SOME THEOREMS IN FOURIER ANALYSIS ON SYMMETRIC SETS

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Let $\mathbb{R}$ be the real line and $A = A(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ which are the Fourier transforms of functions in $L^1(\mathbb{R})$. $A(\mathbb{R})$ is a Banach Algebra when it is given the $L^1(\mathbb{R})$ norm. For a closed $F \subseteq \mathbb{R}$ one defines $A(F)$ as the restrictions of $f \in A$ to $F$ with the norm of $g \in A(F)$ the infimum of the norms of elements of $A$ whose restrictions are $g$. Let $F_r \subseteq \mathbb{R}$ be of the form

$$F_r = \{\sum_1^\infty \varepsilon_j r(j): \varepsilon_j \text{ either 0 or 1}\}.$$  

This paper shows that if

$$\sum (r(j+1)/r(j))^2 < \infty \quad \text{and} \quad \sum (s(j+1)/s(j))^2 < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. We also show that, in some sense square summability is the best possible criterion. In the course of the proof we show that $F_r$ is a set of synthesis and uniqueness if

$$\sum (r(j+1)/r(j))^2 < \infty.$$  

This is almost a converse to a theorem of Salem.

We shall also consider sets $E_m \subseteq \prod_1^\infty \mathbb{Z}_{m(j)}$ of the form

$$E_m = \{x: j^{th} \text{ coordinate is 0 or 1}\}.$$

The $E_m$ will have analogous properties to the $F_r$ that will depend on the $m(j)$.

The original work on isomorphisms of the algebras was done in [2] where Beurling and Helson show that any automorphism of $A$ must arise from a map $\varphi$ by $f \circ \varphi$ where $\varphi(x) = ax + b$. For restriction algebra the situation is more complex. In [5] it is shown that an isomorphism between $A(F_1)$ and $A(F_2)$ of norm one must be given by $f \rightarrow f \circ \varphi$ where $\varphi: F_2 \rightarrow F_1$ is continuous and $e^{i\varphi}$ is a restriction to $F_2$ of a character of the discrete reals. Further if $F_2$ is thick in some appropriate sense the character is continuous. However, McGehee [11] gives examples of $F_1$ and $F_2$ for which the restriction algebras $A(F_1)$ and $A(F_2)$ are isomorphic under an isomorphism induced by a discontinuous character. Meyer [12] has shown that if

$$\sum r(j+1)/r(j) < \infty \quad \text{and} \quad \sum s(j+1)/s(j) < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. For appropriate $r(j)$ this is an example of an isomorphism induced by a $\varphi$ with $e^{i\varphi}$ not even a discontinuous character. He also showed that under these hypothesis $F_r$ was a set of synthesis and uniqueness.
DEFINITIONS AND NOTATIONS. For background material and notation not defined here we refer the reader to [7] and [15].

In this paper $G$ will always be a locally compact abelian group with dual group $\Gamma$. If $g$ and $\gamma$ are elements of $G$ and $\Gamma$ respectively, the value of the character $\gamma$ at the point $g$ will be denoted by $(\gamma, g)$.

When we have a sequence of compact abelian groups $G_j$, we shall denote their direct product (complete direct sum [15]) by $\Pi G_j$. If $\Gamma_j$ is the dual of $G_j$, then the direct sum [15] $\Sigma \Gamma_j$ is the dual of $\Pi G_j$. The $j^{\text{th}}$ coordinate of elements $g$ of $\Pi G_j$ or $\gamma$ of $\Sigma \Gamma_j$ will be denoted by $g_j$ and $\gamma_j$. One has:

$$(\gamma, g) = \Pi (\gamma_j, g_j)$$

where all but a finite number of elements in the product are 1.

We shall be dealing with the following basic groups:

(i) The multiplicative circle group will be denoted by $T$. $T$ shall be identified with the unit interval by $x \in [0, 1) \rightarrow \exp(x)$ where $\exp(x) = e^{2\pi ix}$. The additive group of integers $\mathbb{Z}$ is the dual group of $T$. If $x \in [0, 1)$ represents an element of $T$ and $n \in \mathbb{Z}$ then $(n, x) = \exp(nx)$.

(ii) $R$ will denote the additive group of reals. $R$ is isomorphic to its dual under the pairing given by

$$(y, x) = \exp(xy),$$

$x, y \in R$.

(iii) $\mathbb{Z}_n$ for $n \geq 2$ will denote the additive group of integers mod $n$. $\mathbb{Z}_n$ is also isomorphic to its dual under the pairing given by

$$(r, s) = \exp(rs/n),$$

$r, s \in \mathbb{Z}_n$.

Any nonzero regular translation invariant measure on a locally compact abelian group $G$ is called a Haar measure. If $\mu_G$ and $\mu_\Gamma$ are Haar measures on $G$ and its dual group $\Gamma$ respectively, the Fourier transform $\hat{f}$ of $f$ in $L^1(\Gamma, \mu_\Gamma)$ is defined by

$$\hat{f}(g) = \int f(\gamma)(\gamma, g)d\mu_\Gamma$$

for $g \in G$. The inversion theorem gives

$$\int f(g)(\gamma, -g)d\mu_G = C f(\gamma).$$

We shall normalize $\mu_G$ and $\mu_\Gamma$ so that $C = 1$. If $G$ is compact we can place $\mu_G(G) = 1$ and if $\Gamma$ is discrete $\mu_\Gamma(\gamma) = 1$ for $\gamma \in \Gamma$. $L^1(G)$ will denote $L^1(G, \mu_G)$ for a normalized Haar measure.
For \( f, h \in L^1(\Gamma) \) define the convolution \( f * h \) by
\[
f * h(\gamma) = \int_{x \in \Gamma} f(\gamma - \lambda)h(\lambda)d\mu_{\Gamma}.
\]
In [15] it is shown that \( L^1(\Gamma) \) is a commutative Banach algebra under convolution and for \( g \in G \)
\[
\hat{f} \hat{h}(g) = \hat{f}(g)\hat{h}(g).
\]
We denote by \( M(G) \) the space of all regular, complex valued Borel measures on \( G \) of finite total variation. In [15] the Fourier transform \( \hat{\mu} \) of \( \mu \in M(G) \) and the convolution \( \mu * \nu \) of measures in \( M(G) \) are defined. It is shown that \( M(G) \) is a Commutative Banach Algebra under convolution and
\[
\hat{\mu} \hat{\nu}(\gamma) = \hat{\mu}(\gamma) \hat{\nu}(\gamma)
\]
for \( \gamma \in \Gamma \).

Let \( A = A(G) \) be defined by
\[
A(G) = \{ \hat{f} : f \in L^1(\Gamma) \}.
\]
\( A(G) \) is a Banach algebra under pointwise multiplication and with norm \( \| \cdot \|_A \) defined by \( \| \hat{f} \|_A = \| f \|_{L^1(\Gamma)} \) and is isomorphic to \( L^1(\Gamma) \) under \( * \). For a closed set \( E \subseteq G \) define the restriction algebra
\[
A(E) = \{ \hat{f}/E : f \in L^1(\Gamma) \}
\]
with norm \( \| \cdot \|_{A(E)} \) defined by
\[
\| h \|_{A(E)} = \inf \{ \| \hat{f} \|_A : \hat{f}/E = h \}.
\]
\( A(E) \) is again a Banach algebra under pointwise multiplication. Set
\[
I(E) = \{ \hat{f} : \hat{f}/E = 0 \text{ and } f \in L^1(\Gamma) \}
\]
\( A(E) \) can be identified with the quotient algebra \( A(G)/I(E) \).

