EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

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Denote by $S$ a complex (nondegenerate) Banach space. Suppose that $T$ is a transformation from a subset of $S$ to $S$. A complex number $\lambda$ is said to be in the resolvent of $T$ if $(\lambda I - T)^{-1}$ exists, has domain $S$ and is Fréchet differentiable (i.e., if $p$ is in $S$ there is a unique continuous linear transformation $F = [(\lambda I - T)^{-1}](p)$ from $S$ to $S$ so that

$$\lim_{q \to p} |q - p|^{-1} |(\lambda I - T)^{-1}q - (\lambda I - T)^{-1}p - F(q - p)|| = 0$$

and locally Lipschitzian everywhere on $S$. A complex number is said to be in the spectrum of $T$ if it is not in the resolvent of $T$.

Suppose in addition that the domain of $T$ contains an open subset of $S$ on which $T$ is Lipschitzian.

**Theorem.** $T$ has a (nonempty) spectrum.

If $T$ is a continuous linear transformation from $S$ to $S$, then the notion of resolvent and spectrum given here coincides with the usual one ([11], p. 209, for example). Such a transformation $T$ is, of course, Lipschitzian on all of $S$ and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by $C$.

**Lemma.** Suppose that $d > 0$, $p$ is in $S$, $Q$ is a transformation from a subset of $S$ to $S$, $D$ is an open set containing $p$ which is a subset of the domain $Q$, $Q$ is Lipschitzian on $D$ and $(I-cQ)^{-1}$ exists and has domain $S$ if $c$ is in $C$ and $|c| < d$. Then,

$$\lim_{c \to 0} (I - cQ)^{-1}p = p$$

**Proof.** Denote by $M$ a positive number so that $||Qr - Qs|| \leq M||r - s||$ if $r$ and $s$ are in $D$. Suppose $\epsilon > 0$. Denote by $\delta$ a number so that $0 < \delta < \min(\epsilon, 1/2)$ and $\{q \in S: ||q - p|| \leq \delta\}$ is a subset of $D$. Denote by $\delta'$ a positive number so that $\delta'(\max(M, ||QP||)) < \delta/2$. Denote by $c$ a member of $C$ so that $|c| < \min(\delta', d)$. Denote $(I-cQ)^{-1}p$ by $q$, denote $p$ by $q_0$ and $p + cQQ_{n-1}$ by $q_n$, $n = 1, 2, \ldots$.

Then, $||q_n - q_0|| = ||p + cQQ_0 - q_0|| = |c| ||QQ_0|| < \delta/2$. Suppose that $k$ is a positive integer so that

$$||q_m - q_{m-1}|| < (\delta/2)^m, m = 1, 2, \ldots, k.$$
Then \( \| q_n - p \| \leq \sum_{j=0}^{n-1} \| q_{j+1} - q_j \| \leq \sum_{j=0}^{n-1} (\delta/2)^{j+1} < \delta, \) \( m = 0, 1, \ldots, k \)
and hence
\[
\| q_{k+1} - q_k \| = \| cQq_k - cQq_{k-1} \|
\leq |c| M \| q_k - q_{k-1} \|
\leq |c| M(\delta/2)^k \leq (\delta/2)^{k+1}.
\]
Hence \( \| q_n - q_{n-1} \| \leq (\delta/2)^n, \) \( n = 1, 2, \ldots \) and therefore \( q_1, q_2, \ldots \) converges to a point \( r \) of \( S. \) Note that \( \| q_{n+1} - p \| \leq \sum_{j=0}^n (\delta/2)^{j+1} < \delta, \) \( n = 1, 2, \ldots \) so that \( \| r - p \| \leq \delta \) and hence \( r \) is in \( D. \) But \( \| r - (p + cQr) \| = \| r - q_{n+1} \| + (p + cQq_n) - (p + cQr) \| \leq \| r - q_{n+1} \| + |c| \| Qq_n - Qr \| \leq \| r - q_{n+1} \| + |c| M \| q_n - r \| \to 0 \) as \( n \to \infty. \) Hence \( r = p + cQr, \) i.e., \( (I - cQ)r = p, \) i.e., \( r = (I - cQ)^{-1}p = q. \) Hence, \( \| (I - cQ)^{-1}p - p \| \leq \delta < \varepsilon. \) This proves the lemma.

