PONTRYAGIN SQUARES IN THE THOM SPACE OF A BUNDLE

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The object of this note is to determine the action of the Pontryagin squares in the cohomology of the Thom space of a vector bundle. This computation is then applied to the case of the normal bundle of a manifold imbedded in Euclidean space to give simplified proofs of some theorems of Mahowald.

The first of Mahowald's theorems [3] was inspired by some 1940 results of Whitney [9], who showed that in certain cases the Euler class (with twisted integer coefficients) of the normal bundle of a non-orientable surface imbedded in Euclidean 4-space could be nonzero. This contrasts with the well-known theorem that the Euler class of the normal bundle of an orientable manifold in Euclidean space is always zero.

2. Notation and statement of results. For any space \( X \), we will use integral cohomology \( H^q(X, \mathbb{Z}) \); cohomology with integers mod \( n \) as coefficients, \( H^q(X, \mathbb{Z}_n) \); cohomology with twisted integer coefficients, \( H^q(X, \mathbb{Z}) \); cohomology with twisted integers mod \( n \) coefficients, \( H^q(X, \mathbb{Z}_n) \); and rational cohomology, \( H^q(X, \mathbb{Q}) \). In the third and fourth cases the local system of groups which is used for coefficients will be determined by the Stiefel-Whitney class \( w_i \in H^i(X, \mathbb{Z}_2) \). Note that for the case \( n = 2 \), we have

\[ H^q(X, \mathbb{Z}_2) = H^q(X, \mathbb{Z}) \, , \]

since a cyclic group of order 2 admits no nontrivial automorphisms.

Let \( (E, p, B, S^{n-1}) \) be an \( (n-1) \)-sphere bundle over the base space \( B \) with structure group \( 0(n) \). We will use the following notation for characteristic classes of such a bundle:

Stiefel-Whitney classes:

\[ w_i \in H^i(B, \mathbb{Z}_2) \, , \quad 1 \leq i \leq n \]

\[ W_i \in H^i(B, \mathbb{Z}) \, , \quad 1 \leq i \leq n, \ i \text{ odd} \, . \]

Pontrjagin classes:

\[ p_i \in H^{2i}(B, \mathbb{Z}) \, , \quad 1 \leq i \leq n/2 \, . \]

Euler class:

\[ X_n \in H^n(B, \mathbb{Z}) \, , \quad (\text{If } n \text{ is odd, then } X_n = W_n) \]
Let \((A, \pi, B, D^n)\) be the associated \(n\)-dimensional disc bundle; we will call the pair \((A, E)\) or the single space \(A/E\) the Thom space of the bundle. The Thom class, \(U \in H^*(A, E, \mathbb{Z})\), has twisted integer coefficients; by taking cup products with \(U\), we obtain the Thom isomorphism (see Thom [6]).

\[
\begin{align*}
H^q(A, \mathbb{Z}) &\approx H^{q+n}(A, E, \mathbb{Z}) , \\
H^q(A, \mathbb{Z}_2) &\approx H^{q+n}(A, E, \mathbb{Z}_2) , \\
H^q(A, \mathbb{Z}_n) &\approx H^{q+n}(A, E, \mathbb{Z}_n) ,
\end{align*}
\]

Recall also that the projection \(\pi: A \to B\) is a deformation retraction, and hence induces isomorphisms of cohomology groups with any coefficients (even local coefficients!). For the sake of convenience, we will often identify the cohomology groups of \(A\) and \(B\) by means of this isomorphism; similarly we will identify the cohomology groups of the pair \((A, E)\) and the space \((A/E)\) (except in dimension 0) with ordinary coefficients (the local coefficient systems \(\mathbb{Z}\) and \(\mathbb{Z}_n\) do not exist in the space \(A/E\)).

