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INJECTIVE ENDOMORPHISMS OF VARIETIES AND SCHEMES

JAMES AX

It is shown that every injective endomorphism of a scheme Y of finite type over a scheme X is surjective. The proof is easily reduced to the case where X is field which in turn follows from the analogous result for algebraic varieties. This result is proved using model theoretic methods to transfer the corresponding and trivially true fact about finite fields.

A finiteness property (Corollary 1 to the theorem) of algebraic varieties observed in [1, § 14] is that every injective endomorphism of a variety is surjective. Our purpose here is to establish a generalization of this result.

THEOREM. *Let Y be a scheme of finite type over a scheme X. Let $Y \xrightarrow{\varphi} Y$ be an X-morphism. If φ is injective then φ is surjective.*

COROLLARY 1. *Let Y be an algebraic variety over an algebraically closed field k. Let $Y \xrightarrow{\varphi} Y$ be a morphism. Assume that the induced mapping $\varphi(k)$ of the k-valued points $Y(k)$ of Y to $Y(k)$ is injective. Then $\varphi(k)$ is surjective.*

COROLLARY 2. *Let $R \xrightarrow{\mu} R$ be a homomorphism of a finitely generated algebra to itself. If for each prime ideal p of $\mu(R)$ there is at most one prime q of R such that $q \cap \mu(R) = p$ then for each p there is precisely one such q.*

The proof of our main result goes through its Corollary 1 whose proof in [1] follows certain involved considerations about finite fields which, although they suggested the existence of such a result and motivated its proof, are completely unnecessary. We give afresh in § 1 a brief proof of Corollary 1. The main new point to be established is the special case of the theorem when $X = \text{Spec}(k)$, k a field. This is accomplished by ascent to the algebraic closure.

Several possible extensions of the theorem suggest themselves. Of course there exist well-known examples of injective analytic endomorphisms which are not surjective, for example in [3, Chap. III, § 1], an isomorphism of $\mathbf{C}^2$ onto an open nondense subset of $\mathbf{C}^2$ is defined. In § 4 we exhibit a Dedekind domain not having the finiteness property.

1. Injective endomorphisms of algebraic varieties. In this section, k denotes an algebraically closed field. Let Y be an algebraic variety and $Y \xrightarrow{\varphi} Y$ a morphism, both defined over k. φ defines a mapping (also called φ) from the underlying topological space of Y (also called Y) to itself. φ also defines a map φ^c from the closed points Y^c of Y to Y^c. In Lemma 1 of § 2 it is shown that φ is injective (respectively: surjective) if and only if φ^c is injective (respectively: surjective) . In the present situation Y^c can be identified with the k-valued points $Y(k)$ of Y over k.

Proof of Corollary 1. The complete case being trivial, the most interesting case is when Y is affine; to simplify notation, we assume Y affine (cf. the remark following the proof). Then there exist positive integers n, t, d and polynomials $g_1, \cdots, g_t, f_1, \cdots, f_n \in k[X_1, \cdots, X_n] = k[X]$ such that: Y is (isomorphic to) Spec $(k[X]/\langle g_1, \cdots, g_t \rangle)$; $f_1, \cdots, f_n$ define a k-morphism Spec $(k[X]) \to$ Spec $(k[X])$ inducing $Y \xrightarrow{\varphi} Y$; $\deg g_\tau, \deg f_\nu \leqq d, \tau = 1, \cdots, t, \nu = 1, \cdots, n$.

Let $E = E_{ntd}$ be the following statement about an arbitrary field K:

if $G_1, \cdots, G_t, F_1, \cdots, F_n \in K[X_1, \cdots, X_n]$ are such that

(a) $\deg G_\tau, \deg F_\nu \leqq d$ for $\tau = 1, \cdots, t$ and $\nu = 1, \cdots, n$; and

(b) if $x \in K^n$ is such that $G_\tau(x) = 0$ for $\tau = 1, \cdots, t$ then $G_\tau(F_1(x), \cdots, F_n(x)) = 0$ for $\tau = 1, \cdots, t$; and

(c) if $x, y \in K^n$ are such that $G_\tau(x) = G_\tau(y) = 0$ for $\tau = 1, \cdots, t$ and $F_\nu(x) = F_\nu(y)$ for $\nu = 1, \cdots, n$ then $x = y$;

then for all $x \in K^n$ such that $G_\tau(x) = 0$ for $\tau = 1, \cdots, t$ there exists $w \in K^n$ such that $G_\tau(w) = 0$ for $\tau = 1, \cdots, t$ and such that $x_\nu = F_\nu(w)$, $\nu = 1, \cdots, n$.

It suffices to prove that E holds when K is algebraically closed. Briefly, E says that for all choices of Y and of a polynomial mapping of K^n to itself inducing $Y(K) \xrightarrow{\lambda} Y(K)$ and for all $v \in Y(K)$ there exist $w, x, y \in Y(K)$ such that either $\lambda(w) = v$ or $\lambda(x) = \lambda(y)$. From this we see that E is an elementary property, i.e., there exists an elementary statement $\mathscr{E} = \mathscr{E}_{ntd}$ such that E holds for K if and only if $\mathscr{E}$ is true in K. Moreover from our brief description of E it is seen that $\mathscr{E}$ can be taken to be of universal-existential type, i.e., $\mathscr{E}$ is in the normal form $\forall X_1 \cdots \forall X_a \exists X_{a+1} \cdots \exists X_b \mathscr{F}$ where $\mathscr{F}$ is quantifier free. This last fact means that if E holds for each member of an ascending sequence of fields than E holds for the union of the sequence. This is also easily verified directly.

E is true when K is finite since an injective mapping of a finite set to itself is surjective. Since the algebraic closure of a finite field is an ascending union of finite fields, E is true when K is the algebraic closure of a finite field. Now E being an elementary statement is true in one algebraically closed field if and only if it is true every algebraically closed field of the same characteristic [8, § 5.8] or [7, § 9, p. 109]. Thus E is true in every algebraically closed field of prime characteristic. By a corollary [7, § 6, p. 111] to the compactness theorem [7, Proposition 2, p. 100], E is true in every algebraically closed field, as desired. A more algebraic version of this proof would proceed by observing that E is true for ultraproducts of fields for which it is true. Thus E is true for every ultraproduct of the algebraic closures of finite fields. Since every algebraically closed field of cardinality the continuum can be so obtained [7, § 0, p. 67], the result follows again by the Lefshetz Principle.

REMARK. A detailed proof for Y an abstract variety would be similar but with more complicated notation. It would necessitate considering a finite affine cover of Y, $Y = \bigcup_{i=1}^{a} Y_i$ with $Y_i \cap Y_j = Y_{ij}$ affine. Then we would need to consider affine imbeddings of the Y_i and Y_{ij} and polynomials defining the imbeddings as well as the maps including φ. Then there exists an elementary statement $E = E_{adN}$ corresponding to the case where the above polynomials are all of degree at most d in at most N variables. The proof then continues as before.

2. Injective and surjective morphisms. Let k be a field and let V, W be schemes of finite type over k. Let V^c denote the closed points of V. Any Spec(k)-morphism $V \xrightarrow{\psi} W$ induces a mapping $\psi^c = (\psi \mid V^c)\colon V^c \to W^c$.

LEMMA 1. *ψ is injective $\Leftrightarrow$ ψ^c is injective, ψ is surjective $\Leftrightarrow$ ψ^c is surjective.*

Proof. Let closures be denoted by bars. Let $v \in V$, $w \in Y$. Then $\psi(v) = w$ if and only if $\psi^c(\{\bar{v}\}^c)$ is dense in $\{\bar{w}\}^c$. The lemma is a straightforward consequence of this characterization of ψ by ψ^c and dimension theory.

We denote the function field of a variety V by $K(V)$. If Y is a scheme and x is a point of Y then $\kappa(x)$ is the residue class field of the local ring at x.

LEMMA 2. *Let $V \xrightarrow{\psi} W$ be a* Spec (k)*-morphism of reduced and geometrically irreducible* Spec (k)*-schemes. Assume ψ is dominating and injective. Then ψ is purely inseparable.*

Proof. The domination of ψ allows us to regard $L = K(W)$ as a subfield of $N = K(V)$. The injectivity of ψ implies that N/L is a finite algebraic extension; we must show this extension is purely inseparable. By passage to affine opens we can assume V and W are affine: $V = \operatorname{Spec} B$, $W = \operatorname{Spec} A$. Let M be the maximum separable extension of L contained in N. There exists $\theta \in N$ such that $M = L(\theta)$. We can find an $a \in A - \{0\}$ with the following properties: $\theta \in B[1/a]$; $B[1/a]$ is integral over $A[1/a]$; $A[1/a]$ is normal; and $A[1/a, \theta]$ is unramified over $A[1/a]$. For all maximal ideals p of $C = A[1/a]$ we have

$$[M: L] = \sum_q [\kappa(q): \kappa(p)]$$

where the sum is over all maximal ideals q of $D = C[\theta]$ for which $q \cap C = p$. Since $B[1/a]$ contains D and is integral over C, the injectivity of ψ implies there is precisely one such maximal ideal q_p. To complete the proof it suffices to find p such that $[\kappa(q_p): \kappa(p)] = 1$. Let $f \in L[X]$ be the monic irreducible polynomial for θ over L. Since C is normal, $f \in C[X]$.

We assert $f(C) \not\subseteq C^*$, i.e., that there exists $c \in C$ such that $f(c)$ is a nonunit of C, provided that $C \neq k$ (if $C = k$ then $L = M = k$ and there is nothing more to prove); i.e., provided $m = \dim W \geqq 1$. By Noether normalization, C is a finite integral extension of $k[Y] = k[Y_1, \cdots, Y_m]$. Let $\mathcal{N}$ denote the norm map $L \to k(Y)$. $\mathcal{N}$ defines multiplicative maps $C \to k[Y]$, $C^* \to k[Y]^* = k^*$, and $C[X] \to k[Y][X]$. $g = \mathcal{N}(f)$ is monic polynomial in X with coefficients in $k[Y]$. Hence there exists $c \in k[Y] \subseteqq C$ such that $g(c) \in k[Y] \sim k^*$. Thus $\mathcal{N}(f(c)) = \mathcal{N}(f)(c) = g(c) \in k[Y] \sim k[Y]^*$. Thus $f(c)C \sim C^*$ which establishes our assertion.

Let p be any maximal ideal containing $f(c)$. The C-homomorphism $C[X] \xrightarrow{\mu} C$ defined by $\mu(X) = c$ composed with the natural surjection $C \xrightarrow{\nu} \kappa(p)$ gives a k-homomorphism $C[X] \xrightarrow{\nu \circ \mu} \kappa(p)$ with kernel W generated by $X - c$ and p. Since $f(c) \in p, f(X) \in W$ and so $\nu \circ \mu$ defines a k-homomorphism $D = C[X]/(f) \xrightarrow{\rho} \kappa(p)$. The kernel of ρ is a maximal ideal of D above p, i.e., q_p and ρ induces an inverse of the natural inclusion $\kappa(p) \hookrightarrow \kappa(q_p)$. This completes the proof.

Remark. The case of Lemma 2 corresponding to k algebraically closed is well-known [5, Th. 3, p. 115]. In the general case the crucial point is to establish the existence of a closed point of degree 1. The existence of a closed point of degree 1 can actually be established in greater generality. For example, if $V \xrightarrow{\psi} W$ is any nonconstant k-morphism of algebraic varieties (defined over any field k), then there exists closed points q of V and p of W such that $\psi(q) = p$ and $\kappa(q) \approx_k \kappa(p)$.

3. Reduction to varieties. In this section we show how the theorem follows from the special case where $X = \mathrm{Spec}\,(k)$, k a perfect field and where Y is variety defined over k, i.e., Y is a reduced and geometrically irreducible scheme of finite type over k.

Let x be a point of X, O_x its local ring and $\kappa(x)$ the residue class field of O_x. Let F_x be the fibre of $Y \xrightarrow{\pi} X$ above $x; F_x$ can be defined as the fibre product of π and the canonical map $\mathrm{Spec}\,(\kappa(x)) \rightarrow X$. Thus we get a universal commutative diagram

$$\begin{array}{ccc} F_x & \xrightarrow{\lambda_x} & Y \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\,(\kappa(x)) & \longrightarrow & X\,. \end{array}$$

The map λ_x is a bijection of the points of F_x onto the points $y \in Y$ for which $\pi(y) = x$. The commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & Y \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array} \qquad \text{induces} \qquad \begin{array}{ccc} F_x & \xrightarrow{\varphi_x} & F_x \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathrm{Spec}\,(\kappa(x)) & \end{array}$$

and since φ is injective so is φ_x. Using that $Y \xrightarrow{\pi} X$ is of finite type we deduce that π_x is of finite type; indeed the only finiteness condition we need about π is that π_x is of finite type for all $x \in X$. As $Y = \bigcup \lambda_x(F_x)$, it suffices to prove that every φ_x is surjective. This shows that it suffices to prove the main theorem in the special case where $X = \mathrm{Spec}\,(k)$, k a field.

Let $\sqrt{k}$ denote the perfect closure of k. Then

$$X x_{\mathrm{Spec}(k)} \mathrm{Spec}\,(\sqrt{k}\,) \longrightarrow X$$

is bijective so that we can assume k perfect. If $Y_{\mathrm{red}} \xrightarrow{j} Y$ is the canonical map of the (maximal) closed reduced subscheme of Y to Y then j is a bijection so that we can assume Y is reduced.

We now show that we can assume that Y is irreducible over k, and hence a k-variety. Let $Y = \bigcup_{\tau=1}^{t} Y_\tau$ where the Y_τ are k-varieties. We proceed by induction on $d = \dim Y = \max_\tau \dim Y_\tau$. We arrange notation so that $Y_1, \cdots, Y_s$ are all the components of dimension d. Since φ is injective, $\varphi(\bigcup_{\sigma=1}^{s} Y_\sigma) \subseteqq \bigcup_{\sigma=1}^{s} Y_\sigma$. For all $\sigma \in [1, s]$ as Y_σ is irreducible there exists $p(\sigma) \in [1, s]$ such that $\varphi(Y_\sigma) \subseteqq Y_{p(\sigma)}$. The mapping $[1, s] \xrightarrow{p} [1, s]$ is, by dimension theory (cf. § 2) and the injectivity of φ, itself an injection, i.e., a permutation of $[1, s]$. Thus there exists a positive integer e such that $p^{(e)}$ is the trivial permutation of $[1, s]$. Replacing φ by $\varphi^{(e)}$ we have that p is already trivial. Thus $(\varphi \mid Y_\sigma)\colon Y_\sigma \rightarrow Y_\sigma$ is injectve. Hence assuming the theorem established *for k-varieties we have that each $(\varphi \mid Y_\sigma)$ is surjective for $\sigma \in [1, s]$.* This implies that $\varphi(\bigcup_{\tau=s+1}^{t} Y_\tau) \subseteqq \bigcup_{\tau=s+1}^{t} Y_\tau$ and so by inductive hypothesis, $\varphi(\bigcup_{\tau=s+1}^{t} Y_\tau) = \bigcup_{\tau=s+1}^{t} Y_\tau$. Therefore φ is surjective.

Still assuming k is perfect let $Y \xrightarrow{\pi} \operatorname{Spec}(k)$ with Y a k-variety. Let k' denote the relative algebraic closure of k in $K(Y)$, the function field of Y. Then we have the factorization $[Y \rightarrow \operatorname{Spec}(k)] = [Y \xrightarrow{\pi'} \operatorname{Spec}(k') \rightarrow \operatorname{Spec}(k)]$. Here $Y \xrightarrow{\pi'} \operatorname{Spec}(k')$ gives Y the structure of a variety defined over k' but $Y \xrightarrow{\varphi} Y$ need not be a $\operatorname{Spec}(k')$-morphism. Nevertheless φ induces a k-automorphism α of k'. Since $[k' : k] < \infty$ there exists a positive integer f such that α^f is trivial on k'. Then φ^f is a $\operatorname{Spec}(k')$-morphism and our reduction is complete.

4. **Completion of the proof.** We now complete the proof of the main theorem by proving it when Y is an algebraic variety defined over a field k and ψ is a k-morphism. Let Y' denote the normalization of $Y x_{\operatorname{Spec}(k)} \operatorname{Spec}(\tilde{k})$ where $\tilde{k}$ is an algebraic closure of k. We have a commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{\psi'} & Y' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ Y & \xrightarrow{\psi} & Y . \end{array}$$

Since τ is surjective with finite fibres and ψ is injective, ψ' has finite fibres. By Zariski's Main Theorem [6, p. 414], there exists a factorization $Y' \hookrightarrow Z \xrightarrow{\sigma} Y'$ of ψ where σ is finite and $Y' \hookrightarrow Z$ is an isomorphism of Y' onto a (dense) open of Z. By Lemma 2, ψ is purely inseparable. Hence so are ψ' and σ. As Y' is normal, σ is a *bijection. Thus ψ' is injective. By the corollary to main theorem which we have already established in* § 1, *ψ' is surjective. τ is also surjective and hence so is ψ. This completes the proof.*

AN EXAMPLE. The implication "one-one implies onto" seems to be a persistent property of self-mappings of objects with "some finiteness" conditions. Indeed this property is for abstract sets the definition of finite. *The "first" set not satisfying this is the positive integers* **P** with the mapping $p \rightarrow p + 1$. This can be made into an example of a nice Noetherian ring R and a homomorphism $R \xrightarrow{\varphi} R$ which gives a nonsurjective injection $\operatorname{Spec}(R) \rightarrow \operatorname{Spec}(R)$. Indeed, let $R = \mathbf{C}[t, (t-c)^{-1} : c \in \mathbf{C} \sim \mathbf{P}]$ and let φ be defined by $t \rightarrow t + 1$. Then as point sets, $\mathbf{P} = \operatorname{Spec}(R)$ and the mapping is as above. Since R is the intersection of the discrete valuation rings $\mathbf{C}[t]_q$ where q runs through the prime ideals $(t-c)\mathbf{C}[t]$ for $c \in \mathbf{P}$, R is a Dedekind domain.

We would like to mention some recent proofs of these results. In conversation, G. Shimura showed how to obtain Corollary 1 by using reduction modulo p to again reduce the result to the case where k is the algebraic closure of a finite field. It seems that this technique would also be able to directly establish the case of the main result where $X = \operatorname{Spec}(k)$, k a field (from which the theorem is easily deduced). A third proof was given by A. Borel [4]. His proof is cohomological and proves Corollary 1 at least in characteristic zero. While the prime characteristic case would be difficult to establish by this technique in complete generality, Borel was able to extend his method to prove a real analogue of Corollary 1. More recently S. Lichtenbaum gave a direct proof of the theorem when $X = \operatorname{Spec}(k)$, k a field and Y is affine using the Mordell-Weil Theorem. Finally we should mention that the first and only previous result of this kind was obtained by A. Bialynicki-Birula and M. Rosenlicht who gave a simple proof in [2] of the special case of Corollary 1 when Y is affine space $\mathbf{A}^n$.

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A GENERALIZATION OF THE WEINSTEIN-ARONSZAJN FORMULA

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This paper uses a technique of abstract spectral theory to reduce the study of certain eigenvalues, which are not necessarily isolated, to the case of isolated eigenvalues. By this method the Weinstein-Aronszajn formula for the change in multiplicity of an isolated eigenvalue of a self adjoint operator under a finite dimensional perturbation is extended.

The hypotheses of this generalization are studied in the abstract and also by demonstrative example.

The central result of this work is Corollary 3 to Theorem 1 which gives a generalization of the Weinstein-Aronszajn formula for the change of multiplicity of an eigenvalue under a finite dimensional perturbation. The reader should observe that the hypotheses of that corollary are trivially satisfied in the case of an isolated eigenvalue.

Although the hypotheses of this main result are easy to understand from the point of view of abstract spectral theory, there are obvious questions about their computability and about their applicability. These two questions are investigated in § 1 and § 3. Section 1 describes some Hilbert space geometries under which the hypotheses are satisfied. Section 3 gives two elementary examples.

The actual technique used to prove Theorem 1 is to remove a small deleted interval about the eigenvalue from the spectrum of the operator. This is accomplished by replacing the original Hilbert space with the orthogonal complement of the subspace causing the spectrum in that deleted interval. Such a constructive process requires the handling of complicated technical details. Then we apply the theory for isolated eigenvalues and we deduce from that conclusion some conclusions with the original hypotheses. This new technique, which seems to be very general in nature, is probably the most interesting feature of this paper.

1. Preliminaries.

NOTATION. Throughout this paper T_0 will be a self adjoint (not necessarily bounded) unperturbed operator and $V = \sum_{j=1}^{r} \langle \cdot, \phi_j \rangle c_j \phi_j$ is a self adjoint perturbation; both operators are defined on a complex Hilbert space H. So $T = T_0 + V$ is defined on the dense domain of T_0 and we write $R(z) = (T - zI)^{-1}$, $R_0(z) = (T_0 - zI)^{-1}$. The spectral measures and the resolutions of identify of the two operators T_0, T

are denoted $E_0(\cdot)$, $E(\cdot)$ and $E_0(t)$, $E(t)$, respectively. $\mathscr{R}(V)$ means the range of V and the Weinstein-Aronszajn matrix is denoted by $W(z) = [I + VR_0(z)]/\mathscr{R}(V)$ while $\omega(z) = \det W(z)$ is the $W-A$ determinant.

MAIN HYPOTHESES. This paper is concerned with a generalization of isolated eigenvalues which in many instances includes the so called embedded eigenvalues (eigenvalues which belong to an interval which is wholly contained in the spectrum). If λ_0 is not in the essential spectrum of T_0, i.e., λ_0 is in the resolvent set or λ_0 is an isolated eigenvalue of finite multiplicity, then by the stability of the essential spectrum under compact perturbations (the Weyl Theorem, see p. 367 of [5]) we get that λ_0 is not in the essential spectrum of $T_0 + V$. Thus there exists a $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ with $\delta > 0$ such that $E_0(D_\delta) = 0$ and $E(D_\delta) = 0$. Necessarily

$$(*)\ E_0(D_\delta)E(\{\lambda_0\}) = 0 \quad \text{and} \quad (**)\ E(D_\delta)E_0(\{\lambda_0\}) = 0\ .$$

However, the converse of the last statement is not true; in fact both (*) and (**) may be satisfied while λ_0 is actually an embedded eigenvalue of T_0. If both (*) and (**) are satisfied we say that λ_0 is *quasi-isolated*. Since (*) and (**) depend on V we should say "quasi-isolated with respect to V." However, we shall abuse notation and use the shorter phrase.

All isolated eigenvalues are quasi-isolated. The following propositions will demonstrate some of the Hilbert space geometries which produce quasi-isolated eigenvalues. These constructions exploit the easy fact that the spectral measure of a direct sum operator is the direct sum of the spectral measures of the operators in the sum. Thus if $T_0 = T_1 \oplus T_2$ is self adjoint and defined on $H = H_1 \oplus H_2$, then $E_0(D)H_i \subset H_i$ and in fact $E_0(D)/H_i$ is the spectral measure for T_i.

PROPOSITION 1. *Let $T_0 = T_1 \oplus T_2$ be self adjoint and let $V = V_1 \oplus V_2$ be a compact self adjoint operator on $H = H_1 \oplus H_2$. If λ_0 is not a point in the essential spectrum of T_1 and λ_0 is not an eigenvalue for T_2, then $E(D_\delta)E_0(\{\lambda_0\}) = 0$ for all sufficiently small δ.*

Proof. Let $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ for $\delta > 0$. Since λ_0 does not become a point of the essential spectrum of $T_1 + V_1$ there is some positive δ such that $E(D_\delta)H_1 = \{0\}$. Then $E(D_\delta)H = E(D_\delta)(H_1 \oplus H_2) \subset H_2$ since $E(D_\delta)H_2 \subset H_2$. By hypothesis $E_0(\{\lambda_0\})H \subset H_1$. Thus $E_0(\{\lambda_0\})H$ is orthogonal to $E(D_\delta)H$ and we have $E(D_\delta)E_0(\{\lambda_0\}) = 0$.

PROPOSITION 2. *Let T_0 be self adjoint and let V be a finite dimensional self adjoint operator on H. Let $\{\tau_j\}$ be a basis for $E(\{\lambda_0\})H$. If for each τ_j there exists some $\delta(j) > 0$ such that*

$$E_0(D_{\delta(j)})V\tau_j = 0\ , \quad \textit{then} \quad E_0(D_\delta)E(\{\lambda_0\}) = 0$$

for all sufficiently small δ.

Proof. Since V is finite dimensional for only a finite subset of $\{\tau_j\}$ can V be nonzero. Let $\{\tau_1, \cdots, \tau_p\}$ be that subset; so $V\tau_j = 0$ for $j \geqq p + 1$. Let $0 < \delta < \delta(j)$ then

$$0 = \langle E_0(D_{\delta(j)})V\tau_j\ V\tau_j\rangle \geqq \langle E_0(D_\delta)V\tau_j,\ V\tau_j\rangle = \|E_0(D_\delta)V\tau_j\|^2 \geqq 0\ .$$

Thus $E_0(D_\delta)V\tau_j = 0$ for all j. Hence

($\dagger$) $E_0(D_\delta)VE(\{\lambda_0\})H = \{0\}$.

If $\tau \in E(\{\lambda_0\})H$, then $0 = (T - \lambda_0)\tau = V\tau + (T_0 - \lambda_0)\tau$ or $V\tau = (\lambda_0 - T_0)\tau$. Using ($\dagger$) above we see that

$$0 = E_0(D_\delta)V\tau = E_0(D_\delta)(\lambda_0 - T_0)\tau = (\lambda_0 - T_0)E_0(D_\delta)\tau\ .$$

This says that $E_0(D_\delta)\tau$ which is conspicuously a vector from $E_0(D_\delta)H$ is a λ_0-eigenvector of T_0. Since $E_0(\{\lambda_0\})E_0(D_\delta) = E_0(\{\lambda_0\} \cap D_\delta) = E_0(\phi) = 0$ the above is only possible if $E_0(D_\delta)\tau = 0$. By the arbitrariness of τ we have shown $E_0(D_\delta)E(\{\lambda_0\})H = \{0\}$ or $E_0(D_\delta)E(\{\lambda_0\}) = 0$.

COROLLARY 1. *Let T_0 and V be self adjoint operators on H. If $E_0(D_\delta)H \subset \ker V$, then $E_0(D_\delta)E(\{\lambda_0\}) = 0$.*

Proof. Taking orthogonal complements of $E_0(D_\delta)H$ and ker V while using that $E_0(D_\delta)$ and V are self adjoint we get that ker $E_0(D_\delta) \supset \mathscr{R}(V)$. So the hypotheses of the preceding proposition are trivially satisfied.

2. A generalization of the Weinstein-Aronszajn formula.

In this section we shall prove a generalization of the formula given by Weinstein and Aronszajn and extended by Kuroda for the change in the multiplicity of an eigenvalue under perturbation. As in the work of Weinstein and Aronszajn the perturbation will be finite dimensional. Because we use Kuroda's form of the $W-A$ theorem the restriction that the operators be self adjoint is essential. For Kuroda in [4] uses the notion of algebraic multiplicity of an eigenvalue while the notion used here is that of geometric multiplicity. For self adjoint operators the two notions coincide.

Before giving the generalization of the Weinstein-Aronszajn-Kuroda theory, we reformulate the theory in a manner appropriate for this work. Proofs of the following facts may be found in [3, pp. 244–250].

The Weinstein-Aronszajn formula for isolated eigenvalues. Clearly $\mathscr{R}(V)$, the range of the perturbation, is invariant under the operator $I + VR_0(x)$; so it makes sense to consider $\omega(z) = \det\{I + VR_0(z)/\mathscr{R}(V)\}$ and the usual definition is available since $\mathscr{R}(V)$ is finite dimensional. We define a multiplicity function for a self adjoint operator S by

$$\nu(\zeta, S) = \begin{cases} 0 \text{ if } \zeta \text{ is in the resolvent set of } S \\ \text{dimension of the eigenspace for } S \\ \quad \text{if } \zeta \text{ is an isolated eigenvalue} \\ \infty \text{ otherwise.} \end{cases}$$

We define the multiplicity of $\omega(z)$ at ζ by

$$\nu(\zeta, \omega) = \begin{cases} k \text{ if } \zeta \text{ is a zero of } \omega(z) \text{ of order } k \\ -k \text{ if } \zeta \text{ is a pole of } \omega(z) \text{ of order } k \\ 0 \text{ otherwise .} \end{cases}$$

Then the $W-A$ formula is just $\nu(\zeta, T_0 + V) = \nu(\zeta, T_0) + \nu(\zeta, \omega)$ for $\zeta \in D$ where D is a region of the complex plane such that the only spectra of T_0 and $T_0 + V$ in D are isolated eigenvalues.

