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EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

JOHN WILLIAM NEUBERGER

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J. W. NEUBERGER

Denote by S a complex (nondegenerate) Banach space. Suppose that T is a transformation from a subset of S to S . A complex number λ is said to be in the resolvent of T if $(\lambda I - T)^{-1}$ exists, has domain S and is Fréchet differentiable (i.e., if p is in S there is a unique continuous linear transformation $F = [(\lambda I - T)^{-1}]'(p)$ from S to S so that

$$\lim_{q \rightarrow p} \|q - p\|^{-1} \|(\lambda I - T)^{-1}q - (\lambda I - T)^{-1}p - F(q - p)\| = 0$$

and locally Lipschitzean everywhere on S . A complex number is said to be in the spectrum of T if it is not in the resolvent of T .

Suppose in addition that the domain of T contains an open subset of S on which T is Lipschitzean.

THEOREM. T has a (nonempty) spectrum.

If T is a continuous linear transformation from S to S , then the notion of resolvent and spectrum given here coincides with the usual one ([1], p. 209, for example). Such a transformation T is, of course, Lipschitzean on all of S and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by C .

LEMMA. Suppose that $d > 0$, p is in S , Q is a transformation from a subset of S to S , D is an open set containing p which is a subset of the domain Q , Q is Lipschitzean on D and $(I - cQ)^{-1}$ exists and has domain S if c is in C and $|c| < d$. Then,

$$\lim_{c \rightarrow 0} (I - cQ)^{-1}p = p.$$

Proof. Denote by M a positive number so that $\|Qr - Qs\| \leq M \|r - s\|$ if r and s are in D . Suppose $\varepsilon > 0$. Denote by δ a number so that $0 < \delta < \min(\varepsilon, 1/2)$ and $\{q \in S: \|q - p\| \leq \delta\}$ is a subset of D . Denote by δ' a positive number so that $\delta'(\max(M, \|Qp\|)) < \delta/2$. Denote by c a member of C so that $|c| < \min(\delta', d)$. Denote $(I - cQ)^{-1}p$ by q , denote p by q_0 and $p + cQq_{n-1}$ by q_n , $n = 1, 2, \dots$.

Then, $\|q_1 - q_0\| = \|p + cQq_0 - q_0\| = |c| \|Qq_0\| < \delta/2$. Suppose that k is a positive integer so that

$$\|q_m - q_{m-1}\| < (\delta/2)^m, \quad m = 1, 2, \dots, k.$$

Then $\|q_m - p\| \leq \sum_{j=0}^{m-1} \|q_{j+1} - q_j\| \leq \sum_{j=0}^{m-1} (\delta/2)^{j+1} < \delta$, $m = 0, 1, \dots, k$ and hence

$$\begin{aligned} \|q_{k+1} - q_k\| &= \|cQq_k - cQq_{k-1}\| \\ &\leq |c| M \|q_k - q_{k-1}\| \\ &\leq |c| M (\delta/2)^k \leq (\delta/2)^{k+1}. \end{aligned}$$

Hence $\|q_n - q_{n-1}\| \leq (\delta/2)^n$, $n = 1, 2, \dots$ and therefore q_1, q_2, \dots converges to a point r of S . Note that $\|q_{n+1} - p\| \leq \sum_{j=0}^n (\delta/2)^{j+1} < \delta$, $n = 1, 2, \dots$ so that $\|r - p\| \leq \delta$ and hence r is in D . But $\|r - (p + cQr)\| = \|(r - q_{n+1}) + (p + cQq_n) - (p + cQr)\| \leq \|r - q_{n+1}\| + |c| \|Qq_n - Qr\| \leq \|r - q_{n+1}\| + |c| M \|q_n - r\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $r = p + cQr$, i.e., $(I - cQ)r = p$, i.e., $r = (I - cQ)^{-1}p = q$. Hence, $\|(I - cQ)^{-1}p - p\| \leq \delta < \varepsilon$. This proves the lemma.

Proof of theorem. Suppose the statement of the theorem is false. Then T has an inverse since if not, 0 would be in the spectrum of T . Denote by D an open set on which T is defined and is Lipschitzian. Denote by p a point of D different from $-T(0)$.

Define $f(\lambda)$ to be $(\lambda I - T)^{-1}p$ for all λ in C . Suppose b is in C . If q is in S and different from p denote

$$(1/\|q - p\|)[(bI - T)^{-1}q - (bI - T)^{-1}p] - [(bI - T)^{-1}]'(p)(q - p)$$

by $L(q)$. Denote by $L(p)$ the zero element of S and note that $\lim_{p \rightarrow q} L(q) = L(p)$ since $(bI - T)^{-1}$ is Fréchet differentiable at p . Denote $(bI - T)^{-1}$ by Q . If λ is in C , then

$$(\lambda I - T) = [I - (b - \lambda)(bI - T)^{-1}](bI - T)$$

and, since both $(\lambda I - T)^{-1}$ and $(bI - T)^{-1}$ exist and have domain S , it follows that $[I - (b - \lambda)(bI - T)^{-1}]^{-1} = [I - (b - \lambda)Q]^{-1}$ has the same properties and $(\lambda I - T)^{-1} = Q[I - (b - \lambda)Q]^{-1}$.

Hence, if λ is in C ,

$$\begin{aligned} f(\lambda) - f(b) &= Q[I - (b - \lambda)Q]^{-1}p - Qp \\ &= Q'(p)[[I - (b - \lambda)Q]^{-1}p - p] \\ &\quad + \|[I - (b - \lambda)Q]^{-1}p - p\| L([I - (b - \lambda)Q]^{-1}p). \end{aligned}$$

But $[I - (b - \lambda)Q]^{-1}p - p = (b - \lambda)Q[I - (b - \lambda)Q]^{-1}p$ so

$$\begin{aligned} (\lambda - b)^{-1}[f(\lambda) - f(b)] &= -Q'(p)Q[I - (b - \lambda)Q]^{-1}p \\ &\quad + (|b - \lambda|/|\lambda - b|) \|Q[I - (b - \lambda)Q]^{-1}p\| \\ &\quad \times L([I - (b - \lambda)Q]^{-1}p) \rightarrow -Q'(p)Qp \end{aligned}$$

as $\lambda \rightarrow b$ since $\lim_{\lambda \rightarrow b} [I - (b - \lambda)Q]^{-1}p = p$. Hence,

$$f'(b) = -[(bI - T)^{-1}]'(p)(bI - T)^{-1}p.$$

Now $\lim_{c \rightarrow 0} (I - cT)^{-1}p = p$. Denote by δ a positive number so that if $|c| \leq \delta$, then $\|(I - cT)^{-1}p\| \leq \|p\| + 1$. Then if λ is in C and $|\lambda| \geq 1/\delta$, $\|f(\lambda)\| = \|(\lambda I - T)^{-1}p\| = |1/\lambda| \|(I - (1/\lambda)T)^{-1}p\| \leq \delta(\|p\| + 1)$. Hence f is bounded. So, by Liouville's theorem ([1], p. 129, for example), f is constant, i.e., there is a point q in S such that if λ is in C , $(\lambda I - T)^{-1}p = f(\lambda) = q$, and so $\lambda q = p + Tq$. Hence it must be that $q = 0$, i.e., $p = -T(0)$, a contradiction. This establishes the theorem.

The author considers it likely that the statement of the theorem is true if the condition (in the definition of resolvent) that $(\lambda I - T)^{-1}$ be locally Lipschitzean is dropped.

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