The dual space of \( A(G) \) is denoted by \( PM \) (or \( PM(G) \)). Its elements are called pseudomeasures. Each \( S \in PM \) can be identified with a function \( \hat{S} \in L^\infty(\Gamma) \) as follows. The action of \( S \in PM \) as a linear functional on \( \hat{f} \in A(G) \) is given by
\[
(S, \hat{f}) = \int_{\Gamma} f(\gamma) \hat{S}(\gamma)d\mu_{\Gamma}.
\]
We shall denote by \( \| S \|_{PM} \) the \( L^\infty(\Gamma) \) norm of \( \hat{S} \). Thus \( PM \) under \( \| \cdot \|_{PM} \) is identical with \( L^\infty(\Gamma) \) under the sup norm.

Since \( A(E) \) is the quotient of \( A(G) \) by \( I(E) \), the dual of \( A(E) \) consists of those \( S \in PM \) which annihilate every function in \( I(E) \).
We shall denote this dual of $A(E)$ by $N(E)$. If $N(E)$ is the set of all $S \in PM$ with supp $S \subseteq E$ [7, p. 161], then $E$ is said to be a set of synthesis. The set of all $\mu \in M(G)$ with support in $E$ we denote by $M(E)$. $M(E)$ can be considered a subspace of $N(E)$ with $(\mu, \hat{f}) = \int \hat{f} \, d\hat{\mu}$. The two definitions for $\hat{\mu}$ coincide.

If $G_1$ and $G_2$ are locally compact abelian groups and $E_1$ and $E_2$ are closed subsets of $G_1$ and $G_2$ respectively, we say that $\Phi: A(E_1) \to A(E_2)$ is an isomorphism into if and only if it is an injective algebraic homomorphism and is continuous. If the range of $\Phi$ is dense in $A(E_2)$, there exists a continuous $\varphi: E_2 \to E_1$ with $\Phi f = f \circ \varphi$ [9]. We always denote the adjoint of $\Phi$ taking $N(E_2)$ into $N(E_1)$ by $\Phi^*$. Symmetric sets in $R$ are defined as follows. For any sequence $r = \{r(j): j = 1, \cdots\}$ of positive reals with the property

$$\sum_k r(j) < r(k - 1)$$

we define the subset $F_r$ of $R$ by

$$F_r = \left\{ \sum_{j=1}^{\infty} \epsilon_j r(j): \epsilon_j \text{ either 0 or 1} \right\}.$$

The representation of the elements of $F_r$ as an infinite sum is unique. For each positive integer $k$, the subset $F_r^k$ of $F_r$ is defined by

$$F_r^k = \left\{ \sum_{j=1}^{k} \epsilon_j r(j): \epsilon_j \text{ either 0 or 1} \right\}.$$

We define the subspace $N_i(F_r)$ of $N(F_r)$ by

$$N_i(F_r) = \bigcup_{k=1}^{\infty} M(F_r^k).$$

For any given sequence $m = \{m(j): j = 1, 2, \cdots\}$ of positive integers we define the subset $E_m$ of $\Pi \mathbb{Z}_{m(j)}$ by

$$E_m = \left\{ x: x \in \Pi \mathbb{Z}_{m(j)}; x_j \text{ either 0 or 1} \right\}.$$

For each positive integer $k$ the subset $E_m^k$ of $E_m$ is defined by

$$E_m^k = \left\{ x: x \in E_m; x_j = 0 \text{ if } j > k \right\}.$$

Define the subspace $N_i(E_m)$ of $N(E_m)$ by

$$N_i(E_m) = \bigcup_{k=1}^{\infty} M(E_m^k).$$

For $r$ and $m$ as above there is a standard homeomorphism $\varphi: E_m \to F_r$ which takes $x \to \sum x_j r(j)$. Let the inverse of $\varphi$ be called $\psi$. 

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We shall frequently write $E$ for $E_m$, $E^k$ for $E^k_m$, $F$ for $F_r$, and $F^k$ for $F^k_r$ when the respective sequences are clear.

Throughout this work $\varepsilon_j$ will always denote a quantity that may take on the values 0 or 1.

1. The symbols $r$ and $m$ shall always denote $\{r(j): j = 1, 2, \ldots\}$ and $\{m(j): j = 1, 2, \ldots\}$ respectively. $F_r$ and $E_m$ will then represent the previously defined sets with $\varphi: E_m \rightarrow F_r$ and $\psi: F_r \rightarrow E_m$ the standard homeomorphisms. The maps $\varphi$ and $\psi$ induce maps between $N_i(E_m)$ and $N_i(F_r)$ which we shall again denote by $\varphi$ and $\psi$. The maps have the form

$$\varphi(\mu)(\{\varphi(x)\}) = \mu([x])$$

for $\mu \in N_i(E)$, and

$$\psi(\mu)(\{\psi(x)\}) = \mu([x])$$

for $\mu \in N_i(F)$.

If $x = \langle \varepsilon_1, \cdots, \varepsilon_k, 0, \cdots \rangle$ is an element of $E^k_m$ and $\mu \in M(E^k)$ set

$$a(\varepsilon_1, \cdots, \varepsilon_k) = \mu([x]).$$

If $y = \sum_i^k \varepsilon_j r(j)$ is an element of $F^k_r$ and $\nu \in M(F^k)$ set

$$b(\varepsilon_1, \cdots, \varepsilon_k) = \nu([y]).$$

We see that

$$\|\mu\|_{PM} = \sup_{\varepsilon_1, \cdots, \varepsilon_k} |\sum a(\varepsilon_1, \cdots, \varepsilon_k) \xi_1^{\varepsilon_1} \cdots \xi_k^{\varepsilon_k}|$$

where $\xi_j$ is an arbitrary $m(j)$ root of unity and the sum is taken over all combinations with $\varepsilon_j$ being 0 or 1. Similarly

$$\|\nu\|_{PM} = \sup_x \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left( x \sum_{i=1}^k \varepsilon_j r(j) \right) \right|$$

where $x \in R$.

For any $\mu \in N_i(E)$ we define

$$\|\mu\|_{MAX} = \sup_{\theta_1, \cdots, \theta_k} \left| \sum a(\varepsilon_1, \cdots, \varepsilon_k) \exp \left( \sum \varepsilon_j \theta_j \right) \right|$$

where $\theta_j \in R$. Define $\|\nu\|_{MAX}$ for $\nu \in N_i(F)$ by

$$\|\nu\|_{MAX} = \sup_{\theta_1, \cdots, \theta_k} \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left( \sum \varepsilon_j \theta_j \right) \right|.$$
It is clear that $||\mu||_{PM} \leq ||\mu||_{MAX}$ and $||\nu||_{PM} \leq ||\nu||_{MAX}$. For any standard homeomorphism $\phi$ we have

$$||\phi\mu||_{PM}/||\mu||_{PM} \leq ||\mu||_{MAX}/||\mu||_{PM}.$$ 

Similarly

$$||\psi\nu||_{PM}/||\nu||_{PM} \leq ||\nu||_{MAX}/||\nu||_{PM}.$$ 

One should note that if $r$ is a sequence of reals independent mod 1 over the rationals, Kronecher’s Theorem [4, p. 99] implies that $||\nu||_{MAX} = ||\nu||_{PM}$ for $\nu \in N_r(E_r)$.

