, by definition of \( \sigma \), \( x_{2n-1} \not\in \sigma(V) \). Hence \( x_{2n-1} \not\in \sigma(V) + f(V) \), which is a contradiction. Thus \( \lambda \not\in E \), as desired. However \( \lambda \in \sigma R \), \( \sigma R \subset R \). Hence \( E \) is not a two-sided ideal. Therefore, by Theorem 2.3, \( R \) is not a \( q \)-ring. This completes the proof.

We remark that the above theorem is also a consequence of Theorem 2.4.

The right (left) socle of a ring \( R \) is defined to be the sum of all minimal right (left) ideals of \( R \). It is well known that in a semi-prime ring \( R \), the right and left socles of \( R \) coincide, and we denote any of them by \( \text{soc} \ R \).

**Lemma 2.11.** A semi-prime \( q \)-ring \( R \) with zero socle is strongly regular.

Proof. Let \( M \) be a maximal right ideal in \( R \). Either \( M \) is a direct summand of \( R \) or \( M \) is large in \( R \). If \( M \) is a direct summand of \( R \), then its complement is a minimal right ideal. This implies that \( \text{soc} \ R \not= 0 \), a contradiction. Therefore, every maximal right ideal is large, hence two-sided. By Lemma 2.8, \( J(R) = 0 \). Thus \( R \) is isomorphic to a subdirect sum of division rings, which implies that \( R \) has no nonzero nilpotent elements. Since \( R \) is regular, by Theorem 2.9, \( R \) is strongly regular.
LEMMA 2.12. A prime q-ring has nonzero socle.

Proof. Let R be a prime q-ring. If possible, let soc R = 0. By the above lemma, R is strongly regular. Hence R is a division ring, and soc R = R contradicting our assumption. Therefore soc R ≠ (0).

THEOREM 2.13. A prime ring R is a q-ring if and only if R is simple Artinian.

Proof. By Theorem 2.9, and the above lemma, R is a prime regular ring with nonzero socle. Hence, by Johnson ([2], Th. 3.1), \( \hat{R} = \text{Hom}_D(V, V) \), where V is some vector space over a division ring D. But then \( R = \text{Hom}_D(V, V) \), since R is right self injective. By Theorem 2.10, V has finite dimension over D. Let \( (V; D) = n \). Then \( R \cong D^n \), completing the proof.

LEMMA 2.14. Let \( \{R_a\}_{a \in I} \) be a finite set of rings. Then the direct sum \( \sum_{a \in I} \bigoplus R_a \) is a q-ring if and only if each \( R_a \) is a q-ring.

The proof is obvious.

That Lemma 2.14 is not true for an infinite number of rings is shown by the following example which is due to Storrer.

EXAMPLE 2.15. Let R be a 2 × 2-matrix ring over a field F. Let \( \{R_a\}_{a \in I} \) be an infinite family of copies of R and let \( S = \pi R_a, a \in I \). Let E be the right ideal of S consisting of those elements [\( x_a \)] of S such that all but finite \( x'_a \)'s are matrices with first row zero. Since \( R_a \subseteq E \) for all \( a \in I, E \) is a large right ideal of S. To show that E is not two-sided, consider \([x_a] \in E \) where \( x_a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) for all \( a \in I \). Let \([y_a] \in S \) be such that \( y_a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) for all \( a \in I \). Then \([y_a][x_a] = [z_a] \), where \( z_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). But then \([z_a] \in E \), and E is not two-sided. Hence, by Theorem 2.3, S is not a q-ring.

Example 2.15 also suggests the following.

THEOREM 2.16. Let \( \{R_a\}_{a \in I} \) be a family of simple Artinian rings and let R be their complete direct sum. Then R is a q-ring if and only if all \( R_a \)'s excepting a finite number of them are division rings.

The above theorem shows, in particular, that a regular q-ring may not be Artinian.

LEMMA 2.17. Let R be a semi-prime q-ring such that soc R is
large in $R$. Then $R$ is a complete direct sum of simple Artinian rings.

Proof. Since soc $R$ is large, every nonzero right ideal of $R$ contains a minimal right ideal. Also $R$ is regular, by Theorem 2.9. Hence by Johnson ([2], Th. 3.1), $R$ is a complete direct sum of rings $R_i$, where each $R_i$ is the ring of all linear transformations of some vector space $V_i$ over a division ring $D_i$. But then by Lemma 2.14 and Theorem 2.10, each $R_i$ is a simple Artinian ring. This completes the proof.

In the following two theorems we assume that every ring has a unity element which may be equal to zero.

**Theorem 2.18.** Let $R$ be a semi-prime $q$-ring. Then $R = S \oplus T$, where $S$ is a complete direct sum of simple Artinian rings and $T$ is a semi-prime $q$-ring with zero socle.