The obvious epimorphism \(\rho_n: Z \twoheadrightarrow \mathbb{Z}_n\) and monomorphism \(\theta: \mathbb{Z}_2 \rightarrow \mathbb{Z}_4\) induce homomorphisms of cohomology groups which will be denoted as follows:

\[
\begin{align*}
\rho_n: H^q(X, Z) &\longrightarrow H^q(X, \mathbb{Z}_n) , \\
\tilde{\rho}_n: H^q(X, \mathbb{Z}) &\longrightarrow H^q(X, \mathbb{Z}_n) , \\
\theta: H^q(X, \mathbb{Z}_2) &\longrightarrow H^q(X, \mathbb{Z}_4) , \\
\tilde{\theta}: H^q(X, \mathbb{Z}_2) &\longrightarrow H^q(X, \mathbb{Z}_4) .
\end{align*}
\]

For convenience, we will let \(U_2 = \tilde{\rho}_2(U)\), the Thom class reduced mod 2.

Our main concern will be the Pontryagin squaring operation,

\[
\mathcal{P}: H^q(X, \mathbb{Z}_2) \longrightarrow H^{2q}(X, \mathbb{Z}_4) .
\]

If \(q\) is odd, the Pontryagin square can be expressed in terms of simpler cohomology operations. (see formula (4.2) below); this is not true for \(q\) even. For a list of papers describing this operation, see the first paragraph of [7]. Our main result is the following, which describes the Pontryagin square of the mod 2 Thom class, \(U_2\).

**Theorem I.** Let \((E, p, B, S^{n-1})\) be a (not necessarily orientable) \((n-1)\)-sphere bundle with structure group \(O(n), n\) even. Then

\[
\mathcal{P}(U_2) = [\tilde{\rho}_2(X_2) + \tilde{\theta}(w_1 \cdot w_{n-1})] \cdot U .
\]

As a corollary, we obtain the following result which was proved by Whitney [9] in 1940 for the case \(n = 2\); the general case is due
to Mahowald, [3, Th. I]:

**COROLLARY 1.** Let $M^n$ be a compact, connected, nonorientable $n$-manifold ($n$ even) which is imbedded differentiably in $\mathbb{R}^{2n}$. Then the twisted Euler class of the normal bundle, $X_n$, satisfies the following condition:

$$\bar{\rho}_i(X_n) + \bar{\vartheta}(\bar{w}_i\bar{w}_{n-1}) = 0.$$  

(Here $\bar{w}_i$ denotes the $i$th dual Stiefel-Whitney class of $M^n$.)

In particular, if $\bar{w}_i\bar{w}_{n-1} \neq 0$ (which can only happen if $n$ is a power of 2, cf. [4]) then $X_n \neq 0$. Apparently this is the only general result known about the twisted Euler class of the normal bundle to a non-orientable manifold.

The corollary may be derived from the theorem as follows: Let $(E, p, B, S^{n-1})$ denote the normal sphere bundle of the imbedding, and $(A, \pi, B, D^n)$ the associated disc bundle. It is well known that the top homology group of the Thom space,

$$H_{2n}(A/E, \mathbb{Z}) = H_{2n}(A, E, \mathbb{Z}),$$

is infinite cyclic, and the Hurewicz homomorphism

$$\pi_{2n}(A/E) \rightarrow H_{2n}(A/E)$$

is an epimorphism. From this it follows that $\langle \mathcal{P}(U_2), x \rangle = 0$ for any $x \in H_{2n}(A/E, \mathbb{Z})$, and hence $\mathcal{P}(U_2) = 0$. Applying the formula for $\mathcal{P}(U_2)$ in Theorem I, we obtain the corollary.

Next, we give formulas for the Pontryagin square of an arbitrary mod 2 cohomology class of even degree in the Thom space of a vector bundle.