In order to achieve isomorphisms between certain quotient algebras we shall first study the ratios $||\mu||_{MAX}/||\mu||_{PM}$ and $||\nu||_{MAX}/||\nu||_{PM}$.

**Lemma 1.1.** If $\sum (1/m(j))^2 < \infty$ then there is a $C$ depending only on $m$ so that $||\mu||_{MAX}/||\mu||_{PM} \leq C$ for all nonzero $\mu \in N_1(E_m)$.

**Proof.** For each $k$, since $M(E^k)$ is finite dimensional, there is a smallest constant $A(k)$ so that $||\mu||_{MAX}/||\mu||_{PM} \leq A(k)$ for all nonzero $\mu \in M(E^k)$. We shall show that there are constants $C_k$ with $2IC_k < \infty$ so that $A(k)/A(k - 1) \leq C_k$.

The quotient $||\mu||_{PM}/||\mu||_{MAX}$ is equal to

\[
\sup \left| \frac{\sum \left[ (a(\xi_1, \ldots, \xi_k, 0) + \alpha(\xi_1, \ldots, \xi_k, 1))\xi_k^{1} \cdots \xi_k^{k-1} \right] }{\sum \left[ (\alpha(\xi_1, \ldots, \xi_k, 0) + \alpha(\xi_1, \ldots, \xi_k, 1))Z_k(\xi_1^{1} \cdots \xi_k^{k-1}) \right] } \right|
\]

where $\xi_j$ are $m(j)$ roots of unity and $Z_j$ are complex numbers of modulus 1. By a division and multiplication $||\mu||_{PM}/||\mu||_{MAX}$ becomes

\[
\sup \left| \frac{\sum \left[ (a(\cdots, 0) + a(\cdots, 1))\xi_k^{1} \cdots \xi_k^{k-1} \right] }{\sum \left[ (a(\cdots, 0) + a(\cdots, 1))Z_k(\xi_1^{1} \cdots Z_k^{k-1}) \right] } \right|
\]

\[
\times \left| \frac{\sum \left[ (a(\cdots, 0) + a(\cdots, 1))Z_k(\xi_1^{1} \cdots Z_k^{k-1}) \right] }{\sum \left[ (a(\cdots, 0) + a(\cdots, 1))Z_k(\xi_1^{1} \cdots \xi_k^{k-1}) \right] } \right|
\]

The factor used in division and multiplication in (1.3) is nonzero. If it were zero $||\mu||_{PM}$ would be zero and hence $\mu$ would be zero. The fraction on the left of (1.3) is greater than or equal to $1/A(k - 1)$. Choose $z_j = y_j$ so that the maximum of the denominator in (1.2) is achieved. The fraction on the right in (1.3) is greater than or equal to
\begin{align}
(1.4) \quad \left| 1 + \frac{\sum [a(\cdots, 1)(\xi_k - y_k) y_{i_1} \cdots y_{i_{k-1}}]}{\sum [(a(\cdots, 0) + a(\cdots, 1)) y_k y_{i_1} \cdots y_{i_{k-1}}]} \right| .
\end{align}

If \( \sum a(\cdots, 1)y_{i_1} \cdots y_{i_{k-1}} \) is zero (1.4) is equal to one. Otherwise set \( e^{iz} = \xi_k/y_k \) and (1.4) is equal to
\begin{align}
(1.5) \quad \left| 1 + \frac{e^{iz} - 1}{\sum [a(\cdots, 0)y_{i_1} \cdots y_{i_{k-1}}]/y_k \sum a(\cdots, 1)y_{i_1} \cdots y_{i_{k-1}}} + 1 \right| .
\end{align}

However, in order that the choice \( z_j = y_j \) give \( \| \mu \|_{\text{MAX}} \), the quotient
\[
\frac{\sum a(\cdots, 0)y_{i_1} \cdots y_{i_{k-1}}}{y_k \sum a(\cdots, 1)y_{i_1} \cdots y_{i_{k-1}}}
\]

must be a real positive real number. Call that number \( s \) and (1.5) becomes
\[
\left| 1 + \frac{(\cos x - 1) + i \sin x}{s + 1} \right|
\]

which is greater than or equal to
\[
1 - x^2/2 .
\]

For an appropriate \( \xi_k, |x| \) is less than or equal to \( 2\pi/m(k) \).

From the above calculation we get
\[
\| \mu \|_{F_0}/\| \mu \|_{\text{MAX}} \geq \frac{(1 - 2\pi^2/(m(k))^2)}{A(k - 1)}
\]
and therefore
\[
A(k) \leq A(k - 1) \cdot \left(1 + \frac{C^1}{(m(k))^2} \right),
\]

for some absolute constant \( C^1 \) and for all \( m(k) \) sufficiently large. Since \( \sum (1/m(j))^2 < \infty \) the theorem is proven.

For the symmetric sets \( F \), we shall need the following lemma similar to Lemma 1.1.

\textbf{Lemma 1.6.} Suppose that \( \sum (r(j + 1)/r(j))^2 < 1/24 \). Choose a real number \( x_0 \) and define the interval \( I \) to be
\[
\left\{ x : |x - x_0| < 2\left(\sum_{j=1}^{k} 1/r(j)\right) \right\} .
\]

There is then a constant \( C_1 \) independent of \( k \) and \( x_0 \), so that
\[
\| \nu \|_{\text{MAX}}/\sup |\hat{\nu}(x)| < C_1, \text{ for all nonzero } \nu \in M(F^+_k) .
\]
Proof. Fix $k$ and choose a nonzero $v \in M(F^k_r)$. There exists real numbers $\theta_1, \cdots, \theta_k$ less than or equal to one, for which

$$||v||_{\text{max}} = |\sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp(\sum \varepsilon_j \theta_j)| .$$

Define the functions $\hat{\nu}_k, \cdots, \hat{\nu}_2, \hat{\nu}_1 = \hat{\nu}$ on $R$ by

$$\hat{\nu}_j(x) = \sum \left[ b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left( \sum_{i=1}^{j-1} \varepsilon_j \theta_j \right) \exp \left( \sum_{j=1}^{k} \varepsilon_j r(j) \right) \right] .$$

Let us estimate $\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/||v||_{\text{max}}$ where

$$I_1 = \left\{ x: |x - x_0| \leq \sum_{k=1}^{j} (2/r(j)) \right\} .$$

There is an $x'_0$ within $(1/r(k))$ of $x_0$ for which $x'_0 \cdot r(k) = \theta_k \pmod{1}$. Pick $x_i$ within $1/r(k - 1)$ of $x'_0$ so that $x_i \cdot r(k - 1) = \theta_{k-1} \pmod{1}$. Then

$$\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/||v||_{\text{max}} \geq |\hat{\nu}_{k-1}(x_i)|/||v||_{\text{max}} .$$