Proof. Let $F = \text{soc} R$. Since $R^l = 0$, $\hat{F} = \{x \in R : xE \subset F \text{ for some large right ideal } E \text{ of } R\}$. Then it is immediate that $\hat{F}$ is a two-sided ideal in $R$. Since $R$ is self injective, $\hat{F} = eR$ for some idempotent $e$. Then $e$ is central, since $R$ is regular. Let $S = eR$ and $T = (1 - e)R$. Hence $R = S \oplus T$. By Lemma 2.14, both $S$ and $T$ are $q$-rings. Further, it can be easily verified that (i) $S$ is a semi-prime ring, soc $S = F$, and $F$ is large in $S$, and (ii) $T$ is a semi-prime ring with zero socle. By the above lemma $S$ is a complete direct sum of simple Artinian rings, completing the proof.

As a consequence of Lemma 2.11, Theorem 2.16 and Theorem 2.18 we have the following.

**Theorem 2.19.** A semi-prime ring $R$ is a $q$-ring if and only if $R = A \oplus B$, where $A$ is a right self injective duo ring and $B$ is semi-simple Artinian.

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<table>
<thead>
<tr>
<th>Authors</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>James Burton Ax</td>
<td><em>Injective endomorphisms of varieties and schemes</em></td>
<td>1</td>
</tr>
<tr>
<td>Richard Hindman Bouldin</td>
<td><em>A generalization of the Weinstein-Aronszajn formula</em></td>
<td>9</td>
</tr>
<tr>
<td>John Martin Chadam</td>
<td><em>The asymptotic behavior of the Klein-Gordon equation with external potential. II</em></td>
<td>19</td>
</tr>
<tr>
<td>Rina Hadass</td>
<td><em>On the zeros of the solutions of the differential equation</em></td>
<td>33</td>
</tr>
<tr>
<td>John Sollion Hsia</td>
<td><em>Integral equivalence of vectors over local modular lattices. II</em></td>
<td>47</td>
</tr>
<tr>
<td>Robert Hughes</td>
<td><em>Boundary behavior of random valued heat polynomial expansions</em></td>
<td>61</td>
</tr>
<tr>
<td>Surender Kumar Jain, Saad H. Mohamed and Surjeet Singh</td>
<td><em>Rings in which every right ideal is quasi-injective</em></td>
<td>73</td>
</tr>
<tr>
<td>T. Kawata</td>
<td><em>On the inversion formula for the characteristic function</em></td>
<td>81</td>
</tr>
<tr>
<td>Erwin Kleinfeld</td>
<td><em>On right alternative rings without proper right ideals</em></td>
<td>87</td>
</tr>
<tr>
<td>Robert Leroy Kruse and David Thomas Price</td>
<td><em>On the subring structure of finite nilpotent rings</em></td>
<td>103</td>
</tr>
<tr>
<td>Marvin David Marcus and Stephen J. Pierce</td>
<td><em>Symmetric positive definite multilinear functionals with a given automorphism</em></td>
<td>119</td>
</tr>
<tr>
<td>William Schumacher Massey</td>
<td><em>Pontryagin squares in the Thom space of a bundle</em></td>
<td>133</td>
</tr>
<tr>
<td>William Schumacher Massey</td>
<td><em>Proof of a conjecture of Whitney</em></td>
<td>143</td>
</tr>
<tr>
<td>John William Neuberger</td>
<td><em>Existence of a spectrum for nonlinear transformations</em></td>
<td>157</td>
</tr>
<tr>
<td>Stephen E. Newman</td>
<td><em>Measure algebras on idempotent semigroups</em></td>
<td>161</td>
</tr>
<tr>
<td>K. Chandrasekhar Rao</td>
<td><em>Matrix transformations of some sequence spaces</em></td>
<td>171</td>
</tr>
<tr>
<td>Robert Bruce Schneider</td>
<td><em>Some theorems in Fourier analysis on symmetric sets</em></td>
<td>175</td>
</tr>
<tr>
<td>Ulrich F. K. Schoenwaelder</td>
<td><em>Centralizers of abelian, normal subgroups of hypercyclic groups</em></td>
<td>197</td>
</tr>
<tr>
<td>Jerrold Norman Siegel</td>
<td><em>G-spaces, H-spaces and W-spaces</em></td>
<td>209</td>
</tr>
<tr>
<td>Robert Irving Soare</td>
<td><em>Cohesive sets and recursively enumerable Dedekind cuts</em></td>
<td>215</td>
</tr>
<tr>
<td>Kwok-Wai Tam</td>
<td><em>Isometries of certain function spaces</em></td>
<td>233</td>
</tr>
<tr>
<td>Awadhesh Kumar Tiwary</td>
<td><em>Injective hulls of semi-simple modules over regular rings</em></td>
<td>247</td>
</tr>
<tr>
<td>Eldon Jon Vought</td>
<td><em>Concerning continua not separated by any nonaposyndetic subcontinuum</em></td>
<td>257</td>
</tr>
<tr>
<td>Robert Breckenridge Warfield, Jr.</td>
<td><em>Decompositions of injective modules</em></td>
<td>263</td>
</tr>
</tbody>
</table>