In what follows it will be convenient to have the $W-A$ formula in a slightly different form. A statement clearly equivalent to the one given above is the following: there exists an integer k such that $(\zeta - z)^k\omega(z)$ is bounded above and bounded away from zero in some neighborhood of ζ and $k = \nu(\zeta, T_0) - \nu(\zeta, T_0 + V)$. This statement follows from the well known behavior of a meromorphic function in every neighborhood of a pole and in every neighborhood of a zero. In fact the integer k can be specified by $0 < m \leqq \gamma^k \,|\, \omega(\zeta + i\gamma) \,|\, \leqq M < +\infty$ for $\gamma \leqq \gamma_0$ for some positive γ_0 where ζ is real. In the following we shall use this determination of k.

THEOREM 1. *Let P be the orthogonal projection onto $\mathscr{R}(V)$ which has $\{\phi_j\}_{j=1}^r$ as an orthonormal basis. Let T_0 and $V = \sum_{j=1}^r \langle \cdot, \phi_j \rangle c_j\phi_j$ be self adjoint operators on the complex Hilbert space H. If there exists a $\delta > 0$ such that*

(1) *$E(D_\delta)E_0(\{\lambda_0\}) = 0$ with $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$*
and

(2) *$\| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \leqq M < +\infty$ for all sufficiently small γ, say $\gamma < \gamma_0$, then the following are true:*

(a) *There exists an integer k such that*

$$(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det[(I + VR_0(z))/\mathscr{R}(V)]$$

is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small and

(b) *$\nu(\lambda_0, T_0 + V) \geqq \nu(\lambda_0, T_0) - k$ where $\nu(\zeta, S)$ is the multiplicity of ζ as an eigenvalue for S.*

Step 1. Let $Q_\delta = I - E(D_\delta)$. Then $\nu(\lambda_0, T - Q_\delta VQ_\delta) \geqq \nu(\lambda_0, T_0)$.

Proof. It is sufficient to show that every solution of $T_0\tau = \lambda_0\tau$ is a solution of $(T - Q_\delta VQ_\delta)\tau = \lambda_0\tau$. If τ is a solution of $T_0\tau = \lambda_0\tau$, then $\tau \in E_0(\{\lambda_0\})H$ and by hypothesis (1) $E(D_\delta)\tau = 0$ or $Q_\delta\tau = \tau$ for δ sufficiently small. Thus $(T - Q_\delta VQ_\delta)\tau = (TQ_\delta - Q_\delta V)\tau = Q_\delta T_0\tau = \lambda_0\tau$ as required.

Step 2. Let P be the orthogonal projection onto $\mathscr{R}(V)$. For all sufficiently small $\delta > 0$ $\{Q_\delta\phi_j\}$ is a basis for $\mathscr{R}(-Q_\delta VQ_\delta)$ and in this basis the matrix of $I + (-Q_\delta VQ_\delta)R(z)$ restricted to $\mathscr{R}(-Q_\delta VQ_\delta)$ is identical with the matrix of $[P(Q_\delta - Q_\delta VQ_\delta R(z))]/\mathscr{R}(V)$ in the basis $\{\phi_j\}$.

Proof. First let us note that a straightforward consequence of the measure-theoretic properties of $E(\cdot)$ is that $E(D_\delta) \to 0$ strongly as $\delta \to 0$ and so $Q_\delta \to I$ strongly as $\delta \to 0$.

So for each ϕ_j, $1 \leqq j \leqq r$, there exists a $\delta(j)$ such that $\|(I - Q_\delta)\phi_j\| < 1/2r$ for $\delta \leqq \delta(j)$. If $\delta < \delta(j)$ for all j and $V = \sum_{k=1}^r \beta_k\phi_k$ and $1 = \|V\|^2 = \sum_{k=1}^r |\beta_k|^2$ then $|\beta_k| \leqq 1$ for each k, $1 \leqq k \leqq r$, and

$$\begin{aligned}\|Q_\delta v\| &\geqq \|Iv\| - \|(I - Q_\delta)v\| \geqq 1 - \left\|\sum_{k=1}^r \beta_k(I - Q_\delta)\phi_k\right\| \\ &\geqq 1 - \sum_{k=1}^r |\beta_k|\,\|(I - Q_\delta)\phi_k\| > 1 - (1/2r)\sum_{k=1}^r |\beta_k| \\ &\geqq 1 - (1/2r)r = 1/2\ .\end{aligned}$$

So $Q_\delta v \neq 0$. Thus ker $Q_\delta/\mathscr{R}(V) = \{0\}$ or $Q_\delta/\mathscr{R}(V)$ is one to one for all sufficiently small δ. Since $\{\phi_j\}$ is a basis for $\mathscr{R}(V)$ it must be that $\{Q_\delta\phi_j\}$ is a linearly independent set for all δ sufficiently small. We note that

$$-Q_\delta VQ_\delta = -Q_\delta\left(\sum_{j=1}^r \langle Q_\delta\cdot, \phi_j\rangle c_j\phi_j\right) = \sum_{j=1}^r \langle\cdot, Q_\delta\phi_j\rangle(-Q_\delta c_j\phi_j)\ ,$$

Clearly Span $\{Q_\delta\phi_j\} = \mathscr{R}(-Q_\delta VQ_\delta)$ and since $\{Q_\delta\phi_j\}$ is a linearly independent set for all sufficiently small δ, we get that $\{Q_\delta\phi_j\}$ is a basis for $(-Q_\delta VQ_\delta)$ for all sufficiently small δ.

A straightforward computation gives that both

$$\langle[I - Q_\delta VQ_\delta R(z)]Q_\delta\phi_i, Q_\delta\phi_j\rangle \quad \text{and} \quad \langle P[Q_\delta - Q_\delta VQ_\delta R(z)]\phi_i, \phi_j\rangle$$

are equal to $\langle Q_\delta\phi_i, \phi_j\rangle - \langle VR(z)Q_\delta\phi_i, Q_\delta\phi_j\rangle$ for $i, j = 1, \cdots, r$.

Step 3. For $\nu = \nu(\lambda_0, T) - \nu(\lambda_0, T - Q_\delta VQ_\delta)$, $(\lambda_0 - z)^\nu \det\{P(Q_\delta - Q_\delta VQ_\delta R(z))/\mathscr{R}(V)\}$ is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small.

Proof. First we note that we may add to the hypotheses of the theorem the conclusion of the preceding step. This condition is guaranteed by simply choosing δ sufficiently small.

Consider $T/Q_\delta H$ as an unperturbed operator and $-Q_\delta VQ_\delta/Q_\delta H$ as a perturbation. Since $Q_\delta = I - E((\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta))$ it is clear that λ_0 is isolated from the spectrum of $T/Q_\delta H$. Thus we can apply the classical $W-A$ formula and we observe that the $W-A$ matrix is $[I + (-Q_\delta VQ_\delta)R(z)]/\mathscr{R}(-Q_\delta VQ_\delta)$. The $W-A$ theorem asserts the existence and uniqueness of an integer ν such that

$$(\lambda_0 - z)^\nu \det\{[I + (-Q_\delta VQ_\delta)R(z)]/\mathscr{R}(-Q_\delta VQ_\delta)\}$$

is bounded above and bounded away from zero for all $z = \lambda_0 + ir$ with γ sufficiently small and $\nu = \nu(\lambda_0, T) - \nu(\lambda_0, T - Q_\delta VQ_\delta)$. By the previous step we see that this ν is the unique integer such that

$$(\lambda_0 - z)^\nu \det\{[P(Q_\delta - Q_\delta VQ_\delta R(z))]/\mathscr{R}(V)\}$$

is bounded above and is bounded away from zero for all $z = \lambda_0 + i\gamma$ with γ sufficiently small. Thus Step 3 is proved.

Step 4.

$$[PQ_\delta(T - Q_\delta VQ_\delta - z)(T - z)^{-1}]/\mathscr{R}(V)[(T - z)(T_0 - z)^{-1}]/\mathscr{R}(V)$$

converges to $I_{\mathscr{R}(V)}$ in the norm topology as $\delta \to 0$ and the convergence is uniform in γ for all sufficiently small γ, say $\gamma < \gamma_0$. Thus the determinant of the composite operator is bounded above and is bounded away from zero for all sufficiently small δ uniformly in $z = \lambda_0 + i\gamma$ for $\gamma < \gamma_0$.

Proof. Note that $\mathscr{R}(V)$ is invariant under $(T - z)(T_0 - z)^{-1} = I + VR_0(z)$ and observe the following simplification

$$\begin{aligned}
&[PQ_\delta(T - Q_\delta VQ_\delta - z)(T - z)^{-1}]\mathscr{R}(V)[(T - z)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = [PQ_\delta(T - Q_\delta VQ_\delta - z)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = PQ_\delta[(T_0 - z)(T_0 - z)^{-1} + (V - Q_\delta VQ_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = P[Q_\delta + (Q_\delta V - Q_\delta VQ_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\\
&\quad = P[Q_\delta + Q_\delta V(I - Q_\delta)(T_0 - z)^{-1}]/\mathscr{R}(V)\,.
\end{aligned}$$

Because $\mathscr{R}(V)$ is finite dimensional it would suffice for the conclusion of Step 4 to show convergence in the strong topology of $PQ_\delta/\mathscr{R}(V)$ to $I_{\mathscr{R}(V)}$ and $PQ_\delta V(I - Q_\delta)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)$ to 0 as approaches zero. The first limit is established by taking $x \in \mathscr{R}(V)$ and noting

$$\|(I - PQ_\delta)x\| = \|(P - PQ_\delta)x\| = \|PE(D_\delta)x\| \leqq \|E(D_\delta)x\|$$

and recalling that $E(D_\delta)$ converges strongly to 0. To establish the second limit observe

$$VE(D_\delta)R_0(\lambda_0 + i\gamma)x = \sum_{k=1}^{r} < E(D_\delta)R_0(\lambda_0 + i\gamma)x, \phi_k)c_k\phi_k$$
$$= \sum_{k=1}^{r} \langle x, PR_0(\lambda_0 - i\gamma)E(D_\delta)\phi_k\rangle c_k\phi_k$$

and $\| PR_0(\lambda_0 - i\gamma)E(D_\delta)\phi_k \| \leqq \| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \, \| E(D_\delta)\phi_k \| \leqq M \| E(D_\delta)\phi_k \|$. Since r is finite this proves the second limit is 0 and thus it proves the conclusion of Step 4.

Step 5. Let ν be the same integer as in Step 3. Then

$$(\lambda_0 - z)^{-\nu} \det \{(I + VR_0(z)/\mathscr{R}(V)\}$$

is bounded above and bounded away from zero for $z = \lambda_0 + i\gamma$ and $\gamma < \gamma_0$. This proves the conclusion of the theorem.

Proof. Note the following equation

$$\begin{aligned}[\gamma^{\nu} \det \{P(Q_\delta - Q_\delta VQ_\delta R(\lambda_0 + i\gamma))/\mathscr{R}(V)\}] \\ \times [\gamma^{-\nu} \det \{(T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)\}] \\ = \det \{PQ_\delta(T - Q_\delta VQ_\delta - \lambda_0 - i\gamma)R(\lambda_0 + i\gamma)/\mathscr{R}(V) \\ \times (T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma)/\mathscr{R}(V)\} .\end{aligned}$$

By Step 3 the first bracketed factor is bounded above and bounded away from zero for $\gamma < \gamma_0$; by Step 4 the right side or second line of the equation is bounded above and bounded away from zero for $\gamma < \gamma_0$. Thus ν must be the unique integer such that the second bracketed factor is bounded above and bounded away from zero for $\gamma < \gamma_0$. Since $(T - \lambda_0 - i\gamma)R_0(\lambda_0 + i\gamma) = I + VR_0(\lambda_0 + i\gamma)$ this proves the first assertion of step 5.

By Step 3, $\nu = \nu(\lambda_0, T) - (\lambda_0, T - Q_\delta VQ_\delta)$. Recalling from Step 1 that $\nu(\lambda_0, T - Q_\delta VQ_\delta) \geqq \nu(\lambda_0, T_0)$ we get $\nu(\lambda_0, T) - \nu \geqq \nu(\lambda_0, T_0)$. By letting $k = -\nu$ we see that part (b) of the theorem is proved. Since part (a) was proved in the above paragraph this concludes the proof of Theorem 1.

COROLLARY 1. *Let* (1) *and* (2) *of Theorem* 1 *be satisfied. If* $\omega(\lambda_0 + i\gamma)$ *is bounded above for all sufficiently small* γ *then* λ_0 *is an eigenvalue for* $T_0 + V$ *with multiplicity at least as great as its multiplicity for* T_0.

Proof. This is immediate from Theorem 1.

By using the symmetry between the perturbed and the unperturbed operators—i.e., add $(-V)$ to $(T_0 + V)$ and get T_0—we can get a result symmetric to Theorem 1.

COROLLARY 2 (*to Theorem* 1). *Let* λ_0 *be an eigenvalue of* T_0. *If there exists* $\delta > 0$ *such that*

(1) $E_0(D_\delta)E(\{\lambda_0\}) = 0$ *with* $D_\delta = (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ *and*

(2) $\| PR(\lambda_0 - i\gamma)/E_0(D_\delta)\mathscr{R}(V) \| \leqq M < +\infty$ *for all sufficiently small* γ, *say* $\gamma < \gamma_0$, *then the following are true*:

(a) *there exists a unique integer* k *such that* $(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det\{I + VR_0(z)/\mathscr{R}(V)\}$ *is bounded above and bounded away from zero for all* $z = \lambda_0 + i\gamma$ *and* γ *sufficiently small, and*

(b) $\nu(\lambda_0, T_0) \geqq \nu(\lambda_0, T_0 + V) + k$.

Proof. By a direct application of Theorem 1 considering $T_0 + V$ as the unperturbed operator and $(-V)$ as the perturbation one gets the existence of an integer $-k$ such that $(\lambda_0 - z)^{-k} \det\{I - VR(z)/\mathscr{R}(V)\}$ is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ and γ sufficiently small. Also

$$\nu(\lambda_0, (T_0 + V) - V) \geqq \nu(\lambda_0, (T_0 + V)) - (-k)$$

or

$$\nu(\lambda_0, T_0) \geqq \nu(\lambda_0, T_0 + V) + k\ .$$

$$1 = \{(\lambda_0 - z)^{-k} \det[I - VR(z)]/\mathscr{R}(V)\} \times \{(\lambda_0 - z)^k \det[I + VR_0(z)]/\mathscr{R}(V)\}$$

is obviously bounded above and bounded away from zero everywhere. Since the first factor in braces has this property also, it must be that the second factor in braces has this property. This proves the corollary.

COROLLARY 3. (*The generalized Weinstein-Aronszajn formula.*) *If* λ_0 *is a quasi-isolated eigenvalue for* T_0 *and if there exist positive numbers* γ_0 *and* M *such that* $\| PR_0(\lambda_0 - i\gamma)/E(D_\delta)\mathscr{R}(V) \| \leqq M$ *and* $\| PR(\lambda_0 - i\gamma)/E_0(D_\delta)\mathscr{R}(V) \| \leqq M$ *for* $\gamma < \gamma_0$ *then the following are true*:

(a) *there exists a unique integer* k *such that*

$$(\lambda_0 - z)^k\omega(z) = (\lambda_0 - z)^k \det\{I + VR_0(z)/\mathscr{R}(V)\}$$

is bounded above and bounded away from zero for all $z = \lambda_0 + i\gamma$ *and* $\gamma < \gamma_0$, *and*

(b) $\nu(\lambda_0, T) = \nu(\lambda_0, T_0) - k$.

Proof. Simply apply Theorem 1 and Corollary 2 both.

3. Examples. In this section two examples of the preceding theory will be given; however the verification that they are examples will only be outlined. Following each example, the significant facts about that example will be given.

No effort has been made to render the examples as general as possible; in fact many arbitrary choices have been made. The examples do demonstrate how the theory can be applied and Example 1 shows that the generalization is a proper generalization of the $W-A$ theorem.

EXAMPLE 1. Let H_1 and H_2 be the spaces of square integrable functions on the interval $(1, 2)$ with the measures $du(t)$ and dt, respectively. Here dt is Lebesgue measure while $du(t)$ agrees with dt on $(1, 5/4) \cup (7/4, 2)$ and $u([5/4, 3/2) \cup (3/2, 7/4]) = 0$ and $u(\{3/2\}) = 1$. Let $T_0(f_1(t), f_2(t)) = (tf_1(t), tf_2(t))$ where $(f_1(t), f_2(t))$ is an element of $H = H_1 \oplus H_2$. We obviously have a spectral representation space for T_0 and it is clear that $3/2$ is an eigenvalue for T_0 with a corresponding eigenvector $(\chi_{\{3/2\}}(t), 0) = \psi_0$. Set $V = \langle \cdot, \phi_1 \rangle c_1\phi_1 + \langle \cdot, \phi_2 \rangle c_2\phi_2$ with $\phi_1 = (1, 0)$, $\phi_2 = (0, 1)$.

Fact 1. $3/2$ is quasi-isolated.

Fact 2.

$$\langle R_0(z)\phi_1, \phi_1 \rangle = \int_{(1,2)} (t - z)^{-1} du(t) = (3/2 - z)^{-1} + \int_{(1,5/4) \cup (7/4,2)} (t - z)^{-1} dt$$

and $\langle R_0(z)\phi_2, \phi_2 \rangle = \int_{(1,2)} (t - z)^{-1} dt$ and $\int_{(1,2)} (t - \lambda - i\gamma)^{-1} dt$ approaches as a limit $ln[(2 - \lambda)/(\lambda - 1)] + \pi i$ provided $\lambda \in (1, 2)$ and γ approaches 0 from the right.

Fact 3. If $\omega(z)$ is the $W-A$ determinant $i\gamma\, \omega({}_3/2 + i\gamma)$ is bounded above and is bounded away from zero for $|\gamma|$ sufficiently small.

Fact 4. For all $|\gamma|$ sufficiently small $\| PR(\lambda_0 - i\gamma)/E_0(D_{1/4})\mathscr{R}(V) \| \leqq M$ and thus by Corollary 2 to Theorem 1 we get $\nu(3/2, T_0 + V) = 0$.

Note. Although $3/2$ is quasi-isolated it is an embedded eigenvalue. Still the change in the multiplicity is given by the formula involving the $W-A$ determinant. Finally there is no triviality involved in the example in the sense that H has no proper subspace reducing T_0 and containing $\mathscr{R}(V)$.

EXAMPLE 2. Let H, H_1, H_2, and T_0 be as in Example 1. In the definition of V change ϕ_1 to $(\chi_D(t), 0)$ where $D = (1, 5/4) \cup (7/4, 2)$.

Fact 1. $3/2$ is quasi-isolated.

Fact 2.

$$\langle R_0(z)\phi_1, \phi_1 \rangle = \int_D (t - z)^{-1} du(t) = \int_{(1,5/\ ) \cup (7/4,2)} (t - z)^{-1} dt$$

and $\langle R_0(z)\phi_2, \phi_2\rangle = \int_{(1,2)} (t - z)^{-1}dt$ which approaches as a limit

$$ln(2 - \lambda)/(\lambda - 1)] + i\pi \quad \text{for} \quad \lambda \in (1, 2) .$$

Fact 3. If $\omega(z)$ is the $W-A$ determinant then $\omega(3/2 + i\gamma)$ is bounded above and is bounded away from 0 for $|\gamma|$ sufficiently small.

Fact 4. The 3/2-eigenvector of T_0, $(\chi_{\{3/2\}}(t), 0)$ is in ker V. Thus it is a 3/2-eigenvector for $T = T_0 + V$ and $\nu(3/2, T) \geqq 1$.

Fact 5. For all $|\gamma|$ sufficiently small

$$\| PR(\lambda_0 - i\gamma)/E_0(D_{1/4})\mathscr{R}(V) \| \leqq M .$$

Using Corollary 2 in addition to the preceding fact we get $\nu(3/2, T) = 1$.

Note. This example is guilty of some triviality since a 3/2-eigenvector of T_0 is in the kernel of V. Nevertheless the quasi-isolated eigenvalue 3/2 is embedded and is preserved by the finite dimensional perturbation. Also the new multiplicity is given by the $W-A$ formula.

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THE ASYMPTOTIC BEHAVIOR OF THE KLEIN-GORDON EQUATION WITH EXTERNAL POTENTIAL, II

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Let $U_0(t)$ and $U(t)$ be the one-parameter groups governing the time development of solutions of the Klein-Gordon equation, $\Box\varphi = m^2\varphi$, and the perturbed equation, $\Box\varphi = m^2\varphi + V(\vec{x})\varphi$, respectively. In a previous work the author obtained sufficient conditions on the potential $V(\vec{x})$ which guaranteed the existence of the wave operators, $W_\pm := s - \lim U(-t)U_0(t)$ as $t \to \pm\infty$. Here it is shown that if, in addition, the associated (Schrödinger) wave operators, $W_\pm^S := s - \lim e^{i(m^2I+V-\Delta)t}e^{-i(m^2I-\Delta)t}$ as $t \to \infty$, are complete and the Invariance Theorem is valid then the $W_\pm$ are also complete and are isometries. Finally, these results are used to show that the scattering operator, $W_\mp^{-1}W_-$, is unitarily implemented in Fock space.

The similarity between the wave operators $W_\pm$ and $W_\pm^S$ observed in [1] as far as their existence theories are concerned, is clearly reaffirmed in their completeness theories. Indeed, the proof of the above results is based on the development of an explicit relationship between these wave operators. Connections of this sort were observed by Birman [3, p. 114, § 5] for abstract differential equations of the form $\varphi_{tt} + A\varphi = 0$. Sufficient conditions for such a relationship in this more general framework were obtained by Kato [4, §§ 9, 10] and used to study both potential and obstacle scattering for the wave equation [4, § 11].

In this investigation of the Klein-Gordon equation the argument will be directed so as to take best advantage of the above general results of Kato. However some generalizations will be necessary in order to establish the cited results on the Lorentz-invariant as well as the finite-energy solution spaces of the Klein-Gordon equation. Because a specific equation is being considered some simplification of Kato's arguments will also be possible.

1. Preliminaries. In this section the concepts discussed above are given precise definitions. Some related results which are directly used in the proofs of the main theorems are also included in summarized form.

Suppose Δ is the Laplacian in three dimensions and A^2 is the self-adjoint realization of $m^2I - \Delta$ on $L^2(E^3)$. Throughout this paper V is taken to be a real-valued function of three (space) variables and in

$L^p(E^3)$ for some $2 \leqq p \leqq \infty$.[1] With these hypotheses on V it is a fairly standard result that the perturbed operator, $A^2 + V$, is self-adjoint with $D(A^2 + V) = D(A^2) = D(\Delta)$. This self-adjoint realization of $A^2 + V$ will be denoted by B^2.

So that fractional powers of the above operators can be compared we ask that the perturbation satisfy a restriction on the size of its negative part:

(i) $\| V_- \|_q < M(q)$ for any $q \geqq 3/2$ (including ∞) where $M(q)$ is a constant depending only on q and m.

REMARK. More specifically $M(q) = \text{constant}.\, m^{(3-2q)/q}$ where the constant is that appearing in the Sobolev inequalities [6, p. 125]. The precise value of $M(q)$ is inessential in what follows. All that is needed is that the q-norm of V_- is sufficiently small for at least one $q \geqq 3/2$.

PROPOSITION 1.1. *For perturbations V, as above, satisfying condition* (i), *the self-adjoint operators A^θ, B^θ satisfy*

$$(1) \qquad m^\theta \| \varphi \| \leqq \| A^\theta \varphi \| \leqq C_1^\theta \| B^\theta \varphi \| \leqq C_2^\theta \| A^\theta \varphi \|$$

for all $\varphi \in D(B^\theta) = D(A^\theta)$ and all $0 \leqq \theta \leqq 1$. In addition

$$(2) \qquad C_2^{-\theta} \| A^{-\theta} \varphi \| \leqq C_1^{-\theta} \| B^{-\theta} \varphi \| \leqq \| A^{-\theta} \varphi \| \leqq m^{-\theta} \| \varphi \|$$

for all $\varphi \in L^2(E^3)$ and all $0 \leqq \theta \leqq 1$. C_1 and C_2 are constants depending on V, m, p and q.

Proof. [1, Lemma 2.4, Th. 2.5].

In order to discuss the solution spaces of the $K - G$ equation we shall first write it in its equivalent vector-valued form

$$(3) \qquad i\frac{d}{dt}\begin{pmatrix} \varphi \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \dot{\varphi} \end{pmatrix}$$

which has as its formal solution

$$(4) \qquad \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{pmatrix} = U_0(t)\begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0) \end{pmatrix} = \begin{pmatrix} \cos At & A^{-1}\sin At \\ -A\sin At & \cos At \end{pmatrix}\begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0) \end{pmatrix}$$

where $\varphi(0), \dot{\varphi}(0)$ are the Cauchy data at $t = 0$. Indeed, it is a fairly well known fact that equation (4) rigorously defines the solution of the $K - G$ equation on $H(A, \alpha)$ (defined below) in the sense that $t \rightarrow U_0(t)$ is a one-parameter group of unitary transformations on $H(A, \alpha)$ with infinitesmal generator $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$. The solution spaces $H(A, \alpha)$

[1] $\| \ \|_p$ will denote the usual norm in $L^p(E^3)$. However, for notational convenience $\| \ \|_2$ will be replaced by $\| \ \|$ and the associated inner product will be written as $(,)$.

are described in the following.

DEFINITION. For each $\alpha \in \boldsymbol{R}$, the complex Hilbert space $H(A, \alpha)$ is the completion of $D(A^{\alpha}) \oplus D(A^{\alpha-1})$ with respect to the inner product

$$(\Phi, \Psi)_{A,\alpha} = \left(\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right)_{A,\alpha} = (A^{\alpha}\varphi_1, A^{\alpha}\psi_1) + (A^{\alpha-1}\alpha_2, A^{\alpha-1}\psi_2) \,.$$

As a direct sum $H(A, \alpha)$ will be written as $D[A^{\alpha}] \oplus D[A^{\alpha-1}]$.

REMARK. Our primary interest is in the finite energy $(H(A, 1))$ and the Lorentz-invariant $(H(A, \frac{1}{2}))$ solution spaces of the $K - G$ equation. We shall handle both simultaneously by proving the main results on $H(A, \theta)$ for all $0 \leqq \theta \leqq 1$. For θ in this range it can be checked that the above completion is only required in the second summand of $H(A, \theta)$. In fact, except for the norm, $D[A^{\theta-1}]$ is isomorphic to the Sobolev space $W^{\theta-1,2}(E^3)$ and hence contains non-$L^2(E^3)$ elements.

Condition (i) insures that B^2, like A^2, is a nonnegative (self-adjoint) operator. For this reason the above discussion can be repeated with A replaced by B to obtain the dynamical propagators $U(t)$ on the solution spaces, $H(B, \theta)$, of the perturbed $K - G$ equation. The following observation, which is a direct consequence of Proposition 1.1, will be convenient in the next section.

PROPOSITION 1.2. *With the hypothesis of Proposition* 1.1 $H(A, \theta)$ *and* $H(B, \theta)$ *are isomorphic as linear spaces for each* $0 \leqq \theta \leqq 1$ *and the norms satisfy*

$$K_1 \|\cdot\|_{A,\theta} \leqq \|\cdot\|_{B,\theta} \leqq K_2 \|\cdot\|_{A,\theta} \tag{5}$$

where K_1 *and* K_2 *are constants depending on* C_1 *and* C_2. *It follows that* $U_0(t)$: $H(B, \theta) \to H(B, \theta)$ *and* $U(t)$: $H(A, \theta) \to H(A, \theta)$ *are uniformly bounded.*

The above result allows us to form products of the finite-time propagators even though they were defined on a priori different spaces and hence define the wave operators.

DEFINITION. The (free-to-physical) wave operators $W_{\pm}$ are given by

$$W_{\pm} = s - \lim_{t \to \pm\infty} U(-t) U_0(t)$$

whenever this strong limit exists on all of $H(A, \theta)$.

REMARK. The existence of the strong limit is demanded on all of $H(A, \theta)$ because the generator of $U_0(t)$, $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$, is spectrally absolutely continuous (c.f. Lemma 2.2 to follow). For notational convenience the θ-dependence of $W_\pm$ is deleted since the conditions obtained are valid for all $0 \leqq \theta \leqq 1$.

If one further restriction is made on V,

(ii) $V \in L^p(E^3)$ for any $2 \leqq p < 3$,

then the following existence theorem can be proved [1, Th. 4.1].

