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**PONTRYAGIN SQUARES IN THE THOM SPACE OF A BUNDLE**

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## PONTRYAGIN SQUARES IN THE THOM SPACE OF A BUNDLE

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The object of this note is to determine the action of the Pontryagin squares in the cohomology of the Thom space of a vector bundle. This computation is then applied to the case of the normal bundle of a manifold imbedded in Euclidean space to give simplified proofs of some theorems of Mahowald.

The first of Mahowald's theorems [3] was inspired by some 1940 results of Whitney [9], who showed that in certain cases the Euler class (with twisted integer coefficients) of the normal bundle of a non-orientable surface imbedded in Euclidean 4-space could be nonzero. This contrasts with the well-known theorem that the Euler class of the normal bundle of an orientable manifold in Euclidean space is always zero.

2. Notation and statement of results. For any space  $X$ , we will use integral cohomology  $H^q(X, \mathbf{Z})$ ; cohomology with integers mod  $n$  as coefficients,  $H^q(X, \mathbf{Z}_n)$ ; cohomology with twisted integer coefficients,  $H^q(X, \mathscr{Z})$  cohomology with twisted integers mod  $n$  coefficients,  $H^q(X, \mathscr{Z}_n)$ ; and rational cohomology,  $H^q(X, Q)$ . In the third and fourth cases the local system of groups which is used for coefficients will be determined by the Stiefel-Whitney class  $w_1 \in H^1(X, \mathbf{Z}_2)$ . Note that for the case  $n=2$ , we have

$$H^q(X, \mathscr{Z}_2) = H^q(X, \mathbf{Z}_2) .$$

since a cyclic group of order 2 admits no nontrivial automorphisms.

Let  $(E, p, B, S^{n-1})$  be an  $(n-1)$ -sphere bundle over the base space  $B$  with structure group  $O(n)$ . We will use the following notation for characteristic classes of such a bundle:

Stiefel-Whitney classes:

$$w_i \in H^i(B, \mathbf{Z}_2) , \quad 1 \leq i \leq n$$

$$W_i \in H^i(B, \mathscr{Z}) , \quad 1 \leq i \leq n, \quad i \text{ odd} .$$

Pontryagin classes:

$$p_i \in H^{4i}(B, \mathbf{Z}) , \quad 1 \leq i \leq n/2 .$$

Euler class:

$$X_n \in H^n(B, \mathscr{Z}) . \quad (\text{If } n \text{ is odd, then } X_n = W_n.)$$

Let  $(A, \pi, B, D^n)$  be the associated  $n$ -dimensional disc bundle; we will call the pair  $(A, E)$  or the single space  $A/E$  the *Thom space* of the bundle. The *Thom class*,  $U \in H^n(A, E, \mathcal{L})$ , has twisted integer coefficients; by taking cup products with  $U$ , we obtain the Thom isomorphism (see Thom [6]).

$$\begin{aligned} H^q(A, \mathcal{L}) &\approx H^{q+n}(A, E, Z) , \\ H^q(A, Z) &\approx H^{q+n}(A, E, \mathcal{L}) , \\ H^q(A, \mathcal{L}_n) &\approx H^{q+n}(A, E, Z_n) , \text{ etc.} \end{aligned}$$

Recall also that the projection  $\pi: A \rightarrow B$  is a deformation retraction, and hence induces isomorphisms of cohomology groups with any coefficients (even local coefficients!). For the sake of convenience, we will often identify the cohomology groups of  $A$  and  $B$  by means of this isomorphism; similarly we will identify the cohomology groups of the pair  $(A, E)$  and the space  $(A/E)$  (except in dimension 0) with ordinary coefficients (the local coefficient systems  $\mathcal{L}$  and  $\mathcal{L}_n$  do not exist in the space  $A/E$ ).