As a function of $x$, $\hat{\nu}_k(x)$ is the Fourier Stieltjes transform of a measure $\nu_k$ having support in $[0, r(k)]$. Now,

$$|\hat{\nu}_{k-1}(x_i)|/||v||_{\text{max}} = |\hat{\nu}_k(x_i)|/||\hat{\nu}_k(x'_0)|$$

$$= \left| 1 + \sum \frac{\hat{\nu}'_k(x'_0)(x_i - x'_0)}{\hat{\nu}_k(x'_0)} (x_i - x'_0) + \frac{\hat{\nu}''(x'_0)(x_i - x'_0)^2}{2} + \cdots \right| .$$

$|\hat{\nu}_k|^2$ has a maximum at $x'_0$. Therefore, if $\hat{\nu}_k = f + ig$, with $f$ and $g$ real, $f \cdot f' + g \cdot g' = 0$ at $x'_0$. But, at $x'_0$,

$$\frac{\hat{\nu}_k'}{\hat{\nu}_k} = \frac{f'}{f} + \frac{ig'}{f} + ig = (ff' + gg' + i(fg' - f'g))/f^2 + g^2 ,$$

which is purely imaginary. Therefore,

$$|\hat{\nu}_{k-1}(x_i)|/||v||_{\text{max}} \geq 1 - \left| \frac{\hat{\nu}''(x'_0)(x_i - x'_0)^2}{2} + \cdots \right| .$$

If a measure $\mu$ has support in $[0, \delta]$ a theorem of Bernstein [1, p. 138] shows that for all $x$

$$|\hat{\mu}'(x)| \leq \delta ||\mu||_{PM}$$

and hence its $n$th derivative $\hat{\mu}^{(n)}$ has

$$|\hat{\mu}^{(n)}(x)| \leq \delta^n ||\mu||_{PM} .$$

Since $\nu_k$ has support in $[0, r(k)]$ we obtain

$$|\hat{\nu}_{k-1}(x_i)|/||v||_{\text{max}} \geq 1 - (r(k)^2/r(k - 1)^2) .$$
In effect, we have just shown that there is an \( x_1 \in I_i \) for which

\[
\| \nu \|_{\text{MAX}} / | \hat{\nu}_{k-1}(x_i) | \leq 1 + 2(r(k)/r(k - 1))^2.
\]

Assume that for some \( j < k - 1 \) there is an \( x_j \in I_j = \{ x : | x - x_0 | \leq \sum_{k-j}^k (2/r(l)) \} \) for which

\[
\| \nu \|_{\text{MAX}} / | \hat{\nu}_{k-j}(x_j) | \leq \prod_{l=k-j}^\infty (1 + 24(r(l + 1)/r(l))^2).
\]

We shall show there is then an \( x_{j+1} \in I_{j+1} \) for which

\[
\| \nu \|_{\text{MAX}} / | \hat{\nu}_{k-(j+1)}(x_{j+1}) | \leq \prod_{l=k-j-1}^\infty (1 + 24(r(l + 1)/r(l))^2).
\]

Consider \( S = \{ x : | x - x_j | \leq 1/r(k - (j + 1)) \} \). If \( | \hat{\nu}_{k-j} | \) does not have a relative maximum in \( S \) greater than or equal to \( | \hat{\nu}_{k-j}(x_j) | \), then \( | \hat{\nu}_{k-j} | \) would be greater than or equal to \( | \hat{\nu}_{k-j}(x_j) | \) on some interval in \( S \) of length equal to \( 1/r(k - (j + 1)) \). However there would be an \( x_{j+1} \) in the interval for which \( x_{j+1} \cdot r(k - (j + 1)) = \theta_{k-(j+1)} \) (mod 1) and hence \( \hat{\nu}_{k-(j+1)}(x_{j+1}) = \hat{\nu}_{k-j}(x_{j+1}) \), which implies the induction step. Let us assume therefore that there is an \( x' \) where

\[
| x' - x_0 | \leq (1/r(k - (j + 1)) + \sum_{k-j}^k 2/r(l)),
\]

\( | \hat{\nu}_{k-j}(x') | \geq | \hat{\nu}_{k-j}(x_j) | \) and at which \( | \hat{\nu}_{k-j} | \) has a relative maximum. As before, choosing \( x_{j+1} \) within \( 1/r(k - (j + 1)) \) of \( x' \) and satisfying \( x_{j+1} \cdot r(k - (j + 1)) = \theta_{k-(j+1)} \) gives

\[
| \hat{\nu}_{k-(j+1)}(x_{j+1})/\hat{\nu}_{k-j}(x_j') | = | \hat{\nu}_{k-j}(x_{j+1})/\hat{\nu}_{k-j}(x_j') | \geq \sum_1^\infty 1 - \left| \frac{\hat{\nu}''_{k-j}(x_j')(x_{j+1} - x_j')^2}{2} + \ldots \right|.
\]

\( \hat{\nu}_{k-j} \) as a function of \( x \) is the Fourier Stieltjes of a measure \( \nu_{k-j} \) having support in \([0, 2r(k - j)]\). Since \( \| \nu_{k-j} \|_{PM} \leq \| \nu \|_{\text{MAX}} \), the previously stated theorem of Bernstein gives

\[
| \hat{\nu}^{(n)}_{k-j}(x') | \leq (2r(k - j))^n \| \nu \|_{\text{MAX}}.
\]

However

\[
\| \nu \|_{\text{MAX}} \leq \left[ \prod_{l=k-j}^\infty (1 + 24(r(l + 1)/r(l))^2) \right] \times | \hat{\nu}_{k-j}(x_j') | \\
\leq e^{24\Sigma}(r(l+1)/r(l))^2 \cdot | \hat{\nu}_{k-j}(x_j') | \\
\leq 3 | \hat{\nu}_{k-j}(x_j') |
\]
Since $\Sigma (r(l + 1)/r(l))^2 \leq (1/24)$. Therefore in (1.8),

$$|\hat{\varphi}_{k-j+1}(x_{j+1})/\hat{\varphi}_{k-j}(x_j)| \geq 1 - 12(r(k-j)/r(k-(i+1))^2$$
and hence (1.7) is true, finishing the induction.

Lemma 1.6 in its present form is an adaptation and extension of a lemma of Meyer [12]. Previously we had much more stringent conditions on the $r$, to arrive at a similar conclusion to Lemma 1.6.

To utilize the Lemmas 1.1 and 1.6 to obtain isomorphisms of restriction algebras we shall introduce some functional analysis.

Let $V$ represent a Banach Space and $V^*$ its dual. For $r > 0$ let $B_r = \{ t: t \in V^*, \| t \| \leq r \}$. A set $O \subseteq V^*$ is said to be open in the bounded topology on $V^*$ if and only if $O \cap B_r$ is open in the relative weak* topology of $B_r$ for all $r > 0$. For a distribution of the bounded topology the reader should consult [6, p. 427].

**Lemma 1.10.** Let $V, W$ be Banach spaces with duals $V^*$ and $W^*$. Let $K \subset V^*$ be a weak* dense subspace of $V^*$. Suppose that $T: K \rightarrow W$ is linear and continuous when $K$ has the topology induced by the bounded topology on $V^*$ and $W^*$ has the weak topology. Then there exists a bounded linear transformation $S: W \rightarrow V$ for which $T = S^*/K$.

**Proof.** For each $w \in W$, define the linear functional $T_w$ on $K$ by

$$T_w(t) = Tt(w).$$

Each $T_w$ is continuous in the topology induced by the bounded topology of $V^*$ which is a locally convex topology by Corollary 5, page 428 of [6]. Hence by the Hahn-Banach theorem there exists an extension $\tilde{T}_w$ of $T_w$ to all of $V^*$, continuous in the bounded topology of $V^*$.

