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ON THE INVERSION FORMULA FOR THE CHARACTERISTIC FUNCTION

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In the inversion formula

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt$$

for the characteristic function $f(t)$ of a distribution function $F(x)$, the limit of the symmetric integral is used. The purpose of this paper is to give a necessary and sufficient condition for the existence of the asymmetric improper integral $\lim_{T,T'\to\infty} \int_{-T'}^{T}$ on the right of the above formula.

Let $F(x)$ be a probability distribution function and $f(t)$ the corresponding characteristic function,

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) \ . \tag{1.1}$$

We assume in this note that $F(x)$ is standardized so that

$$F(x) = \tfrac{1}{2}[F(x + 0) + F(x - 0)] \ . \tag{1.2}$$

The well known inversion formula states that

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt \tag{1.3}$$

for every $-\infty < x < \infty$.

It is also known that the symmetric integral of the right hand side cannot be replaces by the improper integral $\lim_{T,T'\to\infty} \int_{-T'}^{T'}$ (T, T' going to infinity independently).

Actually we may easily see that

$$\mathrm{Re}\left(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt\right) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt} - 1}{-it} f(t)dt\right), \tag{1.4}$$

and

$$\mathrm{Im}\left(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt\right) = -\mathrm{Im}\left(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt} - 1}{-it} f(t)dt\right), \tag{1.5}$$

and hence $\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt$ cancels out its imaginary part.

The real part (1.4) always converges to $\frac{1}{2}[F(x) - F(0)]$. This gives

the proof of (1.3). (See [1], pp. 263-264).

However the imaginary part (1.5) does not necessarily converge without some condition on $F(x)$. This is why the limit of the symmetric integral in (1.3) cannot be replaced by the general improper integral.

2. The condition for the existence of the improper integral.

We shall give the necessary and sufficient condition for the existence of the limit of (1.5).

THEOREM 1. *In order that the limit of* (1.5) *when* $T \to \infty$ *exists, it is necessary and sufficient that the integral*

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{G(u, x) - G(u, 0)}{u} du \tag{2.1}$$

exists where,

$$G(u, x) = F(u + x) - F(-u + x) \tag{2.2}$$

and if (2.1) *exists*

$$\lim_{T \to \infty} \mathrm{Im}\left(\frac{1}{2\pi} \int_0^T \frac{e^{-ixt} - 1}{-it} f(t) dt\right) = \int_0^{\infty} \frac{G(u, x) - G(u, 0)}{u} du \,. \tag{2.3}$$

It must be noted that the integral of the right hand side of (2.3) exists in the neighborhood of the infinity. In fact $G(u, x) - G(u, 0) = [F(u + x) - F(u)] - [F(-u + x) - F(-u)]$ and $F(u + x) - F(u) \in L$, $(-\infty, \infty)$ for every fixed x.

We shall now prove the theorem.

Let

$$I(x, T) = \mathrm{Im}\left(\frac{1}{2\pi} \int_0^T \frac{e^{-ixt} - 1}{-it} f(t) dt\right) .$$

We then easily see that

$$\begin{aligned} I(x, T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T \frac{\sin xt \sin ut - (1 - \cos xt) \cos ut}{t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T \frac{\cos (u - x)t - \cos ut}{t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T dt \int_{u-x}^{u} \sin vt dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_{u-x}^{u} \frac{1 - \cos vT}{v} dv \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos vT}{v} dv \int_{v}^{v+x} dF(u)$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} [G(v, x) - G(v, 0)] \frac{1-\cos vT}{v} dv .$$

As was mentioned before, $G(v, x) - G(v, 0) \in L_1(-\infty, \infty)$. Hence the Riemann-Lebesgue lemma shows that

$$\lim_{T\to\infty} \int_{\varepsilon}^{\infty} [G(v, x) - G(v, 0)] \frac{\cos vT}{v} dv = 0$$

for any $\varepsilon > 0$. Therefore we may write

$$I(x, T) = \int_{0}^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v} (1 - \cos vT) dv + \int_{\varepsilon}^{\infty} \frac{G(v, x) - G(v, 0)}{v} dv + o(1) \tag{2.4}$$

as $T \to \infty$, for a fixed $\varepsilon > 0$.

Now we shall show the sufficiency of the condition of the theorem. Let $\varepsilon > 0$ be arbitrary but fixed. Write

$$\int_{0}^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v} (1 - \cos vT) dv = \int_{0}^{1/T} + \int_{1/T}^{\varepsilon} = K_1 + K_2 , \tag{2.5}$$

say. We have

$$|K_1| \leqq \int_{0}^{1/T} |G(v, x) - G(v, 0)| \frac{1-\cos vt}{v} dv \leqq CT \int_{0}^{1/T} |G(v, x) - G(v, 0)| \, dv , \tag{2.6}$$

for some constant C.