**Proof of theorem.** Suppose the statement of the theorem is false. Then \( T \) has an inverse since if not, \( 0 \) would be in the spectrum of \( T. \)
Denote by \( D \) an open set on which \( T \) is defined and is Lipschitzian.
Denote by \( p \) a point of \( D \) different from \( -T(0). \)
Define \( f(\lambda) \) to be \( (\lambda I - T)^{-1}p \) for all \( \lambda \) in \( C. \) Suppose \( b \) is in \( C. \)
If \( q \) is in \( S \) and different from \( p \) denote
\[
\frac{1}{\| q - p \|} \{ \| (bI - T)^{-1}q - (bI - T)^{-1}p \| - \| (bI - T)^{-1}p \| \}
\]
by \( L(q). \) Denote by \( L(p) \) the zero element of \( S \) and note that \( \lim_{q \to p} L(q) = L(p) \) since \( (bI - T)^{-1} \) is Fréchet differentiable at \( p. \) Denote \( (bI - T)^{-1} \) by \( Q. \) If \( \lambda \) is in \( C, \) then
\[
(\lambda I - T) = [I - (b - \lambda)(bI - T)^{-1}](bI - T)
\]
and, since both \( (\lambda I - T)^{-1} \) and \( (bI - T)^{-1} \) exist and have domain \( S, \) it follows that \( [I - (b - \lambda)(bI - T)^{-1}]^{-1} = [I - (b - \lambda)Q]^{-1} \) has the same properties and \( (\lambda I - T)^{-1} = Q[I - (b - \lambda)Q]^{-1}. \)
Hence, if \( \lambda \) is in \( C, \)
\[
f(\lambda) - f(b) = Q[I - (b - \lambda)Q]^{-1}p - Qp
\]
\[
= Q'(p)[I - (b - \lambda)Q]^{-1}p - (bI - T)^{-1}p
\]
\[
+ \| (I - (b - \lambda)Q)^{-1}p - p \| L([I - (b - \lambda)Q]^{-1}p) .
\]
But \( [I - (b - \lambda)Q]^{-1}p - p = (b - \lambda)Q[I - (b - \lambda)Q]^{-1}p \) so
\[
(b - \lambda)^{-1}[f(\lambda) - f(b)]
\]
\[
= -Q'(p)Q[I - (b - \lambda)Q]^{-1}p
\]
\[
+ (|b - \lambda||/\lambda - b)| Q[I - (b - \lambda)Q]^{-1}p ||
\]
\[
\times L([I - (b - \lambda)Q]^{-1}p) \to Q'(p)Qp
\]
as \( \lambda \to b \) since \( \lim_{\lambda \to b} [I - (b - \lambda)Q]^{-1}p = p. \) Hence,
Now \( \lim_{c \to 0} (I - cT)^{-1}p = p \). Denote by \( \delta \) a positive number so that if \( |c| \leq \delta \), then \( \| (I - cT)^{-1}p \| \leq \| p \| + 1 \). Then if \( \lambda \) is in \( C \) and \( |\lambda| \geq 1/\delta \), \( \| f(\lambda) \| = \| (\lambda I - T)^{-1}p \| = |1/\lambda| \| (I - (1/\lambda)T)^{-1}p \| \leq \delta(\| p \| + 1) \). Hence \( f \) is bounded. So, by Liouville’s theorem ([1], p. 129, for example), \( f \) is constant, i.e., there is a point \( q \) in \( S \) such that if \( \lambda \) is in \( C \), \( (\lambda I - T)^{-1}p = f(\lambda) = q \), and so \( \lambda q = p + Tq \). Hence it must be that \( q = 0 \), i.e., \( p = -T(0) \), a contradiction. This establishes the theorem.

The author considers it likely that the statement of the theorem is true if the condition (in the definition of resolvent) that \( (\lambda I - T)^{-1} \) be locally Lipschitzian is dropped.

**Reference**


Received December 12, 1968. The author is an Alfred P. Sloan Research Fellow.

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