The obvious epimorphism  $\rho_n: Z \rightarrow Z_n$  and monomorphism  $\theta: Z_2 \rightarrow Z_4$  induce homomorphisms of cohomology groups which will be denoted as follows:

$$\begin{aligned} \rho_n: H^q(X, Z) &\longrightarrow H^q(X, Z_n) , \\ \tilde{\rho}_n: H^q(X, \mathcal{L}) &\longrightarrow H^q(X, \mathcal{L}_n) , \\ \theta: H^q(X, Z_2) &\longrightarrow H^q(X, Z_4) , \\ \tilde{\theta}: H^q(X, Z_2) &\longrightarrow H^q(X, \mathcal{L}_4) . \end{aligned}$$

For convenience, we will let  $U_2 = \tilde{\rho}_2(U)$ , the Thom class reduced mod 2.

Our main concern will be the Pontryagin squaring operation,

$$\mathcal{P}: H^q(X, Z_2) \longrightarrow H^{2q}(X, Z_4) .$$

If  $q$  is odd, the Pontryagin square can be expressed in terms of simpler cohomology operations. (see formula (4.2) below); this is not true for  $q$  even. For a list of papers describing this operation, see the first paragraph of [7]. Our main result is the following, which describes the Pontryagin square of the mod 2 Thom class,  $U_2$ .

**THEOREM I.** *Let  $(E, p, B, S^{n-1})$  be a (not necessarily orientable)  $(n - 1)$ -sphere bundle with structure group  $O(n)$ ,  $n$  even. Then*

$$\mathcal{P}(U_2) = [\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1 \cdot w_{n-1})] \cdot U .$$

As a corollary, we obtain the following result which was proved by Whitney [9] in 1940 for the case  $n = 2$ ; the general case is due

to Mahowald, [3, Th. I]:

**COROLLARY 1.** *Let  $M^n$  be a compact, connected, nonorientable  $n$ -manifold ( $n$  even) which is imbedded differentiably in  $R^{2n}$ . Then the twisted Euler class of the normal bundle,  $X_n$ , satisfies the following condition:*

$$\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1}) = 0 .$$

(Here  $\bar{w}_i$  denotes the  $i$ th dual Stiefel-Whitney class of  $M^n$ .)

In particular, if  $\bar{w}_1\bar{w}_{n-1} \neq 0$  (which can only happen if  $n$  is a power of 2, cf. [4]) then  $X_n \neq 0$ . Apparently this is the only general result known about the twisted Euler class of the normal bundle to a non-orientable manifold.

The corollary may be derived from the theorem as follows: Let  $(E, p, B, S^{n-1})$  denote the normal sphere bundle of the imbedding, and  $(A, \pi, B, D^n)$  the associated disc bundle. It is well known that the top homology group of the Thom space,

$$H_{2n}(A/E, Z) = H_{2n}(A, E, Z) ,$$

is infinite cyclic, and the Hurewicz homomorphism

$$\pi_{2n}(A/E) \longrightarrow H_{2n}(A/E)$$

is an epimorphism. From this it follows that  $\langle \mathcal{P}(U_2), x \rangle = 0$  for any  $x \in H_{2n}(A/E, Z)$ , and hence  $\mathcal{P}(U_2) = 0$ . Applying the formula for  $\mathcal{P}((U_2)$  in Theorem I, we obtain the corollary.

Next, we give formulas for the Pontryagin square of an arbitrary mod 2 cohomology class of even degree in the Thom space of a vector bundle.

**THEOREM II.** *Let  $(E, p, B, S^{n-1})$  be an  $(n - 1)$ -sphere bundle with structure group  $O(n)$ , and let  $x \in H^m(B, Z_2)$ ,  $m + n$  even. Then if  $m$  and  $n$  are both even,*

$$\begin{aligned} \mathcal{P}(U_2x) = \{ & \mathcal{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1})] \\ & + \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\} \cdot U \end{aligned}$$

while if  $m$  and  $n$  are odd,

$$\begin{aligned} \mathcal{P}(U_2x) = \{ & \mathcal{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1} + w_1^2w_{n-2})] \\ & + \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\} \cdot U . \end{aligned}$$

As a corollary, we derive a necessary condition due to Mahowald [3] for the imbeddability of an orientable manifold in Euclidean space

of dimension  $4k$  with codimension  $n$ .

