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## **RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE**

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## RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE

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It is well known that if every right ideal of a ring  $R$  is injective, then  $R$  is semi simple Artinian. The object of this paper is to initiate the study of a class of rings in which each right ideal is quasi-injective. Such rings will be called  $q$ -rings. It is shown by an example that a  $q$ -ring need not be even semi prime. A number of important properties of  $q$ -rings are obtained.

Throughout this paper, unless otherwise stated, we assume that every ring has unity  $1 \neq 0$ . If  $M$  is a right  $R$ -module, then  $\hat{M}$  will denote the injective hull of  $M$ . For any positive integer  $n$ ,  $R_n$  will denote the ring of all  $n \times n$  matrices over the ring  $R$ .  $R^d$ ,  $J(R)$  and  $B(R)$  will denote the right singular ideal, the Jacobson radical and the prime radical respectively. A ring  $R$  is said to be a right duo ring if every right ideal of  $R$  is two-sided. Left duo rings are defined symmetrically. By a duo ring we mean a ring which is both right and left duo ring.

It is shown that  $R_n (n > 1)$  is a  $q$ -ring if and only if  $R$  is semi-simple Artinian. Some of the main results are: (i) a prime  $q$ -ring is simple Artinian, (ii) a semi-prime  $q$ -ring is a direct sum of two rings  $S$  and  $T$ , where  $S$  is a complete direct sum of simple Artinian rings, and  $T$  is a semi-prime  $q$ -ring with zero socle, and (iii) a semi-prime  $q$ -ring is a direct sum of two rings  $A$  and  $B$ , where  $A$  is a right self injective duo ring, and  $B$  is semi-simple artinian.

2. Let  $R$  be a right self injective ring. If  $B$  is any right ideal of  $R$ , then  $\hat{B} = eR$  for some idempotent  $e$  of  $R$ . Let  $K = \text{Hom}_R(\hat{B}, \hat{B})$ . Then  $K \cong eRe$ . In fact every element in  $K$  can be realized by the left multiplication of some element of  $eRe$ . By Johnson and Wong ([3], Theorem 1.1)  $B$  is a quasi injective as a right  $R$ -module if and only if  $KB = B$ . Hence  $B$  is quasi injective if and only if  $B = KB = (eRe)B = (eR)(eB) = \hat{B}B$ . Hence every two-sided ideal in a right self injective ring is quasi-injective. So, the following is immediate.

2.1. Every commutative self injective ring is a  $q$ -ring.

Now, we give an example of a  $q$ -ring which is not semi-prime.

EXAMPLE 2.2. Let  $Z$  be the ring of integers. Set  $R = Z/(4)$ . It is trivial that  $R$  is a  $q$ -ring. But  $R$  is not semi-prime, since its only proper ideal is nilpotent.

In fact,  $Z/(n)$  is a  $q$ -ring for every integer  $n > 1$ , since it is self injective (cf. Levy [5]). Also we remark that  $Z/(n)$  has nonzero nilpotent ideals if  $n$  is not square free.

Next we prove

**THEOREM 2.3.** *The following are equivalent*

- (1)  $R$  is a  $q$ -ring
- (2)  $R$  is right self injective, and every right ideal of  $R$  is of the form  $eI$ ,  $e$  is an idempotent in  $R$ ,  $I$  is a two sided ideal in  $R$ .
- (3)  $R$  is right self-injective, and every large right ideal of  $R$  is two sided.

*Proof.* Assume (1). Therefore  $R$  is right self injective. Let  $B$  be any right ideal of  $R$ . Then  $\hat{B} = eR$  for some idempotent  $e$ . Since  $B$  is quasi injective  $B = \hat{B}B = eRB = eI$ , where  $I = RB$ , the smallest two-sided ideal of  $R$  containing  $B$ . Hence (1) implies (2).

Assume (2). Let  $A$  be a large right ideal of  $R$ . Then  $A = eI$ ,  $e^2 = e$ ,  $I$  is a two sided ideal. Since  $A \cap (1 - e)R = 0$ ,  $(1 - e)R = 0$ . This implies that  $e = 1$ . Hence  $A = I$ , proving (3).

Now assume (3). Let  $B$  be a right ideal of  $R$ . If  $K$  is a complement of  $B$ , then  $B \oplus K$  is large in  $R$ . By assumption  $B \oplus K$  is a two-sided ideal in  $R$ , hence quasi-injective. This implies  $B$  is a quasi-injective, completing the proof.