THEOREM 1.3. *If V is real-valued and satisfies conditions* (i) *and* (ii) *then $W_\pm$ exist on $H(A, \theta)$ for each $0 \leqq \theta \leqq 1$.*

2. Main results.

In this section the isometric nature and the completeness of $W_\pm: H(A, \theta) \to H(B, \theta)$ will be established for perturbations which satisfy the additional conditions

(iii) $W_\pm^S = s - \lim_{t\to\pm\infty} e^{iB^2t}e^{-iA^2t}$ are complete;

(iv) $W_\pm^S = s - \lim_{t\to\pm\infty} e^{i\varphi(B^2)t}e^{-i\varphi(A^2)t}$ for φ as in Invariance Theorem.[2]

The method of proof will be to establish a relationship between $W_\pm$ and $W_\pm^S$ by using the ideas concerning identification operators proved by Kato [4, §§ 9, 10]. Indeed the proof will be directed so as to take best advantage of these general results of Kato.

We begin by considering the transformation $\Gamma(A, \theta): H(A, \theta) \to L^2(E^3) \oplus L^2(E^3)$ formally defined by the equation

$$\Gamma(A, \theta) = \frac{1}{\sqrt{2}}\begin{pmatrix} A^\theta & iA^{\theta-1} \\ A^\theta & -iA^{\theta-1} \end{pmatrix}.$$

This transformation, which is the analog of one considered by Birman [3, p. 114, § 5] and Kato [4, p. 335, 8.9], will provide us with a unitary operator which "diagonalizes" $U_0(t)$ in an operationally convenient way.

LEMMA 2.1. *For each $0 \leqq \theta \leqq 1$,*

$$\Gamma(A, \theta): D(A^\theta) \oplus D(A^{\theta-1}) (\subset H(A, \theta)) \to L^2(E^3) \oplus L^2(E^3)$$

defined above has a unique unitary extension

$$\tilde{\Gamma}(A, \theta): H(A, \theta) \to L^2(E^3) \oplus L^3(E^3) .$$

In addition

[2] The strongest version of condition (iv) required is with $\varphi(\lambda) = \lambda^{\theta/2}, 0 \leqq \theta \leqq 1$. This is not an operationally weaker condition, however, since the full Invariance Theorem [5, p. 544-7] must be used to determine conditions on V for it to occur.

$$(6) \qquad \tilde{\Gamma}(A, \theta)U_0(t)\tilde{\Gamma}(A, \theta)^{-1} = e^{-iAt} \oplus e^{iAt} .$$

Proof. For $\Phi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in D(A^\theta) \oplus D(A^{\theta-1})$, a straight-forward computation using the defining equation gives $\| \Gamma(A, \theta)\Phi \| = \| \Phi \|_{A,\theta}$.[3] Furthermore

$$\Gamma(A, \theta)(D(A^\theta) \oplus D(A^{\theta-1})) = R(A^\theta) \oplus R(A^\theta) = L^2(E^3) \oplus L^2(E^3) .$$

Thus the isometry $\Gamma(A, \theta)$ has a unique extension to one with domain the $H(A, \theta)$-closure of $D(A^\theta) \oplus D(A^{\theta-1})$ (i.e., all of $H(A, \theta)$) and range $L^2(E^3) \oplus L^2(E^3)$. This unitary extension is

$$\tilde{\Gamma}(A, \theta) = \frac{1}{\sqrt{2}}\begin{pmatrix} A^\theta & i\widetilde{A^{\theta-1}} \\ A^\theta & -i\widetilde{A^{\theta-1}} \end{pmatrix} \quad \text{where} \quad \widetilde{A^{\theta-1}}: D[A^{\theta-1}] \to L^2(E^3)$$

is the unitary transformation defined by $\widetilde{A^{\theta-1}}\varphi = A^{\theta-1}\varphi$ for all $\varphi \in L^2(E^3) \subset D[A^{\theta-1}]$. A simple algebraic computation shows that

$$\tilde{\Gamma}(A, \theta)U_0(t) = \{e^{-iAt} \oplus e^{iAt}\}\tilde{\Gamma}(A, \theta)$$

on a suitable dense set from which the relation (6) follows by continuity.

Before applying the above to the problem at hand we shall obtain a more precise description of the absolutely continuous part of the generator of $U_0(t)$ $\left(\text{i.e., of } \begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix} \text{ on } H(A, \theta)\right)$ since it is at the basis of the completeness problem for $W_\pm$. In particular we shall relate the subspace of absolute continuity of $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ to that of A by means of an adaption to the present situation of a result of Kato [4, p. 355, Lemma 8.1].

LEMMA 2.2. *Let $P_{A,\theta}$ and Q_A denote (the projections in $H(A, \theta)$ and $L^2(E^3)$ onto) the subspaces of absolute continuity of* $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ *and A respectively then the following conditions are equivalent:*

(a) $\Phi \in P_{A,\theta}$;

(b) $\tilde{\Gamma}(A, \theta)\Phi \in Q_A \oplus Q_A$;

(c) $A^\theta\varphi_1 \in Q_A$ *and* $\widetilde{A^{\theta-1}}\varphi_2 \in Q_A$.

Proof. Since Q_A is a closed linear subspace of $L^2(E^3)$ [5, p. 516,

[3] The norm $(\|\cdot\|^2 + \|\cdot\|^2)^{1/2}$ in $L^2(E^3) \oplus L^2(E^3)$ is also denoted by $\|\cdot\|$ since there is no possibility of confusion.

Th. 1.5] (b) and (c) are clearly equivalent. Suppose $E(\lambda)$ and $e(\lambda)$ are the spectral projections for A and $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ respectively. Then equation (6) is equivalent to $\tilde{\Gamma}(A, \theta)e(S)\tilde{\Gamma}(A, \theta)^{-1} = \{E(S) \oplus -E(S)\}$ for all Borel sets $S \supset \boldsymbol{R}$. Thus

$$\| e(S)\Phi \|_{A,\theta} = \| \{E(S) \oplus -E(S)\}\tilde{\Gamma}(A, \theta)\Phi \|$$

from which the equivalence of (a) and (b) is immediate.

REMARK 1. Because $m^2I - \Delta$ is spectrally absolutely continuous, A and hence $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$, is likewise. This motivates the definition of $W_\pm$ in the previous section.

REMARK 2. Clearly if condition (i) is satisfied (so that B^θ is a nonnegative self-adjoint operator for each $0 \leqq \theta \leqq 1$), the above two results can be proved with A replaced by B. In general, however, B will not be spectrally absolutely continuous so that $P_{B,\theta} \neq I$.

Returning to the main problems we now indicate how the above may be used to provide a connection between the quasi-relativistic wave operators $W_\pm$ and the nonrelativistic wave operators $W_\pm^S$. This will be accomplished by comparing each to the wave operator

$$W'_\pm := s - \lim_{t \to \pm\infty} U(-t)\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta)U_0(t) .$$

The requirement that the identification operator [4, p. 343, 1.2 and p. 346, Definition 3.1] $\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta) \in B(H(A, \theta), H(B, \theta))$ is satisfied, since $\tilde{\Gamma}(A, \theta)$ and $\tilde{\Gamma}(B, \theta)$ are unitary.

THEOREM 2.3. *If the perturbation V satisfies conditions* (i) *and* (iv), *then*

(a) *$W'_\pm$ exist if and only if $W_\pm^S$ exist;*

(b) *$W'_\pm$ are complete if and only if $W_\pm^S$ are complete.*

Proof. Relation (6) for A, and the corresponding one for B can be used to obtain

$$\tilde{\Gamma}(B, \theta)U(-t)\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta)U_0(t)\tilde{\Gamma}(A, \theta)^{-1} = \{e^{iBt}e^{-iAt} \oplus e^{-iBt}e^{iAt}\} .$$

Because the Γ-operators are bounded with bounded inverse, standard results on strong limits can be used on the above equation to give

$$\tilde{\Gamma}(B, \theta)W'_\pm\tilde{\Gamma}(A, \theta)^{-1} = s - \lim_{t \to \pm\infty} \{e^{iBt}e^{-iAt} \oplus e^{-iBt}e^{iAt}\} = W_\pm^S \oplus W_\mp^S . \tag{7}$$

The last equality follows from the invariance condition (iv). This establishes part (a). Similarly, part (b) follows from (7) and the equivalence of the first two statements in Lemma 2.2.

REMARK. The existence and completeness of $W_\pm^S$ are equivalent to the same questions for the more familiar wave operators,

$$s - \lim_{t \to \pm\infty} e^{i(V-\Delta)t} e^{-i(-\Delta)t} ,$$

since the associated prewave operators are identical. In particular, the existence of the latter is assured for potentials which satisfy condition (ii) [5, p. 534–5]; the completeness follows if $V \in L^1(E^3) \cap L^2(E^3)$ [5, p. 546, Example 4.10]. The proof of the completeness shows that condition (iii) and (iv) are closely related. It is interesting to distinguish them, however, since the latter is used for other purposes (e.g., in equation (7) and in a more essential manner in Lemma 2.5 to follow).

All that remains then is to show that $W_\pm = W'_\pm$. This will require condition (i), (iv) and the existence of $W_\pm^S$ (e.g., condition (ii)) in an explicit way. We now state this as a theorem, the proof of which is rather lengthy, and as a result, will proceed as a sequence of lemmas.

THEOREM 2.4. *If V satisfies conditions* (i), (ii) *and* (iv) *then $W_\pm = W'_\pm$ in the sense that the existence of one implies the existence of the other and their equality.*

Proof. A straightforward application of Theorem 4.2 of [4] shows that sufficient conditions for the equality of $W_\pm$ and $W'_\pm$ are

(a) $\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta)$ and $I \in B(H(A, \theta), H(B, \theta))$, and

(b) $\text{s} - \lim_{t \to \pm\infty} (\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta) - I)U_0(t) = 0$ on $H(A, \theta)$.

The first part of (a) has already been noticed to be true if condition (i) is satisfied. The second part follows from Proposition 1.2 which likewise requires condition (i). In addition $U_0(t): H(A, \theta) \to H(B, \theta)$ is uniformly bounded by K_2 (c.f. Proposition 1.2). Thus it suffices to establish (b) on a dense subset of $H(A, \theta)$; say $D(A) \oplus L^2(E^3)$. For $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(A) \oplus L^2(E^3)$,

$$\begin{aligned}
&(\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta) - I)U_0(t)\Phi \\
&= \left[\frac{1}{2}\begin{pmatrix} B^{-\theta} & B^{-\theta} \\ -iB^{1-\theta} & iB^{1-\theta} \end{pmatrix}\begin{pmatrix} A^{\theta} & iA^{\theta-1} \\ A^{\theta} & -iA^{\theta-1} \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]U_0(t)\Phi \qquad (8) \\
&= \begin{pmatrix} B^{-\theta}A^{\theta} - I & 0 \\ 0 & B^{1-\theta}A^{\theta-1} - I \end{pmatrix}\begin{pmatrix} \varphi_0(t) \\ \varphi_0(t) \end{pmatrix}
\end{aligned}$$

where $\varphi_0(t)$ is the solution of the $K-G$ equation with Cauchy data φ_1, φ_2 at $t=0$, and $\dot{\varphi}_0(t)$ is its time derivative. Thus

$$\begin{aligned} &\| (\tilde{\Gamma}(B,\theta)^{-1}\tilde{\Gamma}(A,\theta) - I)U_0(t)\Phi \|^2_{B,\theta} \\ &\quad = \| B^\theta(B^{-\theta}A^\theta - I)\varphi_0(t)\|^2 + \| B^{\theta-1}(B^{1-\theta}A^{\theta-1} - I)\dot{\varphi}_0(t)\|^2 \\ &\quad = \|(A_\theta - B^\theta)\varphi_0(t)\|^2 + \|(A^{\theta-1} - B^{\theta-1})\dot{\varphi}_0(t)\|^2 . \end{aligned}$$

The last equation follows from Proposition 1.1 (i.e., $D(A) = D(B) \subset D(A^\theta) = D(B^\theta)$) and the fact that $D(A) \oplus L^2(E^3)$ is invariant under $U_0(t)$. Thus (b) is implied by $\|(A^\theta - B^\theta)\varphi_0(t)\|$ and $\|(A^{\theta-1} - B^{\theta-1})\dot{\varphi}_0(t)\| \to 0$ as $t \to \pm\infty$.

We now reduce the conditions, step-by-step, to one which is much more amenable [4, p. 361, Condition 10.2 and Th. 10.5]. Let

$$\Phi_\pm = \frac{1}{2}\begin{pmatrix} (W^S_\pm + W^S_\mp), & i(W^S_\pm - W^S_\mp) \\ -i(W^S_\pm - W^S_\mp), & (W^S_\pm + W^S_\mp) \end{pmatrix}\Phi$$

and $\begin{pmatrix}\varphi_\pm(t) \\ \dot{\varphi}_\pm(t)\end{pmatrix} = U(t)\Phi_\pm$.

LEMMA 2.5. *Under the hypothesis of Theorem* 2.4, $\|A^\theta\varphi_0(t) - B^\theta\varphi_\pm(t)\|$ *and* $\|A^{\theta-1}\dot{\varphi}_0(t) - B^{\theta-1}\dot{\varphi}_\pm(t)\|$ *tend to zero as* $t \to \pm\infty$.

Proof. As previously observed the hypothesis implies the existence of $W^S_\pm$ which, by the invariance condition equals $s - \lim_{t\to\pm\infty} e^{iB^\theta t}e^{-iA^\theta t}$ for each $\theta \geqq 0$ (in particular for $0 \leqq \theta \leqq 1$). Now

$$\begin{aligned} A^\theta\varphi_0(t) &= A^\theta(\cos At\varphi_1 + A^{-1}\sin At\varphi_2) \\ &= \frac{1}{2}e^{-iAt}(A^\theta\varphi_1 + iA^{\theta-1}\varphi_2) + \frac{1}{2}e^{iAt}(A^\theta\varphi_1 - iA^{\theta-1}\varphi_2) . \end{aligned}$$

But the existence of $W^S_\pm$ implies that $s - \lim_{t\to\pm\infty}(e^{-iAt} - e^{-iBt}W^S_\pm) = 0$ and $W^S_\pm A^\theta = B^\theta W^S_\pm$ (using the invariance condition and the fact that $Q_A = I$). It is clear then that

$$\begin{aligned} \Big\| A^\theta\varphi_0(t) - \Big\{ &\frac{1}{2}e^{-iBt}(B^\theta W^S_\pm\varphi_1 + iB^{\theta-1}W^S_\pm\varphi_2) \\ &+ \frac{1}{2}e^{iBt}(B^\theta W^S_\mp\varphi_1 - iB^{\theta-1}W^S_\mp\varphi_2)\Big\}\Big\| \end{aligned}$$

tends to zero as $t \to \pm\infty$. A straightforward algebraic computation shows that the term in braces is $B^\theta\varphi_\pm(t)$. This establishes the first part of the lemma and the second part can be proved similarly.

By writing

$$\|(A^\theta - B^\theta)\varphi_0(t)\| = \|A^\theta\varphi_0(t) - B^\theta\varphi_\pm(t) + B^\theta\varphi_\pm(t) - B^\theta\varphi_0(t)\|$$
$$\leqq \|A^\theta\varphi_0(t) - B^\theta\varphi_\pm(t)\| + \|B^\theta(\varphi_0(t) - \varphi_\pm(t))\| ,$$

it is clear that $\|(A^\theta - B^\theta)\varphi_0(t)\| \to 0$ as $t \to \pm\infty$ if $\|B^\theta(\varphi_0(t) - \varphi_\pm(t))\| \to 0$ as $t \to \pm\infty$. Similarly $\|(A^{\theta-1} - B^{\theta-1})\dot\varphi_0(t)\| \to 0$ as $t \to \pm\infty$ if

$\|B^{\theta-1}(\dot\varphi_0(t) - \dot\varphi_\pm(t))\| \to 0$ as $t \to \pm\infty$.

LEMMA 2.6. *Under the hypothesis of Theorem* 2.4,

$$\|B^\theta(\varphi_0(t) - \varphi_\pm(t))\| \quad \text{and} \quad \|B^{\theta-1}(\dot\varphi_0(t) - \dot\varphi_\pm(t))\| \to 0$$

as $t \to \pm\infty$ *if* $\|B(\varphi_0(t) - \varphi_\pm(t))\| \to 0$ *as* $t \to \pm\infty$.

Proof. Since $\Phi \in D(A) \oplus L^2(E^3)$, $\varphi_0(t)$ and $\varphi_\pm(t) \in D(A) = D(B)$ [8, p. 614, Th. 2.1]. But $\|B^\theta\psi\| = \|B^{\theta-1}B\psi\| \leqq (mC_1^{-1})^{\theta-1}\|B\psi\|$ for all $\psi \in D(B)$ by Proposition 1.1, which establishes the first part. The second part follows directly from the existence of $W_\pm^S$ and (iv). To see this write

$$\begin{aligned} \dot\varphi_0(t) &= -A\sin At\varphi_1 + \cos At\varphi_2 \\ &= -\frac{i}{2}e^{-iAt}(A\varphi_1 + i\varphi_2) + \frac{i}{2}e^{iAt}(A\varphi_1 - i\varphi_2) , \end{aligned} \tag{10}$$

and

$$\dot\varphi_\pm(t) = -\frac{i}{2}e^{-iBt}W_\pm^S(A\varphi_1 + i\varphi_2) + \frac{i}{2}e^{iBt}W_\mp^S(A\varphi_1 - i\varphi_2) . \tag{11}$$

Thus $\|\dot\varphi_0(t) - \dot\varphi_\pm(t)\| \to 0$ as $t \to \pm\infty$ if $s - \lim(e^{-iAt} - e^{-iBt}W_\pm^S) = 0$ as $t \to \pm\infty$ which follows from conditions (ii) and (iv). The proof is completed by again observing that

$$\|B^{\theta-1}(\dot\varphi_0(t) - \dot\varphi_\pm(t))\| \leqq (mC_1^{-1})^{\theta-1}\|\dot\varphi_0(t) - \dot\varphi_\pm(t)\| .$$

LEMMA 2.7. $\|B(\varphi_0(t) - \varphi_\pm(t))\| \to 0$ *as* $t \to \pm\infty$ *if* $\|Ve^{-iBt}W_\pm^S\psi\| \to 0$ *as* $t \to \pm\infty$ *for all* $\psi \in D(A^2)$.

Proof. This is essentially condition (e) of Theorem 10.5 of [4]. A careful examination of the proof shows that it suffices to have $s - \lim_{t\to\pm\infty} Ve^{-iBt} = 0$ on $\{W_\pm^S\psi;\, \psi \in D(A^2)\}$ rather than on all of $D(B^2) \cap Q_B$. Condition (iv) is used in the present formulation but in a rather inessential way.

LEMMA 2.8. *Under the hypotheses of Theorem* 2.4,

$$\|V(e^{-iBt}W_\pm^S - e^{-iAt})\psi\| \to 0$$

as $t \to \pm\infty$ *for all* $\psi \in D(A^2)$.

Proof. Since $\psi \in D(A^2)$, $e^{-iAt}\psi$ and $e^{-iBt}W_\pm^S\psi \in D(A^2) = D(B^2)$ [8, p. 614, Th. 2.1]. Now

$$\| V(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| \leqq \| V \|_p \| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|_q$$

where $q = 2p(p-2)^{-1}$. The last term is estimated using inequalities of the Sobolev type [6, p. 125] to obtain

$$\| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|_q \leqq \text{constant} \| (-\Delta)(e^{-iBt}W_\pm^S - e^{-iAt})\psi \|^\gamma \cdot \| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|^{1-\gamma} \tag{12}$$

where $\gamma = 3(2p)^{-1}$. The result will now follow if it can be shown that the first term on the right in (12) is uniformly bounded in t and the second tends to zero as $t \to \pm\infty$. The second requirement follows from the existence of $W_\pm^S$ and the invariance condition provided $\gamma < 1$ or $p > 3/2$ which is guaranteed by the hypothesis. Turning to the second requirement,

$$\begin{aligned} \| (-\Delta)(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| &\leqq \| A^2(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| \\ &\leqq \| A^2 e^{-iBt}W_\pm^S\psi \| + \| A^2\psi \| . \end{aligned}$$

To show that the first term on the right of the above inequality is bounded recall [1, Th. 2.1] that if $V \in L^p(E^3)$ for any $p \geqq 2$, there exist constants $a < 1$ and b, such that for $\chi \in D(A^2)$,

$$\| B^2\chi - A^2\chi \| = \| V\chi \| \leqq a \| A^2\chi \| + b \| \chi \| .$$

Hence

$$\| A^2\chi \| \leqq (1-a)^{-1}(\| B^2\chi \| + b \| \chi \|) . \tag{13}$$

Applying (13) to the above and using well-known properties of $W_\pm^S$ one obtains

$$\begin{aligned} \| A^2 e^{-iBt}W_\pm^S\psi \| &\leqq (1-a)^{-1}(\| B^2 e^{-iBt}W_\pm^S\psi \| + b \| e^{-iBt}W_\pm^S\psi \|) \\ &\leqq (1-a)^{-1}(\| A^2\psi \| + b \| \psi \|) \end{aligned}$$

which proves the lemma.

Clearly, the above result reduces the proof of Theorem 2.4 to showing that $\| Ve^{-iAt}\psi \| \to 0$ as $t \to \pm\infty$ for all $\psi \in D(A^2)$.

LEMMA 2.9. *If* $V \in L^p(E^3)$ *for any* $2 \leqq p < \infty$, *then* $\| Ve^{-iAt}\psi \| \to 0$ *as* $t \to \pm\infty$ *for all* $\psi \in D(A^2)$.

Proof. We first show that it suffices to prove the result on a core of A^2 (i.e., a set $\mathscr{C} \subset D(A^2)$ such that for each $\psi \in D(A^2)$, there exists a sequence $\{\psi_n\} \subset \mathscr{C}$ such that $\| A^2(\psi - \psi_n) \| + \| \psi - \psi_n \| \to 0$ as

$n \to \infty$). If ψ and ψ_n are as above then the observation follows from

$$\begin{aligned} \| Ve^{-iAt}\psi \| &= \| Ve^{-iAt}(\psi - \psi_n + \psi_n) \| \\ &\leqq \| Ve^{-iAt}(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| \\ &\leqq \| A^2 e^{-iAt}(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| \\ &\leqq \| A^2(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| . \end{aligned}$$

Of course, the above computation requires $V \in L^p(E^3)$ for any $p \geqq 2$ so that $\| V\chi \| \leqq \| A^2\chi \|$ for all $\chi \in D(A^2)$.

In particular take $\mathscr{C} = \mathfrak{F}C_c^\infty(E^3)$ (i.e., the image under Fourier transformation of $C_c^\infty(E^3)$). $\mathscr{C}$ is a core for A^2 if and only if $C_c^\infty(E^3)$ is a core for $M_{k^2+m^2}$ [5, p. 300]. The latter condition is true since $M_{k^2+m^2}$ maps $C_c^\infty(E^3)$ onto $C_c^\infty(E^3)$ [5, p. 166, 5.19]. All that remains then is to show that $\| Ve^{-iAt}\psi \| \to 0$ as $t \to \pm\infty$ for all $\psi \in \mathscr{C}$. Now

$$\| Ve^{-iAt}\psi \| \leqq \| V \|_p \| e^{-iAt}\psi \|_q$$

where $q = 2p(p-2)^{-1}$. But $\| e^{-iAt}\psi \|_r = 0(|t|^{-3\{(1/2)-(1/r)\}})$ as $|t| \to \infty$ for each $2 \leqq r \leqq \infty$ and each $\psi \in \mathscr{C}$ by a variant of Proposition 4.2 of [1] which is a direct consequence of a result of Segal [7, p. 95, Lemma 3]. Thus the decay is established if $q > 2$ or $2 \leqq p < \infty$.

The above results can be used in a fairly obvious manner to prove the result indicated at the beginning of this section; namely,

THEOREM 2.10. *If conditions* (i)–(iv) *are satisfied then the* $W_\pm$ *are complete.*

REMARK. A careful examination of the above proofs shows that condition (ii) is used only to show that $W_\pm^S$ exist. Thus the above theorem is valid if condition (ii) is replaced by the weaker condition

(ii)′ $W_\pm^S$ exist.

Indeed the same change gives an alternate formulation of the existence Theorem 1.3. This result is more appealing from the viewpoint of the similarly of $W_\pm^S$ and $W_\pm$ but the proof requires the very restrictive condition (iv). It is interesting however, that condition (i) is present in both versions.

One further result which follows from the above is the isometric nature of the $W_\pm$. More specifically,

THEOREM 2.11. *If conditions* (i), (ii)′ *and* (iv) *are satisfied then for each* $0 \leqq \theta \leqq 1$, $W_\pm: H(A, \theta) \to H(B, \theta)$ *are isometries.*

Proof. Theorems 2.3 and 2.4 give

$$W_{\pm} = \tilde{\Gamma}(B, \theta)^{-1}\{W_{\pm}^{S} \oplus W_{\mp}^{S}\}\tilde{\Gamma}(A, \theta) \,, \tag{14}$$

from which the result immediately follows since the Γ-operators are unitary and the $W_{\pm}^{S}$ are isometries.

3. Application. In this section the preceding results will be used to show that the scattering operator, $S = W_{+}^{-1} W_{-}$, is unitarily implementable in the free representation of the quantized Klein-Gordon field with mass m. We shall introduce only the most basic concepts here and direct the reader to [2] and the references therein for a more detailed and systematic discussion.

The unique, relativistically invariant, classical dynamical system associated with the K $-$ G field in three space consists of the real Hilbert space $H_r(A, \frac{1}{2})$ (the real part of $H(A, \frac{1}{2})$) and the nondegenerate, skew-symmetric bilinear form $\mathrm{Re}(J\cdot, \cdot)_{A,1/2}$ where $J = \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$. A transformation on $H_r(A, \frac{1}{2})$ which preserves the above form is called symplectic. It is well-known that the symplectic transformations form a group. By means of a straightforward algebraic computation [e.g., 2, p. 391, Lemma 3.4], it can be shown that both $U_0(t)$ and $U(t)$, and hence the prewave operators $W(t)$, are symplectic. In addition, it is not difficult to show that strong limits of symplectic operators are likewise symplectic. Thus $W_{\pm}$ and S are symplectic in the above sense.

A quantization of the above classical $K - G$ field is basically a map $\Phi \longrightarrow Q(\Phi)$ from $H_r(A, \frac{1}{2})$ into unitary operators on a complex Hilbert space $\mathscr{H}$ which satisfy the Weyl (exponentiated) form of the commutation relations. The most familiar of these, and the one with which we shall deal, is called the Fock-Cook quantization. It will be denoted by Q_0 on $\mathscr{H}_0$. If $T: H_r(A, \frac{1}{2}) \longrightarrow H_r(A, \frac{1}{2})$ is symplectic then $\Phi \longrightarrow Q_0(T\Phi)$ is another quantization. If it is unitarily equivalent to the Fock-Cook quantization, T is said to be unitarily implementable (in the free representation of the $K - G$ field with mass m). This situation occurs if and ond only if T, as an operator on $H_r(A, \frac{1}{2})$, is bounded with bounded (everywhere defined) inverse such that $T^*T - I$ is Hilbert-Schmidt [2, p. 388, Corollary 2.3].

THEOREM 3.1. *S is unitarily implementable in the free representation of the $K - G$ field with mass m if conditions* (i)–(iv) *are satisfied.*

Proof. Since $W_{\pm}$ are complete, $D(W_{+}^{-1}) = R(W_{+}) = R(W_{-}) = P_{B,1/2}$, and hence S is well defined on $H(A, \frac{1}{2})$. In addition, since $R(W_{+}^{-1}) = D(W_{+}) = H(A, \frac{1}{2})$, the image of $H(A, \frac{1}{2})$ under S is all of $H(A, \frac{1}{2})$. Furthermore, the isometric nature of $W_{\pm}: H(A, \frac{1}{2}) \longrightarrow H(B, \frac{1}{2})$ implies that $S: H(A, \frac{1}{2}) \longrightarrow H(A, \frac{1}{2})$ is an isometry, and hence unitary. Thus

$S: H_r(A, \frac{1}{2}) \rightarrow H_r(A, \frac{1}{2})$ is orthogonal and the required conditions for unitary implementability are satisfied trivially.

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ON THE ZEROS OF THE SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(n)}(z) + p(z)y(z) = 0$.