**COROLLARY 2.** *Let  $M$  be a compact, connected, orientable manifold of dimension  $q$  which is differentiably imbedded in Euclidean space of dimension  $q + n = 4k$ . Then for any  $x \in H^m(M, Z_2)$ , where  $m = 1/2(q - n)$ , we must have*

$$\bar{w}_{n-1}xSq^1x = 0 .$$

*Proof of corollary.* One applies Theorem II with  $B = M$  and  $(E, p, B, S^{n-1})$  the normal bundle of the imbedding. Since  $M$  is assumed orientable,  $\bar{w}_1 = 0, \bar{w}_n = 0, X_n = 0$ , and  $\bar{W}_n = 0$ . Exactly as in the proof of the previous corollary we know that  $\mathcal{P}(U_2 \cdot x) = 0$  in this case. Thus we conclude that

$$\theta(\bar{w}_{n-1}xSq^1x) = 0$$

for any  $x \in H^m(M, Z_2)$ . Since  $M$  is orientable, the homomorphism

$$\theta: H^q(M, Z_2) \longrightarrow H^q(M, Z_2)$$

is a monomorphism, and therefore we must have  $\bar{w}_{n-1}xSq^1x = 0$ , as desired.

Perhaps the neatest application of this corollary is to prove that  $q$ -dimensional real projective space does not imbed in  $R^{2^{q-2}}$  for  $q = 2^r + 1$ . A discussion of the possibilities of using this theorem to prove non-imbedding results is given in § 5.

**COROLLARY 3.** *Let  $M$  be a compact, connected, nonorientable manifold of dimension  $q$  which is differentiably imbedded in Euclidean space of dimension  $q + n = 4k, q$  and  $n$  even. Then for any element  $x \in H^m(M, Z_2)$ , where  $m = (1/2)(q - n)$ , we must have*

$$\mathcal{P}(x) \cdot [\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1})] + \tilde{\theta}(\bar{w}_{n-1}xSq^1x) = 0 .$$

This is a generalization of Corollary 1, and the proof is similar. Presumably this theorem would enable one to prove in certain cases that  $\tilde{\rho}_4(X_n) \neq 0$ , and hence  $X_n \neq 0$ , but the author knows of no examples to illustrate this possibility. Perhaps the most likely case in which this theorem could be applied is the case where  $n = q - 4, m = 2$ .

**3. Proof of Theorem I.** As is usual in such cases, one only need prove Theorem I in the case of the universal example, where  $B = BO(n), n$  even. Then  $E$  has the same homotopy type as  $BO(n - 1)$ . Consider the following commutative diagram for this universal example:

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\hat{\delta}} & H^*(A, E, Z_k) & \xrightarrow{j^*} & H^*(A, Z_k) & \xrightarrow{i^*} & H^*(E, Z_k) \xrightarrow{\hat{\delta}} \dots \\
 & & \uparrow 1 & \nearrow & \uparrow \pi^* & & \uparrow 2 \\
 & & H^*(A, \mathcal{Z}_k) & & & & \\
 & & \uparrow \pi^* & & & & \\
 \dots & \xrightarrow{\psi} & H^*(B, \mathcal{Z}_k) & \xrightarrow{\mu} & H^*(B, Z_k) & \xrightarrow{p^*} & H^*(E, Z_k) \xrightarrow{\psi} \dots
 \end{array}$$

The top line of this diagram is the mod  $k$  cohomology sequence of the pair  $(A, E)$  while the bottom line is the Gysin sequence of fibration. All vertical arrows are isomorphisms; arrow No. 1 denotes the Thom isomorphism, and arrow No. 2 is the identity. It is well known that in these exact sequences for the case  $k=2$  (i.e., mod2 cohomology), the following statements are true:

- $p^*$  and  $i^*$  are epimorphisms,
- $\mu$  and  $j^*$  are monomorphisms, and
- $\psi$  and  $\hat{\delta}$  are zero.