**THEOREM 2.4.** *Let  $n > 1$  be an integer. Then  $R_n$  is a  $q$ -ring if and only if  $R$  is semi-simple Artinian.*

*Proof.* Suppose that  $R$  is not semi-simple Artinian. By Lambek ([4], Proposition 2, p. 61), there exists a large right ideal  $B$  of  $R$  such that  $B \neq R$ . Let  $e_{ij}$ ,  $1 \leq i, j \leq n$  be the matrix units of  $R_n$  and let  $E = \{\sum a_{ij}e_{ij} : a_{ij} \in B, 1 \leq j \leq n \text{ and } a_{ij} \in R, 1 \leq i, j \leq n\}$ . It is clear that  $E$  is a right ideal in  $R_n$ . But  $E$  is not two-sided, for  $e_{nn} \in E$  and  $e_{1n}e_{nn} = e_{1n} \notin E$ . Now, we prove that  $E$  is a large right ideal in  $R_n$ . Let  $0 \neq x = \sum_{i,j=1}^n b_{ij}e_{ij}$ . If  $b_{1j} = 0, 1 \leq j \leq n$ , then  $x \in E$ . So, let  $b_{1k} \neq 0$  for some  $k$ . Since  $B$  is large in  $R$ , there exists  $a \in R$  such that  $0 \neq b_{1k}a \in B$ . Then,

$$x(ae_{kk}) = (\sum_{i,j=1}^n b_{ij}e_{ij})(ae_{kk}) = \sum_{i=1}^n b_{ik}ae_{ik} \in E.$$

Hence,  $0 \neq x(ae_{kk}) \in E$ . Therefore  $E$  is a large right ideal in  $R_n$  which is not two-sided, and by Theorem 2.3,  $R_n$  is not a  $q$ -ring. This proves "only if" part. Other part is obvious.

We are now ready to show the existence of right self injective rings which are not  $q$ -rings.

EXAMPLE 2.5. Let  $R$  be a right self injective ring which is not semi-simple (we can take  $R = Z/(4)$ ). Let  $n > 1$  be an integer. By Utumi ([6], Th. 8.3)  $R_n$  is right self injective. But  $R_n$  is not a  $q$ -ring, by the above theorem.

Next we prove

THEOREM 2.6. *A simple ring is a  $q$ -ring if and only if it is Artinian.*

*Proof.* Let  $R$  be a simple  $q$ -ring. Let  $B$  be a large right ideal in  $R$ . Then  $B$  is two-sided, and hence  $B = R$ . This proves that  $R$  does not contain any proper large right ideal. Hence  $R$  is Artinian. The converse is trivial.

Now, we give an example of a right self injective simple ring which is not a  $q$ -ring.

EXAMPLE 2.7. Let  $S$  be a noncommutative integral domain which is not a right Öre domain (cf. Goldie [1]). Let  $R = \hat{S}$ . Then  $R$  is a right self injective simple regular ring which is not Artinian. By the above theorem  $R$  is not a  $q$ -ring.

LEMMA 2.8. *Let  $R$  be a  $q$ -ring. Then  $B(R)$  is essential in  $J(R)$  as a right  $R$ -module.*

*Proof.* Since  $R$  is self injective,  $J(R) = R^d$ , by Utumi ([6], Lemma 4.1). Let  $0 \neq x \in J(R)$ . There exist a large right ideal  $E$  of  $R$  such that  $xE = 0$ . Then  $xE \subset P$  for every prime ideal  $P$  of  $R$ . Since  $R$  is a  $q$ -ring,  $E$  is two-sided. This implies that either  $x \in P$  or  $E \subset P$ .

Let  $\{P_i\}_{i \in I}$  be the set of all prime ideals of  $R$  such that  $x \in P_i$  for every  $i \in I$ , and  $\{P_j\}_{j \in J}$  be the set of all prime ideals of  $R$  such that  $x \notin P_j$  for every  $j \in J$ . Let  $X = \bigcap_{i \in I} P_i$ , and  $Y = \bigcap_{j \in J} P_j$ .  $X \neq 0$ , since  $0 \neq x \in X$ . On the other hand,  $E \subset P_j$  for every  $j \in J$ . Thus  $E \subset Y$ , which implies that  $Y$  is large in  $R$ . Therefore  $B(R) = X \cap Y \neq (0)$ . Moreover, there exists  $a \in R$  such that  $0 \neq xa \in Y$ . This implies that  $0 \neq xa \in X \cap Y = B(R)$ , completing the proof.