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In this paper sufficient conditions for disconjugacy and for nonoscillation of the equation $y^{(n)}(z) + p(z)y(z) = 0$ are given. For $n = 2m$ a theorem ensuring that no solution of this equation has two zeros of multiplicity m is obtained. Here the invariance of the equation under linear transformations of z is used.

In [6] Nehari considered the equation

$$y^{(n)}(z) + p_{n-1}(z)y^{(n-1)}(z) + \cdots + p_0(z)y(z) = 0 , \tag{1}$$

where the analytic functions $p_i(z)$, $i = 0, \cdots, n-1$ are regular in a given domain D, and obtained a disconjugacy theorem for bounded convex domains and a nonoscillation theorem for the unit disk. Equation (1) is called *disconjugate* in a domain D, if no nontrivial solution of (1) has more than $(n-1)$ zeros in D. (The zeros are counted by their multiplicity). The equation is called nonoscillatory in D, if no nontrivial solution has an infinite number of zeros in D.

In this paper we obtained related results for a special case of (1); i.e., for the equation

$$y^{(n)}(z) + p(z)y(z) = 0 , \tag{2}$$

where the analytic function $p(z)$ is regular in the unit disk.

Section 1 deals with the invariance of equation (2), where $p(z)$ is analytic in a general domain, under the linear transformation

$$\zeta = \frac{az + b}{cz + d} , \qquad ad - bc \neq 0 , \tag{3}$$

(Theorem 1). The invariance of

$$y''(z) + p(z)y(z) = 0 \tag{4}$$

played an important role in Nehari's results on this second order equation [3; 5].

In § 2 we obtain sufficient conditions for disconjugacy and nonoscillation of equation (2) in the unit disk (Theorem 2 and Theorem 4 respectively). From Theorem 2 and the invariance of (2) under the linear transformations (3) we get a sufficient condition for the disconjugacy of (2) in non-Euclidean disks (Theorem 3).

In § 3 we deal with equations of even order $n = 2m$, and obtain a condition on $p(z)$, which ensures that no solution of (2) has two zeros of multiplicity m. For the proof of this Theorem 5 we apply Theorem 1 and the method used in [5].

1. Invariance under linear transformations.

THEOREM 1. *The equation*

$$\frac{d^n y}{dz^n} + p(z)y(z) = 0 \tag{2}$$

is transformed by the linear mapping

$$\zeta = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{3'}$$

into an equation of the same form

$$\frac{d^n w_1}{d\zeta^n} + P_1(\zeta)w_1(\zeta) = 0. \tag{2'}$$

Here

$$w_1(\zeta) = (a - c\zeta)^{n-1} w(\zeta) \tag{5}$$

and

$$P_1(\zeta) = \left(\frac{dz}{d\zeta}\right)^n P(\zeta) = (a - c\zeta)^{-2n} P(\zeta), \tag{6}$$

where

$$w(\zeta) = y(z) = y\left(\frac{d\zeta - b}{-c\zeta + a}\right) \tag{7}$$

and

$$P(\zeta) = p(z) = p\left(\frac{d\zeta - b}{-c\zeta + a}\right). \tag{8}$$

Proof. It is easily verified that

$$(a - c\zeta)^{n+1} \frac{d^n w_1}{d\zeta^n} = \frac{d^n y}{dz^n}.$$

Applying this and (5)—(8) to equation (2) we obtain

$$\frac{d^n y}{dz^n} + p(z)y(z) = \frac{d^n y}{dz^n} + p(z)w_1(\zeta)(a - c\zeta)^{1-n}$$
$$= \left[\frac{d^n w_1}{d\zeta^n} + P_1(\zeta)w_1(\zeta)\right](a - c\zeta)^{n+1} ,$$

which proves the statement of our theorem.

The assumption $ad - bc = 1$ in (3′) was made just for convenience. In the general case (3), formula (6) has to be replaced by

$$P_1(\zeta) = \left(\frac{dz}{d\zeta}\right)^n P(\zeta) = \frac{(a - c\zeta)^{-2n}}{(ad - bc)^{-n}} P(\zeta) .$$

The converse of Theorem 1 is also true: the only transformations $\zeta = \psi(z)$, which leave the form of equation (2), for $n \geqq 3$, invariant are the linear transformations (3). This follows from a theorem of Wilczynski [11, p. 26]. For $n = 2$ equation (4) is invariant for any univalent transformation $\zeta = \psi(z)$; however if $\psi(z)$ is not linear, the connection between $p(z)$ and $P_1(\zeta)$ is more complicated than (6).

2. Disconjugacy and nonoscillation.

THEOREM 2. *Let the analytic function $p(z)$ be regular in $|z| < 1$. If*

(9) $$|p(z)| \leqq \frac{n!}{(1 - |z|)(1 + |z|)^{n-1}}, \quad |z| < 1 ,$$

then equation

(2) $$y^{(n)}(z) + p(z)y(z) = 0$$

is disconjugate in $|z| < 1$.

We remark that for $n = 2$, (9) becomes

$$|p(z)| \leqq \frac{2}{1 - |z|^2} , \qquad |z| < 1 ,$$

which is a condition of Pokornyi [8; 5] for disconjugacy of equation (4) in the unit disk.

In the case of equation (2) and $|z| < 1$, the general theorem [6, p. 328] gives that

$$\overline{\lim_{r\to 1}} \frac{(2r)^{n-1}}{(n - 1)!} \int_{|\zeta|=r} |p(\zeta)| \, |d\zeta| < 2$$

implies the disconjugacy of (2) in $|z| < 1$.

Using [4, p. 127, Ex. 8] this corollary to Nehari's theorem follows from Theorem 2.

As the function $f_n(r) = n!/(1-r)(1+r)^{n-1}$ is monotonic decreasing in $0 \leqq r \leqq (n-2)/n$, it follows by the maximum principle that, for $n > 2$, (9) is equivalent to

$$|p(z)| \leqq \frac{n!}{(1-|z|)(1+|z|)^{n-1}}, \quad \frac{n-2}{n} \leqq |z| < 1 .$$

Proof. For proving this theorem we use "divided differences" [6; 7, Chapter 1]. We denote by $[z, z_1, \cdots, z_k]$ the k – th divided difference of $y(z)$, i.e., we set

$$[z, z_1] = \frac{y(z) - y(z_1)}{z - z_1},$$

$$[z, z_1, \cdots, z_k] = \frac{[z, z_1, \cdots, z_{k-1}] - [z_1, z_2, \cdots, z_k]}{z - z_k}, \quad k = 2, \cdots, n .$$

If C is a closed contour in the unit disk, such that $z, z_1, \cdots, z_n$ are in the interior of C, then it follows from the definition that

$$[z, z_1, \cdots, z_n] = \frac{1}{2\pi i}\int_C \frac{y(\zeta)}{(\zeta - z)(\zeta - z_1)\cdots(\zeta - z_n)}\,d\zeta .$$

The right hand side is defined also when some of the z_i's coincide and may thus serve as a definition of the left hand side also in that case (where the divided differences would have to be defined with the help of derivatives). Clearly then $[z, z_1, \cdots, z_n]$ is continuous in all its arguments. Moreover, if $y(z_1) = \cdots = y(z_n) = 0$, we obtain

$$[z, z_1, \cdots, z_n] = \frac{y(z)}{\prod_{i=1}^{n}(z - z_i)} . \tag{10}$$

To prove the theorem, assume now, by negation, that (2) has a nontrivial solution $y(z)$ which vanishes at the n-points $z_1, \cdots, z_n$ of the open unit disk E. These n points cannot all coincide, as $y(z^*) = y'(z^*) = \cdots = y^{(n-1)}(z^*) = 0$ implies $y \equiv 0$. Therefore there are at least two distinct points. Let H be the convex hull of the points $z_1, \cdots, z_n$. H is therefore either a segment or a convex polygon.

Let z be any point in H; we use now Hermite's formula for the divided difference of $y(z)$ [7, p. 9]

$$[z, z_1, \cdots, z_n] = \int \cdots \int y^{(n)}(t_0 z + t_1 z_1 + \cdots + t_n z_n)\,dt_1 \cdots dt_n , \tag{11}$$

where the integral is extended over the n dimensional simplex of volume $1/n!$ given by

$$t_i \geqq 0 \qquad i = 0, \cdots, n \,; \qquad \sum_{i=0}^{n} t_i = 1 \,. \tag{12}$$

We remark that formula (11) is proved in [7, p. 9] only made the assumption that all the z_i's are distinct. As however both sides are continuous in $z_1, \cdots, z_n$, this formula is valid also in the case where some of the z_i's coincide. The point $\zeta = t_0 z + \cdots + t_n z_n$, where the t_i satisfy (12), belongs to the convex hull of the $n+1$ points $z, z_1, \cdots z_n$, and as $z \in H$, it follows that $\zeta \in H$.

From (10), (11) and (2) it follows that

$$\frac{y(z)}{\prod_{i=1}^{n} (z - z_i)} = -\int_{t_1 \cdots t_n} \cdots \int p(\zeta) y(\zeta) dt_1 \cdots dt_n \,, \tag{13}$$

where $\zeta = t_0 z + t_1 z_1 + \cdots + t_n z_n \in H$. Let ζ_0 be a point, or one of the points, in which $|\,p(z)y(z)\,|$ attains its maximum in H. (This maximum is positive, otherwise $p(z)y(z) \equiv 0$, and as $y(z) \not\equiv 0$, it follows that $p(z) \equiv 0$. Equation (2) becomes $y^{(n)}(z) = 0$, which is clearly disconjugate). As

$$|\,p(\zeta_0)y(\zeta_0)\,| \geqq |\,p(z)y(z)\,| \,, \qquad z \in H \,, \tag{14}$$

it follows by (13) that for every $z \in H$,

$$|\,y(z)\,| \leqq \prod_{i=1}^{n} |\,z - z_i\,| \frac{|\,y(\zeta_0)\,|\,|\,p(\zeta_0)\,|}{n!} \,.$$

Choosing now $z = \zeta_0$ and using $y(\zeta_0) \neq 0$ we obtain

$$|\,p(\zeta_0)\,| \prod_{i=1}^{n} |\,\zeta_0 - z_i\,| \geqq n! \,. \tag{15}$$

We prove that for ζ_0 satisfying (14),

$$\prod_{i=1}^{n} |\,\zeta_0 - z_i\,| < (1 - |\,\zeta_0\,|)(1 + |\,\zeta_0\,|)^{n-1} \,; \tag{16}$$

(cf [10, Th. 2)].

Let us assume first that the convex hull H of $z_1, \cdots, z_n$ is a polygon. Then, by the maximum principle, ζ_0 is on the boundary of H. Therefore ζ_0 is on a segment, the endpoints of which are two of the n given points $z_1, \cdots, z_n$. We denote these points by z_1, z_2 Clearly,

$$|\,\zeta_0 - z_i\,| < 1 + |\,\zeta_0\,| \,. \qquad i = 3, \cdots, n \,. \tag{17}$$

Denoting by z_1^*, z_2^* the endpoints $|z_1^*| = |z_2^*| = 1$ of the chord determined by z_1 and z_2, we obtain

$$(18) \qquad |\zeta_0 - z_1||\zeta_0 - z_2| < |\zeta_0 - z_1^*||\zeta_0 - z_2^*| .$$

As the product of the segments of a chord through ζ_0 depends only on ζ_0, we have

$$|\zeta_0 - z_1^*||\zeta_0 - z_2^*| = (1 - |\zeta_0|)(1 + |\zeta_0|) .$$

This and (18) give

$$(19) \qquad |\zeta_0 - z_1||\zeta_0 - z_2| < (1 - |\zeta_0|)(1 + |\zeta_0|) .$$

(17) and (19) imply (16).

If H is a segment and ζ_0 one of the points of the segment in which $|p(z)y(z)|$ becomes maximum, then we denote by z_1, z_2 the endpoints of H and by z_1^*, z_2^* the endpoints of the corresponding chord. (17) and (19) hold and therefore (16) is again valid.

(15), which followed from the assumption that (2) is not disconjugate in $|z| < 1$, and (16) imply

$$(20) \qquad |p(\zeta_0)| \geqq \frac{n!}{\prod_{i=1}^{n} |\zeta_0 - z_i|} > \frac{n!}{(1 - |\zeta_0|)(1 + |\zeta_0|)^{n-1}} ,$$

which contradicts assumption (9). This contradiction concludes the proof of the theorem.

For the proof of the next theorem it is convenient to state some simple consequences of Theorem 2. The transformation $\zeta = z/\rho$ maps $|z| < \rho$ on $|\zeta| < 1$, and equation (2) is transformed into (2′) with $P_1(\zeta) = \rho^n p(z)$. As (2) is disconjugate in $|z| < \rho$ if (2′) is disconjugate in $|\zeta| < 1$, we obtain a sufficient condition for disconjugacy of (2) in $|z| < \rho$, namely

$$|p(z)| \leqq \frac{n!}{(\rho - |z|)(\rho + |z|)^{n-1}} , \quad |z| < \rho .$$

Using the minimum of the function $n!/(\rho - r)(\rho + r)^{n-1}$ for $0 \leqq r < \rho$, we obtain another, weaker, sufficient condition for disconjugacy of (2) in $|z| < \rho$,

$$(21) \qquad |p(z)| \leqq \frac{n!}{(n-1)^{n-1}} \left(\frac{n}{2\rho}\right)^n , \quad |z| < \rho .$$

We remark that for $\rho = 1, n = 2$ the value of the constant in (21) is 2. The exact constant in this case is $\pi^2/4$ [3, Th. 2].

THEOREM 3. *Let the analytic function $p(z)$ be regular in $|z| < 1$*

and assume that there exists ρ, $0<\rho<1$, *such that*

$$|p(z)|(1-|z|^2)^n \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n(1-\rho^2)^n, \quad |z|<1 . \tag{22}$$

Then equation (2) *is disconjugate in every non-Euclidean disk of radius* $1/2 \log[(1+\rho)/(1-\rho)]$.

Proof. Let ρ satisfy (22) and let G be a given disk in $|z|<1$ with non-Euclidean radius $1/2\log[(1+\rho)/(1-\rho)]$. By mapping the unit disk on itself, G can be mapped onto a disk G_1 given by $|\zeta|<\rho$. Equation (2) is transformed into (2′). As for linear mappings $\zeta=\zeta(z)$ of the unit disk on itself

$$\left|\frac{d\zeta}{dz}\right| = \frac{1-|\zeta|^2}{1-|z|^2} ,$$

we obtain

$$|P_1(\zeta)| = |p(z)|\left|\frac{dz}{d\zeta}\right|^n = |p(z)|\frac{(1-|z|^2)^n}{(1-|\zeta|^2)^n} . \tag{23}$$

From (23) together with (22) it follows that

$$|P_1(\zeta)| \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n\frac{(1-\rho^2)^n}{(1-|\zeta|^2)^n} ,$$

which for $|\zeta|<\rho$ gives

$$|P_1(\zeta)| \leqq \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n .$$

By (21), this is a sufficient condition for disconjugacy of (2′) in G_1, $|\zeta|<\rho$, and therefore (2) is disconjugate in G. Theorem 3 is thus proved.

This theorem can be stated as follows: if

$$|p(z)|(1-|z|^2)^n \leqq C < \infty , \qquad |z|<1 , \tag{24}$$

then equation (2) is disconjugate in every non-Euclidean disk of radius $1/2\log[(1+\rho_0)/(1-\rho_0)]$, where $\rho_0 = g^{-1}(C)$ and

$$g(\rho) = \frac{n!}{(n-1)^{n-1}}\left(\frac{n}{2\rho}\right)^n(1-\rho^2)^n .$$

$g(\rho)$ is a monotonic decreasing function. Therefore the smallest C satisfying (24) gives the biggest non-Euclidean radius.

For $n=2$ non-Euclidean disks of disconjugacy were considered in [2] and [9].

THEOREM 4. *Assume that the analytic function* $p(z)$ *is regular in* $|z| < 1$. *Let* $n \geqq 3$ *and let* C *be a positive constant. If*

$$|p(z)| \leqq \frac{C}{1 - |z|}, \qquad |z| < 1, \tag{25}$$

then equation (2) *is nonoscillatory in* $|z| < 1$.

In the case $n = 2$, equation (4) is nonoscillatory in $|z| < 1$, if there exists x_1, $0 < x_1 < 1$, such that

$$|p(z)| \leqq \frac{2}{1 - |z|^2}, \quad x_1 < |z| < 1. \tag{26}$$

Proof. Assume that equation (2) has a solution with an infinite number of zeros in the unit disk. We can then find a sequence of zeros $z_1, z_2, \cdots$ tending to z^* on the boundary, $|z^*| = 1$. For any ρ, $0 < \rho < 1$, let $G(\rho)$ be the intersection of the disk $|z - z^*| < \rho$ with the unit disk. Any $G(\rho)$ contains an infinite number of zeros. Denote n of these zeros by $z_1, \cdots, z_n$. As in the proof of Theorem 2, we denote the convex hull of these n points by H and choose $\zeta_0 \in H$ such that (14) holds. We choose z_1 and z_2 as in that proof; (15) and (19) are again valid.

If $n \geqq 3$, then clearly

$$|\zeta_0 - z_i| < 2\rho \qquad i = 3, \cdots, n.$$

Using this and (19) we obtain

$$\prod_{i=1}^{n} |\zeta_0 - z_i| < (1 - |\zeta_0|)(1 + |\zeta_0|)(2\rho)^{n-2} < (1 - |\zeta_0|)2^{n-1}\rho^{n-2}. \tag{27}$$

From (15) and (27) it follows that

$$|p(\zeta_0)| > \frac{n!}{(1 - |\zeta_0|)2^{n-1}\rho^{n-2}}. \tag{28}$$

For any given C, we can find ρ such that

$$\frac{n!}{2^{n-1}\rho^{n-2}} > C. \tag{29}$$

From (28) and (29) we obtain a contradiction to our assumption (25), which completes the proof of the first part of the theorem ($n \geqq 3$).

For $n = 2$, we choose ρ such that $\rho = 1 - x_1$. (15) and (19) imply

$$|p(\zeta_0)| > \frac{2}{1 - |\zeta_0|^2}. \tag{30}$$

As $x_1 < |\zeta_0| < 1$, (30) contradicts (26), which completes the proof of the second part of Theorem 4 ($n = 2$).

By [9, Th. 1] the condition

$$|p(z)| \leqq \frac{1}{(1-|z|^2)^2}, \ |z| > x_0, \qquad 0 < x_0 < 1$$

is sufficient for nonoscillation of (4) in $|z| < 1$; hence the second part ($n = 2$) of Theorem 4 follows from this theorem.

Nehari has given a nonoscillation theorem for the general equation (1) in any bounded convex domain. In the case of the unit disk and the special equation (2) his sufficient condition becomes

$$\lim_{r\to 1} \int_0^{2\pi} |p(re^{i\theta})|\, d\theta < \infty . \tag{31}$$

This sufficient condition (31) implies our condition (25). (See [4, p. 127, Ex. 8]).

3. Equations of even order $n = 2m$; nonexistence of solutions with two zeoros of multiplicity m.

THEOREM 5. *Let the analytic function $p(z)$ be regular in $|z|<1$. The equation*

$$y^{(2m)}(z) + (-1)^{m+1}p(z)y(z) = 0 \tag{32}$$

has no solution having two zeros of multiplicity m in $|z| < 1$ if

$$|p(z)| \leqq P(|z|) , \tag{33}$$

where $P(x)$ is a function with the following properties:

(a) *$P(x)$ is positive and continuous for $-1 < x < 1$;*

(b) *$P(-x) = P(x)$;*

(c) *$(1-x^2)^{2m}P(x)$ is nonincreasing if x varies from 0 to 1;*

(d) *the differential equation*

$$u^{(2m)}(x) + (-1)^{m+1}P(x)u(x) = 0 \tag{34}$$

has no solution with two zeros of multiplicity m in $-1 < x < 1$.

Proof. (cf. [5]). Suppose the theorem is false and there exists a solution of (32) with zeros of multiplicity m at α and $\beta(|\alpha| < 1, |\beta| < 1, \alpha \neq \beta)$. The circle passing through α and β and orthogonal to $|z| = 1$ is divided by $|z| = 1$ into two arcs. We denote the arc inside $|z| < 1$ by C. Without loss of generality, we may assume that C is in the upper half plane and symmetric with respect to the imaginary axis. The linear transformation

(35) $$z = \frac{\zeta + i\rho}{1 - i\rho\zeta}, \qquad 0 \leqq \rho < 1,$$

maps $|z| < 1$ on $|\zeta| < 1$ and C on the linear segment $-1 < \zeta < 1$. With the aid of Theorem 1 and (23), equation (32) is transformed into the equation

(36) $$w^{(2m)}(\zeta) + (-1)^{m+1} q(\zeta) w(\zeta) = 0,$$

with

(37) $$|q(\zeta)| = |p(z)| \left| \frac{dz}{d\zeta} \right|^{2m} = |p(z)| \frac{(1 - |z|^2)^{2m}}{(1 - |\zeta|^2)^{2m}}.$$

It follows from (35) that $|\zeta| \leqq |z|$ if $-1 < \zeta < 1$. Hence, by assumption (c) it follows that

$$(1 - |z|^2)^{2m} P(|z|) \leqq (1 - |\zeta|^2)^{2m} P(|\zeta|), \quad -1 < \zeta < 1.$$

Combining this with (33) and (37) we obtain

(38) $$|q(\zeta)| \leqq P(|\zeta|), \qquad -1 < \zeta < 1.$$

Thus, our assumption that (32) has a solution with two zeros at α and β of multiplicity m implies that (36) has a solution $w(\zeta)$ possessing two zeros of multiplicity m at a and b, $-1 < a < b < 1$. Let $w(\zeta)$ be this solution. Multiplying equation (36) by $\bar{w}(\zeta)$ and integrating from a to b along the real axis, we obtain

$$\int_a^b w^{(2m)}(x)\bar{w}(x)dx + (-1)^{m+1} \int_a^b q(x)\,|w(x)|^2\,dx = 0.$$

Integrating by parts m times and noting that all the integrated parts vanish, we get

$$\int_a^b w^{(m)}(x)\bar{w}^{(m)}(x)dx = \int_a^b q(x)\,|w(x)|^2\,dx.$$

By (38) and assumption (b) it follows that

(39) $$\int_a^b |w^{(m)}(x)|^2\,dx \leqq \int_a^b P(x)\,|w(x)|^2\,dx.$$

If we write $w(x) = \sigma(x) + i\tau(x)$, both σ and τ have zeros of multiplicity m at a and b and we have $|w^{(m)}|^2 = [\sigma^{(m)}]^2 + [\tau^{(m)}]^2$. (39) becomes

(40) $$\int \{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\} dx \leqq \int_a^b P(x)[\sigma^2(x) + \tau^2(x)]dx.$$

Let now λ be the lowest eigenvalue of the real differential system given by

$$u^{(2m)}(x) + (-1)^{m+1}\lambda P(x)u(x) = 0 \tag{41}$$

with $a \leqq x \leqq b$, $-1 < a < b < 1$, and the boundary conditions

$$u(a) = u'(a) = \cdots = u^{(m-1)}(a) = 0$$
$$u(b) = u'(b) = \cdots = u^{(m-1)}(b) = 0 .$$

As σ and τ are admissible comparison functions for this problem, it follows by Rayleigh's inequality that

$$\begin{aligned} \lambda \int_a^b P(x)\sigma^2(x)dx &\leqq \int_a^b [\sigma^{(m)}(x)]^2 dx \\ \lambda \int_a^b P(x)\tau^2(x)dx &\leqq \int_a^b [\tau^{(m)}(x)]^2 dx . \end{aligned} \tag{42}$$

Combining (42) with (40) we obtain

$$\int_a^b \{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\}dx \leqq \frac{1}{\lambda}\int_a^b \{[\sigma^{(m)}(x)]^2 + [\tau^{(m)}(x)]^2\}dx . \tag{43}$$

Hence, $\lambda \leqq 1$. If $\lambda = 1$, then equation (41) becomes (34), and the first eigenfunction of the corresponding system contradicts assumption (d). If $\lambda < 1$, we take $a < c < b$ and consider equation (41) for $a \leqq x \leqq c$, with the boundary conditions

$$u(a) = u'(a) = \cdots = u^{(m-1)}(a) = 0$$
$$u(c) = u'(c) = \cdots = u^{(m-1)}(c) = 0 .$$

Let $\lambda_p(c)$ be the first eigenvalue of this system. By the minimum characterization,

$$\lambda_p(c) = \operatorname{Min} \frac{\int_a^c v^{(m)2}(x)dx}{\int_a^c Pv^2(x)dx} , \tag{44}$$

where the minimum is taken over the class of all functions $v(x)$ in C^m (or D^m) satisfying

$$v(a) = v'(a) = \cdots = v^{(m-1)}(a) = 0$$
$$v(c) = v'(c) = \cdots = v^{(m-1)}(c) = 0 .$$

Hence, $\lambda_p(c)$ is increasing as c goes from b to a. From (44) it follows that

$$\lambda_p(c) \geqq \lambda_k(c) , \tag{45}$$

where k is a constant satisfying

$$k > P(x) > 0 \quad \text{in} \quad [a, b] .$$

Denoting $c - a = l$ and $lt = x - a$, the system

$$u^{(2m)}(x) + (-1)^{m+1}\lambda k u(x) = 0$$
$$u(a) = \cdots = u^{(m-1)}(a) = 0$$
$$u(c) = \cdots = u^{(m-1)}(c) = 0$$

is transformed into the system

$$u^{(2m)}(t) + (-1)^{m+1}\Lambda k u(t) = 0$$
$$u(0) = \cdots = u^{(m-1)}(0) = 0$$
$$u(1) = \cdots = u^{(m-1)}(1) = 0 .$$

Denoting the first eigenvalue of this system by Λ_k, it follows that

$$\Lambda_k = \lambda_k(c) l^{2m} . \tag{46}$$

From (45) and (46) it follows that as c goes to a $(l \to 0)$, $\lambda_p(c)$ tends to ∞. Hence, there exists a value c_1, $a < c_1 < b$, such that $\lambda(c_1) = 1$, and we again obtain a contradiction to our assumption (d). This completes the proof of Theorem 5.

For $m = 1$ Theorem 5 reduces to [5, Th. 1].

We bring now some examples. For $m = 2$, i.e. for the differential equation of the fourth order,

$$y^{(4)}(z) - p(z)y(z) = 0 ,$$

the following functions may serve as examples in Theorem 5 :

$$P_1(x) = (0.753\,\pi)^4 = 31.28 \cdots , \tag{47}$$

$$P_2(x) = \frac{9}{(1 - x^2)^4} \tag{48}$$

and

$$P_3(x) = \frac{24}{(1 - x^2)^2} . \tag{49}$$

$P_1(x)$, $P_2(x)$ and $P_3(x)$ clearly satisfy assumptions (a), (b), (c) of the theorem. In order to show that $P_1(x)$ satisfies assumption (d), we consider the equation

$$u^{(4)}(x) - k^4 u(x) = 0 ,$$

which has $u(x) = C_1 \cos kx + C_2 \sin kx + C_3 \cos hkx + C_4 \sin hkx$ as general solution. The requirement $u(\pm 1) = u'(\pm 1) = 0$ implies $\tan hk = \pm \tan k$, the smallest solution of which is $k = 2.3650 = 0.753\,\pi$. The equation

$$u^{(4)}(x) - (0.753\,\pi)^4 u(x) = 0$$

has therefore a solution with double zeros at ± 1; in other words, the first eigenvalue λ_1 of the system

$$
\begin{aligned}
&u^{(4)}(x) - \lambda(0.753\,\pi)^4 u(x) = 0 \\
&u(\pm 1) = u'(\pm 1) = 0
\end{aligned} \tag{50}
$$

equals 1. As for any $a, b, -1 < a < b < 1$, the eigenvalues of the system

$$
\begin{aligned}
&u^{(4)}(x) - \lambda(0.753\,\pi)^4 u(x) = 0 \\
&u(a) = u'(a) = u(b) = u'(b) = 0
\end{aligned} \tag{51}
$$

are greater than the eigenvalues of (50), the system (51) cannot have an eigenvalue equal to 1. $P_1(x)$ thus satisfies assumption (d).

The following inequalities due to Beesack [1, p. 494]

$$\int_{-1}^{1} v''^2 dx > \int_{-1}^{1} \frac{9v^2}{(1-x^2)^4}\,dx\,, \quad v \in D''\,, \quad v(\pm 1) = v'(\pm 1) = 0$$

unless $v = A(1-x^2)^{3/2}$, and

$$\int_{-1}^{1} v''^2 dx > \int_{-1}^{1} \frac{24v^2}{(1-x^2)^2}\,dx\,, \quad v \in D''\,, \quad v(\pm 1) = v'(\pm 1) = 0$$

unless $v = A(1-x^2)^2$, imply that $P_2(x)$ and $P_3(x)$ satisfy assumption (d).