We assert that these statements are also true in case  $k = 4$ . In order to prove this, it suffices to prove that  $j^*$  is a monomorphism, and for this purpose consider the following commutative diagram:

$$\begin{array}{ccc}
 0 \longrightarrow & H^{q-1}(A, E, Z_2) & \xrightarrow{j_2} & H^{q-1}(A, Z_2) \\
 & \downarrow Sq^1 & & \downarrow Sq^1 \\
 0 \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) \\
 & \downarrow \theta & & \downarrow \theta \\
 \dots \longrightarrow & H^q(A, E, Z_4) & \xrightarrow{j_4} & H^q(A, Z_4) \\
 & \downarrow \eta & & \downarrow \eta \\
 0 \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) .
 \end{array}$$

The vertical lines are exact sequences corresponding to the following short exact sequence of coefficients:

$$0 \longrightarrow Z_2 \xrightarrow{\theta} Z_4 \xrightarrow{\eta} Z_2 \longrightarrow 0 .$$

Let  $x \in H^q(A, E, Z_4)$  and assume that  $j^*(x) \equiv j_4(x) = 0$ . Therefore

$$j_2 \eta(x) = \eta j_4(x) = 0$$

and since  $j_2$  is a monomorphism,  $\eta(x) = 0$ . By exactness, there exists an element  $y \in H^q(A, E, Z_2)$  such that

$$\theta(y) = x .$$

Since  $\theta j_2(y) = 0$ , there exists an element  $z \in H^{q-1}(A, Z_2)$  such that

$$Sq^1(z) = j_2(y) .$$

We wish to show that  $z$  can be chosen so that  $z \in \text{image } j_2$ . For this purpose, recall that we are identifying  $H^*(A, Z_2)$  with  $H^*(B, Z_2) = Z_2[w_1, w_2, \dots, w_n]$ ; using this identification, the image of  $j^*$  is the ideal generated by  $w_n$ . We may split  $H^*(A, Z_2)$  into the (vector space) direct sum of this ideal and a supplementary subspace as follows: one subspace is spanned by all monomials which have  $w_n$  as a factor, the other subspace is spanned by those monomials which do not have  $w_n$  as a factor. It is readily verified that the homomorphism

$$Sq^1: H^*(A, Z_2) \longrightarrow H^*(A, Z_2)$$

maps each of these summands into itself (this depends on the fact that  $n$  is even). Since  $j_2(y)$  belong to this ideal generated by  $w_n$ , we can choose  $z$  so it also belongs to this ideal. Therefore  $z = j_2(u)$  for some element  $u \in H^{q-1}(A, E, Z_2)$ . It follows that

$$j_2(y - Sq^1u) = 0 .$$

Since  $j_2$  is a monomorphism,  $y = Sq^1u$ , and

$$x = \theta(y) = \theta Sq^1u = 0$$

as asserted.

Next, let  $X_n \in H^n(BO(n), \mathcal{Z})$  denote the Euler class ( $n$  even). We assert that

$$X_n^2 = p_{n/2} \in H^{2n}(BO(n), Z) .$$

To prove this, we make use of the fact that all torsion in  $H^*(BO(n), Z)$  is of order 2 (cf. Borel and Hirzebruch, [2]). Hence it suffices to prove that the following two equations:

$$\begin{aligned} \rho_2(X_n^2) &= \rho_2(p_{n/2}) \text{ and} \\ \rho_0(X_n^2) &= \rho_0(p_{n/2}) , \end{aligned}$$

where  $\rho_0$  is the homomorphism of cohomology groups induced by the coefficient map  $Z \rightarrow Q$ .