By Theorem 6, page 428 of [6], $\tilde{T}_w$ is continuous in the weak* topology on $V^*$. Hence there exists an element $v \in V$ such that $T_w(t) = t(v)$ for all $t \in K$. Since $K$ is assumed weak* dense in $V^*$, the element $v$ is determined by $w$. Define $S: W \rightarrow V$ by $S(w) = v$. $S$ is linear. Since $K$ is weak* dense $S$ is closed. Therefore by the Closed Graph Theorem $S$ is bounded. If $t \in K, w \in W$

$$S^*t(w) = t(S(w)) = Tt(w),$$

which completes the proof.

It is clear that $N_1(E_m)$ and $N_1(F_r)$ are weak* dense in $N(E_m)$ and $N(F_r)$, respectively. By studying the continuity of the standard maps between $N_1(E_m)$ and $N_1(F_r)$, we shall be able to use Lemma 1.10 to
obtain isomorphisms between $A(E_m)$ and $A(F_r)$ for certain classes of sequences $m$ and $r$.

Choose $\mu \in N_i(E)$. For each $k$ we define an approximating measure $\mu_k$ in $M(E^k)$ by

$$\mu_k([x]) = \sum_{y \in D} \mu([y])$$

where $x \in E^k$ and $D = \{y: y \in E$ and $y_j = x_j$ for $j \leq k\}$. Let

$$I^k = \{\gamma: \gamma \in \Sigma Z(m(j)) \text{ and } \gamma_j = 0 \text{ if } j > k\}.$$ 

If $\gamma \in I^k \hat{\mu}_k(\gamma) = \hat{\mu}(\gamma)$. It is easy to see that

$$||\mu_k||_{PM} = \sup_{\gamma \in I^k} |\hat{\mu}_k(\gamma)|.$$

To each $\lambda \in M(E^k)$ we associate the measure $\lambda'$ in $M(E^k)$ defined by

$$\lambda'([x]) = \begin{cases} 0 & \text{if } x_k = 0 \\ \lambda([x]) & \text{if } x_k = 1 \end{cases}.$$ 

It is not hard to see that

$$||\lambda'||_{PM} \leq 2 ||\lambda||_{PM}.$$

Choose $\nu \in N_i(F)$. For each $k$ define an approximating measure $\nu_k$ in $M(F^k)$ by

$$\nu_k([x]) = \sum_{y \in D} \nu([y])$$

where $x = \sum_{j=1}^{k} x_j r(j)$ and $D = \{y: y = \Sigma \varepsilon_j r(j) \text{ and } \varepsilon_j = x_j \text{ for } j \leq k\}$.

To each $\beta \in M(F^k)$ we associate the measure $\beta'$ in $M(F^k)$ defined by

$$\beta'(x) = \begin{cases} 0 & \text{if } x = \sum_{j=1}^{k} \varepsilon_j r(j) \text{ and } \varepsilon_k = 0 \\ 1 & \text{if } x = \sum_{j=1}^{k} \varepsilon_j r(j) \text{ and } \varepsilon_k = 1 \end{cases}.$$ 

We are now ready to prove the following theorem.

**Theorem 1.11.** If $\Sigma(1/m(j))^2 < \infty$ and $\Sigma(r(j + 1)/r(j))^2 < \infty$ then $A(E_m)$ is isomorphic to $A(F_r)$.

We shall break the proof into two lemmas.

**Lemma A.** Let $F_r$ be any symmetric set. Let $\Sigma(1/m(j))^2 < \infty$ and $\varphi: E_m \rightarrow F_r$ the standard homeomorphism. Then there is an iso-
morphism into $\Phi: A(F_r) \to A(E_m)$ given by

$$\Phi(f) = f \circ \varphi, \quad f \in A(F_r).$$

**Proof.** We shall study the continuity properties of

$$\varphi: N_1(E) \to N_1(F).$$

For $f \in A(F)$ define

$$U_{\varepsilon,f} = \{\nu: \nu \in N_1(F) \text{ and } |(\nu, f)| < \varepsilon\}.$$

To establish that $\varphi$ is continuous from the bounded weak* topology of $N_1(E)$ to the weak* topology of $N_1(F)$ it is sufficient to prove that the zero element of $N_1(E)$ is an interior point of $\varphi^{-1}(U_{\varepsilon,f})$ (i.e., that $\varphi$ is continuous at 0). This follows at once if we prove that given $a$ and $\varepsilon$, there exists $\delta, k$ such that if for $\mu \in N_1(E)$

$$\|\mu\|_{PM} \leq a \quad \text{and} \quad |\hat{\mu}(\gamma)| < \delta \quad \text{for} \quad \gamma \in \Gamma^k$$

$$\varphi(\mu) \quad \text{is an element of} \quad U_{\varepsilon,f}.$$

In view of Lemma 1.1 (1.12) follows if we can show that given $a, \varepsilon,$ and $M$ then there exists $\delta, k$ such that for $\mu \in N_1(E)$,

$$\|\mu\|_{PM} \leq a \quad \text{and} \quad |\hat{\mu}(\gamma)| < \delta \quad \text{for} \quad \gamma \in \Gamma^k$$

then

$$|\varphi(\mu)(x)| < \varepsilon \quad \text{for} \quad |x| \leq M.$$

We first estimate $|\varphi(\mu) - \varphi(\mu_k)|$ for $\mu \in M(E^*)$.

$$|\varphi(\mu)(x) - \varphi(\mu_k)(x)| \leq \sum_{j=0}^{s-1} |\varphi(\mu_{j+1})(x) - \varphi(\mu_j)(x)|$$

$$\leq \sum_{j=0}^{s-1} |\exp(-xr(j+1)) - 1| \cdot \|\varphi(\mu_{j+1})\|_{PM}.$$

By Lemma 1.1, for any $s$

$$|\varphi(\mu)(x) - \varphi(\mu_k)(x)| \leq 4\pi C \cdot |x| \cdot \|\mu\|_{PM} \cdot \sum_{k=1}^{\infty} r(j).$$

For $\mu$ with $\|\mu\|_{PM} \leq a$, pick $\delta < \varepsilon/2C$ where $C$ is the constant of Lemma 1.1 and choose $k$ so that $4\pi C a \sum_{k=1}^{\infty} r(j) < \varepsilon/2$. If $|\hat{\mu}(\gamma)| < \delta$ for $\gamma \in \Gamma^k$, then $\|\mu_k\|_{PM} < \delta$ and by Lemma 1.1 $\|\varphi(\mu_k)\|_{PM} < \varepsilon/2$. If $|x| \leq M$, then $|\varphi(\mu)(x) - \varphi(\mu_k)(x)| < \varepsilon/2$ so

$$|\varphi(\mu)(x)| < \varepsilon, \quad \text{for} \quad |x| \leq M.$$

The conditions of Lemma 1.10 are satisfied so $\varphi = \Phi^*$ for some
linear $\Phi : A(F) \to A(E)$. For $\mu \in N_1(E)$ and $f \in A(F)$

$$(\Phi f, \mu) = (f, \varphi(\mu)).$$

Therefore if $x \in \bigcup_i^\infty E_i$

$$\Phi f(x) = f(\varphi(x)).$$

Since $\varphi, f$ and $\Phi f$ are continuous, $\Phi$ is the linear map wanted.