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INTEGRAL EQUIVALENCE OF VECTORS OVER LOCAL MODULAR LATTICES, II

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In an earlier paper in this Journal we have shown that the integral equivalence problem for vectors in a modular lattice L on a dyadic local field F can be determined, for $\dim L \neq 4$, 5, 6, by inspecting the numbers represented in F by the characteristic sets which are canonically associated to the given vectors. The purpose of this paper is to remove this dimensional restriction of L. In addition, we shall discuss the effective determination of integral equivalences amongst vectors as well as derive some "cancellation" results. Finally, we prove, as expected, that this same improvement carries over in the characteristic two situation.

The presentation of the results contatined herein shall be as follows:

1. Preliminary observations.
2. Statement and proof of the main theorem.
3. Effective computability.
4. Cancellation theorems.
5. Characteristic two case.

We shall adhere to the same terminology and notations as those contained in [2]. The following data will be fixed throughout this paper. L is a unimodular lattice, u and v are two maximal (primitive) vectors in L having the same quadratic length $Q(u) = Q(v) = \delta$. Integral equivalence between u and v shall always be denoted by $u \sim v$.

1. Preliminary observations. For any maximal vector $w \in L$, the characteristic set of w in L is defined as

$$\mathfrak{M}_w = \{x \in L \,|\, B(x, w) = 1\} .$$

It is easy to see that

$$\mathfrak{M}_w = \hat{w} + \langle w \rangle^{\perp} = \{\hat{w} + y \,|\, y \in \langle w \rangle^{\perp}\}$$

where $\hat{w}$ is *any* vector in $\mathfrak{M}_w$.

NOTATION 1.1. Almost always when we write

$$\mathcal{O}x + \mathcal{O}y \cong A(\alpha, \beta)$$

we mean that

$$(*) \qquad Q(x) = \alpha\ , \qquad Q(y) = \beta\ , \qquad B(x, y) = 1\ .$$

However, sometimes—and the context under which it occurs will be clear—it may simply mean that the lattice $\mathcal{O}x + \mathcal{O}y$ is isometric to $A(\alpha, \beta)$ without necessarily implying that the basis vectors $\{x, y\}$ satisfy $(*)$.

LEMMA 1.2. *$\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$ does not, in general, imply $u \sim v$.*

Proof. Let $L = \mathcal{O}x + \mathcal{O}u = A(a', 0)$ where a' is a norm generator for $\mathcal{G}L$. Suppose $v = \varepsilon[u - (2/a')x]$ for some unit ε, and $y = -\varepsilon^{-1}x$. Then,

$$L = \mathcal{O}y + \mathcal{O}v \cong A(\varepsilon^{-2}a', 0)\ .$$

Clearly then,

$$\langle u\rangle^{\perp} = \mathcal{O}u \cong \mathcal{O}v = \langle v\rangle^{\perp}\ .$$

Any isometry $\sigma \in O(L)$ sending u onto v takes x onto, say,

$$\sigma(x) = \alpha y + \beta v, \alpha, \beta \in \mathcal{O}\ .$$

But, $B(\sigma(x), v) = B(x, u) = 1$ implies $\alpha = 1$. The length of $\sigma(x)$ must, on the other hand, be a' so that

$$(**) \qquad a'\left(\frac{1 - \varepsilon^2}{\varepsilon^2}\right) + 2\beta = 0\ .$$

This equation $(**)$, of course, does not always admit integral solution for β when norm generator a' and unit ε can be arbitrary. Thus, we can not expect, in general, to have $u \sim v$ with just requiring their orthogonal complements to be isometric.

1.3. Unless otherwise specified, $\dim L \geqq 4$ shall be assumed throughout the rest of this paper. To avoid excessive repetitions, let us fix a few more notations here. For any primitive vector $w \in L$, and any vector $\bar{w} \in \mathfrak{M}_w$, put

$$L(w; \bar{w}) = \mathcal{O}w + \mathcal{O}\bar{w}\ ; \qquad M(w; \bar{w}) = L(w; \bar{w})^{\perp}\ .$$

LEMMA 1.4. *Let $\dim L$ be arbitrary. $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$ implies $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$.*

Proof. Pick $\bar{u} \in \mathfrak{M}_u$, and $\bar{v} \in \mathfrak{M}_v$ such that $Q(\bar{u}) = Q(\bar{v})$. Let D be the common discriminant of $L(u; \bar{u})$ and $L(v; \bar{v})$. Then,

$$\langle u\rangle^{\perp} = M(u;\bar{u}) \perp \mathcal{O}\langle u-\delta\bar{u}\rangle \cong M(u;\bar{u}) \perp \langle \delta D\rangle$$
$$\langle v\rangle^{\perp} = M(v;\bar{v}) \perp \mathcal{O}\langle v-\delta\bar{v}\rangle \cong M(v;\bar{v}) \perp \langle \delta D\rangle .$$

An application of Witt's theorem yields

$$F\langle u\rangle^{\perp} \cong F\langle v\rangle^{\perp} .$$

Also, it is not difficult to see that hypothesis of the lemma implies the equality of the norm groups (via 93: 21, [4]) may be assumed:

$$\mathcal{G}M(u;\bar{u}) = \mathcal{G}M(v;\bar{v}) = \mathcal{G}^*$$

when $\dim L \geqq 5$; here, $\mathcal{G}^* = \mathcal{G}(\langle u\rangle^{\perp}) = \mathcal{G}(\langle v\rangle^{\perp})$. Hence, by O'Meara's theorem on modular lattices (93: 16, [4]) $M(u;\bar{u})$ is isometric to $M(v;\bar{v})$. So let $\dim L = 4$. Adjoin the hyperbolic lattice $A(0,0)$ to L and call the enlarged lattice L'. Then, $\langle u\rangle^{\perp}$ is isometric to $\langle v\rangle^{\perp}$ in L'. But,

$$\langle u\rangle^{\perp}(\text{in } L') \cong \langle u\rangle^{\perp}(\text{in } L) \perp A(0,0) ,$$

and similarly for $\langle v\rangle^{\perp}$. Cancelling $A(0,0)$ gives the desired result. When $\dim L \leqq 3$, the proof is entirely trivial.

REMARK 1.5. The proof of Lemma 1.4 is one without using the fact that $Q(\mathfrak{M}_u)$ equals $Q(\mathfrak{M}_v)$ implies $u \sim v$ for large enough dimension of L as we did in Corollary 4.2, [2].

LEMMA 1.6. *If* $|\delta| = 0, 1$, *then* $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$ *implies* $u \sim v$.

Proof. By Lemma 1.4, $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$. If δ is an unit, then everything is clear. Otherwise, let $\delta = 0$. Let $\bar{u}$ and $\bar{v}$ be the two vectors as in Lemma 1.4, then we have the radical splittings

$$\langle u\rangle^{\perp} = \operatorname{Rad}\langle u\rangle^{\perp} \perp M(u;\bar{u}) = \mathcal{O}u \perp M(u;\bar{u})$$
$$\langle v\rangle^{\perp} = \operatorname{Rad}\langle v\rangle^{\perp} \perp M(v;\bar{v}) = \mathcal{O}v \perp M(v;\bar{v}) .$$

But, in this case

$$\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp} \text{ if and only if } M(u;\bar{u}) \cong M(v;\bar{v}) .$$

In view of this Lemma 1.6, we shall henceforth, unless otherwise noted, assume that $|\delta|$ is neither 0 nor 1.

1.7. In the proof of Theorem 4.4, [2], an important fact used was Lemma 4.5, [2], whose proof can be much simplified by observing that in the case when both u and v are Type I vectors, $Q(\mathfrak{M}_u)$ equals $Q(\mathfrak{M}_v)$ implies u and v are of the same parity, and also $\langle u\rangle^{\perp}$ is isometric to $\langle v\rangle^{\perp}$ via Lemma 1.4. Therefore, $u \sim v$ by Proposition 3.5,

[2]. Hence, Lemma 4.5, [2] becomes easy to see.

2. Statement and proof of the main theorem.

MAIN THEOREM 2.1. *Let L be an unimodular lattice over a dyadic local field; then, two maximal (primitive) vectors in L having the same quadratic length are integrally equivalent if and only if their characteristic sets represent the same numbers in the field.*

NOTATIONS 2.2. Put $\mathscr{G}^* = \mathscr{G}(\langle u\rangle^\perp) = \mathscr{G}(\langle v\rangle^\perp)$. For any $\bar{u} \in \mathfrak{M}_u$, we set

$$\lambda_{\bar{u}} = \delta Q(\bar{u}) - 1 ,$$

the discriminant of $\mathscr{O}\bar{u} + \mathscr{O}u$. The letters a, b shall always be used for norm and weight (base) generator of $\mathscr{G}L$ respectively; similarly, a^* and b^* for the norm group $\mathscr{G}^*$.

2.3. In view of Theorem 4.1, [2], it suffices to prove for $\dim L = 4, 5, 6$. Proposition 3.5, [2] allows us to assume that both u and v are Type II vectors.

2.4. Because u and v are Type II vectors, we may further suppose henceforth that $\langle u\rangle^\perp$ (hence, also $\langle v\rangle^\perp$) are not "depleted" in the sense of [1]. For, if not, then the norm group of $M(u; \bar{u})$ will be equal to $\mathscr{G}^*$, $\forall \bar{u} \in \mathfrak{M}_u$. Consequently, the integral equivalence between u and v may be readily deduced from the hypothesis of the Main Theorem. So, in particular, $a^*b^* \sim \pi$ (i.e., $\operatorname{ord} a^* + \operatorname{ord} b^*$ is odd).

2.5. Roughly, we first observe that the number δ may be assumed to have a special feature. Using this "reduction lemma," we settle the 4-dimensional case by computational means; in the case of $\dim L$ equals five, we shall show that the hypothesis, and hence also the conclusion, of Theorem 4.4, [2] is satisfied. Finally, the $\dim L = 6$ case falls through by a modified argument tailored after the 5-dimensional situation.

LEMMA 2.6. *If the quadratic defect $\mathscr{D}(\delta\lambda_{\bar{u}}a^*)$ is strictly contained in the ideal $a^*b^*\mathscr{O}$ for some norm generator a^* of $\mathscr{G}^*$, then $\mathscr{D}(\delta\lambda_u a') \subsetneqq a^*b^*\mathscr{O}$ for every norm generator a' of $\mathscr{G}^*$.*

Proof. Put $\delta\lambda_u = a^*t^2 + b^*t^2 + b^*\alpha; t, \alpha \in \mathscr{O}$. Now, u is a Type II vector implies $|t| < 1$. The hypothesis together with the fact that $a^*b^* \sim \pi$ yields $|\alpha| < 1$. Now then,

$$a' = a^*\varepsilon^2 + b^*M, |\varepsilon| = 1, M \in \mathcal{O}$$

and

$$(\circ) \qquad \delta\lambda_{\bar{u}}a' = a^{*2}t^2\varepsilon^2 + a^*b^*\alpha\varepsilon^2 + a^*t^2b^*M + b^{*2}\alpha M .$$

The quadratic defect of the right hand side of ($\circ$) is clearly contained in a $a^*b^*\mathcal{O}$ since each of the last three terms is in it.

COROLLARY 2.7. *If* $\exists\bar{u} \in \mathfrak{M}_u$ *such that for some* a^*, *we have* $\mathcal{D}(\delta\lambda_{\bar{u}}a^*) = a^*b^*\mathcal{O}$ *then,* $\mathcal{D}(\delta\lambda_{\bar{u}}a') = a^*b^*\mathcal{O}$ *for every norm generator* a' *of* $\mathcal{G}^*$.

COROLLARY 2.8. *If* $\exists\bar{u} \in M_u$ *such that for some* a^*, *we have* $\mathcal{D}(\delta\lambda_{\bar{u}}a^*) \subsetneqq a^*b^*\mathcal{O}$, *then for every* $x \in \mathfrak{M}_u$ *such that* $\lambda_{\bar{u}} \in \lambda_x\mathcal{U}^2$ ($\mathcal{U}$ *denotes the group of units in* $\mathcal{O}$), *the inequality below is valid*

$$\mathcal{D}(\delta\lambda_x a') \subsetneqq a^*b^*\mathcal{O} , \qquad a' \textit{ norm generators of } \mathcal{G}^* .$$

REDUCTION LEMMA 2.9. *We may henceforth assume that*

$$\mathcal{D}(\delta\lambda_{\bar{u}}a') = a^*b^*\mathcal{O}$$

for every $\bar{u} \in \mathfrak{M}_u$ *and every norm generator* a' *of* $\mathcal{G}^*$. *(Of course, the same goes for all* $\bar{v} \in \mathfrak{M}_v$ *and all* a' *of* $\mathcal{G}^*$.)

Proof. Since u is a Type II vector so that, by definition, for each vector $\tilde{u} \in \mathfrak{M}_u$, the sublattice $M(u; \tilde{u})$ has norm ideal equal to $a^*\mathcal{O}$. If there exists a vector $\bar{u} \in \mathfrak{M}_u$ with the property that

$$\mathcal{D}(\delta\lambda_{\bar{u}}a^*) \subsetneqq a^*b^*\mathcal{O} ,$$

then Corollary 2.8 together with a simple computation of the weight ideal $\mathcal{W}(\langle u\rangle^\perp)$ of $\langle u\rangle^\perp$ tell us that $\mathcal{W}M(u; \bar{u})$ equals $b^*\mathcal{O}$. In other words, we have an equality of the norm groups

$$\mathcal{G}M(u; \bar{u}) = \mathcal{G}^* .$$

Now, pick any $\bar{v} \in \mathfrak{M}_v$ with $Q(\bar{v}) = Q(\bar{u})$. Since $\lambda_{\bar{v}} \in \lambda_{\bar{u}}\mathcal{U}^2$, Corollary 2.8 implies that the norm group of $M(v; \bar{v})$ must equal to $\mathcal{G}^*$ also. Hence,

$$M(v; \bar{v}) \cong M(u; \bar{u})$$

by Witt and O'Meara. Thus, $u \sim v$.

2.10. What Lemma 2.9 says, in effect, is that for any norm generator a^* and weight generator b^* of $\mathcal{G}^*$, and any vector $\bar{u} \in \mathfrak{M}_u$, the number δ has the special feature that

$$\delta\lambda_{\bar{u}} = a^* t^2 + b^* \gamma$$

where γ is an unit. (Of course, t is not an unit by Type II-ness.)

2.11. Proof of the Main Theorem for $\dim L = 4$.

Proof. The following claims may be readily proved and we do not give the proofs here:

(I) For any binary nondepleted unimodular lattice K, whenever K is represented as $K \cong A(\alpha, \gamma)$ either α or γ must be a norm generator for $\mathscr{G}K$ and furthermore, the quadratic defect $\mathscr{D}(\alpha\gamma)$ must be $\mathscr{N}L \cdot \mathscr{W}L$.

(II) If K has same hypothesis as in statement (I), and if, say, α is a norm generator for $\mathscr{G}K$ and $Q(x) = \alpha$, for some $x \in K$, then $K = \mathscr{O}x + \mathscr{O}y = A(\alpha, \beta)$—where one may take, if needed, β to be a weight (base) generator.

SUBLEMMA. *Suppose $K \cong \mathscr{C}(a, b)$ with $ab \sim \pi$ and $b \notin 2\mathscr{O}$—in Riehm's notation, see* [6]*—and if L is any binary unimodular lattice such that FK (i.e., $F \otimes_{\mathscr{O}} K$) is isometric to FL, and $a' \in Q(L)$ is a norm generator for both $\mathscr{G}L$ and $\mathscr{G}K$, then $K \cong L$.*

Proof. Let $Q(x) = a'$, for some primitive vector $x \in L$. FL isometric to FK implies the discriminant of K equals that of L so that if we write the common discriminant as $-(1 + a)$ whose defect is $\alpha\mathscr{O}$, then

$$L \cong \mathscr{C}(a', -\alpha a'^{-1}) .$$

It is easy to see $\operatorname{ord}(-\alpha a'^{-1}) = \operatorname{ord}(b)$. Hence, their weights (and therefore their norm groups as well) are equal.

SUBLEMMA. *There exists a vector $\bar{u} \in \mathfrak{M}_u$ such that $M(u; \bar{u})$ has norm group equaling to $\mathscr{G}^*$.*

Proof. Choose any $\tilde{u} \in \mathfrak{M}_u$. If the norm group of $M(u; \tilde{u})$ is not $\mathscr{G}^*$, we put

$$\langle u \rangle^{\perp} \cong M(u; \tilde{u}) \perp \langle \delta\lambda_{\tilde{u}} \rangle ,$$

where, since L is 4-dimensional $M(u; \tilde{u})$ is isometric to $A(a^*, \beta)$. (*N. B.* We used Type II-ness here.) Performing an ou-transform (see [5]) changing β to $\beta \perp \delta\lambda_{\tilde{u}}$ we see we endup with

$$\langle u \rangle^{\perp} \cong M(u; \bar{u}) \perp \langle \delta\lambda_{\bar{u}} \rangle ,$$

where

$$M(u; \bar{u}) \cong A(a^*, \beta \perp \delta\lambda_{\bar{u}})$$

whose norm group is obviously that of $\mathscr{G}^*$.

Using the results collected in this § 2.11, the proof of the dim $L = 4$ case is apparent. Indeed, choose such a vector $\bar{u}$ in $\mathfrak{M}_u$ as mentioned in the sublemma. Choose a $\bar{v} \in \mathfrak{M}_v$ having $Q(\bar{u}) = Q(\bar{v})$. Since 2.4 allows us to assume $\langle u \rangle^\perp$ is not depleted, one of the sublemmas implyies immediately that $M(u; u) \cong M(v; \bar{v})$.

2.12. Proof of the Main Theorem for dim $L = 5$. In view of Theorem 4.4, [2], it is sufficient to prove the following statement.

PROPOSITION. *Let dim L be either 5 or 6, u be a Type II vector in L with $\langle u \rangle^\perp$ being nondepleted; and $Q(u) = \delta$ is a number satisfying the equation in § 2.10. Then, there exists a vector $\bar{u} \in \mathfrak{M}_u$ such that $M(u; \bar{u})$ is isotropic.*

Proof. Recall $\mathscr{G}(\langle u \rangle)^\perp = \mathscr{G}^* = a^*\mathscr{O}^2 + b^*\mathscr{O}$, for any norm generator a^* and weight generator b^* of $\mathscr{G}(\langle u \rangle^\perp)$. Pick any $\tilde{u}$ from $\mathfrak{M}_u$. We see by 93: 21, [4], the sublattice $M(u; \tilde{u})$ may be assumed to have norm group equal to $\mathscr{G}^*$.

If dim $L = 5$ then by 93: 18, [4],

$$M(u; \tilde{u}) \cong A(b', 4\rho b'^{-1}) \perp \langle -d\Delta \rangle$$

where $\Delta = 1 - 4\rho$. Here again, b' can be *any* weight generator! And d is the discriminant of $M(u; \tilde{u})$. But, clearly by a suitable op-transformation on $M(u; \tilde{u})$, we can have the following splitting:

$$(*) \qquad M(u; \tilde{u}) \cong A(b' \perp (-d\Delta)s^2, 4\rho b'^{-1}) \perp \langle \varepsilon \rangle$$

for some unit ε; s can be any integer.

Now, § 2.10 tells us that

$$-\delta\lambda_{\tilde{u}} = (-d\Delta)t^2 + \tilde{b} ,$$

where $\tilde{b}$ is a weight generator and t is not an unit because of Type II-ness. All we have to do next is to let b' to equal to $\tilde{b}$ and apply the above op-transform so that $s = t$. Finally, apply another op-transform on $\langle u \rangle^\perp$, this time, changing $b' \perp (-d\Delta)s^2$ into

$$b' \perp (-d\Delta)s^2 \perp \delta\lambda_{\tilde{u}} = 0$$

and the resulting picture looks like

$$\langle u \rangle^\perp \cong A(0, 0) \perp \langle \mu \rangle \perp \langle \delta\lambda_{\bar{u}} \rangle$$

for some unit μ and some $\bar{u} \in \mathfrak{M}_u$. (*N.B.* We get surely an hyperbolic component because $4\rho b'^{-1}$ lies in $2\mathcal{O}$.) This is, of course, the vector $\bar{u}$ that does the job for us.

The proof of the case for $\dim L = 6$ is almost identical except there the original sublattice $M(u; \bar{u})$ looks like

$$M(u; \tilde{u}) \cong A(b', 4\rho b'^{-1}) \perp A(a', [1 - d\Delta]a'^{-1})$$

for arbitrarily chosen norm generator a' and weight generator b'.

REMARK 2.13. An important observation to be made in the proof of the proposition in § 2.12 is that under the conditions given in that proposition, one can always derive a vector $\bar{u} \in \mathfrak{M}_u$ such that the sublattice $M(u; \bar{u})$ has an hyperbolic component. This is the key to the short proof the $\dim L = 6$ case of our Main Theorem to be given below. Our first proof for this 6-dimensional situation involved long and elaborate arguments treating the vectors "case by case"; that is, considering them when they are both $\mathcal{N}$- ($\mathcal{G}$-) regular, irregular, ... etc. Yet, it is precisely by looking at them at such detailed level that enabled us to realize the necessity for some result like our "Reduction Lemma", and hence the equality in § 2.10.

2.14. Proof of the Main Theorem for $\dim L = 6$.

Proof. By § 2.13, choose a vector $\bar{u} \in \mathfrak{M}_u$ such that $M(u; \bar{u})$ incorporates an hyperbolic component. [*N.B.* Strictly speaking, the existence of such a vector $\bar{u} \in \mathfrak{M}_u$ has thus far been verified only when $M(u; \bar{u})$ assumes the so-called "K-form", see 93: 18, [4]; that is

$$M(u; \bar{u}) \cong A(b', 4b'^{-1}) \perp A(a', [1 - d\Delta]a'^{-1}) .$$

It is not difficult to see, however, that if $M(u; \bar{u})$ assumes the "J-form":

$$M(u; \bar{u}) \cong A(b', 0) \perp A(a', -\alpha a'^{-1})$$

where α is that integer such that

$$d = 1 + \alpha , \qquad \mathcal{D}(d) = \alpha\mathcal{O} ,$$

then, an entirely analogous argument carries through.]

A word of caution! The temptation here is to cancel the $A(0, 0)$ component in both $\langle u \rangle^{\perp}$ and $\langle v \rangle^{\perp}$, and then claim a "reduction" to the quaternary case. The fallacy is clearly that the resulting characteristic sets in the now smaller lattices need not necessarily represent the same field elements any more! What one can claim instead is that one can indeed find a vector u^* in $\mathfrak{M}_u$ such that $M(u; u^*)$ has the "J-form" because

$$\begin{aligned}\langle u\rangle^{\perp} &\cong A(a^*, \cdots) \perp A(0, 0) \perp \langle \delta\lambda_{\bar{u}}\rangle \\ &\cong A(a^*, \cdots) \perp A(\delta\lambda_{\bar{u}}, 0) \perp \langle \delta\lambda_{u^*}\rangle \qquad \text{for some } u^* \in \mathfrak{M}_u \\ &\cong A(a^*, \cdots) \perp A(b^*, 0) \perp \langle \delta\lambda_{u^*}\rangle .\end{aligned}$$

Pick any v^* from $\mathfrak{M}_v$ such that $Q(v^*)$ equals $Q(u^*)$. Put

$$\mathscr{G}M(v; v^*) = a^*\mathscr{O}^2 + \bar{b}\mathscr{O} ,$$

for some $\bar{b} \in \mathscr{O}$. We now claim $M(v; v^*)$ must also assume a "J-form" This is clear when $a^*\bar{b} \sim 1$. On the other hand, if $a^*\bar{b} \sim \pi$, then $b^*\bar{b} \sim 1$ so that since $FM(u; u^*)$ is isometric to $FM(v; v^*)$, our claim becomes clear. Therefore, we have

$$M(v; v^*) \cong A(a^*, \cdots) \perp A(\bar{b}, 0)$$

and

$$\langle v\rangle^{\perp} \cong M(v; v^*) \perp \langle \delta\lambda_{v^*}\rangle .$$

We are now presented in a situation which is strikingly similar to the 5-dimensional case in Theorem 4.4, [2]. Indeed, if $\bar{b}$ has order greater than that of b^*, a similar op-transformation finishes the proof.

2.15. The proof of the Main Theorem is now complete.

3. Effective computability.

3.1. **Binary case.** Given a maximal vector u with quadratic length $Q(u) = \delta$, it is easy to find a vector $\bar{u}$ from $\mathfrak{M}_u$. Do the same for $\bar{v}$. Compute $Q(\bar{u})$ and $Q(\bar{v})$ and see if they are congruent modulo $\omega\mathscr{O}$, where ω, as usual, denotes $\max\{\delta, 2\}$. If they are, then it is easily verified that u and v must be of the same parity so that Theorem 2.1, [2], tells us $u \sim v$. If not, obviously u and v are not of the same parity. Hence, u and v are not integrally equivalent. Since the vectors $\bar{u}$ and $\bar{v}$ are arbitrarily chosen, we see the actual computation involved for checking integral equivalence in $\dim L = 2$ is quite minimal.

3.2. Computationally, it is not always a pleasant task to determine $Q(\mathfrak{M}_u)$ for a given vector u. Fortunately, for sufficiently large dimension of the given lattice L, say $\dim L \geqq 5$, there is a good remedy. We have, indeed the following result.

THEOREM. *Suppose one can find a single pair of vectors $\bar{u} \in \mathfrak{M}_u$ and $\bar{v} \in \mathfrak{M}_v$ such that $Q(\bar{u}) = Q(\bar{v})$, and suppose further we have*

$$\mathscr{G}(\langle u\rangle^{\perp}) = \mathscr{G}(\langle v\rangle^{\perp}) = \mathscr{G}^* ,$$

then, $u \sim v$ *whenever* $\dim L \geqq 5$.

Proof. Express

$$\langle u\rangle^{\perp} \cong M(u; \bar{u}) \perp \langle \delta\lambda_{\bar{u}}\rangle .$$

By 93: 21, [4], there is a Jordan decomposition

$$\langle u\rangle^{\perp} \cong M(u; u^*) \perp \langle \delta\lambda_{u^*}\rangle$$

such that $\lambda_{u^*} = \lambda_{\bar{u}}$, and $\mathscr{G}(M(u; u^*)) = \mathscr{G}^*$. Do the same for vector v. We see then the norm groups for $M(u; u^*)$ and $M(v; v^*)$ are equal and moreover, $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$, whenever u and v are anisotropic vectors. If δ should be zero, then modulo radicals $\langle u\rangle^{\perp}$ is just $M(u; \bar{u})$. Similarly for $\langle v\rangle^{\perp}$. But, $\mathscr{G}(\langle u\rangle^{\perp}) = \mathscr{G}(M(u; \bar{u}))$ so that the hypothesis that the norm groups for $\langle u\rangle^{\perp}$ and $\langle v\rangle^{\perp}$ being equal implies here their isometry.

Suppose, for the moment, that $\dim L \geqq 7$, then $\dim M(u; \bar{u}) \geqq 5$ so that

$$Q(M(u; \bar{u})) = \mathscr{G}(M(u; \bar{u})) .$$

Therefore, the hypothesis of the theorem here implies $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$ and $u \sim v$ by the Main Theorem.

For $\dim L = 6$, again $M(u; \bar{u})$ represents every element of its own norm group by a theorem of Riehm, see Theorem 7.4, [6]. So, once again $u \sim v$ by Main Theorem.

Let $\dim L = 5$. If u is $\mathscr{G}$-regular and δ satisfies condition (D) in the sense defined in § 3, [2], then it is not difficult to see with the help of Corollary 3.3, [2], that $u \sim v$. If δ does not satisfy condition (D), then, since we have already shown that $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$, we deduce $u \sim v$ by Proposition 3.4, [2]. So, let u be $\mathscr{N}$-regular, but $\mathscr{G}$-irregular. If $\mathscr{D}(\delta\lambda_{\bar{u}}a^*) \subsetneqq a^*b^*\mathscr{O} = b^*\mathscr{O}$, then, we can show, by same argument in the proof of Main Theorem, that $\mathscr{G}(M(u; \bar{u}))$ would equal to $\mathscr{G}^*$. Similarly, for $\bar{v}$ and $M(v; \bar{v})$. Thus, $u \sim v$. If, on the other hand,

$$\mathscr{D}(\delta\lambda_{\bar{u}}a^*) = b^*\mathscr{O} ,$$

then, as in § 2.14, we can find a vector u^* such that $M(u; u^*)$ supports an hyperbolic component and again we get

$$Q(M(u; u^*)) = \mathscr{G}(M(u; u^*))$$

by Theorem 7.4, [6]. Everything repeats once more; $u \sim v$ is therefore clear.