$\lim_{v\to 0+} [G(v, x) - G(v, 0)]$ exists since F is nondecreasing and it must be zero, otherwise (2.1) does not exist. Hence the last expression converges to zero.

$$K_1 = o(1), \quad \text{as} \quad T \to \infty . \tag{2.7}$$

Next write

$$\chi(v) = G(v, x) - G(v, 0) . \tag{2.8}$$

Choose ε such that $|\chi(v)| < \delta$ for $|v| \leqq \varepsilon$ for an arbitrary chosen δ. Since $\chi(v)/v$ is of bounded variation in $[1/T, \varepsilon]$, we have, using the second mean value theorem,

$$K_2 = \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} \cos vT dv$$
$$= \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - T\chi\left(\frac{1}{T}\right) \int_{1/T}^{\xi} \cos vT dv - \frac{\chi(\varepsilon)}{\varepsilon} \int_{\varepsilon}^{\varepsilon} \cos vT dv$$

for some $1/T < \xi < \varepsilon$. Thus

$$\left| K_2 - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv \right| \leqq 2\chi\left(\frac{1}{T}\right) + 2\chi(\varepsilon) \leqq 4\delta .$$

Therefore from (2.4) and (2.5)

$$\left| I(x, T) - \frac{1}{2\pi} \int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq \frac{2\delta}{\pi} + o(1) . \tag{2.9}$$

This shows the sufficiency of the condition of the theorem and gives (2.3).

We shall next show the necessity. Define $\chi(v)$ as before. We see that $\chi(v)$ has the limit c as $v \to +0$. If $c \neq 0$, then from (2.4)

$$I(x, T) - c \int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv$$
$$= \int_0^{\varepsilon} \frac{[\chi(v) - c](1 - \cos vT)}{v} dv + \int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv + o(1) .$$

The first integral of the right hand side is handled in the same way as in deriving (2.6) and (2.9) with $\chi(v) - c$ in place of

$$\chi(v) = G(v, x) - G(v, 0) .$$

Actually instead of (2.6) we see that K_1 with $\phi(v) - c$ is bounded by $CT \int_0^{1/T} |\chi(v) - c| dv$ which is $o(1)$. In place of (2.9) we have

$$\left| I(x, T) - \frac{c}{2\pi} \int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv - \frac{1}{2\pi} \int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right.$$
$$\left. - \frac{1}{2\pi} \int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) , \tag{2.10}$$

where C_1 is some constant.

$$\int_0^{\varepsilon} \frac{1 - \cos vT}{v} dv = 2 \int_0^{\varepsilon T} \frac{\sin^2 v/2}{v} dv \geqq C_2 \log \varepsilon T , \tag{2.11}$$

where C_2 is an absolute constant. Choose ε for an arbitrary given $\eta < C_2$ so that $|\chi(v) - c| < \eta$ for $0 < v < \varepsilon$. Then

$$\left| \int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right| \leqq \eta \log \varepsilon T . \tag{2.12}$$

Hence if $I(x, T)$ has a limit as $T \to \infty$, then in view of (2.11) and (2.12), (2.10) implies a contradiction. Hence we have that $c = 0$. Using (2.10), this yields

$$\left| I(x, T) - \frac{1}{2\pi}\int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) .$$

This proves the necessity of the condition.

3. **Remarks.** From Theorem 1, we immediately obtain

THEOREM 2. *In order that*

$$\lim_{T,T'\to\infty} \frac{1}{2\pi}\int_{-T'}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt$$

exists, it is necessary and sufficient that (2.1) *exists for* $\varepsilon > 0$. (*The limit is* $F(x) - F(0)$).

Similar arguments apply to the integral

$$(3.1) \quad J_1(x, T) = \int_1^T \frac{f(t)e^{-ixt}}{it} dt \quad \text{and} \quad J_2(x, T) = \int_{-T}^{-1} \frac{f(t)e^{-ixt}}{it} dt .$$

We easily see that J_1 and J_2 are conjugate complex. We may show that *in order* for $J_1(x, T)$ *or* $J_2(x, T)$ *to converge as* $T \to \infty$, *it is necessary and sufficient that*

$$(3.2) \qquad \int_0^{\varepsilon} \frac{F(u + x) - F(-u + x)}{u} du < \infty$$

for some $\varepsilon > 0$.

(3.1) implies that

$$(3.3) \qquad \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{-ixt}dt = 0$$

which is very well known when $F(x)$ is continuous at x. (3.1) says more than this about the improper integrability of $f(t)$ near infinity with the additional condition (3.2) on $F(x)$.

The sufficiency of (3.2) for the existence of the limits of (3.1) was proved in [2] before.

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