**THEOREM II.** Let $(E, p, B, S^{n-1})$ be an $(n-1)$-sphere bundle with structure group $O(n)$, and let $x \in H^{m}(B, \mathbb{Z}_2)$, $m + n$ even. Then if $m$ and $n$ are both even,

$$\mathcal{P}(U_2x) = \{\mathcal{P}(x)[\bar{\rho}_i(X_n) + \bar{\vartheta}(\bar{w}_i\bar{w}_{n-1})]
+ \bar{\vartheta}[w_{n-1}xSq^i + w_n\bar{w}_{n-1}x] \cdot U,$$

while if $m$ and $n$ are odd,

$$\mathcal{P}(U_2x) = \{\mathcal{P}(x)[\bar{\rho}_i(X_n) + \bar{\vartheta}(w_i\bar{w}_{n-1} + w_i\bar{w}_{n-2})]
+ \bar{\vartheta}[w_{n-1}xSq^i + w_n\bar{w}_{n-1}x] \cdot U.$$

As a corollary, we derive a necessary condition due to Mahowald [3] for the imbeddability of an orientable manifold in Euclidean space.
of dimension $4k$ with codimension $n$.

**Corollary 2.** Let $M$ be a compact, connected, orientable manifold of dimension $q$ which is differentiably imbedded in Euclidean space of dimension $q + n = 4k$. Then for any $x \in H^n(M, \mathbb{Z}_2)$, where $m = \frac{1}{2}(q - n)$, we must have

$$\bar{w}_{n-1} x Sq^1 x = 0.$$  

**Proof of corollary.** One applies Theorem II with $B = M$ and $(E, p, B, S^{n-1})$ the normal bundle of the imbedding. Since $M$ is assumed orientable, $\bar{w}_1 = 0$, $\bar{w}_n = 0$, $X_n = 0$, and $\bar{W}_n = 0$. Exactly as in the proof of the previous corollary we know that $\mathcal{P}(U_z x) = 0$ in this case. Thus we conclude that

$$\theta(\bar{w}_{n-1} x Sq^1 x) = 0$$

for any $x \in H^n(M, \mathbb{Z}_2)$. Since $M$ is orientable, the homomorphism

$$\theta: H^q(M, \mathbb{Z}_2) \longrightarrow H^q(M, \mathbb{Z}_2)$$

is a monomorphism, and therefore we must have $\bar{w}_{n-1} x Sq^1 x = 0$, as desired.

Perhaps the neatest application of this corollary is to prove that $q$-dimensional real protective space does not imbed in $\mathbb{R}^q$ for $q = 2^r + 1$. A discussion of the possibilities of using this theorem to prove non-imbedding results is given in §5.

**Corollary 3.** Let $M$ be a compact, connected, nonorientable manifold of dimension $q$ which is differentiably imbedded in Euclidean space of dimension $q + n = 4k$, $q$ and $n$ even. Then for any element $x \in H^n(M, \mathbb{Z}_2)$, where $m = \frac{1}{2}(q - n)$, we must have

$$\mathcal{P}(x) [\bar{\partial}_i (X_n) + \bar{\partial} (\bar{w}_i \bar{w}_{n-i}) ] + \bar{\partial} (\bar{w}_{n-1} x Sq^1 x) = 0.$$  

This is a generalization of Corollary 1, and the proof is similar. Presumably this theorem would enable one to prove in certain cases that $\bar{\partial}_i (X_n) \neq 0$, and hence $X_n \neq 0$, but the author knows of no examples to illustrate this possibility. Perhaps the most likely case in which this theorem could be applied is the case where $n = q - 4$, $m = 2$.

3. **Proof of Theorem I.** As is usual in such cases, one only need prove Theorem I in the case of the universal example, where $B = BO(n)$, $n$ even. Then $E$ has the same homotopy type as $BO(n - 1)$. Consider the following commutative diagram for this universal example:
The top line of this diagram is the mod $k$ cohomology sequence of the pair $(A, E)$ while the bottom line is the Gysin sequence of fibration. All vertical arrows are isomorphisms; arrow No. 1 denotes the Thom isomorphism, and arrow No. 2 is the identity. It is well known that in these exact sequences for the case $k=2$ (i.e., mod2 cohomology), the following statements are true:

- $p^*$ and $i^*$ are epimorphisms,
- $μ$ and $j^*$ are monomorphisms, and
- $ψ$ and $δ$ are zero.