The theorem is therefore proved.

COROLLARY 3.3. *If* $\dim L \geqq 6$, $\mathscr{G}(\langle u\rangle^{\perp})$ *equals* $\mathscr{G}(\langle v\rangle^{\perp})$ *and there exists a single pair of vectors* $\bar{u} \in \mathfrak{M}_u$ *and* $\bar{v} \in \mathfrak{M}_v$ *such that*

$$Q(\bar{u}) \equiv Q(\bar{v}) \bmod \mathscr{G}(\langle u\rangle^{\perp}) ,$$

then, $u \sim v$.

REMARK 3.4. As in the binary case, if one can concoct a single pair $\bar{u} \in \mathfrak{M}_u$ and $\bar{v} \in \mathfrak{M}_v$ such that $Q(\bar{u})$ is not congruent to $Q(\bar{v})$ modulo $\mathscr{G}^*$, then u can not be integrally equivalent to v. Here again, therefore, the computation is reduced essentially to finding the norm groups for $\langle u\rangle^{\perp}$ and $\langle v\rangle^{\perp}$. The Jordan decompositions involved are rather simple and the associated fundamental invariants can *usually* be read off directly from an arbitrary Jordan splitting. In the cases for $\dim L = 3$, 4, it is expedient to check the classification of the given vectors u and v and then employ the results contained in § 3, [2]. The exceptional cases in these dimensions must be handled via characteristic sets, which for such low dimensions are not computationally unmanageable.

4. Cancellation theorems.

One of the basic results in the study of integral quadratic forms over dyadic local rings is a result of O'Meara's which allows one to (orthogonally) cancel hyperbolic components. Over fields (characteristic not two), the classical Witt's Theorem can be stated in any of the two equivalent forms: the cancellation version and the extension version. The solutions given in this paper and in [2] completes the investigation of one-dimensional integral analogue of Witt Extension Theorem for the case of modular forms over any dyadic local ring. (*N.B.* Over rings, cancellation is not equivalent to extension.) At present, the theory of orthogonally cancelling equivalent forms over rings (even over dyadic local rings) is still practically nonexistent. In this short section, we observe some immediate consequences of our Main Result and others from § 3.

NOTATION 4.1. If T are S and isometric sublattices of a given lattice L and if $t \in T, s \in S$, then we write $t \sim s$ over $[T : S]$ to mean that there is an isometry σ on L such that $\sigma(t) = s$, and $\sigma(T) = S$.

THEOREM 4.2. *Let* $L = K_u \perp J_u = K_v \perp J_v$ *with* $J_u \cong J_v$ *and* $u \in K_u, v \in K_v$. *Moreover,*

$$Q(J_u) \subseteqq Q(\mathfrak{M}_u \textit{ in } K_u)$$
$$Q(J_v) \subseteqq Q(\mathfrak{M}_v \textit{ in } K_v) .$$

Then, $u \sim v$ *over* L *implies* $u \sim v$ *over* $[K_u : K_v]$.

COROLLARY 4.3. *If u is integrally equivalent to v over*

$$K \perp A(0,0) \perp A(0,0)$$

where u and v both lie in K, then $u \sim v$ over $K \perp A(0,0)$. (*N.B.* Thus, if there are two hyperbolic components in the orthogonal complements of the vectors, one can always cancel at least one ef them.)

COROLLARY 4.4. *Let $L = K_u \perp J_u = K_v \perp J_v$ with $J_u \cong J_v$ and $u \in K_u$, $v \in K_v$, $\dim K_u \geqq 6$. Furthermore, $\mathscr{G}J_u \subseteqq \mathscr{G}(\langle u\rangle^{\perp}$ in $K_u)$ and $\mathscr{G}J_v \subseteqq \mathscr{G}(\langle v\rangle^{\perp}$ in $K_v)$. Then, $u \sim v$ over L implies $u \sim v$ over $[K_u : K_v]$.*

5. **Characteristic two case.** Although there is no longer the possibility for L having dimension five, it is not difficult to see that the techniques introduced in the proof of the Main Theorem in §2 carry through here in the characteristic two case—with obvious parallel arguments. Hence, the actual proofs are left as exercises to the readers. (Note, for example, that the case when both u and v are both Type I vectors is once more being taken care of by a result like that of Proposition 3.5, [2]. However, the proof for this proposition must be modified as follows. Pick any $\bar{u}$ from $\mathfrak{M}_u$. Let σ be the isometry sending $\langle u\rangle^{\perp}$ onto $\langle v\rangle^{\perp}$. Put

$$\sigma M(u, \bar{u}) = M(v, \bar{v})$$

for some $\bar{v} \in \mathfrak{M}_v$. If δ is integral, then the hypothesis of u and v being of the same parity implies the equality of norm groups

$$\mathscr{G}L(u, \bar{u}) = \mathscr{G}L(v, \bar{v})\ .$$

Hence, an isometry between the lattices $L(u, \bar{u})$ and $L(v, \bar{v})$ by theorems of Sah and Arf (the characteristic two parallels of O'Meara and Witt). If δ is not integral. Once again, define the space isometry:

$$\Phi\colon FL(u, \bar{u}) \longrightarrow FL(v, \bar{v})$$

sending u onto v and $u + \delta\bar{u}$ onto $\mu v + \delta\bar{v}$, where μ is that number in the ground field which appears from comparing the Arf invariants —instead of the discriminant comparisons as in the characteristic zero situation—of the two lattice. Namely,

$$\delta Q(\bar{u}) = \delta Q(\bar{v}) + \mu^2 + \mu\ .$$

It is readily checked that Φ indeed is an lattice-isometry between $L(u, \bar{u})$ and $L(v, \bar{v})$.

Again, when δ is integral: after seeing $L(u, \bar{u})$ is isometric to

$L(v, \bar{v})$. We apply Theorem 2.1, [2] whose characteristic two analogue was proved in [3].)

Thus, we state the result:

THEOREM. *Let L be any unimodular lattice over the local ring $\mathcal{O} = k[[\pi]]$ of formal power series in one uniformizing variable π and k being a finite field of characteristic two. Two maximal vectors having same quadratic lengths are integrally equivalent if and only if their respective characteristic sets represent the same elements in $\mathcal{O}$.* (*N.B.* The result is of course valid for any $\mathfrak{A}$-modular lattice, $\mathfrak{A}$ a fractional ideal in the quotient field of $\mathcal{O}$.)

Clearly, the discussion about effective computability treads through a parallel course.

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BOUNDARY BEHAVIOR OF RANDOM VALUED HEAT POLYNOMIAL EXPANSIONS

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This paper is concerned with random series of the form $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ where the X_n's are random variables, the a_n's are real numbers, and the v_n's are heat polynomials as introduced by P. C. Rosenbloom and D. V. Widder. The sequences $\{a_n\}$ are assumed to satisfy $\limsup_{n\to\infty} |a_n|^{2/n}(2n/e) = 1$ which implies $\sum_{n=0}^{\infty} a_n v_n(x, t)$ has $|t| < 1$ as its strip of convergence, i.e., it converges to a C^2-solution of the heat equation in $|t| < 1$ and does not converge everywhere in any larger open strip. Associated with each sequence $\{a_n\}$ is its classification number from $[0, 1]$ which indicates how rapidly a_n tends to zero. Assumptions are placed on the random variables which imply that for almost every ω the series $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ has $|t| < 1$ as its strip of convergence.

The main results of the paper are two theorems. The first states that if $\{a_n\}$ has its classification number in $[0, 1/2)$, then for almost every ω the lines $t = 1$ and $t = -1$ form the natural boundary for $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$. The second is concerned with sequences having their classification numbers in $(1/2. 1]$. The conclusion implies that for almost every ω no interval of either of the lines $t = 1$ or $t = -1$ is part of the natural boundary for $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$.

The present work had it original motivation in the study of the boundary behavior of random power series. These are series of the form $\sum_{n=0}^{\infty} a_n(\omega)z^n$ where the a_n's are complex valued random variables and z is a complex number. Reference [1] contains a history of results in this area. One of the early results which helped to motivate the first part of the proof of our Theorem 1 is due to Paley and Zygmund in a 1932 paper [see 6, p. 220]. In this theorem it is assumed that $\sum_{n=0}^{\infty} a_n z^n$ is an ordinary power series with a finite radius of convergence. Letting $\{\phi_n\}$ be the sequence of Rademacher functions, the conclusion is that for almost every ω in $[0, 1]$ the series $\sum_{n=0}^{\infty} \phi_n(\omega)a_n z^n$ has its circle of convergence as its natural boundary.

More recently [see 3] V. L. Shapiro has considered series of the form $\sum_{n=0}^{\infty} X_n(\omega)H_n(x)$ where the X_n's are random variables and

$$\sum_{n=0}^{\infty} H_n(x)$$

is the spherical harmonic representation of a harmonic function in the unit ball. The harmonic continuability across the boundary of the unit ball of the functions $\sum_{n=0}^{\infty} X_n(\omega)H_n(x)$ was investigated. This work further motivated the first part of the proof of our Theorem 1 and influenced our choice of the class of random variables to be considered.

2. Definitions and preliminary comments. For a point (x_0, t_0) in the plane and a number $\rho > 0$ we let

$$S(x_0, t_0; \rho) = \{(x, t) : |x - x_0| < \rho \text{ and } |t - t_0| < \rho\} .$$

If $u(x, t)$ is a C^2-solution to the heat equation in the strip $|t| < \sigma$ we say the line $t = -\sigma$ $(t = \sigma)$ is part of the natural boundary for u in case for every x_0 and every $\rho > 0$ there is no C^2-solution $v(x, t)$ in $S(x_0, -\sigma; \rho)$ $(S(x_0, \sigma; \rho))$ which agrees with $u(x, t)$ where u and v are both defined.

Let E_0 be the set of all sequences $\{a_n\}_{n=0}^{\infty}$ of real numbers. For $r > 0$ let

$$E_r = \{\{a_n\} \in E_0 : |a_n| (2n/e)^{n/2} = O(e^{-n^r}) \text{ as } n \to \infty\} .$$

We call $\sup\{r : \{a_n\} \in E_r\}$ the classification number of $\{a_n\}$.

Explicitly, from [2, p. 222]

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n - 2k)!} \frac{t^k}{k!}, \ n = 0, 1, \cdots . \tag{2.1}$$

In [2, Th. 5.3, p. 231] it was shown that the series $\sum_{n=0}^{\infty} a_n v_n(x, t)$ converges to a C^2-solution of the heat equation in the strip $|t| < \sigma$ where

$$\sigma = (\limsup |a_n|^{2/n}(2n/e))^{-1} \tag{2.2}$$

and that this strip is the largest open strip of convergence of the series. One easily shows that sequences $\{a_n\}$ satisfying

$$\limsup |a_n|^{2/n}(2n/e) = 1$$

have their classification numbers in $[0, 1]$.

We will make repeated use of the following bounds which appear in [4] by S. Täcklind. Assume $u(x, t)$ is continuous on the rectangle $R = \{(x, t) : |x| \leqq \mathscr{L}, 0 \leqq t \leqq T\}$, is a C^2-solution to the heat equation in the interior of R, and μ is an upper bound for $|u(x, t)|$ on R; then $u(x, t)$ is in class C^∞ on the interior of R and for $n = 2, 3, \cdots, |x| < \mathscr{L}$, and $0 < t \leqq T$

$$\begin{aligned} \left|\frac{\partial^n u}{\partial x^n}(x, t)\right| &\leqq \frac{\mu}{2\sqrt{\pi}} \frac{2^{(n+3)/2}}{t^{n/2}} \Gamma((n + 1)/2) \\ &+ \frac{\mu}{\sqrt{\pi}} \left(\frac{\pi}{2}\right)^{5/2} \frac{2^{3n/2}}{(\mathscr{L} - |x|)^n} \Gamma(n + 1) . \end{aligned} \tag{2.3}$$

3. THEOREM 1. *Let* $\{X_n\}_{n=0}^{\infty}$ *be a sequence of symmetric independent random variables defined on the complete probability space* $(\Omega, \mathscr{F}, P)$ *and satisfying*

(i) *there exists a number* M *such that*

$$\int_{\Omega} |X_n(\omega)|^2 dP(\omega) \leqq M \text{ for } n = 0, 1, \cdots, \text{ and}$$

(ii) *there exists* $N > 0$ *such that*

$$N \leqq \int_{\Omega} |X_n(\omega)| \, dP(\omega), \; n = 0, 1, \cdots .$$

Assume $\{a_n\}$ *satisfies* $\limsup |a_n|^{2/n}(2n/e) = 1$ *and has its classification number in* $[0, 1/2)$. *Then for almost every* ω *in* Ω *the lines* $t = 1$ *and* $t = -1$ *form the natural boundary for*

$$u_\omega(x, t) = \sum_{n=0}^{\infty} X_n(\omega) a_n v_n(x, t) .$$

Proof. Letting $\Omega' = \{\omega \varepsilon \Omega : \sum_{n=0}^{\infty} X_n(\omega) a_n v_n(x, t)$ converges in the strip $|t| < 1\}$, we will first show $P(\Omega') = 1$. Clearly

$$[\limsup |X_n|^{2/n} \leqq 1] \supseteq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [|X_n| \leqq n M^{1/2}]$$

and by the Borel-Cantelli Lemma the last set has probability 1 since $P[|X_n| > nM^{1/2}] \leqq 1/n^2$ from (i). Hence

$$P\{\omega : \limsup |X_n(\omega) a_n|^{2/n}(2n/e) \leqq 1\} = 1$$

which by (2.2) shows $P(\Omega') = 1$.

The following fact is essentially a merger of Lemma 1 from [3] and a special case of Lemma 2 from [3]. There exist numbers Λ in $(0, 1)$ and $B > 0$ with the following property: for $E \in \mathscr{F}$ with $P(E) > \Lambda$ there is a positive integer n_0 such that for $n \geqq n_0$, every sequence $\{c_j\}_{j=0}^{\infty}$ of real numbers, and $k \geqq 1$ we have

$$\sum_{j=n}^{n+k} c_j^2 \leqq B \int_E \left\{ \sum_{j=n}^{n+k} c_j X_j(\omega) \right\}^2 dP(\omega) . \tag{3.1}$$

We will show that for almost every ω the line $t = -1$ is part of the natural boundary for u_ω and will use this in the proof for the line $t = 1$.

Assume it is false that for a.e. ω in Ω the line $t = -1$ is part of the natural boundary for u_ω. The first part of the argument we give in order to obtain a contradiction is analogous to parts of the proof of Theorem 1 in [3] by V. L. Shapiro. We will employ (2.3), (3.1), and an asymptotic estimate for heat polynomials from [2] in order to obtain conditions on the sequence $\{a_n\}$ which contradict the fact that the classification number of $\{a_n\}$ is in $[0, 1/2)$.

Let $E = \{\omega \in \Omega' : t = -1$ is not part of the natural boundary for $u_\omega\}$. Then either (i) $E \notin \mathscr{F}$, or (ii) $E \in \mathscr{F}$ and $P(E) > 0$. Using the fact that the real line is separable and the countable additivity of the probability P, it follows that there exist a real number x_0 and $\rho_0 > 0$ such that $E_1 = \{\omega \in E$: there is a C^2-solution to the heat equation in $S(x_0, -1; \rho_0)$ which agrees with u_ω where they are both defined$\}$ satisfies either (i) $E_1 \notin \mathscr{F}$, or (ii) $E_1 \in \mathscr{F}$ and $P(E_1) > 0$. For $i = 1, 2, \cdots$ define

$$E_{2,i} = \left\{\omega \in \Omega' : \left|\frac{\partial^m u_\omega}{\partial x^m}(x, t)\right| \leqq i^m m^m \text{ for } (x, t) \text{ in } S\left(x_0, -1; \frac{\rho_0}{2}\right), \right.$$
$$\left. |t| < 1, \text{ and } m = i, i+1, \cdots\right\}$$

and let $E_2 = \bigcup_{i=1}^\infty E_{2,i}$. E_2 is in the tail σ-field generated by the independent X_n's. From (2.3) it follows that $E_1 \subseteq E_2$. By Kolmolgorov's zero-one law $P(E_2) = 1$. Let A and B be as in (3.1). Take i_0 sufficiently large that $P(E_{2,i_0}) > A$ and let n_0 correspond to E_{2,i_0} as in (3.1). Now let $m \geqq \max\{n_0, i_0\}$ and let (x, t) be in $S(x_0, -1; \rho_0/2)$ with $|t| < 1$. Then by (3.1) for $k = 1, 2, \cdots$

$$\sum_{n=m}^{m+k}\left[\frac{n!}{(n-m)!} a_n v_{n-m}(x, t)\right]^2$$
$$\leqq B\int_{E_{2,i_0}}\left[\sum_{n=m}^{m+k}\frac{n!}{(n-m)!} a_n v_{n-m}(x, t) X_n(\omega)\right]^2 dP(\omega) .$$

Making use of the independence and symmetry of the random variables and of condition (i) we see that the integrand of the last integral is Cauchy in the variable k in $L^1(\Omega)$ and thus in $L^1(E_{2,i_0})$. Hence

$$\sum_{n=m}^{\infty}\left[\frac{n!}{(n-m)!} a_n v_{n-m}(x, t)\right]^2$$
$$\leqq B\int_{E_{2,i_0}}\left[\sum_{n=m}^{\infty}\frac{n!}{(n-m)!} a_n v_{n-m}(x, t) X_n(\omega)\right]^2 dP(\omega)$$
$$= B\int_{E_{2,i_0}}\left|\frac{\partial^m u_\omega}{\partial x^m}(x, t)\right|^2 dP(\omega) \leqq B i_0^{2m} m^{2m}$$

with the last inequality following from the definition of E_{2,i_0}. We conclude that for every $m \geqq \max\{n_0, i_0\}$, every $n \geqq m$, and every (x, t) in $S(x_0, -1; \rho_0/2)$ with $|t| < 1$; we have

$$\frac{n!}{(n-m)!}|a_n|\,|v_{n-m}(x, t)| \leqq B^{1/2} i_0^m m^m . \tag{3.2}$$

It follows from Theorem 3.1 of [2] that there exist numbers A and l_0 such that for $n \geqq l_0$

$$\sup_{|x-x_0|<\rho_0/2} |v_n(x, -1)| \geqq A[2n/e]^{n/2} .$$

Thus from (3.2) we have for $n > m + l_0 > m \geqq \max\{n_0, i_0\}$

$$|a_n| \frac{n!}{(n-m)!} A[2(n-m)/e]^{(n-m)/2} \leqq B^{1/2} i_0^m m^m .$$

Employing Stirling's theorem we see there is a number C such that for $n > m + l_0 > m \geqq \max\{n_0, i_0\}$

$$(3.3) \qquad |a_n|(2n/e)^{n/2} \leqq \left[\frac{Cm}{\sqrt{n-m}}\right]^m \cdot ((n-m)/n)^{(n+1)/2} .$$

Let r be a number which is strictly greater than the classification number of $\{a_n\}$ and strictly less than 1/2. Let m be related to n by $m = [4n^r] + 1$ where the brackets denote the greatest integer function. Then from (3.3), for sufficiently large n,

$$(3.4) \qquad |a_n|(2n/e)^{n/2} \leqq (1 - 4/n^{1-r})^{(n^{1-r}/4)\cdot 2 \cdot n^r} .$$

For large enough n, $(1 - 4/n^{1-r})^{(n^{1-r}/4)\cdot 2} \leqq 1/e$ and thus from (3.4) we have for such n, $|a_n|(2n/e)^{n/2} \leqq 1/e^{n^r}$. Hence $\{a_n\} \in E_r$ which contrandicts the fact that r is strictly greater than the classification number of $\{a_n\}$ and concludes the proof for the line $t = -1$.

For the last part of the proof we find it convenient to introduce the probability space $(R^\omega, \mathscr{A}', \mu')$ which we now describe.

$$R^\omega = \prod_{n=0}^{\infty} R_n$$

where each R_n is the set of real numbers. Let $\mathscr{A}_0$ be the field of all subsets of R^ω of the form $B \times (\prod_{n=n_0+1}^{\infty} R_n)$ where n_0 is a positive integer and B is a Borel set in $\prod_{n=0}^{n_0} R_n$. Let $\mathscr{A}$ be the σ-field generated by $\mathscr{A}_0$. Let μ be the probability on $(R^\omega, \mathscr{A})$ which is induced by the X_n's. Then $(R^\omega, \mathscr{A}', \mu')$ is the completion of $(R^\omega, \mathscr{A}, \mu)$.

Let $\{\eta_i\}_{i=0}^{\infty}$ be a sequence of ± 1's. Define $T: R^\omega \to R^\omega$ by

$$T((\xi_0, \xi_1, \cdots)) = (\eta_0\xi_0, \eta_1\xi_1, \cdots) .$$

Notice that

$$\mu\left(\prod_{n=0}^{n_0} (a_n, b_n] \times \prod_{n=n_0+1}^{\infty} R_n\right) = \prod_{n=0}^{n_0} P[X_n \in (a_n, b_n]]$$
$$= \prod_{n=0}^{n_0} P[X_n \in \eta_n(a_n, b_n]] = \mu\left(T\left(\prod_{n=0}^{n_0} (a_n, b_n] \times \prod_{n=n_0+1}^{\infty} R_n\right)\right)$$

where we have used both the independence and symmetry of the X_n's. From this it follows that for $A \in \mathscr{A}'$, $\mu'(A) = \mu'(T(A))$. We will make use of this fact twice in the remainder of this proof.

To finish the proof it suffices to show that for a.e. $p \in R^\omega$ the line $t = 1$ is part of the natural boundary for

$$u_p(x, t) = \sum_{n=0}^{\infty} \pi_n(p) a_n v_n(x, t)$$

where the π_n's are the projection random variables. Suppose this is false. From the first paragraph of the present proof we know $R^{\omega\prime} = \{p \in R^\omega: \sum_{n=0}^{\infty} \pi_n(p) a_n v_n(x, t)$ converges in $|t| < 1\}$ has μ'-measure 1. Now let $F = \{p \in R^{\omega\prime}: t = 1$ is not part of the natural boundary for $u_p\}$. Then either (i) $F \notin \mathscr{A}'$, or (ii) $F \in \mathscr{A}'$ and $\mu'(F) > 0$. It follows that there exist numbers a, b, ρ with $a < b$ and $\rho > 0$ such that $F_1 = \{p \in R^{\omega\prime}$: there is a function $v_p(x, t)$ which is continuous on $a \leqq x \leqq b$, $0 \leqq t \leqq 1 + \rho$; is a C^2-solution to the heat equation for $a < x < b$, $0 < t < 1 + \rho$; and agrees with $u_p(x, t)$ in $a \leqq x \leqq b, 0 \leqq t < 1\}$ satisfies either (i) $F_1 \notin \mathscr{A}'$, or (ii) $F_1 \in \mathscr{A}'$ and $\mu'(F_1) > 0$. But $F_1 = \{p \in R^{\omega\prime}$: $\lim_{t \uparrow 1} u_p(a, t)$ and $\lim_{t \uparrow 1} u_p(b, t)$ both exist$\}$. F_1 is in the tail σ-field generated by the independent π_n's. From the zero-one law, $\mu'(F_1) = 1$.

Either $a \neq 0$ or $b \neq 0$ and for definiteness we assume $a \neq 0$. Then $F_2 = \{p \in R^{\omega\prime}: \lim_{t \uparrow 1} u_p(a, t)$ exists$\}$ has $\mu'(F_2) = 1$. Let $T: R^\omega \to R^\omega$ be defined by $T((\xi_0, \xi_1, \cdots)) = (\xi_0, -\xi_1, \xi_2, -\xi_3, \cdots)$. By our earlier comments concerning such mappings we have $\mu'(F_2 \cap T(F_2)) = 1$. For $p \in R^{\omega\prime}$ and $|t| < 1$ one checks that $u_{T(p)}(-a, t) = u_p(a, t)$. Hence for $p \in F_2 \cap T(F_2)$, $\lim_{t \uparrow 1} u_p(-a, t)$ and $\lim_{t \uparrow 1} u_p(a, t)$ both exist. Thus for $p \in F_2 \cap T(F_2)$ there is a function $w_p(x, t)$ which is continuous in $|x| \leqq a$, $0 \leqq t \leqq 2$; is a C^2-solution to the heat equation in $|x| < a$, $0 < t < 2$; and agrees with u_p in $|x| \leqq a, 0 \leqq t < 1$. For $p \in F_2 \cap T(F_2)$ and $0 \leqq t \leqq 2$ let $\phi_p(t) = w_p(0, t)$ and $\psi_p(t) = (\partial w_p/\partial x)(0, t)$. Then, employing (2.3), we see that ϕ_p and ψ_p are in class $C\{(2n)!\}$ on $[0, 2]$ (a function f is in class $C\{(2n)!\}$ on an interval I if f is in class C^∞ on I and there exist constants β and B such that for every t in I, $|f^{(n)}(t)| \leqq \beta B^n (2n)!$, $n = 0, 1, \cdots$).

Now let $T': R^\omega \to R^\omega$ be defined by

$$T'((\xi_0, \xi_1, \cdots)) = (\xi_0, \xi_1, -\xi_2, -\xi_3, \xi_4, \xi_5, -\xi_6, -\xi_7, \cdots) .$$

Then for $p \in R^{\omega\prime}$ and $|t| < 1$, $u_p(0, t) = u_{T'(p)}(0, -t)$ and

$$\frac{\partial u_p}{\partial x}(0, t) = \frac{\partial u_{T'(p)}}{\partial x}(0, -t) .$$

For p in the almost sure set $T'(F_2 \cap T(F_2))$ we have $T'(p) \in F_2 \cap T(F_2)$ and we define ϕ'_p and ψ'_p on $[-2, 0]$ by $\phi'_p(t) = \phi_{T'(t)}(-t)$ and

$$\psi'_p(t) = \psi_{T'(p)}(-t)$$

thereby obtaining class $C\{(2n)!\}$ extensions of $u_p(0, t)$ and $(\partial u_p/\partial x)(0, t)$ on $[-1, 0]$. Thus for $p \in T'(F_2 \cap T(F_2))$

$$u'_p(x, t) = \sum_{n=o}^{\infty} \frac{\phi_p'^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\psi_p'^{(n)}(t)x^{2n+1}}{(2n+1)!}$$

provides a solution to the heat equation which is a C^2-extension of u_p into some rectangle $|x| < r$, $-2 < t < 0$ which contradicts the first part of the proof.

4. THEOREM 2. *Let* $\{X_n\}$ *be a sequence of independent random variables over a probability space* $(\Omega, \mathscr{F}, P)$ *which satisfies* (i) *and* (ii) *of Theorem* 1. *Assume* $\{a_n\}$ *satisfies* $\limsup |a_n|^{2/n}(2n/e) = 1$ *and has its classification number in* $(1/2, 1]$. *Then for almost every* ω *in* Ω *the following holds:* $|t| < 1$ *is the strip of convergence of* $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ *which for every* $\mathscr{L} > 0$ *can be extended as a* C^2*-solution of the heat equation into* $\{|t| < 1\} \cup \{|x| < \mathscr{L}\}$.

Proof. We will first show for almost every ω in Ω that $|t| < 1$ is the strip of convergence of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$. By (2.2) we must show that almost surely $\limsup |X_n(\omega)a_n|^{2/n}(2n/e) = 1$. The argument given in the first part of the proof of Theorem 1 shows that almost surely the last limit superior does not exceed 1. Let $\{n_j\}$ be a strictly increasing sequence of positive integers such that

$$\lim_{j\to\infty} |a_{n_j}|^{2/n_j}(2n_j/e) = 1 .$$

Then $\limsup |X_n(\omega)a_n|^{2/n}(2n/e) \geqq \limsup_{j\to\infty} |X_{n_j}(\omega)a_{n_j}|^{2/n_j}(2n_j/e) \geqq \lim \sup_{j\to\infty} |X_{n_j}(\omega)|^{2/n_j}$ which by the zero-one law is equal to some number c almost surely. Suppose $c < 1$. Then $X_{n_j} \to 0$ almost surely. By (ii) for $A > 0$ and $j = 0, 1, \cdots$

$$N \leqq \int_{[|X_{n_j}|\leqq A]} |X_{n_j}(\omega)| \, dP(\omega) + A^{-1} \int_{[|X_{n_j}|>A]} |X_{n_j}(\omega)|^2 dP(\omega) .$$

By the Lebesgue dominated convergence theorem the next to the last integral tends to 0 as j tends to ∞. From (i) the last term is uniformly bounded by $A^{-1}M$. Thus for every $A > 0, N \leqq A^{-1}M$ which is a contradiction. We conclude that $c \geqq 1$. Thus almost surely

$$\limsup |X_n(\omega)a_n|^{2/n}(2n/e) \geqq 1$$

which concludes the proof that almost surely this limit superior is 1.