As to the first equation, it is well known that  $\rho_2(X_n) = w_n$  and  $\rho_2(p_i) = w_{2i}^2$ , hence

$$\rho_2(X_n^2) = w_n^2 = \rho_2(p_{n/2}) .$$

To prove the second equation, consider the following commutative diagram.

$$\begin{array}{ccc} H^{2n}(BO(n), Z) & \xrightarrow{f^*} & H^{2n}(BSO(n), Z) \\ \downarrow \rho_0 & & \downarrow \rho_0 \\ H^{2n}(BO(n), Q) & \xrightarrow{f^*} & H^{2n}(BSO(n), Q) . \end{array}$$

Here  $f: BSO(n) \rightarrow BO(n)$  is the 2-fold covering induced by the inclusion of  $SO(n)$  in  $O(n)$ . It is well known that  $\rho_0 f^*(X_n^2) = \rho_0 f^*(p_{n/2})$  and that  $f^*$  is a monomorphism on rational cohomology (see Borel and Hirzebruch [2]). Hence  $\rho_0(X_n^2) = \rho_0(p_{n/2})$  as required.

With these preliminaries out of the way, we will now prove Theorem I by consideration of the following commutative diagram:

$$\begin{array}{ccc} H^n(A, E, Z_2) & \xrightarrow{j_2} & H^n(A, Z_2) \\ \downarrow \varphi & & \downarrow \varphi \\ H^{2n}(A, E, Z_4) & \xrightarrow{j_4} & H^{2n}(A, Z_4) . \end{array}$$

It is well known that  $j_2(U_2) = w_n$ , and according to Thomas [8], Theorem C,

$$\mathcal{P}(w_n) = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n) .$$

Since  $j_4$  is a monomorphism, it suffices to prove that

$$j_4\{\tilde{\rho}_4(X_n) + [\tilde{\theta}(w_1 w_{n-1})] \cdot U\} = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n)$$

in order to complete the proof. Now

$$\tilde{\rho}_4(X_n) \cdot U = \rho_4(X_n \cdot U)$$

and

$$\begin{aligned} j_4\{\tilde{\rho}_4(X_n) \cdot U\} &= j_4 \rho_4(X_n \cdot U) = \rho_4 j(X_n \cdot U) \\ &= \rho_4(X_n^2) = \rho_4(p_{n/2}) \end{aligned}$$

since  $j(U) = X_n$ . Similarly,

$$[\tilde{\theta}(w_1 w_{n-1})] \cdot U = \theta(w_1 w_{n-1} \cdot U_2) = \theta(w_1 Sq^{n-1} U_2) ,$$

hence

$$\begin{aligned} j_4\{\tilde{\theta}(w_1 w_{n-1}) \cdot U\} &= j_4 \theta(w_1 Sq^{n-1} U_2) \\ &= \theta j_2(w_1 Sq^{n-1} U_2) \\ &= \theta(w_1 Sq^{n-1} w_n) \end{aligned}$$

since  $j_2(U_2) = w_n$ . This completes the proof.

4. **Proof of Theorem 2.** The proof is a routine application of the following two formulas. For the first formula, assume that  $X$  is



a topological space,  $u \in H^m(X, Z_2)$ ,  $v \in H^n(X, Z_2)$ , and  $m \equiv n \pmod 2$ ; then the Pontryagin square of the cup product  $uv$  is given by the following formula:

$$(4.1) \quad \mathcal{P}(uv) = (\mathcal{P}u)(\mathcal{P}v) + \theta[(Sq^{m-1}u)vSq^1v + uSq^1u(Sq^{n-1}v)] .$$