**Lemma B.** Let $F_r$ be a symmetric set with $\sum (r(j + 1)/r(j))^2 < \infty$. Let $\psi : F_r \to E_m$ be the standard homeomorphism of $F_r$ with some $E_m$. Then there is an isomorphism into $\overline{\psi} : A(E_m) \to A(F_r)$ given by

$$\overline{\psi}(f) = f \circ \psi, \quad f \in A(E_m).$$

**Proof.** There is an $l$ so that $\sum_{i+1}^\infty (r(j + 1)/r(j))^2 < 1/24$. $F$ is a union of $2^l$ sets which are translations of the set $F' = \{x : x = \sum_{i+1}^\infty \varepsilon_j r(j)\}$. It is therefore sufficient to prove the theorem for $F'$. For convenience, assume $F_r$ has the property $\sum_{i+1}^\infty (r(j + 1)/r(j))^2 < 1/24$. We shall show as in Lemma A that $\psi : N_1(F_r) \to N_1(E_m)$ has the required continuity properties to be the adjoint of a continuous linear map $\overline{\psi} : A(E_m) \to A(F_r)$ satisfying $\overline{\psi}(f) = f \circ \psi$.

Using Lemmas 1.6 and 1.10 as in Lemma A, it is enough to show that if $a, \varepsilon, M$ are given, then there exists $\delta, x_1, \ldots, x_t$ so that the following holds.

If $\nu \in N_1(F)$, $\|\nu\|_{PM} \leq a$ and $\hat{\nu}(x_j) < \delta$ for $j = 1, \ldots, t$, then $|\hat{\psi}(\nu)(\gamma)| < \varepsilon$ for $\gamma \in \Gamma^M$.

Choosing $\nu \in N_1(F)$ with $\|\nu\|_{PM} \leq a$ and estimating $|\hat{\nu} - \hat{\nu}_k|$ gives

$$|\hat{\nu}(x) - \hat{\nu}_k(x)| \leq \sum \frac{1}{k} |\hat{\nu}_{j+1}(x) - \hat{\nu}_j(x)|$$

$$\leq \sum_{k} \left| \exp \left( -\varepsilon r(j + 1) \right) - 1 \right| \|\nu_j\|_{PM}.$$

Lemma 1.1 and 1.6 show that the $PM$ norm on $N_1(F_r)$ and $N_1(E_m)$ are equivalent when $\sum (1/m(j))^2 < \infty$. Hence

$$|\hat{\nu}(x) - \hat{\nu}_k(x)| \leq 4\pi x C r^{\infty} \sum_{k+1}^\infty r(j)$$

$$\leq 8\pi |x| C C a \cdot r(k + 1).$$

An easy consequence of the condition $\sum (r(j + 1)/r(j))^2 < 1/24$ is that

$$\lim_{k \to \infty} 8\pi C a \cdot \left( \sum_{k} 2/r(j) \right) \cdot r(k + 1) = 0.$$
Pick $k \geq M$ large enough so that
\[ 8\pi C_1 C \left( \sum_{i=1}^{k} \frac{2}{r(j)} \right) r(k + 1) < \varepsilon / 4C_1. \]

Then
\[
(1.14) \quad |\hat{v}(x) - \hat{v}_k(x)| < \varepsilon / 4C_1
\]
for $|x| < \sum_i^k (2/r(j))$. By Lemma 1.6 there is an $x_0$ with
\[ |x_0| < \sum_{i=1}^{k} (2/r(j)) \]
so that for $v_k \in M(F^k)$
\[ ||v_k||_{\text{MAX}} / ||\hat{v}_k(x_0)|| < C_1. \]

By a theorem of Bernstein [1, p. 138]
\[ |\hat{v}_k(x_0) - \hat{v}_k(x_0)| \leq C_1 |\hat{v}_k(x_0)| \left( \sum_{i=1}^{\infty} r(j) \right) |x_0 - x_0|. \]

Therefore, if $|x_0 - x_0| < 1/2(\sum r(j)) \cdot C_1$
\[
(1.15) \quad ||v_k||_{\text{MAX}} / ||\hat{v}_k(x_0)|| \leq 2C_1.
\]

Choose for $i = 1, \ldots, t; x_i$ with $|x_i| \leq \sum_i^k (2/r(j))$ so that for every $x$ with $|x| \leq \sum_i^k (2/r(j))$ there is an $x_j$ with $|x - x_j| < 1/2(\sum r(j)) \cdot C_1$.
If $|\hat{v}(x_j)| < \varepsilon / 4C_1$ for $x_j, j = 1, \ldots, t$, then $|\hat{v}_k(x_j)| < \varepsilon / 2C_1$ by (1.14), and by (1.15) $||v_k||_{\text{MAX}} < \varepsilon$. Consequently, $||\psi(v_k)||_{F,M} < \varepsilon$. Since $k > M$ we see that $|\psi(v)(\gamma)| < \varepsilon$ for $\gamma \in F^{\gamma}$.

As in Lemma A, the continuity conditions of Lemma 1.10 are satisfied and
\[ \overline{F}(f) = f \circ \psi. \]

Theorem 1.11 is an immediate consequence of Lemmas A and B. Meyer [12] has proven that if $\Sigma (r(j + 1)/r(j)) < \infty$ and
\[ \Sigma (s(j + 1)/s(j)) < \infty \]
then $A(F_r') \cong A(F_r')$. Lemmas 1.6 was an analogue and improvement on his main lemma which allowed us to obtain the theorem with square summability.

If $\mathcal{r}_s(j) = \{ e^{-j} \cdot 2^{-j^2} \}$ then every $A(F_r)$ and $A(E_m)$ with
\[ \Sigma (r(j + 1)/r(j)) < \infty \quad \text{and} \quad \Sigma (1/m(j))^2 < \infty \]
is isomorphic to $A(F_{r,s})$. The isomorphisms are given by
\[ f \rightarrow f \circ \varphi. \]
where $f$ is in an appropriate restriction algebra and $\varphi$ one of the standard homeomorphisms. We shall call an isomorphism between any two restriction algebras induced in this manner a standard isomorphism. If $A(F_\alpha)$ or $A(E_\alpha)$ is isomorphic to $A(F_{\alpha_0})$ by standard isomorphisms, $F_\alpha$ or $E_\alpha$ will then be said to belong to the class $M_\alpha$. One should note that for $\mu \in N_\alpha(F_{\alpha_0})$, $\| \mu \|_{PM} = \| \mu \|_{MAX}$.

Define sets of multiplicity and uniqueness as in [7, p. 52]. In [7, p. 100] it is shown that if $\alpha \in [0, 1/2)$ one can construct sets $F_\alpha$ of multiplicity with $r(j + 1)/r(j) = 0(j^{-\alpha})$. The next theorem shows, in particular, that if $r(j + 1)/r(j) = 0(j^{-\alpha})$ with $\alpha \in (1/2, \infty)$ then $F_\alpha$ is a set of uniqueness.

**Theorem 1.16.** Suppose that $\Sigma(r(j + 1)/r(j))^2 < \infty$. Then $F_\alpha$ is a set of synthesis and there is a constant $B$ so that for all $S \in N(F_\alpha)$

$$\| S \|_{PM} \leq B \lim | \hat{S}(x) | .$$

Hence $F_\alpha$ is a set of uniqueness.