In order to establish Theorem 2 for the line $t = 1$ we first construct a function which is C^∞ on the closed strip $|t| \leqq 1$ and has a heat polynomial expansion in $|t| < 1$. Let r be a number which is strictly greater than 1/2 and strictly less than the classification number of $\{a_n\}$. For $n = 0, 1, \cdots$ define $\alpha_n = (2n)e^{-n^r}$. Define f on $[-1, 1]$ by $f(t) = \sum_{k=0}^{\infty} \alpha_k t^k$. We will show this definition makes sense and obtain some bounds on the derivatives of f.

Let n be a nonnegative integer. Differentiating $\sum_{k=0}^{\infty} \alpha_k t^k$ term by term n times yields $\sum_{k=n}^{\infty} k!/(k-n)!\, \alpha_k t^{k-n}$. For $|t| \leqq 1$ the k^{th} term of this series is dominated by $2\ k^{n+1}e^{-k^r}$. One checks that

$$g_n(x) = x^{n+1}e^{-x^r}$$

is increasing on $(0, (n + 1/r)^{1/r})$ and decreasing on $((n + 1/r)^{1/r}, \infty)$. Hence

$$\sum_{k=n}^{\infty} k^{n+1}e^{-k^r} \leqq \int_n^{\infty} g_n(x)dx + g_n\left(\left(\frac{n+1}{r}\right)^{1/r}\right) \leqq 3\Gamma((n+2)/r)/r \,.$$

We conclude that f is a C^∞-function with $|f^{(n)}(t)| \leqq 6\Gamma((n + 2/r)/r$ for $n = 0, 1, \cdots$ and $|t| \leqq 1$.

Now define

$$u(x, t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{f^{(n+1)}(t)x^{2n+1}}{(2n+1)!} \,. \tag{4.1}$$

Because of the bounds obtained in the preceding paragraph it can be shown that the series of (4.1) can be differentiated term by term and that $u(x, t)$ is a C^∞-solution to the heat equation in the closed strip $|t| \leqq 1$. Since both $u(0, t)$ and $\partial u/\partial x(0, t)$, as functions of t on $(-1, 1)$, are given by their Maclaurin expansions, u has a heat polynomial expansion in $|t| < 1$ (see [5]). Thus

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} b_n v_n(x, t) \,, \\ b_{2n} &= f^{(n)}(0)/(2n)! \,, \\ b_{2n+1} &= f^{(n+1)}(0)/(2n+1)! \,. \end{aligned} \tag{4.2}$$

According to the first paragraph of the proof of Theorem 1, $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [|X_n| \leqq nM^{1/2}]$ has probability 1. Let ω be in this almost sure set. Let k_0 be a positive integer such that for $n \geqq k_0$, $|X_n(\omega)| \leqq nM^{1/2}$. Since r is less than the classification number of $\{a_n\}$, there is a number K such that $|a_n|(2n/e)^{n/2} \leqq Ke^{-n^r}$, $n = 1, 2, \cdots$. Using Stirling's theorem we have for $2n \geqq k_0$

$$b_{2n}(4n/e)^n \geqq |X_{2n}(\omega)a_{2n}|\,(4n/e)^n(1/2)^{3/2}/KM^{1/2} \,.$$

Similarly for $2n + 1 \geqq k_0$

$$b_{2n+1}(2(2n+1)/e)^{(2n+1)/2} \geqq |X_{2n+1}(\omega)a_{2n+1}|\,(2(n+1)/e)^{(2n+1)/2}e^{-1/2}/KM^{1/2} \,.$$

Letting $K' = K(Me)^{1/2}$ we have

$$|X_n(\omega)a_n| \leqq K'b_n \text{ for } n \geqq k_0 .$$

Let $\mathscr{L} > 0$. Then for $0 < t < 1$ we have

$$\left|\frac{\partial}{\partial t} \sum_{n=k_0}^{\infty} X_n(\omega)a_n v_n(\pm\mathscr{L}, t)\right| = K' \sum_{n=k_0}^{\infty} b_n n(n-1)\,|v_{n-2}(\pm\mathscr{L}, t)|$$

$$\leqq K' \sum_{n=k_0}^{\infty} b_n n(n-1) v_{n-2}(\mathscr{L}, t) \leqq K' \frac{\partial u}{\partial t}(\mathscr{L}, 1) < \infty .$$

Thus $\lim_{t\uparrow 1} \sum_{n=k_0}^{\infty} X_n(\omega)a_n v_n(\pm\mathscr{L}, t)$ both exist as is easily seen from the mean value theorem and the Cauchy criterion. Hence we can obtain an extension of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ into

$$\{(x, t): |t| < 1\} \cup \{(x, t): |x| < \mathscr{L}, 0 < t\}$$

which is a C^2-solution of the heat equation. (Notice at this point that we can also obtain an extension which is a bounded C^2-solution in $\{(x, t): |x| < \mathscr{L}, 0 \leqq t\}$.) Since ω was from the almost sure set

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [|X_n| \leqq nM^{1/2}] ,$$

this establishes the result for the line $t = 1$.

We now turn to the line $t = -1$. Define $\{Y_n\}_{n=0}^{\infty}$ on Ω by $Y_{2n} = (-1)^n X_{2n}$ and $Y_{2n+1} = (-1)^n X_{2n+1}$. Then, applying the first part of the proof, there is a set F in $\mathscr{F}$ with $P(F) = 1$ such that for ω in F and $\mathscr{L} > 0$ the solution $v_\omega(x, t) = \sum_{n=0}^{\infty} Y_n(\omega)a_n v_n(x, t)$ can be extended into $\{|t| < 1\} \cup \{|x| < \mathscr{L}$ and $0 < t\}$ so as to be a bounded C^2-solution of the heat equation in $\{(x, t): |x| < \mathscr{L}$ and $0 < t\}$. One easily checks that for ω in F,

$$\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(0, t) = \sum_{n=0}^{\infty} Y_n(\omega)a_n v_n(0, -t)$$

and $\sum_{n=1}^{\infty} X_n(\omega)a_n n v_{n-1}(0, t) = \sum_{n=1}^{\infty} Y_n(\omega)a_n n v_{n-1}(0, -t)$. Using these facts and (2.3) we see that for ω in F and $\mathscr{L} > 0$ the functions $\phi(t) = \sum_{n=0}^{\infty} X_n(\omega)a_n v_n(0, t)$ and $\psi(t) = \sum_{n=1}^{\infty} X_n(\omega)a_n n v_{n-1}(0, t)$ on $(-1, 1)$ possess sufficiently well behaved extensions ϕ' and ψ' to $(-\infty, 1)$ that

$$\sum_{n=0}^{\infty} \frac{\phi'^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\psi'^{(n)}(t)x^{2n+1}}{(2n+1)!}$$

is an extension of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ in $|t| < 1$ to

$$\{(x, t): |t| < 1\} \cup \{(x, t): |x| < \mathscr{L} \text{ and } -\infty < t < 1\} .$$

5. **Examples.** The first example will show that our two theorems are best possible with respect to the allowable values of the classification number.

EXAMPLE 1. We will take [0, 1] with Lebesgue measure as the probability space and the sequence of Rademacher functions, $\{\phi_n\}_{n=0}^{\infty}$, for the random variables.

For $k = 0, 1, \cdots$ define $\alpha_k = e^{-\sqrt{k}}$. Then, as in the proof of Theorem 2, defining f on $[-1, 1]$ by $f(t) = \sum_{k=0}^{\infty} \alpha_k t^k$ yields a C^∞-function whose n^{th} derivative on $[-1, 1]$ is bounded in absolute value by $6\Gamma(2(2n+1))$. In the strip $|t| < 1$ define $u(x, t) = \sum_{n=0}^{\infty} (f^{(n)}(t)x^{2n})/(2n)!$. To see that this definition makes sense and that term by term partial differentiation is permitted, we note that for every closed interval $I \subsetneqq (-1, 1)$, f is in class $C\{n!\}$ on I. Because of the bounds on the derivatives of f we see from the defining series for u that u may be extended as a C^∞-solution of the heat equation to

$$\{|t| < 1\} \cup \{(x, 1): |x| < 1\} .$$

Since $u(0, t)$ and $\partial u/\partial x(0, t)$ are both given by their Maclaurin expansions in $|t| < 1$, u possesses a heat polynomial expansion in the strip $|t| < 1$ (see [5]). Thus for $|t| < 1$, $u(x, t) = \sum_{n=0}^{\infty} a_n v_n(x, t)$; $a_{2n} = (e^{-\sqrt{n}} n!)/(2n)!$, $a_{2n+1} = 0$. One checks that $\limsup |a_n|^{2/n}(2n/e) = 1$. Also it is easily seen that limit $|a_{2n}|(4n/e)^n e^{\sqrt{2n}} = \infty$ which implies $\{a_n\} \notin E_{1/2}$ and thus the classification number of $\{a_n\}$ is in $[0, 1/2]$. As in the proof of Theorem 2, $\lim_{t \uparrow 1} u_\omega(\pm 1/2, t)$ both exist for every ω in [0, 1]. Thus for every $\omega \in [0, 1]$ the line $t = 1$ is not part of the natural boundary for $u_\omega(x, t)$. Using Theorem 1, we conclude that the classification number of $\{a_n\}$ is 1/2 and that in Theorem 1 we cannot replace [0, 1/2) by [0, 1/2] as the allowable range for the classification number.

We will next show that the conclusion of Theorem 2 does not hold for $\sum_{n=0}^{\infty} \phi_n(\omega) a_n v_n(x, t)$. Assume there is a set A in [0, 1] with $m(A) = 1$ such that for each ω in A no interval of the line $t = 1$ is part of the natural boundary for $u_\omega(x, t)$. Thus for ω in A, $g_\omega(x) = \lim_{t \uparrow 1} u_\omega(x, t)$ is well defined and is the restriction of an entire function to the real axis (this last assertion can be seen by employing (2.3)). Thus for ω in A, $\limsup (|g_\omega^{(n)}(0)|/n!)^{1/n} = 0$. For ω in A, $|g_\omega^{(2n+1)}(0)| = 0$ and $|g_\omega^{(2n)}(0)| = |\sum_{k=2n}^{\infty} \phi_k(\omega) a_k (k!/(k-2n)!) v_{k-2n}(0, 1)| = |\sum_{k=n}^{\infty} \phi_{2k}(\omega)(k!/(k-n)!) e^{-\sqrt{k}}|$. Thus for ω in A,

$$\limsup \left[\frac{\left| \sum_{k=n}^{\infty} \phi_{2k}(\omega) \dfrac{k!}{(k-n)!} e^{-\sqrt{k}} \right|}{(2n)!} \right]^{1/n} = 0 .$$

Let $\delta > 0$. For $m = 0, 1, \cdots$ let

$$F_m = \left\{ \omega \in A: \left(\left| \sum_{k=n}^{\infty} \phi_{2k}(\omega) \frac{k!}{(k-n)!} e^{-\sqrt{k}} \right| \Big/ (2n)! \right)^{1/n} \right.$$
$$\left. \leqq \delta \text{ for } n = m, m+1, \cdots \right\}$$

and note $F_m \uparrow A$. Let A and B be two numbers associated with the sequence $\{\phi_{2n}\}_{n=0}^{\infty}$ as in (3.1). Let m_0 be sufficiently large that $m(F_{m_0}) > A$. Let n_0 be an integer larger than m_0 with n_0 corresponding to F_{m_0} as in (3.1). Thus for $n \geqq n_0$ and $k \geqq 1$

$$(5.1)\quad \sum_{j=n}^{n+k}\left[\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right]^2 \leqq B\int_{F_{m_0}}\left(\sum_{j=n}^{n+k}\phi_{2j}(\omega)\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right)^2 dm(\omega)\,.$$

As in the proof of Theorem 1, letting k tend to ∞ yields (5.1) with $n+k$ replaced by ∞. Using the definition of F_{m_0}, we have

$$\sum_{j=n}^{\infty}\left[\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right]^2 \leqq B((2n)!\,\delta^n)^2\,,$$

for $n \geqq n_0$. From this we conclude that

$$\limsup\left[\frac{\left[\sum_{k=n}^{\infty}\left(\frac{k!}{(k-n)!}e^{-\sqrt{k}}\right)^2\right]^{1/2}}{(2n)!}\right]^{1/n} = 0\,.$$

On the other hand, letting L denote this last limit superior, we have

$$L \geqq \limsup \left[\frac{\left[\sum_{k=n}^{\infty}(k-n)^{2n}\exp(-2\sqrt{k-n})\exp(-(2\sqrt{k}-2\sqrt{k-n}))\right]^{1/2}}{(2n)!}\right]^{1/n}.$$

But $\exp(-(2\sqrt{k}-2\sqrt{k-n})) \geqq e^{-2\sqrt{n}}$ for $k \geqq n$ and $\lim(e^{-\sqrt{n}})^{1/n} = 1$. Hence $L \geqq \limsup((\sum_{k=0}^{\infty}k^{2n}e^{-2\sqrt{k}})^{1/2}/(2n)!)^{1/n}$. Define h_n on $(0,\infty)$ by $h_n(x) = x^{2n}e^{-2\sqrt{x}}$. One checks that h_n is increasing on $(0,(2n)^2)$ and decreasing on $((2n)^2,\infty)$. Thus $\sum_{k=0}^{\infty}k^{2n}e^{-2\sqrt{k}} \geqq \int_0^{\infty}h_n(x)dx - h_n((2n)^2) = (\Gamma(4n+2) - 2(4n)^{4n}e^{-4n})/(2\cdot 4^{2n})$. Thus

$$L \geqq \frac{1}{4}\limsup\left[\left(\frac{\Gamma(4n+2)}{(4n)!} - \frac{2(4n)^{4n}e^{-4n}}{(4n)!}\right)((4n)!/((2n)!)^2)\right]^{1/2n} > 0\,.$$

This is a contradiction. Hence in Theorem 2 we cannot replace (1/2, 1] by [1/2, 1] as the allowable range for the classification number.

The next example shows that in Theorem 1 we cannot omit the symmetry of the random variables.

EXAMPLE 2. Let $k(x,t) = e^{-x^2/4t}/(4\pi t)^{1/2}$ for $t > 0$ and define

$$u(x,t) = k(x,t+1)$$

in the strip $|t| < 1$. Then [2, Th. 4.2, p. 227]

$$u(x, t) = (4\pi)^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 4^n} v_{2n}(x, t) .$$

Let $\{a_n\}_{n=0}^{\infty}$ be defined by $a_{2n} = (-1)^n/n!\, 4^n$ and $a_{2n+1} = 0$. One easily checks that $\limsup |a_n|^{2/n}(2n/e) = 1$ and that the classification number of $\{a_n\} = 0$. Let $X_n = 1, n = 0, 1, \cdots$ on some complete probability space. Then for every ω, u_ω can be continued above the line $t = 1$.

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RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE

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It is well known that if every right ideal of a ring R is injective, then R is semi simple Artinian. The object of this paper is to initiate the study of a class of rings in which each right ideal is quasi-injective. Such rings will be called q-rings. It is shown by an example that a q-ring need not be even semi prime. A number of important properties of q-rings are obtained.

Throughout this paper, unless otherwise stated, we assume that every ring has unity $1 \neq 0$. If M is a right R-module, then $\hat{M}$ will denote the injective hull of M. For any positive integer n, R_n will denote the ring of all $n \times n$ matrices over the ring R. R^{Δ}, $J(R)$ and $B(R)$ will denote the right singular ideal, the Jacobson radical and the prime radical respectively. A ring R is said to be a right duo ring if every right ideal of R is two-sided. Left duo rings are defined symmetrically. By a duo ring we mean a ring which is both right and left duo ring.

It is shown that $R_n (n > 1)$ is a q-ring if and only if R is semi-simple Artinian. Some of the main results are: (i) a prime q-ring is simple Artinian, (ii) a semi-prime q-ring is a direct sum of two rings S and T, where S is a complete direct sum of simple Artinian rings, and T is a semi-prime q-ring with zero socle, and (iii) a semi-prime q-ring is a direct sum of two rings A and B, where A is a right self injective duo ring, and B is semi-simple artinian.

2. Let R be a right self injective ring. If B is any right ideal of R, then $\hat{B} = eR$ for some idempotent e of R. Let $K = \mathrm{Hom}_R(\hat{B}, \hat{B})$. Then $K \cong eRe$. In fact every element in K can be realized by the left multiplication of some element of eRe. By Johnson and Wong ([3], Theorem 1.1) B is a quasi injective as a right R-module if and only if $KB = B$. Hence B is quasi injective if and only if $B = KB = (eRe)B = (eR)(eB) = \hat{B}B$. Hence every two-sided ideal in a right self injective ring is quasi-injective. So, the following is immediate.

2.1. Every commutative self injective ring is a q-ring.

Now, we give an example of a q-ring which is not semi-prime.

EXAMPLE 2.2. Let Z be the ring of integers. Set $R = Z/(4)$. It is trivial that R is a q-ring. But R is not semi-prime, since its only proper ideal is nilpotent.

In fact, $Z/(n)$ is a q-ring for every integer $n > 1$, since it is self injective (cf. Levy [5]). Also we remark that $Z/(n)$ has nonzero nilpotent ideals if n is not square free.

Next we prove

THEOREM 2.3. *The following are equivalent*

(1) *R is a q-ring*

(2) *R is right self injective, and every right ideal of R is of the form eI, e is an idempotent in R, I is a two sided ideal in R.*

(3) *R is right self-injective, and every large right ideal of R is two sided.*

Proof. Assume (1). Therefore R is right self injective. Let B be any right ideal of R. Then $\hat{B} = eR$ for some idempotent e. Since B is quasi injective $B = \hat{B}B = eRB = eI$, where $I = RB$, the smallest two-sided ideal of R containing B. Hence (1) implies (2).

Assume (2). Let A be a large right ideal of R. Then $A = eI$, $e^2 = e$, I is a two sided ideal. Since $A \cap (1-e)R = 0$, $(1-e)R = 0$. This implies that $e = 1$. Hence $A = I$, proving (3).

Now assume (3). Let B be a right ideal of R. If K is a complement of B, then $B \oplus K$ is large in R. By assumption $B \oplus K$ is a two-sided ideal in R, hence quasi-injective. This implies B is a quasi-injective, completing the proof.

THEOREM 2.4. *Let $n > 1$ be an integer. Then R_n is a q-ring if and only if R is semi-simple Artinian.*

Proof. Suppose that R is not semi-simple Artinian. By Lambek ([4], Proposition 2, p. 61), there exists a large right ideal B of R such that $B \neq R$. Let e_{ij}, $1 \leqq i, j \leqq n$ be the matrix units of R_n and let $E = \{\sum a_{ij}e_{ij}: a_{1j} \in B, 1 \leqq j \leqq n \text{ and } a_{ij} \in R, 1 \leqq i, j \leqq n\}$. It is clear that E is a right ideal in R_n. But E is not two-sided, for $e_{nn} \in E$ and $e_{1n}e_{nn} = e_{1n} \notin E$. Now, we prove that E is a large right ideal in R_n. Let $0 \neq x = \sum_{i,j=1}^{n} b_{ij}e_{ij}$. If $b_{1j} = 0, 1 \leqq j \leqq n$, then $x \in E$. So, let $b_{1k} \neq 0$ for some k. Since B is large in R, there exists $a \in R$ such that $0 \neq b_{1k}a \in B$. Then,

$$x(ae_{kk}) = \left(\sum_{i,j=1}^{n} b_{ij}e_{ij}\right)(ae_{kk}) = \sum_{i=1}^{n} b_{ik}ae_{ik} \in E.$$

Hence, $0 \neq x(ae_{kk}) \in E$. Therefore E is a large right ideal in R_n which is not two-sided, and by Theorem 2.3, R_n is not a q-ring. This proves "only if" part. Other part is obvious.

We are now ready to show the existence of right self injective rings which are not q-rings.

EXAMPLE 2.5. Let R be a right self injective ring which is not semi-simple (we can take $R = Z/(4)$). Let $n > 1$ be an integer. By Utumi ([6], Th. 8.3) R_n is right self injective. But R_n is not a q-ring, by the above theorem.

Next we prove

THEOREM 2.6. *A simple ring is a q-ring if and only if it is Artinian.*

Proof. Let R be a simple q-ring. Let B be a large right ideal in R. Then B is two-sided, and hence $B = R$. This proves that R does not contain any proper large right ideal. Hence R is Artinian. The converse is trivial.

Now, we give an example of a right self injective simple ring which is not a q-ring.

EXAMPLE 2.7. Let S be a noncommutative integral domain which is not a right Öre domain (cf. Goldie [1]). Let $R = \hat{S}$. Then R is a right self injective simple regular ring which is not Artinian. By the above theorem R is not a q-ring.

LEMMA 2.8. *Let R be a q-ring. Then $B(R)$ is essential in $J(R)$ as a right R-module.*

Proof. Since R is self injective, $J(R) = R^{\Delta}$, by Utumi ([6], Lemma 4.1). Let $0 \neq x \in J(R)$. There exist a large right ideal E of R such that $xE = 0$. Then $xE \subset P$ for every prime ideal P of R. Since R is a q-ring, E is two-sided. This implies that either $x \in P$ or $E \subset P$.

Let $\{P_i\}_{i \in I}$ be the set of all prime ideals of R such that $x \in P_i$ for every $i \in I$, and $\{P_j\}_{j \in J}$ be the set of all prime ideals of R such that $x \notin P_j$ for every $j \in J$. Let $X = \bigcap_{i \in I} P_i$, and $Y = \bigcap_{j \in J} P_j$. $X \neq 0$, since $0 \neq x \in X$. On the other hand, $E \subset P_j$ for every $j \in J$. Thus $E \subset Y$, which implies that Y is large in R. Therefore $B(R) = X \cap Y \neq (0)$. Moreover, there exists $a \in R$ such that $0 \neq xa \in Y$. This implies that $0 \neq xa \in X \cap Y = B(R)$, completing the proof.

Hence, we have the following

THEOREM 2.9. *A q-ring is regular if and only if it is semi-prime.*

Proof. The result follows by the above lemma, and Utumi ([6], Corollary 4.2).

THEOREM 2.10. *Let V be a vector space over a division ring D, and let $R = \mathrm{Hom}_D(V, V)$. Then R is a q-ring if and only if V is of finite dimension over D.*

Proof. The "if" part is obvious. Conversely, suppose that V is of infinite dimension over D. Let $X = \{x_1, x_2, \cdots\}$ be a denumerable set of linearly independent elements of V. X can be extended to a basis $X \cup Y$ of V. Let F be the ideal in R consisting of all elements of finite rank. Let $\sigma \in R$ be defined by $\sigma(x_{2i}) = x_{2i}$, $\sigma(x_{2i-1}) = 0$ for every i, and $\sigma(y) = 0$ for every $y \in Y$. Let $E = \sigma R + F$. Then $F \subset E$. Since F is a two-sided ideal in R, F is large. Therefore E is a large right ideal in R. We proceed to prove that E is not two-sided. Let $\lambda_1, \lambda_2 \in R$ be defined by: $\lambda_1(x_i) = x_{2i}$ for every i, and $\lambda_1(y) = 0$ for every $y \in Y$, $\lambda_2(x_{2i}) = x_i$, $\lambda_2(x_{2i-1}) = 0$, for every i, and $\lambda_2(y) = 0$ for every $y \in Y$. Let $\lambda = \lambda_2\sigma\lambda_1$. Then $\lambda(x_i) = x_i$ for every i. Hence $X \subset \lambda(V)$. We assert that $\lambda \notin E$; for otherwise, let $\lambda = \sigma r + f$, $r \in R$, $f \in F$. Then $X \subset \lambda(V) = (\sigma r + f)(V) \subset \sigma(V) + f(V)$. But since f is of finite rank, there exists an integer n such that $x_{2n-1} \notin f(V)$. Also, by definition of σ, $x_{2n-1} \notin \sigma(V)$. Hence $x_{2n-1} \notin \sigma(V) + f(V)$, which is a contradiction. Thus $\lambda \notin E$, as desired. However $\lambda \in R\sigma R \subset RE$. Hence E is not a two-sided ideal. Therefore, by Theorem 2.3, R is not a q-ring. This completes the proof.

We remark that the above theorem is also a consequence of Theorem 2.4.

The right (left) socle of a ring R is defined to be the sum of all minimal right (left) ideals of R. It is well known that in a semiprime ring R, the right and left socles of R coincide, and we denote any of them by $\mathrm{soc}\, R$.

LEMMA 2.11. *A semi-prime q-ring R with zero socle is strongly regular.*

Proof. Let M be a maximal right ideal in R. Either M is a direct summand of R or M is large in R. If M is a direct summand of R, then its complement is a minimal right ideal. This implies that $\mathrm{soc}\, R \neq 0$, a contradiction. Therefore, every maximal right ideal is large, hence two-sided. By Lemma 2.8, $J(R) = 0$. Thus R is isomorphic to a subdirect sum of division rings, which implies that R has no nonzero nilpotent elements. Since R is regular, by Theorem 2.9, R is strongly regular.

LEMMA 2.12. *A prime q-ring has nonzero socle.*

Proof. Let R be a prime q-ring. If possible, let $\operatorname{soc} R = 0$. By the above lemma, R is strongly regular. Hence R is a division ring, and $\operatorname{soc} R = R$ contradicting our assumption. Therefore $\operatorname{soc} R \neq (0)$.

THEOREM 2.13. *A prime ring R is a q-ring if and only if R is simple Artinian.*

Proof. By Theorem 2.9, and the above lemma, R is a prime regular ring with nonzero socle. Hence, by Johnson ([2], Th. 3.1), $\hat{R} = \operatorname{Hom}_D(V, V)$, where V is some vector space over a division ring D. But then $R = \operatorname{Hom}_D(V, V)$, since R is right self injective. By Theorem 2.10, V has finite dimension over D. Let $(V: D) = n$. Then $R \cong D_n$, completing the proof.

LEMMA 2.14. *Let $\{R_\alpha\}_{\alpha \in I}$ be a finite set of rings. Then the direct sum $\sum_{\alpha \in I} \oplus R_\alpha$ is a q-ring if and only if each R_α is a q-ring.*

The proof is obvious.

That Lemma 2.14 is not true for an infinite number of rings is shown by the following example which is due to Storrer.

EXAMPLE 2.15. Let R be a 2×2-matrix ring over a field F. Let $\{R_\alpha\}_{\alpha \in I}$ be an infinite family of copies of R and let $S = \pi R_\alpha$, $\alpha \in I$. Let E be the right ideal of S consisting of those elements $[x_\alpha]$ of S such that all but finite x_α's are matrices with first row zero. Since $R_\alpha \subset E$ for all $\alpha \in I$, E is a large right ideal of S. To show that E is not two-sided, consider $[x_\alpha] \in E$ where $x_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for all $\alpha \in I$. Let $[y_\alpha] \in S$ be such that $y_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all $\alpha \in I$. Then $[y_\alpha][x_\alpha] = [z_\alpha]$, where $z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But then $[z_\alpha] \notin E$, and E is not two-sided. Hence, by Theorem 2.3, S is not a q-ring.

Example 2.15 also suggests the following.

THEOREM 2.16. *Let $\{R_\alpha\}_{\alpha \in I}$ be a family of simple Artinian rings and let R be their complete direct sum. Then R is a q-ring if and only if all R_α's excepting a finite number of them are division rings.*

The above theorem shows, in particular, that a regular q-ring may not be Artinian.

LEMMA 2.17. *Let R be a semi-prime q-ring such that* $\operatorname{soc} R$ *is* *large in R. Then R is a complete direct sum of simple Artinian rings.*

Proof. Since $\operatorname{soc} R$ is large, every nonzero right ideal of R contains a minimal right ideal. Also R is regular, by Theorem 2.9. Hence by Johnson ([2], Th. 3.1), R is a complete direct sum of rings R_i, where each R_i is the ring of all linear transformations of some vector space V_i over a division ring D_i. But then by Lemma 2.14 and Theorem 2.10, each R_i is a simple Artinian ring. This completes the proof.

In the following two theorems we assume that every ring has a unity element which may be equal to zero.