For the case where  $m$  and  $n$  are both odd, this formula is given by Thomas [8], formula (10.5); in case  $m$  and  $n$  are even, the formula is given by Nakaoka [5], Theorem III. Our second formula expresses the Pontryagin square of an odd dimensional cohomology class in terms of more usual cohomology operations. Assume  $u \in H^{2q+1}(X, Z_2)$ ; then

$$(4.2) \quad \mathcal{P}(u) = \rho_4\beta Sq^{2q}u + \theta Sq^{2q}Sq^1u ,$$

where  $\beta$  is the Bockstein coboundary operator associated with the exact coefficient sequence  $0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0$ . In particular, if we apply (4.2) to the computation of  $\mathcal{P}(U_2)$  for an  $m$ -dimensional vector bundle,  $m$  odd, and make use of the formula  $Sq^i U_2 = w_i U_2$ , we obtain the formula

$$(4.3) \quad \mathcal{P}(U_2) = [\tilde{\rho}_4(W_m) + \tilde{\theta}(w_1 w_{m-1} + w_1^2 w_{m-2})] \cdot U .$$

The proof of Theorem II is now a direct application of formula (4.1); one also uses Theorem I in case  $m$  and  $n$  are even, and (4.3) in case  $m$  and  $n$  are odd.

5. Critique of corollary 2. We propose to discuss the following question: Under what conditions does Corollary 2 enable one to prove nonimbedding theorems not provable by more standard and/or elementary methods? We will assume, as in the statement of the corollary, that  $M$  is a compact, connected, orientable manifold of dimension  $q$ , that  $\bar{w}_{n-1} \neq 0$ , and

$$q + n \equiv 0 \pmod 4 .$$

We wish to prove that  $M$  can not be imbedded differentiably in Euclidean space of dimension  $q + n$ . We may as well assume that  $\bar{w}_i = 0$  for all  $i > n - 1$ , otherwise the proof would be trivial.

We assert that if  $n$  is even, then for any  $x \in H^m(M, Z_2)$ ,  $m = (1/2)(q - n)$ ,

$$\bar{w}_{n-1} x Sq^1 x = 0$$

under the above hypotheses, and hence Corollary 2 can not be applied to prove nonimbedding results.

*Proof of assertion.* By Lemma 1 of Massey and Peterson [4],

$$\begin{aligned} \bar{w}_{n-1}xSq^1x &= Q^{n-1}(xSq^1x) \\ &= Q^{n-1}(xQ^1x) \\ &= \sum_{i+k=n-1} (Q^i x)(Q^k Q^1 x) . \end{aligned}$$

But

$$Q^j Q^1 = \begin{cases} Q^{j+1} & \text{if } j \text{ is even ,} \\ 0 & \text{if } j \text{ is odd .} \end{cases}$$

Hence

$$\begin{aligned} \bar{w}_{n-1}xSq^1x &= \sum_{i+j=n-1} (Q^i x)(Q^{j+1} x) \\ &= \sum_{i+k=n} (Q^i x)(Q^k x) . \end{aligned}$$

where the summations are restricted to even values of  $j$  and odd values of  $k$  respectively.

If  $n \equiv 0 \pmod 4$ , then  $i$  must also be odd in this sum, and the non-zero terms occur in pairs which cancel. If  $n \equiv 2 \pmod 4$ , then all terms cancel in pairs except for the term where  $i = k = n/2$ , and one sees that in this case

$$\bar{w}_{n-1}xSq^1x = Q^n(x^2) = \bar{w}_n \cdot x^2 .$$

But by our hypothesis,  $\bar{w}_n = 0$ ; hence  $\bar{w}_{n-1}xSq^1x = 0$  in this case also.