We assert that these statements are also true in case $k = 4$. In order to prove this, it suffices to prove that $j^*$ is a monomorphism, and for this purpose consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & & H^{q-1}(A, E, Z_2) & & j_2 & & H^q(A, Z_2) & & \\
& & Sq^1 & & & & Sq^1 & & \\
0 & & H^q(A, E, Z_2) & & j_2 & & H^q(A, Z_2) & & \\
& & & & & & & & \\
\cdots & & H^q(A, E, Z_2) & & j_2 & & H^q(A, Z_2) & & \\
& & & & & & & & \\
& & & & & & & & \\
0 & & H^q(A, E, Z_2) & & j_2 & & H^q(A, Z_2) & & \\
\end{array}
\]

The vertical lines are exact sequences corresponding to the following short exact sequence of coefficients:

\[
0 \rightarrow Z_2 \xrightarrow{θ} Z_4 \xrightarrow{η} Z_2 \rightarrow 0 .
\]

Let $x \in H^q(A, E, Z_2)$ and assume that $j^*(x) \equiv j_2(x) = 0$. Therefore

\[
j_2ηj(x) = ηj_2(x) = 0
\]

and since $j_2$ is a monomorphism, $η(x) = 0$. By exactness, there exists an element $y \in H^q(A, E, Z_2)$ such that $θ(y) = x$.

Since $θj_2(y) = 0$, there exists an element $z \in H^{q-1}(A, Z_2)$ such that
We wish to show that \( z \) can be chosen so that \( z \in \text{image } j_2 \). For this purpose, recall that we are identifying \( H^*(A, \mathbb{Z}_2) \) with \( H^*(B, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_n] \); using this identification, the image of \( j^* \) is the ideal generated by \( w_n \). We may split \( H^*(A, \mathbb{Z}_2) \) into the (vector space) direct sum of this ideal and a supplementary subspace as follows: one subspace is spanned by all monomials which have \( w_n \) as a factor, the other subspace is spanned by those monomials which do not have \( w_n \) as a factor. It is readily verified that the homomorphism

\[
Sq^1: H^*(A, \mathbb{Z}_2) \longrightarrow H^*(A, \mathbb{Z}_2)
\]

maps each of these summands into itself (this depends on the fact that \( n \) is even). Since \( j_2(y) \) belong to this ideal generated by \( w_n \), we can choose \( z \) so it also belongs to this ideal. Therefore \( z = j_2(u) \) for some element \( u \in H^{q-1}(A, E, \mathbb{Z}_2) \). It follows that

\[
j_2(y - Sq^1 u) = 0 .
\]

Since \( j_2 \) is a monomorphism, \( y = Sq^1 u \), and

\[
x = \theta(y) = \theta Sq^1 u = 0
\]
as asserted.

Next, let \( X_n \in H^*(BO(n), \mathbb{Z}) \) denote the Euler class (\( n \) even). We assert that

\[
X_n^+ = p_n/2 \in H^{2n}(BO(n), \mathbb{Z}) .
\]

To prove this, we make use of the fact that all torsion in \( H^*(BO(n), \mathbb{Z}) \) is of order 2 (cf. Borel and Hirzebruch, [2]). Hence it suffices to prove that the following two equations:

\[
\rho_4(X_n^+) = p_n/2 \quad \text{and} \quad \rho_4(X_2^+) = \rho_4(p_n/2) ,
\]

where \( \rho_4 \) is the homomorphism of cohomology groups induced by the coefficient map \( \mathbb{Z} \rightarrow \mathbb{Q} \).

