EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

John William Neuberger
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Denote by \( S \) a complex (nondegenerate) Banach space. Suppose that \( T \) is a transformation from a subset of \( S \) to \( S \). A complex number \( \lambda \) is said to be in the resolvent of \( T \) if \((\lambda I - T)^{-1}\) exists, has domain \( S \) and is Fréchet differentiable (i.e., if \( p \) is in \( S \) there is a unique continuous linear transformation \( F = [(\lambda I - T)^{-1}]'(p) \) from \( S \) to \( S \) so that
\[
\lim_{q \to p} ||q - p||^{-1} ||(\lambda I - T)^{-1}q - (\lambda I - T)^{-1}p - F(q - p)|| = 0
\]
and locally Lipschitzian everywhere on \( S \). A complex number is said to be in the spectrum of \( T \) if it is not in the resolvent of \( T \).

Suppose in addition that the domain of \( T \) contains an open subset of \( S \) on which \( T \) is Lipschitzian.

Theorem. \( T \) has a (nonempty) spectrum.

If \( T \) is a continuous linear transformation from \( S \) to \( S \), then the notion of resolvent and spectrum given here coincides with the usual one ([1], p. 209, for example). Such a transformation \( T \) is, of course, Lipschitzian on all of \( S \) and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by \( C \).

Lemma. Suppose that \( d > 0 \), \( p \) is in \( S \), \( Q \) is a transformation from a subset of \( S \) to \( S \), \( D \) is an open set containing \( p \) which is a subset of the domain \( Q \), \( Q \) is Lipschitzian on \( D \) and \((I - cQ)^{-1}\) exists and has domain \( S \) if \( c \) is in \( C \) and \( |c| < d \). Then,
\[
\lim_{c \to 0} (I - cQ)^{-1}p = p.
\]

Proof. Denote by \( M \) a positive number so that \( ||Qr - Qs|| \leq M ||r - s|| \) if \( r \) and \( s \) are in \( D \). Suppose \( \varepsilon > 0 \). Denote by \( \delta \) a number so that \( 0 < \delta < \min (\varepsilon, 1/2) \) and \( \{q \in S: ||q - p|| \leq \delta \} \) is a subset of \( D \). Denote by \( \delta' \) a positive number so that \( \delta'(\max(M, ||Qp||)) < \delta/2 \). Denote by \( c \) a member of \( C \) so that \( |c| < \min(\delta', \delta) \). Denote \((I - cQ)^{-1}p\) by \( q \), denote \( p \) by \( q_{0} \) and \( p + cQq_{n-1} \) by \( q_{n} \), \( n = 1, 2, \ldots \).

Then,
\[
||q_{1} - q_{0}|| = ||p + cQq_{0} - q_{0}|| = |c|||Qq_{0}|| < \delta/2.
\]
Suppose that \( k \) is a positive integer so that
\[
||q_{m} - q_{m-1}|| < (\delta/2)^{m}, m = 1, 2, \ldots, k.
\]
Then \( \| q_m - p \| \leq \sum_{j=0}^{m-1} \| q_{j+1} - q_j \| \leq \sum_{j=0}^{m-1} (\delta/2)^{j+1} < \delta, \ m = 0, 1, \ldots, k \) and hence

\[
\| q_{k+1} - q_k \| = \| cQq_k - cQq_{k-1} \|
\leq |c| M \| q_k - q_{k-1} \|
\leq |c| M(\delta/2)^k \leq (\delta/2)^{k+1}.
\]

Hence \( \| q_n - q_{n-1} \| \leq (\delta/2)^n, n = 1, 2, \ldots \) and therefore \( q_1, q_2, \ldots \) converges to a point \( r \) of \( S \). Note that \( \| q_{n+1} - p \| \leq \sum_{j=0}^{n} (\delta/2)^{j+1} < \delta, n = 1, 2, \ldots \) so that \( \| r - p \| \leq \delta \) and hence \( r \) is in \( D \). But \( \| r - (p + cQr) \| = \| (r - q_{n+1}) + (p + cQq_n) - (p + cQr) \| \leq \| r - q_{n+1} \| + |c| \| Qq_n - Qr \| \leq \| r - q_{n+1} \| + |c| M \| q_n - r \| \to 0 \) as \( n \to \infty \). Hence \( r = p + cQr \), i.e., \( (I - cQ)r = p \), i.e., \( r = (I - cQ)^{-1}p = q \). Hence, \( \| (I - cQ)^{-1}p - p \| \leq \delta < \varepsilon \). This proves the lemma.