Hence, we have the following

THEOREM 2.9. *A  $q$ -ring is regular if and only if it is semi-prime.*

*Proof.* The result follows by the above lemma, and Utumi ([6], Corollary 4.2).

**THEOREM 2.10.** *Let  $V$  be a vector space over a division ring  $D$ , and let  $R = \text{Hom}_D(V, V)$ . Then  $R$  is a  $q$ -ring if and only if  $V$  is of finite dimension over  $D$ .*

*Proof.* The “if” part is obvious. Conversely, suppose that  $V$  is of infinite dimension over  $D$ . Let  $X = \{x_1, x_2, \dots\}$  be a denumerable set of linearly independent elements of  $V$ .  $X$  can be extended to a basis  $X \cup Y$  of  $V$ . Let  $F$  be the ideal in  $R$  consisting of all elements of finite rank. Let  $\sigma \in R$  be defined by  $\sigma(x_{2i}) = x_{2i}$ ,  $\sigma(x_{2i-1}) = 0$  for every  $i$ , and  $\sigma(y) = 0$  for every  $y \in Y$ . Let  $E = \sigma R + F$ . Then  $F \subset E$ . Since  $F$  is a two-sided ideal in  $R$ ,  $F$  is large. Therefore  $E$  is a large right ideal in  $R$ . We proceed to prove that  $E$  is not two-sided. Let  $\lambda_1, \lambda_2 \in R$  be defined by:  $\lambda_1(x_i) = x_{2i}$  for every  $i$ , and  $\lambda_1(y) = 0$  for every  $y \in Y$ ,  $\lambda_2(x_{2i}) = x_i$ ,  $\lambda_2(x_{2i-1}) = 0$ , for every  $i$ , and  $\lambda_2(y) = 0$  for every  $y \in Y$ . Let  $\lambda = \lambda_2 \sigma \lambda_1$ . Then  $\lambda(x_i) = x_i$  for every  $i$ . Hence  $X \subset \lambda(V)$ . We assert that  $\lambda \notin E$ ; for otherwise, let  $\lambda = \sigma r + f$ ,  $r \in R$ ,  $f \in F$ . Then  $X \subset \lambda(V) = (\sigma r + f)(V) \subset \sigma(V) + f(V)$ . But since  $f$  is of finite rank, there exists an integer  $n$  such that  $x_{2n-1} \notin f(V)$ . Also, by definition of  $\sigma$ ,  $x_{2n-1} \notin \sigma(V)$ . Hence  $x_{2n-1} \notin \sigma(V) + f(V)$ , which is a contradiction. Thus  $\lambda \notin E$ , as desired. However  $\lambda \in R\sigma R \subset RE$ . Hence  $E$  is not a two-sided ideal. Therefore, by Theorem 2.3,  $R$  is not a  $q$ -ring. This completes the proof.

We remark that the above theorem is also a consequence of Theorem 2.4.

The right (left) socle of a ring  $R$  is defined to be the sum of all minimal right (left) ideals of  $R$ . It is well known that in a semi-prime ring  $R$ , the right and left socles of  $R$  coincide, and we denote any of them by  $\text{soc } R$ .

**LEMMA 2.11.** *A semi-prime  $q$ -ring  $R$  with zero socle is strongly regular.*

*Proof.* Let  $M$  be a maximal right ideal in  $R$ . Either  $M$  is a direct summand of  $R$  or  $M$  is large in  $R$ . If  $M$  is a direct summand of  $R$ , then its complement is a minimal right ideal. This implies that  $\text{soc } R \neq 0$ , a contradiction. Therefore, every maximal right ideal is large, hence two-sided. By Lemma 2.8,  $J(R) = 0$ . Thus  $R$  is isomorphic to a subdirect sum of division rings, which implies that  $R$  has no nonzero nilpotent elements. Since  $R$  is regular, by Theorem 2.9,  $R$  is strongly regular.

LEMMA 2.12. *A prime  $q$ -ring has nonzero socle.*

*Proof.* Let  $R$  be a prime  $q$ -ring. If possible, let  $\text{soc } R = 0$ . By the above lemma,  $R$  is strongly regular. Hence  $R$  is a division ring, and  $\text{soc } R = R$  contradicting our assumption. Therefore  $\text{soc } R \neq (0)$ .

THEOREM 2.13. *A prime ring  $R$  is a  $q$ -ring if and only if  $R$  is simple Artinian.*

*Proof.* By Theorem 2.9, and the above lemma,  $R$  is a prime regular ring with nonzero socle. Hence, by Johnson ([2], Th. 3.1),  $\hat{R} = \text{Hom}_D(V, V)$ , where  $V$  is some vector space over a division ring  $D$ . But then  $R = \text{Hom}_D(V, V)$ , since  $R$  is right self injective. By Theorem 2.10,  $V$  has finite dimension over  $D$ . Let  $(V: D) = n$ . Then  $R \cong D_n$ , completing the proof.