As to the first equation, it is well known that \( \rho_4(X_n) = w_n \) and \( \rho_4(p_i) = w_i^2 \), hence

\[
\rho_4(X_n^+) = w_n = \rho_4(p_n/2) .
\]

To prove the second equation, consider the following commutative diagram.
\[ H^{2n}(BO(n), Z) \xrightarrow{f^*} H^{2n}(BSO(n), Z) \]
\[ \xrightarrow{\rho_0} \]
\[ H^{2n}(BO(n), Q) \xrightarrow{f^*} H^{2n}(BSO(n), Q) . \]

Here \( f: BSO(n) \to BO(n) \) is the 2-fold covering induced by the inclusion of \( SO(n) \) in \( O(n) \). It is well known that \( \rho_0 f^*(x) = \rho_0 f^*(p_{n/2}) \) and that \( f^* \) is a monomorphism on rational cohomology (see Borel and Hirzebruch [2]). Hence \( \rho_0(X^*_n) = \rho_0(p_{n/2}) \) as required.

With these preliminaries out of the way, we will now prove Theorem I by consideration of the following commutative diagram:

\[ H^{2n}(A, E, Z) \xrightarrow{j^*} H^{2n}(A, Z) \]
\[ \xrightarrow{\phi} \]
\[ H^{2n}(A, E, Z) \xrightarrow{j^*} H^{2n}(A, Z) . \]

It is well known that \( j^*_2(U) = w_n \), and according to Thomas [8], Theorem C,

\[ \phi(w_n) = \rho_0(p_{n/2}) + \theta(w_n Sq^{n-1} w_n) . \]

Since \( j^* \) is a monomorphism, it suffices to prove that

\[ j^*_2(\tilde{\sigma}(X_n) + [\tilde{\theta}(w_{1, w_{n-1}})] \cdot U) = \rho_0(p_{n/2}) + \theta(w_n Sq^{n-1} w_n) \]

in order to complete the proof. Now

\[ \tilde{\sigma}(X_n) \cdot U = \rho_0(X_n \cdot U) \]

and

\[ j^*_2(\tilde{\sigma}(X_n) \cdot U) = j^*_2 \rho_0(X_n \cdot U) = \rho_0(j(X_n \cdot U)) = \rho_0(X^*_n) = \rho_0(p_{n/2}) \]

since \( j(U) = X_n \). Similarly,

\[ [\tilde{\theta}(w_{1, w_{n-1}})] \cdot U = \theta(w_{1, w_{n-1}} \cdot U) = \theta(w_n Sq^{n-1} U) , \]

hence

\[ j^*_2(\tilde{\theta}(w_{1, w_{n-1}}) \cdot U) = j^*_2 \theta(w_n Sq^{n-1} U) \]
\[ = \theta j^*_2(w_n Sq^{n-1} U) \]
\[ = \theta(w_n Sq^{n-1} w_n) \]

since \( j^*_2(U) = w_n \). This completes the proof.

4. Proof of Theorem 2*. The proof is a routine application of the following two formulas. For the first formula, assume that \( X \) is
a topological space, \( u \in H^m(X, \mathbb{Z}_2) \), \( v \in H^n(X, \mathbb{Z}_2) \), and \( m \equiv n \mod 2 \); then the Pontryagin square of the cup product \( uv \) is given by the following formula:

\[
\mathcal{P}(uv) = (\mathcal{P}u)(\mathcal{P}v) + \theta[(Sq^{m-1}u)vSq^nv + uSq^i u(Sq^{n-1}v)] .
\]

For the case where \( m \) and \( n \) are both odd, this formula is given by Thomas [8], formula (10.5); in case \( m \) and \( n \) are even, the formula is given by Nakaoka [5], Theorem III. Our second formula expresses the Pontryagin square of an odd dimensional cohomology class in terms of more usual cohomology operations. Assume \( u \in H^{2q+1}(X, \mathbb{Z}_2) \); then

\[
\mathcal{P}(u) = \rho_2 \beta Sq^{2q} u + \theta Sq^{2q}Sq^u ,
\]

where \( \beta \) is the Bockstein coboundary operator associated with the exact coefficient sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0 \). In particular, if we apply (4.2) to the computation of \( \mathcal{P}(U_2) \) for an \( m \)-dimensional vector bundle, \( m \) odd, and make use of the formula \( Sq^i U_2 = w_i U_2 \), we obtain the formula

\[
\mathcal{P}(U_2) = [\bar{\beta}_i(W_m) + \bar{\theta}(w_i w_{m-1} + w_i w_{m-2})] \cdot U .
\]

The proof of Theorem II is now a direct application of formula (4.1); one also uses Theorem I in case \( m \) and \( n \) are even, and (4.3) in case \( m \) and \( n \) are odd.