**Proof.** Choose $l$ so that $\Sigma_{i=1}^\infty (r(j + 1)r(j))^2 < 1/24$. Then $F$ is a union of $2^l$ disjoint sets of the form $a(\varepsilon) + F(l)$ where $\varepsilon = \langle \varepsilon_1, \ldots, \varepsilon_\ell \rangle$ and $F(l) = \{ x : x = \Sigma_{i=1}^{\infty} \varepsilon_j r(j) \}$. We can find $2^l$ functions $\varphi_\varepsilon$ in $A(R)$ where $\varphi_\varepsilon = 1$ on $a(\varepsilon) + F(l)$ and $0$ on the other sets. Let $S \in PM$ with support in $F_\alpha$. $S = \Sigma_\varepsilon \varphi_\varepsilon S$ and hence if $\varphi_\varepsilon S \in N(a(\varepsilon) + F(l))$ for each $\varepsilon$, $S \in N(F_\alpha)$. Moreover, for some $\varepsilon$ the inequality

$$\| \varphi_\varepsilon S \|_{PM} \geq 2^{-l} \| S \|_{PM}$$

must hold. If $\| S \|_{PM} > B \lim | \hat{S}(x) |$ we see that

$$\| \varphi_\varepsilon S \|_{PM} \geq \frac{2^{-l}B}{\| \varphi_\varepsilon \|_A} \lim | \varphi_\varepsilon \hat{S}(x) | .$$

We may therefore assume that $\Sigma(r(j + 1)/r(j))^2 < 1/24$.

Lemma 1.6 and [12, Proposition 2.2.3] imply that there is a natural isomorphism $T$ from $A(F_k^\alpha \times [-2r(k + 1), 2r(k + 1)])$ in $A(R \times R)$ to $A(F_k^\alpha + [-2r(k + 1), 2r(k + 1)]$ with norm

$$T \leq (1 - \alpha 4r(k + 1) \cdot (\Sigma^\ell \varepsilon_j r(j)))^{-1}$$

and $\| T^{-1} \| = 1$, where $\alpha \leq 1$ and is independent of $k$. For large enough $k$ the norm is smaller than some constant $B_1$. For each $x \in R$ consider the function $f_x \in A(F_k^\alpha + [-2r(k + 1), 2r(k + 1)])$

$$f_x(y) = \exp(xy) - \exp(x \cdot \Sigma^\ell \varepsilon_j r(j)) \quad \text{for} \quad | y - \Sigma^\ell \varepsilon_j r(j) | \leq 2r(k + 1) .$$

Its image in $A(F_k^\alpha \times [-2r(k + 1), 2r(k + 1)]$ is
\[ f_x(t, y) = \exp (xt) \cdot (\exp (xy) - 1) . \]

Then
\[ \| f_x \|_{A(F_r^\infty + \{1\}]} \leq B_1 \| f_x \|_{A(F_r^{\infty + \{1\}]} \leq B_2 |x| r(k + 1) . \]

Define \( v_k \in M(F_r^\infty) \) by
\[
v_k((\sum \delta(f(j))) = (S|_{2^{k+1}j+1}^{2^{k+1}j+1}))(0) .
\]

where \( S \) is a given element of \( PM \) with support in \( F_r \). Then for sufficiently large \( k \)
\[ |\hat{S}(x) - \hat{v}_k(x)| = |(S, f_x)| \leq B_2 \cdot |x| \cdot \| S \|_{PM} \cdot r(k + 1) . \]

By Lemma 1.6 we have that
\[ \hat{v}_k(x) \rightarrow \hat{S}(x) \forall x \in \mathbb{R} ; \lim \| v_k \|_{PM} \leq C \| S \|_{PM} \]
and hence \( S \in N(F_r) \) and \( F_r \) is a set of synthesis.

For convenience assume that \( \| S \|_{PM} = 1 \) and \( |\hat{S}(0)| > 1/2 \). Suppose that \( |\hat{S}(x)| < \varepsilon \) for \( x > x_0 \). Pick a constant \( k_0 \) so that
\[ (x_0 + 4 \cdot \sum \delta(f(j)))B_2 \| S \|_{PM} \cdot r(k + 1) < \varepsilon \]
for \( k > k_0 \). Then if \( k > k_0 \)
\[ |\hat{v}_k(x)| < 2\varepsilon \]
for all \( x \) satisfying \( |x - x_*| \leq \sum \delta(2/r(j)) \) where \( x_* \) is the center of the interval \([x_0, x_0 + 4 \sum \delta(1/r(j))]\). Since \( |\hat{v}_k(0)| > 1/2 \) Lemma 1.6 shows that
\[ \varepsilon > 1/4C_1 . \]

Theorem 1.16 is essentially methods of McGehee and Meyer utilizing Lemma 1.6.

We next examine the sets \( E_m \). By [15, p. 166] they are sets of synthesis. If \( m(j) = 2 \) for all but a finite number of \( j \), \( E_m \) has positive measure and there is an \( S \in N(E_m) \) with \( \inf_{r} \sup_{r \sim T} |\hat{S}(\gamma)| = 0 \). The following is a converse.

**Theorem 1.17.** Let \( m(j) \) be a sequence of integers with infinitely many \( m(j) \geq 3 \). Then there is a constant \( C \) so that for all \( S \in N(E_m) \)
\[ \| S \|_{PM} \leq C \inf \sup_{r \sim T} |\hat{S}(\gamma)| \]
where \( T \) is any finite set in \( \Sigma Z_{m(j)} \).
Proof. Let $S \in N(E)$ and assume for simplicity that $\|S\|_{P_M} = 1$ and $\hat{S}(0) > 3/4$. Let $\{\mu_k\}$ be the measure defined by
\[
\mu_k(x) = \left( \hat{S} \sum_{j=k+1}^{\infty} \varepsilon_m(j) \right)(0)
\]
where $x = \langle \varepsilon_1, \ldots, \varepsilon_k, 0, 0, \ldots \rangle$. Let $\gamma_s \in \sum \Gamma_{m(j)}$ be that element with
\[
\gamma_s^k = \begin{cases} 0 & \text{if } j \neq s \\ 1 & \text{if } j = s \end{cases}.
\]
Then for $1 \leq s \leq k$
\[
\hat{\mu}_k(\gamma^s) = \sum_{\varepsilon(s) = 0} a(\varepsilon(1), \ldots, \varepsilon(k))
+ \sum_{\varepsilon(s) = 1} a(\varepsilon(1), \ldots, \varepsilon(k)) \exp(1/m(s)) .
\]
If we call $\sum_{\varepsilon(s) = 0} a(\varepsilon(1), \ldots, \varepsilon(k)) = \alpha$
\[
\sum_{\varepsilon(s) = 1} a(\varepsilon(1), \ldots, \varepsilon(k)) = \beta \text{ then } \hat{\mu}_k(0) = \alpha + \beta .
\]
It is easy to see that $\alpha \leq 1$ and $\beta \leq 2$. Therefore
\[
|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(0)| \leq 2 \exp(1/m(s) - 1) \\
\leq 4\pi/m(s) .
\]
Therefore, if $m(s) > 8\pi$
\[
|\hat{\mu}_k(\gamma^s)| > 1/4 .
\]
Let $\tilde{\gamma}^s \in \sum \Gamma_{m(j)}$ be the element with
\[
\tilde{\gamma}^j_s = \begin{cases} 0 & \text{if } j \neq s \\ m(s) - 1 & \text{if } j = s \end{cases}.
\]
Then
\[
\hat{\mu}_k(\tilde{\gamma}^s) = \alpha + \beta \exp(-1/m(s))
\]
and hence
\[
|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s)| = 2\beta \sin(2\pi/m(s)) .
\]
If $3 \leq m(s) < 8\pi$ and $|\hat{\mu}_k(\gamma^s)| < (1/100)$ then $\beta > (1/3)$ and
\[
|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s)| > 1/50
\]
and hence $|\hat{\mu}_k(\gamma^s)| > 1/50$. Therefore we may conclude that for all $k$ either $|\hat{\mu}_k(\gamma^s)|$ or $|\hat{\mu}_k(\tilde{\gamma}^s)|$ is greater than $1/100$ provided $m(s) \geq 3$.