LEMMA 2.14. *Let  $\{R_\alpha\}_{\alpha \in I}$  be a finite set of rings. Then the direct sum  $\sum_{\alpha \in I} \oplus R_\alpha$  is a  $q$ -ring if and only if each  $R_\alpha$  is a  $q$ -ring.*

The proof is obvious.

That Lemma 2.14 is not true for an infinite number of rings is shown by the following example which is due to Storrer.

EXAMPLE 2.15. Let  $R$  be a  $2 \times 2$ -matrix ring over a field  $F$ . Let  $\{R_\alpha\}_{\alpha \in I}$  be an infinite family of copies of  $R$  and let  $S = \pi R_\alpha, \alpha \in I$ . Let  $E$  be the right ideal of  $S$  consisting of those elements  $[x_\alpha]$  of  $S$  such that all but finite  $x'_\alpha$ 's are matrices with first row zero. Since  $R_\alpha \subset E$  for all  $\alpha \in I$ ,  $E$  is a large right ideal of  $S$ . To show that  $E$  is not two-sided, consider  $[x_\alpha] \in E$  where  $x_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  for all  $\alpha \in I$ . Let  $[y_\alpha] \in S$  be such that  $y_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  for all  $\alpha \in I$ . Then  $[y_\alpha][x_\alpha] = [z_\alpha]$ , where  $z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . But then  $[z_\alpha] \notin E$ , and  $E$  is not two-sided. Hence, by Theorem 2.3,  $S$  is not a  $q$ -ring.

Example 2.15 also suggests the following.

THEOREM 2.16. *Let  $\{R_\alpha\}_{\alpha \in I}$  be a family of simple Artinian rings and let  $R$  be their complete direct sum. Then  $R$  is a  $q$ -ring if and only if all  $R'_\alpha$ 's excepting a finite number of them are division rings.*

The above theorem shows, in particular, that a regular  $q$ -ring may not be Artinian.

LEMMA 2.17. *Let  $R$  be a semi-prime  $q$ -ring such that  $\text{soc } R$  is*

large in  $R$ . Then  $R$  is a complete direct sum of simple Artinian rings.

*Proof.* Since  $\text{soc } R$  is large, every nonzero right ideal of  $R$  contains a minimal right ideal. Also  $R$  is regular, by Theorem 2.9. Hence by Johnson ([2], Th. 3.1),  $R$  is a complete direct sum of rings  $R_i$ , where each  $R_i$  is the ring of all linear transformations of some vector space  $V_i$  over a division ring  $D_i$ . But then by Lemma 2.14 and Theorem 2.10, each  $R_i$  is a simple Artinian ring. This completes the proof.

In the following two theorems we assume that every ring has a unity element which may be equal to zero.

**THEOREM 2.18.** *Let  $R$  be a semi-prime  $q$ -ring. Then  $R = S \oplus T$ , where  $S$  is a complete direct sum of simple Artinian rings and  $T$  is a semi-prime  $q$ -ring with zero socle.*

*Proof.* Let  $F = \text{soc } R$ . Since  $R^d = 0$ ,  $\hat{F} = \{x \in R : xE \subset F \text{ for some large right ideal } E \text{ of } R\}$ . Then it is immediate that  $\hat{F}$  is a two-sided ideal in  $R$ . Since  $R$  is self injective,  $\hat{F} = eR$  for some idempotent  $e$ . Then  $e$  is central, since  $R$  is regular. Let  $S = eR$  and  $T = (1 - e)R$ . Hence  $R = S \oplus T$ . By Lemma 2.14, both  $S$  and  $T$  are  $q$ -rings. Further, it can be easily verified that (i)  $S$  is a semi-prime ring,  $\text{soc } S = F$ , and  $F$  is large in  $S$ , and (ii)  $T$  is a semi-prime ring with zero socle. By the above lemma  $S$  is a complete direct sum of simple Artinian rings, completing the proof.

As a consequence of Lemma 2.11, Theorem 2.16 and Theorem 2.18 we have the following.

**THEOREM 2.19.** *A semi-prime ring  $R$  is a  $q$ -ring if and only if  $R = A \oplus B$ , where  $A$  is a right self injective duo ring and  $B$  is semi-simple Artinian.*

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