Thus this method is only of interest in case  $n$  and  $q$  are odd. Perhaps the first case of interest is the case where  $q$  is odd and  $n = q - 2$ . In this case  $m = 1$ ,  $x \in H^1(M, Z_2)$ ,  $Sq^1x = x^2$ , and

$$\bar{w}_{n-1}xSq^1x = Q^{n-1}(x^3) \in H^q(M, Z_2) .$$

The question is, for what values of  $n$  can  $Q^{n-1}(x^3)$  be nonzero? Now it is easy to prove that for any 1-dimensional cohomology class  $x$ ,

$$Q(x) = x + x^2 + x^4 + x^8 + \dots + x^{2^k} + \dots ,$$

(see Atiyah and Hirzebruch [1], pp. 168-169), hence

$$\begin{aligned} Q(x^3) &= (Qx)^3 = x^3 + (x^4 + x^5) + (x^8 + x^9) \\ &\quad + \dots + (x^{2^k} + x^{2^k+1}) + \dots . \end{aligned}$$

Therefore the only case for which  $Q^{n-1}(x^3)$  can possibly be nonzero is the case  $q = n + 2 = 2^k + 1$ , and in this case

$$Q^{n-1}(x^3) = x^q .$$

Thus the example  $M = q$ -dimensional real projective space is typical for this situation.

The next case of interest would be the case  $q$  odd,  $n = q - 6$ ,  $m = 3$ . The author knows no nontrivial examples to illustrate this case.

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James Burton Ax, <i>Injective endomorphisms of varieties and schemes</i> . . . . .	1
Richard Hindman Bouldin, <i>A generalization of the Weinstein-Aronszajn formula</i> . . . . .	9
John Martin Chadam, <i>The asymptotic behavior of the Klein-Gordon equation with external potential. II</i> . . . . .	19
Rina Hadass, <i>On the zeros of the solutions of the differential equation <math>y^{(n)}(z) + p(z) = 0</math></i> . . . . .	33
John Sollion Hsia, <i>Integral equivalence of vectors over local modular lattices. II</i> . . . . .	47
Robert Hughes, <i>Boundary behavior of random valued heat polynomial expansions</i> . . . . .	61
Surender Kumar Jain, Saad H. Mohamed and Surjeet Singh, <i>Rings in which every right ideal is quasi-injective</i> . . . . .	73
T. Kawata, <i>On the inversion formula for the characteristic function</i> . . . . .	81
Erwin Kleinfeld, <i>On right alternative rings without proper right ideals</i> . . . . .	87
Robert Leroy Kruse and David Thomas Price, <i>On the subring structure of finite nilpotent rings</i> . . . . .	103
Marvin David Marcus and Stephen J. Pierce, <i>Symmetric positive definite multilinear functionals with a given automorphism</i> . . . . .	119
William Schumacher Massey, <i>Pontryagin squares in the Thom space of a bundle</i> . . . . .	133
William Schumacher Massey, <i>Proof of a conjecture of Whitney</i> . . . . .	143
John William Neuberger, <i>Existence of a spectrum for nonlinear transformations</i> . . . . .	157
Stephen E. Newman, <i>Measure algebras on idempotent semigroups</i> . . . . .	161
K. Chandrasekhara Rao, <i>Matrix transformations of some sequence spaces</i> . . . . .	171
Robert Bruce Schneider, <i>Some theorems in Fourier analysis on symmetric sets</i> . . . . .	175
Ulrich F. K. Schoenwaelder, <i>Centralizers of abelian, normal subgroups of hypercyclic groups</i> . . . . .	197
Jerrold Norman Siegel, <i>G-spaces, H-spaces and W-spaces</i> . . . . .	209
Robert Irving Soare, <i>Cohesive sets and recursively enumerable Dedekind cuts</i> . . . . .	215
Kwok-Wai Tam, <i>Isometries of certain function spaces</i> . . . . .	233
Awadhesh Kumar Tiwary, <i>Injective hulls of semi-simple modules over regular rings</i> . . . . .	247
Eldon Jon Vought, <i>Concerning continua not separated by any nonaposyndetic subcontinuum</i> . . . . .	257
Robert Breckenridge Warfield, Jr., <i>Decompositions of injective modules</i> . . . . .	263