On $\Gamma^k$, $\hat{\mu}_k$ and $\hat{S}$ are identical. Suppose there is a $t$ so that
for $\gamma \in \Gamma^t$. Pick a $k > t$ so that there is an $s$ with $k > s > t$ for which $m(s) \geq 3$. Then either $|\hat{\mu}_k(\gamma^t)|$ or $|\hat{\mu}_k(\gamma^t)|$ is greater than $1/100$. Hence $|\hat{S}(\gamma^t)|$ or $|\hat{S}(\gamma^t)|$ is greater than $1/100$ contradicting (1.19).

2. In this section we shall exhibit sets $E_m, F_r$ that do not have $A(E_m)$ or $A(F_r)$ isomorphic to $A(F_{r_0})$ by standard isomorphisms. They are then not in the class $M_y$.

The first theorem is a converse to Lemma A.

**Theorem 2.1.** If $\Sigma(1/m(j))^2 = \infty$, then $E_m$ is not an element of the class $M_y$.

**Proof.** It is sufficient to show that

$$\sup_{\nu \in N(E)} \| \mu \|_{\text{MAX}}/\| \mu \|_{PM} = \infty$$

since for $\nu \in N_1(F_{r_0}) \| \nu \|_{PM} = \| \nu \|_{\text{MAX}}$. For each integer $s$, let $x^s \in \Pi Z_{m(j)}$ be that element with $x^x_j = \delta^x_j$. Let $\alpha_s$ be the two point measure

$$\alpha_s[x^s] = \exp(1/3m(s)).$$

For each $k$, define an element $\mu_k$ of $M(E^k)$ by

$$\mu_k = \alpha_1 \ast \cdots \ast \alpha_k.$$

we see that

$$\| \mu_k \|_{\text{MAX}} = 2^k$$

while

$$\| \mu_k \|_{PM} = \sup_{\xi_s} \left| \prod_{s=1}^{k} (1 + \exp(1/(3m(s))) \cdot \xi_s \right|,$$

where the $\xi_s$ are $m(s)$ roots of unity. Since

$$|1 + \exp(1/3m(s))| \geq |1 + \exp(1/3m(s))\xi_s|$$

for $\xi_s$ any $m(s)$ root of unity, and since $\cos(\theta) < 1 - \theta^2/4$ for $\theta < 1$

$$\| \mu_k \|_{PM} \leq 2^k \prod_{s=1}^{k} \cos(\pi/3m(s))$$

$$\leq 2^k \prod_{s=1}^{k} (1 - (1/3m(s))^2).$$

Therefore

$$\| \mu_k \|_{\text{MAX}}/\| \mu_k \|_{PM} \geq 1/ \prod_{s=1}^{k} (1 - (1/3m(s))^2)$$
and since $\Sigma(1/m(s))^2 = \infty$, $\|\mu_k\|_{\text{max}}/\|\mu_k\|_{\text{p.m.}} \to \infty$ as $k \to \infty$.

We have actually shown more than claimed in Theorem 2.1. The proof shows that if $\{r(j)\}$ is any independent sequence and $\Sigma(1/m(j))^2 = \infty$, then $A(E_m)$ is not isomorphic to $A(F'_r)$ by a standard isomorphism.

The next theorem will imply that no condition on the convergence of $(r(j+1)/r(j))$ weaker than

$$\Sigma(r(j+1)/r(j))^2 < \infty,$$

is sufficient for a set $F_r$ to be a member of the class $M_y$.

**Theorem 2.2.** Suppose that $n_j$ is an increasing sequence of integers. Let $b \geq 2$ be an integer and put $r(j) = b^{-n_j}$. If

$$\Sigma(r(j+1)/r(j))^2 = \infty$$

then $F_r$ is not an element of the class $M_y$.

**Proof.** Let us assume for convenience that $\Sigma(r(2j)/r(2j-1))^2 = \infty$ and $b = 10$. We can also assume our set $F$ to be on the circle. For any integer $j$ define the two point measure $\gamma_j$ by

$$\gamma_j[0] = 1, \quad \gamma_j[r(j)] = \exp\left(-\frac{1}{2}\right).$$

For each $k$, define an element $\nu_k$ of $M(F^k)$ by

$$\nu_k = \gamma_1 \cdots \gamma_k.$$

Then for any integer $s$

$$|\mathcal{D}_{2k}(s)| = 2^{2k} \left| \prod_1^{2b} \cos\left(\pi\left(s \cdot 10^{-n_j} - \frac{1}{2}\right)\right) \right|.$$

In this product, consider terms $\delta_j(s)$ of the form

$$\left| \cos\left(\pi\left(s \cdot 10^{-n_{2j-1}} - \frac{1}{2}\right)\right) \cdot \cos\left(\pi\left(s \cdot 10^{-n_{2j}} - \frac{1}{2}\right)\right) \right|.$$

If

$$\left| s \cdot 10^{-n_{2j-1}} - \frac{1}{2} \right| < 1/10 \mod 1,$$

then

$$\left| s \cdot 10^{-n_{2j}} - \frac{1}{2} \right| \geq \frac{1}{10} \cdot (10^{n_{2j-1}}/10^{n_j}) \mod 1.$$
Then

\[ |\hat{\nu}_{2k}(s)| = 2^{2k} \prod_{j=1}^{k} |\hat{\delta}_j(s)| \leq 2^{2k} \prod_{j=1}^{k} (1 - D \cdot (10^{n_j} - 1)^2) , \]

where \( D \) is an absolute constant. Therefore

\[ \|\nu_{2k}\|_{PM} \leq 2^{2k} \prod_{j=1}^{k} (1 - D(r(2j)/r(2j - 1))^2) . \]

However, \( \|\nu_{2k}\|_{\text{MAX}} = 2^{2k} \), so

\[ \|\nu_{2k}\|_{\text{MAX}}/\|\nu_{2k}\|_{PM} \geq \left| \prod_{j=1}^{k} (1 - D(r(2j)/r(2j - 1))^2) \right| . \]

Therefore \( \|\nu_{2k}\|_{\text{MAX}}/\|\nu_{2k}\|_{PM} \to \infty \) as \( k \to \infty \). Hence \( F_r \) is not a member of the class \( M_\nu \). The proof with \( b \neq 10 \) is completely analogous to the proof with \( b = 10 \).

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**BIBLIOGRAPHY**


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