5. Critique of corollary 2. We propose to discuss the following question: Under what conditions does Corollary 2 enable one to prove nonimbedding theorems not provable by more standard and/or elementary methods? We will assume, as in the statement of the corollary, that \( M \) is a compact, connected, orientable manifold of dimension \( q \), that \( \bar{\omega}_{n-1} \neq 0 \), and

\[
q + n \equiv 0 \mod 4 .
\]

We wish to prove that \( M \) can not be imbedded differentiably in Euclidean space of dimension \( q + n \). We may as well assume that \( \bar{\omega}_i = 0 \) for all \( i > n - 1 \), otherwise the proof would be trivial.

We assert that if \( n \) is even, then for any \( x \in H^m(M, \mathbb{Z}_2) \), \( m = (1/2)(q - n) \),

\[
\bar{\omega}_{n-1} x Sq^i x = 0
\]

under the above hypotheses, and hence Corollary 2 can not be applied to prove nonimbedding results.

Proof of assertion. By Lemma 1 of Massey and Peterson [4],
\[ \bar{w}_{n-1} x Sq^i x = Q^{n-1}(x Sq^i x) = Q^{n-1}(x Q^i x) = \sum_{i + k = n - 1} (Q^i x)(Q^k Q^i x). \]

But

\[ Q^i Q^j = \begin{cases} Q^{j+1} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}. \end{cases} \]

Hence

\[ \bar{w}_{n-1} x Sq^i x = \sum_{i + j = n - 1} (Q^i x)(Q^{j+1} x) = \sum_{i + k = n} (Q^i x)(Q^k x). \]

where the summations are restricted to even values of \( j \) and odd values of \( k \) respectively.

If \( n \equiv 0 \mod 4 \), then \( i \) must also be odd in this sum, and the non-zero terms occur in pairs which cancel. If \( n \equiv 2 \mod 4 \), then all terms cancel in pairs except for the term where \( i = k = n/2 \), and one sees that in this case

\[ \bar{w}_{n-1} x Sq^i x = Q^n(x^2) = \bar{w}_{n} \cdot x^2. \]

But by our hypothesis, \( \bar{w}_n = 0 \); hence \( \bar{w}_{n-1} x Sq^i x = 0 \) in this case also.

Thus this method is only of interest in case \( n \) and \( q \) are odd. Perhaps the first case of interest is the case where \( q \) is odd and \( n = q - 2 \). In this case \( m = 1 \), \( x \in H^i(M, \mathbb{Z}_2) \), \( Sq^i x = x^2 \), and

\[ \bar{w}_{n-1} x Sq^i x = Q^{n-1}(x^2) \in H^i(M, \mathbb{Z}_2). \]

The question is, for what values of \( n \) can \( Q^{n-1}(x^2) \) be nonzero? Now it is easy to prove that for any 1-dimensional cohomology class \( x \),

\[ Q(x) = x + x^2 + x^4 + x^8 + \cdots + x^{2k} + \cdots, \]

(see Atiyah and Hirzebruch [1], pp. 168–169), hence

\[ Q(x^2) = (Qx)^2 = x^2 + (x^4 + x^8) + (x^3 + x^9) + \cdots + (x^{2k} + x^{2k+1}) + \cdots. \]

Therefore the only case for which \( Q^{n-1}(x^2) \) can possibly be nonzero is the case \( q = n + 2 = 2k + 1 \), and in this case

\[ Q^{n-1}(x^2) = x^q. \]

Thus the example \( M = q \)-dimensional real projective space is typical for this situation.
The next case of interest would be the case $q$ odd, $n = q - 6$, $m = 3$. The author knows no nontrivial examples to illustrate this case.

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