Proof of theorem. Suppose the statement of the theorem is false. Then \( T \) has an inverse since if not, \( 0 \) would be in the spectrum of \( T \). Denote by \( D \) an open set on which \( T \) is defined and is Lipschitzian. Denote by \( p \) a point of \( D \) different from \(- T(0)\).

Define \( f(\lambda) \) to be \( (\lambda I - T)^{-1}p \) for all \( \lambda \) in \( C \). Suppose \( b \) is in \( C \). If \( q \) is in \( S \) and different from \( p \) denote

\[
(1/\| q - p \|)[(bI - T)^{-1}q - (bI - T)^{-1}p] - [(bI - T)^{-1}]'(p)(q - p)
\]

by \( L(q) \). Denote by \( L(p) \) the zero element of \( S \) and note that \( \lim_{q \to p} L(q) = L(p) \) since \( (bI - T)^{-1} \) is Fréchet differentiable at \( p \). Denote \( (bI - T)^{-1} \) by \( Q \). If \( \lambda \) is in \( C \), then

\[
(\lambda I - T) = [I - (b - \lambda)(bI - T)^{-1}](bI - T)
\]

and, since both \( (\lambda I - T)^{-1} \) and \( (bI - T)^{-1} \) exist and have domain \( S \), it follows that \( [I - (b - \lambda)(bI - T)^{-1}]^{-1} = [I - (b - \lambda)Q]^{-1} \) has the same properties and \( (\lambda I - T)^{-1} = Q[I - (b - \lambda)Q]^{-1} \).

Hence, if \( \lambda \) is in \( C \),

\[
f(\lambda) - f(b) = Q[I - (b - \lambda)Q]^{-1}p - Qp = Q'(p)[I - (b - \lambda)Q]^{-1}p - p] + \| [I - (b - \lambda)Q]^{-1}p - p \| L([I - (b - \lambda)Q]^{-1}p) .
\]

But \( [I - (b - \lambda)Q]^{-1}p - p = (b - \lambda)Q[I - (b - \lambda)Q]^{-1}p \) so

\[
(\lambda - b)^{-1}[f(\lambda) - f(b)] = -Q'(p)Q[I - (b - \lambda)Q]^{-1}p
+ ((b - \lambda)/(\lambda - b)) \| Q[I - (b - \lambda)Q]^{-1}p \|
\times L([I - (b - \lambda)Q]^{-1}p) \to -Q'(p)Qp
\]

as \( \lambda \to b \) since \( \lim_{\lambda \to b}[I - (b - \lambda)Q]^{-1}p = p \). Hence,
\[ f'(b) = -[(bI - T)^{-1}]'(p)(bI - T)^{-1}p. \]

Now \(\lim_{c \to 0} (I - cT)^{-1}p = p\). Denote by \(\delta\) a positive number so that if \(|c| \leq \delta\), then \(\|(I - cT)^{-1}p\| \leq \|p\| + 1\). Then if \(\lambda\) is in \(C\) and \(|\lambda| \geq 1/\delta\), \(\|f(\lambda)\| = \|(\lambda I - T)^{-1}p\| = \|1/\lambda\| \|(I - (1/\lambda)T)^{-1}p\| \leq \delta(\|p\| + 1)\). Hence \(f\) is bounded. So, by Liouville's theorem ([1], p. 129, for example), \(f\) is constant, i.e., there is a point \(q\) in \(S\) such that if \(\lambda\) is in \(C\), \((\lambda I - T)^{-1}p = f(\lambda) = q\), and so \(\lambda q = p + Tq\). Hence it must be that \(q = 0\), i.e., \(p = -T(0)\), a contradiction. This establishes the theorem.

The author considers it likely that the statement of the theorem is true if the condition (in the definition of resolvent) that \((\lambda I - T)^{-1}\) be locally Lipschitzian is dropped.

**Reference**


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