THEOREM 2.18. *Let R be a semi-prime q-ring. Then $R = S \oplus T$, where S is a complete direct sum of simple Artinian rings and T is a semi-prime q-ring with zero socle.*

Proof. Let $F = \operatorname{soc} R$. Since $R^{\Delta} = 0$, $\hat{F} = \{x \in R : xE \subset F$ for some large right ideal E of $R\}$. Then it is immediate that $\hat{F}$ is a two-sided ideal in R. Since R is self injective, $\hat{F} = eR$ for some idempotent e. Then e is central, since R is regular. Let $S = eR$ and $T = (1 - e)R$. Hence $R = S \oplus T$. By Lemma 2.14, both S and T are q-rings. Further, it can be easily verified that (i) S is a semi-prime ring, $\operatorname{soc} S = F$, and F is large in S, and (ii) T is a semi-prime ring with zero socle. By the above lemma S is a complete direct sum of simple Artinian rings, completing the proof.

As a consequence of Lemma 2.11, Theorem 2.16 and Theorem 2.18 we have the following.

THEOREM 2.19. *A semi-prime ring R is a q-ring if and only if $R = A \oplus B$, where A is a right self injective duo ring and B is semi-simple Artinian.*

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ON THE INVERSION FORMULA FOR THE CHARACTERISTIC FUNCTION

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In the inversion formula

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt$$

for the characteristic function $f(t)$ of a distribution function $F(x)$, the limit of the symmetric integral is used. The purpose of this paper is to give a necessary and sufficient condition for the existence of the asymmetric improper integral $\lim_{T,T'\to\infty}\int_{-T'}^{T}$ on the right of the above formula.

Let $F(x)$ be a probability distribution function and $f(t)$ the corresponding characteristic function,

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) \ . \tag{1.1}$$

We assume in this note that $F(x)$ is standardized so that

$$F(x) = \tfrac{1}{2}[F(x+0) + F(x-0)] \ . \tag{1.2}$$

The well known inversion formula states that

$$F(x) - F(0) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt \tag{1.3}$$

for every $-\infty < x < \infty$.

It is also known that the symmetric integral of the right hand side cannot be replaces by the improper integral $\lim_{T,T'\to\infty}\int_{-T'}^{T'}$ (T, T' going to infinity independently).

Actually we may easily see that

$$\mathrm{Re}\Big(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) = \mathrm{Re}\Big(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) , \tag{1.4}$$

and

$$\mathrm{Im}\Big(\frac{1}{2\pi}\int_{0}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) = -\mathrm{Im}\Big(\frac{1}{2\pi}\int_{-T}^{0} \frac{e^{-ixt}-1}{-it} f(t)dt\Big) , \tag{1.5}$$

and hence $\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ixt}-1}{-it} f(t)dt$ cancels out its imaginary part.

The real part (1.4) always converges to $\frac{1}{2}[F(x) - F(0)]$. This gives the proof of (1.3). (See [1], pp. 263-264).

However the imaginary part (1.5) does not necessarily converge without some condition on $F(x)$. This is why the limit of the symmetric integral in (1.3) cannot be replaced by the general improper integral.

2. **The condition for the existence of the improper integral.** We shall give the necessary and sufficient condition for the existence of the limit of (1.5).

THEOREM 1. *In order that the limit of* (1.5) *when* $T \to \infty$ *exists, it is necessary and sufficient that the integral*

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{G(u, x) - G(u, 0)}{u} du \tag{2.1}$$

exists where,

$$G(u, x) = F(u + x) - F(-u + x) \tag{2.2}$$

and if (2.1) *exists*

$$\lim_{T \to \infty} \mathrm{Im}\left(\frac{1}{2\pi} \int_0^T \frac{e^{-ixt} - 1}{-it} f(t) dt\right) = \int_0^\infty \frac{G(u, x) - G(u, 0)}{u} du . \tag{2.3}$$

It must be noted that the integral of the right hand side of (2.3) exists in the neighborhood of the infinity. In fact $G(u, x) - G(u, 0) = [F(u + x) - F(u)] - [F(-u + x) - F(-u)]$ and $F(u + x) - F(u) \in L$, $(-\infty, \infty)$ for every fixed x.

We shall now prove the theorem.

Let

$$I(x, T) = \mathrm{Im}\left(\frac{1}{2\pi} \int_0^T \frac{e^{-ixt} - 1}{-it} f(t) dt\right) .$$

We then easily see that

$$\begin{aligned} I(x, T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T \frac{\sin xt \sin ut - (1 - \cos xt) \cos ut}{t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T \frac{\cos (u - x)t - \cos ut}{t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_0^T dt \int_{u-x}^{u} \sin vt dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dF(u) \int_{u-x}^{u} \frac{1 - \cos vT}{v} dv \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos vT}{v} dv \int_{v}^{v+x} dF(u)$$
$$= \frac{1}{2\pi}\int_{0}^{\infty} [G(v, x) - G(v, 0)] \frac{1-\cos vT}{v} dv .$$

As was mentioned before, $G(v, x) - G(v, 0) \in L_1(-\infty, \infty)$. Hence the Riemann-Lebesgue lemma shows that

$$\lim_{T\to\infty} \int_{\varepsilon}^{\infty} [G(v, x) - G(v, 0)] \frac{\cos vT}{v} dv = 0$$

for any $\varepsilon > 0$. Therefore we may write

$$I(x, T) = \int_0^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v}(1 - \cos vT)dv + \int_{\varepsilon}^{\infty} \frac{G(v, x) - G(v, 0)}{v} dv + o(1) \tag{2.4}$$

as $T \to \infty$, for a fixed $\varepsilon > 0$.

Now we shall show the sufficiency of the condition of the theorem. Let $\varepsilon > 0$ be arbitrary but fixed. Write

$$\int_0^{\varepsilon} \frac{G(v, x) - G(v, 0)}{v}(1 - \cos vT)dv = \int_0^{1/T} + \int_{1/T}^{\varepsilon} = K_1 + K_2 , \tag{2.5}$$

say. We have

$$| K_1 | \leqq \int_0^{1/T} | G(v, x) - G(v, 0) | \frac{1 - \cos vt}{v} dv \leqq CT \int_0^{1/T} | G(v, x) - G(v, 0) | \, dv , \tag{2.6}$$

for some constant C.

$\lim_{v\to 0+} [G(v, x) - G(v, 0)]$ exists since F is nondecreasing and it must be zero, otherwise (2.1) does not exist. Hence the last expression converges to zero.

$$K_1 = o(1), \quad \text{as} \quad T \to \infty . \tag{2.7}$$

Next write

$$\chi(v) = G(v, x) - G(v, 0) . \tag{2.8}$$

Choose ε such that $|\chi(v)| < \delta$ for $|v| \leqq \varepsilon$ for an arbitrary chosen δ. Since $\chi(v)/v$ is of bounded variation in $[1/T, \varepsilon]$, we have, using the second mean value theorem,

$$K_2 = \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} \cos vT dv$$
$$= \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv - T\chi\Big(\frac{1}{T}\Big)\int_{1/T}^{\xi} \cos vT dv - \frac{\chi(\varepsilon)}{\varepsilon}\int_{\xi}^{\varepsilon} \cos vT dv$$

for some $1/T < \xi < \varepsilon$. Thus

$$\left| K_2 - \int_{1/T}^{\varepsilon} \frac{\chi(v)}{v} dv \right| \leqq 2\chi\Big(\frac{1}{T}\Big) + 2\chi(\varepsilon) \leqq 4\delta .$$

Therefore from (2.4) and (2.5)

$$\left| I(x, T) - \frac{1}{2\pi}\int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq \frac{2\delta}{\pi} + o(1) . \tag{2.9}$$

This shows the sufficiency of the condition of the theorem and gives (2.3).

We shall next show the necessity. Define $\chi(v)$ as before. We see that $\chi(v)$ has the limit c as $v \to +0$. If $c \neq 0$, then from (2.4)

$$I(x, T) - c\int_{0}^{\varepsilon} \frac{1 - \cos vT}{v} dv$$
$$= \int_{0}^{\varepsilon} \frac{[\chi(v) - c](1 - \cos vT)}{v} dv + \int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv + o(1) .$$

The first integral of the right hand side is handled in the same way as in deriving (2.6) and (2.9) with $\chi(v) - c$ in place of

$$\chi(v) = G(v, x) - G(v, 0) .$$

Actually instead of (2.6) we see that K_1 with $\phi(v) - c$ is bounded by $CT\int_{0}^{1/T} |\chi(v) - c| dv$ which is $o(1)$. In place of (2.9) we have

$$\left| I(x, T) - \frac{c}{2\pi}\int_{0}^{\varepsilon} \frac{1 - \cos vT}{v} dv - \frac{1}{2\pi}\int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right.$$
$$\left. - \frac{1}{2\pi}\int_{\varepsilon}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) , \tag{2.10}$$

where C_1 is some constant.

$$\int_{0}^{\varepsilon} \frac{1 - \cos vT}{v} dv = 2\int_{0}^{\varepsilon T} \frac{\sin^2 v/2}{v} dv \geqq C_2 \log \varepsilon T , \tag{2.11}$$

where C_2 is an absolute constant. Choose ε for an arbitrary given $\eta < C_2$ so that $|\chi(v) - c| < \eta$ for $0 < v < \varepsilon$. Then

$$\left| \int_{1/T}^{\varepsilon} \frac{\chi(v) - c}{v} dv \right| \leqq \eta \log \varepsilon T . \tag{2.12}$$

Hence if $I(x, T)$ has a limit as $T \to \infty$, then in view of (2.11) and (2.12), (2.10) implies a contradiction. Hence we have that $c = 0$. Using (2.10), this yields

$$\left| I(x, T) - \frac{1}{2\pi}\int_{1/T}^{\infty} \frac{\chi(v)}{v} dv \right| \leqq C_1\delta + o(1) .$$

This proves the necessity of the condition.

3. **Remarks.** From Theorem 1, we immediately obtain

THEOREM 2. *In order that*

$$\lim_{T, T' \to \infty} \frac{1}{2\pi}\int_{-T'}^{T} \frac{e^{-ixt} - 1}{-it} f(t)dt$$

exists, it is necessary and sufficient that (2.1) *exists for* $\varepsilon > 0$. (*The limit is* $F(x) - F(0)$).

Similar arguments apply to the integral

(3.1) $$J_1(x, T) = \int_1^T \frac{f(t)e^{-ixt}}{it} dt \quad \text{and} \quad J_2(x, T) = \int_{-T}^{-1} \frac{f(t)e^{-ixt}}{it} dt .$$

We easily see that J_1 and J_2 are conjugate complex. We may show that *in order* for $J_1(x, T)$ *or* $J_2(x, T)$ *to converge as* $T \to \infty$, *it is necessary and sufficient that*

(3.2) $$\int_0^{\varepsilon} \frac{F(u + x) - F(-u + x)}{u} du < \infty$$

for some $\varepsilon > 0$.

(3.1) implies that

(3.3) $$\lim_{T \to \infty} \frac{1}{2\pi}\int_{-T}^{T} f(t)e^{-ixt}dt = 0$$

which is very well known when $F(x)$ is continuous at x. (3.1) says more than this about the improper integrability of $f(t)$ near infinity with the additional condition (3.2) on $F(x)$.

The sufficiency of (3.2) for the existence of the limits of (3.1) was proved in [2] before.

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ON RIGHT ALTERNATIVE RINGS WITHOUT PROPER RIGHT IDEALS

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It is shown that a right alternative ring R without proper right ideals, of characteristic not two, containing idempotents e and 1, $e \neq 1$, such that $ex = e(ex)$ for all $x \in R$ must be alternative and hence a Cayley vector matrix algebra of dimension 8 over its center.

In the classification of simple right alternative rings of characteristic not two it is still an open question whether there exist any which are not alternative, in contrast to characteristic two, where there do exist division rings which are not alternative [8]. A number of people have worked on this problem and were able to prove the alternative identity whenever they assumed an additional hypothesis such as finite dimensionality [1, 3], other identities [6, 7], or internal conditions on the ring [4, 5, 9]. It seems natural to try to tackle the case where there exists an idempotent $e \neq 1$ in R such that $(e, e, R) = 0$. If one could establish in this case that all simple R of characteristic not 2 are alternative, then this would be a natural generalization of the theorem of Albert [2] for alternative rings, in which he showed that a simple alternative ring with idempotent $e \neq 1$ had to be either associative or a Cayley vector matrix algebra of dimension eight over its center.

In this paper we do not quite achieve this result, for we need to strengthen the hypothesis of simplicity to the assumption that the ring has no proper right ideals. On the other hand there is a good deal of information here that should prove useful in either romoving the hypothesis of $(e, e, R) = 0$, or in constructing an example of a simple, right alternative ring of characteristic not two which is not alternative, if indeed such an example exists.

The main tool here is the fact that $(e, e, R) = 0$ allows a Peirce decomposition into four "subspaces" $R_{i,j}$, $i, j = 0, 1$ as in the associative and alternative cases. The multiplication table for these subspaces differs in six places from the same table for alternative rings. By constructing appropriate right ideals we show in fact that the tables are the same. In the process we reduce the problem to the one studied by M. Humm-Kleinfeld [4], although by that time one can deduce from our work quite readily that indeed R must be alternative.

2. Preliminary identities. In the course of the paper we require a number of identities which are true in arbitrary right alternative rings of characteristic not two:

(1) $(ab, c, d) + (a, b, (c, d)) = a(b, c, d) + (a, c, d)b$.

(2) $(x, ab, a) = (x, b, a)a$.

(3) $([ab]c)b = a([bc]b)$.

Proofs of these identities may be found on page 940 of [5].

(4) $(ab)c = a(bc) + a(cb) - (ac)b$,

also holds as this is the linearization of the right alternative identity.

3. Peirce decomposition.

Henceforth in the paper, we assume that R is a right alternative ring of characteristic not two, and that R contains 1 and an idempotent of $e \neq 1$, such that $(e, e, R) = 0$. If we define $R_{ij} = \{x \in R \mid ex = ix, xe = jx\}$ and $i, j = 0, 1$, then R may be decomposed into a direct sum by $R = R_{11} + R_{10} + R_{01} + R_{00}$. Humm-Kleinfeld has shown on page 166 [4] that the multiplication table of the R_{ij} has the following containment properties:

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	$R_{11} + R_{01}$	R_{10}	R_{10}	0
R_{10}	0	$R_{11} + R_{01}$	R_{11}	R_{10}
R_{01}	R_{01}	R_{00}	$R_{00} + R_{10}$	0
R_{00}	0	R_{01}	R_{01}	$R_{00} + R_{10}$

Thus the first entry gives the information that $(R_{11})^2 \subset R_{11} + R_{01}$, etc. Besides, it is true that $x_{ii}^2 \in R_{ii}$, and whenever $i \neq j$ that $x_{ij}^2 \in R_{ii}$ as well as $x_{ij}^3 = 0$.

Throughout the paper whenever we need to refer to this result we shall use the phrase "it follows from the table that..."

We should bear in mind that in an alternative ring there are six places where stronger assertions can be made. These are: $(R_{11})^2 \subset R_{11}$, $R_{10}^2 \subset R_{01}$, $R_{01}^2 \subset R_{10}$, $(R_{00})^2 \subset R_{00}$, $R_{11}R_{01} = 0$, and $R_{00}R_{10} = 0$.

4. Main section.

LEMMA 1. *In R we have $(R_{11}^2)_{01}R_{10} = 0$, and $(R_{11}^2)_{01}R_{01} \subset R_{10}$.*

Proof. Let $x_{11}, y_{11} \in R_{11}$, $z_{10} \in R_{10}$. From the table it is obvious that $(x_{11}, z_{10}, y_{11}) = 0$. Hence, using the right alternative identity,

$$0 = (x_{11}, y_{11}, z_{10}) = (x_{11}y_{11})z_{10} - x_{11}(y_{11}z_{10}) .$$

Let $x_{11}y_{11} = a_{11} + b_{01}$. Then, by substituting this in the previous equation, it follows that $a_{11}z_{10} + b_{01}z_{10} - x_{11}(y_{11}z_{10}) = 0$, so that $b_{01}z_{10} = x_{11}(y_{11}z_{10}) - a_{11}z_{10} \in R_{10} \cap R_{00} = 0$, by use of the table. Hence, $b_{01}z_{10} = 0$, thus proving the first part. Also let $z_{01} \in R_{01}$. Then

$$(x_{11}, y_{11}, z_{01}) = (x_{11}y_{11})z_{01} - x_{11}(y_{11}z_{01}) = a_{11}z_{01} + b_{01}z_{01} - x_{11}(y_{11}z_{01}).$$

From the right alternative identity it follows that

$$(x_{11}, y_{11}, z_{01}) = -(x_{11}, z_{01}, y_{11}) = -(x_{11}z_{01})y_{11} + x_{11}(z_{01}y_{11}) = x_{11}(z_{01}y_{11}) \in R_{10},$$

using the table. Hence solving the previous equation for $b_{01}z_{01}$, we see that $b_{01}z_{01} = (x_{11}, y_{11}, z_{01}) - a_{11}z_{01} + x_{11}(y_{11}z_{01}) \in R_{10}$, using the table. This completes the proof of the lemma.

DEFINITION. Let $T_{01} = \{x_{01} \in R_{01} \mid x_{01}R_{10} = 0, \text{ and } x_{01}R_{01} \subset R_{10}\}$ and form $T = T_{01} + T_{01}R_{01} + \cdots + (\cdots (T_{01}R_{01})R_{01} \cdots)R_{01} + \cdots +$ where each term except the first is obtained from the preceding by right multiplication by R_{01}.

LEMMA 2. *T is a right ideal of R such that $T \subset R_{01} + R_{10} + R_{11}$.*

Proof. For arbitrary $t_{01} \in T_{01}$, $x_{11} \in R_{11}$, $y_{10} \in R_{10}$ and $z_{01} \in R_{01}$ we have

$$(t_{01}x_{11})y_{10} = (t_{01}, x_{11}, y_{10}) = -(t_{01}, y_{10}, x_{11}) = -(t_{01}y_{10})x_{11} + t_{01}(y_{10}x_{11}) = 0 ,$$

using the right alternative identity and the definition of T_{01}, as well as the table. Also,

$$\begin{aligned}(t_{01}x_{11})z_{01} &= (t_{01}, x_{11}, z_{01}) + t_{01}(x_{11}z_{01}) = (t_{01}, x_{11}, z_{01}) \\ &= -(t_{01}, z_{01}, x_{11}) = -(t_{01}z_{01})x_{11} + t_{01}(z_{01}x_{11}) \\ &= a_{10}x_{11} + t_{01}(z_{01}x_{11}) = t_{01}(z_{01}x_{11}) \in R_{10},\end{aligned}$$

using the same reasons as before. But then $t_{01}x_{11} \in T_{01}$ and thus $T_{01}R_{11} \subset T_{01}$. Also, from the definition of T_{01} it follows almost immediately that $T_{01}R_{10} = 0$, and $T_{01}R_{01} \subset R_{10}$, while the table implies that $T_{01}R_{00} = 0$. Let $P(n)$ be the $n + 1^{st}$ term in the sum that defines T, and let $U(n) = T_{01} + T_{01}R_{01} + \cdots + P(n)$, be the sum of the first $n + 1$ terms in the definition of T. We shall prove by induction that $P(n)R_{11} \subset U(n)$, $P(n)R_{10} \subset U(n)$ and $P(n)R_{00} \subset U(n)$. We have already seen this is true for $n = 0$. Assume it is true for n and then we shall prove it true for $n + 1$. We abbreviate $P(n)$ by simply P. Then using (4), and the table, $(PR_{01})R_{11} \subset P(R_{01}R_{11}) + P(R_{11}R_{01}) + (PR_{11})R_{01} \subset PR_{01} + PR_{10} + (PR_{11})R_{01} \subset P(n + 1) + U(n) + U(n)R_{01} \subset U(n + 1)$. Also similarly,

$$\begin{aligned}(PR_{01})R_{10} \subset P(R_{01}R_{10}) + P(R_{10}R_{01}) + (PR_{10})R_{01} \subset PR_{00} + PR_{11} \\ + (PR_{10})R_{01} \subset U(n) + U(n)R_{01} \subset U(n + 1) ,\end{aligned}$$

and

$$\begin{aligned}(PR_{01})R_{00} \subset P(R_{01}R_{00}) + P(R_{00}R_{01}) + (PR_{00})R_{01} \subset PR_{01} \\ + (PR_{00})R_{01} \subset P(n + 1) + U(n)R_{01} \subset U(n + 1) .\end{aligned}$$

Consequently, $P(n+1)A \subset U(n+1)$ for $A = R_{11}, R_{10}$, and R_{00}. This completes the induction. But then, $TA \subset T$. Of course, also $TR_{01} \subset T$. But then, $TR \subset T$ and, hence, T is a right ideal of R. Also, $P(1) = T_{01}R_{01} \subset R_{10}$ by definition of T_{01}. Hence $P(2) \subset R_{10}R_{01} \subset R_{11}$, and so $P(3) \subset R_{11}R_{01} \subset R_{10}$. Thence $P(2n+1) \subset R_{10}$ and $P(2n) \subset R_{11}$, so that $T \subset T_{01} + R_{10} + R_{11} \subset R_{01} + R_{10} + R_{11}$. This completes the proof of the lemma.

We note that there is complete symmetry if the idempotent e is replaced by the idempotent $1-e$. In terms of the Peirce decomposition this has the effect of simply permuting subscripts. We shall frequently use this play in order to obtain new results from theorems already proved, and justify it by stating that "we may reverse subscripts...." Thus we may assert:

COROLLARY 1. *If R has no proper right ideals then $R_{11}^2 \subset R_{11}$.*

COROLLARY 2. *If R has no proper right ideals then $R_{00}^2 \subset R_{00}$.*

Proof. The right ideal T of Lemma 2 cannot be R since $1 - e \in R_{00}$ would then have to be zero, contrary to assumption. But then $T = 0$, hence $T_{01} = 0$. But Lemma 1 implies that $(R_{11}^2)_{01} \in T_{01}$, so that $(R_{11}^2)_{01} = 0$, hence $R_{11}^2 \subset R_{11}$. But then we may reverse subscripts and obtain the second corollary as well.

In the remainder of the paper we shall assume tacitly that, in addition, R has no proper right ideals, so that we may freely use the results of the last two corollaries.

LEMMA 3. *R_{11} is associative.*

Proof. Let $A = \sum (R_{11}, R_{11}, R_{11}) + R_{11}(R_{11}, R_{11}, R_{11})$. Since $R_{10}R_{11} = 0$ follows from the table, while $R_{11}^2 \subset R_{11}$ because of Corollary 1, we can easily verify that $(R_{10}, R_{11}, R_{11}) = 0$. Select $w_{11}, y_{11}, z_{11}, \in R_{11}$ and $x_{10} \in R_{10}$. Then substitute $a = w_{11}, b = x_{10}, c = y_{11}, d = z_{11}$ in (1), obtaining

$$(w_{11}x_{10}, y_{11}, z_{11}) + (w_{11}, x_{10}, (y_{11}, z_{11})) = w_{11}(x_{10}, y_{11}, z_{11}) + (w_{11}, y_{11}, z_{11})x_{10} .$$

However, by inspection $(R_{11}, R_{10}, R_{11}) = 0$, as a consequence of the table, so that only one term survives in the preceding equation. Thus $(R_{11}, R_{11}, R_{11})R_{10} = 0$. We have already observed that $(R_{11}, R_{10}, R_{11}) = 0$. If we apply the right alternative identity in this situation then it follows that $(R_{11}, R_{11}, R_{10}) = 0$, and hence $(R_{11},(R_{11}, R_{11}, R_{11}), R_{10}) = 0$. Expanding the last associator, thus $R_{11}(R_{11}, R_{11}, R_{11}) \cdot R_{10} = 0$. But then,

$AR_{10} = 0$. Since $A \subset R_{11}$, it follows from the table that $AR_{00} = 0$. Besides, it is well known that even if R_{11} where an arbitrary ring, not necessarily right alternative, that A is always a two-sided ideal of R_{11}, so that $AR_{11} \subset A$. Let us form

$$B = A + AR_{01} + (AR_{01})R_{01} + \cdots + (\cdots (AR_{01})R_{01} \cdots) + \cdots$$

where the n^{th} term is obtained from the preceding by right multiplication by R_{01}, except for $n = 1$. As in the proof of Lemma 2 the reader may easily check that B is a right ideal of R using induction. But the odd terms in the equation defining B are contained in R_{11}, while the even terms are contained in R_{10}, using the table. Hence, $B \subset R_{10} + R_{11}$. Since $B = R$ implies $1 - e = 0$, we must have $B = 0$, hence $A = 0$. Thus R_{11} is associative, completing the proof of the lemma.

COROLLARY. *R_{00} is associative.*

Proof. We may reverse subscripts in the lemma.

LEMMA 4. (i) $R_{11} + R_{01} = R_{10}R_{01} + R_{10}^2 + R_{00}R_{10}$.
(ii) $R_{00} + R_{10} = R_{01}R_{10} + R_{01}^2 + R_{11}R_{01}$.
(iii) $R_{00} = R_{01}R_{10}$.
(iv) $R_{11} = R_{10}R_{01}$.

Proof. Define inductively $R_{10}^n = R_{10}^{n-1}R_{10}$ and form $A = R_{10}R_{01} + R_{10} + \cdots + R_{10}^n + \cdots$. First we aim to show that A must be a right ideal of R. By repeated use of (4) and table we see that

$$(R_{10}R_{01})R_{11} \subset R_{10}(R_{01}R_{11} + R_{11}R_{01}) + (R_{10}R_{11})R_{01} \subset R_{10}R_{01} + R_{10}^2 ,$$
$$(R_{10}R_{01})R_{10} \subset R_{11}R_{10} \subset R_{10}, \; (R_{10}R_{01})R_{01} \subset R_{11}R_{01} \subset R_{10} ,$$
$$(R_{10}R_{01})R_{00} \subset R_{11}R_{00} = 0 ,$$

thus showing that $(R_{10}R_{01})R \subset R_{10}R_{01} + R_{10} + R_{10}^2 \subset A$. Also $R_{10}R_{11} = 0$, $R_{10}R_{10} = R_{10}^2$, $R_{10}R_{01} \subset A$, $R_{10}R_{00} \subset R_{10}$, using the table. But use of (4) and the table shows that

$$(R_{10}^2)R_{11} \subset R_{10}(R_{10}R_{11} + R_{11}R_{10}) + (R_{10}R_{11})R_{10} \subset R_{10}^2, \; R_{10}^2R_{10} = R_{10}^3 ,$$
$$(R_{10}^2)R_{01} \subset R_{10}(R_{10}R_{01} + R_{01}R_{10}) + (R_{10}R_{01})R_{10} \subset R_{10}R_{11} + R_{10}R_{00} + R_{11}R_{10} \subset R_{10} ,$$

while $(R_{10}^2)R_{00} \subset (R_{11} + R_{01})R_{00} = 0$. Now define

$$Q(n) = R_{10}R_{01} + R_{10} + \cdots + R_{10}^n .$$

The above calculations show that $Q(2)B \subset Q(2)$, for $B = R_{11}, R_{01}$ and

R_{00}. Assume inductively that $R_{10}^n B \subset Q(n)$ and we proceed to prove this inclusion for $n+1$ in place of n. Besides the induction hypothesis, our main tools are (4) and the table. $(R_{10}^n R_{10})R_{11} \subset R_{10}^n(R_{10}R_{11} + R_{11}R_{10}) + (R_{10}^n R_{11})R_{10} \subset R_{10}^n R_{10} + (R_{10}^n R_{11})R_{10} \subset R_{10}^{n+1} + Q(n)R_{10} \subset Q(n+1)$. Similarly,

$$(R_{10}^n R_{10})R_{01} \subset R_{10}^n(R_{10}R_{01} + R_{01}R_{10}) + (R_{10}^n R_{01})R_{10} \subset R_{10}^n R_{11} + R_{10}^n R_{00} + Q(n)R_{10} \subset Q(n) + Q(n+1) \subset Q(n+1) .$$

Finally, $(R_{10}^n R_{10})R_{00} \subset R_{10}^n(R_{10}R_{00} + R_{00}R_{10}) + (R_{10}^n R_{00})R_{10} \subset R_{10}^n R_{10} + R_{10}^n R_{01} + Q(n)R_{10} \subset Q(n+1)$. This completes the induction. Armed with this information we are now ready to prove that A is a right ideal of R. Since $Q(2)B \subset Q(2) \subset A$ and $R_{10}^n B \subset Q(n) \subset A$, we see that $AB \subset A$. Since obviously $R_{10}^n R_{10} = R_{10}^{n+1}$, it follows also that $AR_{10} \subset A$. But then $AR \subset A$, and thus A is a right ideal of R. Let us consider first the case $A = 0$. In that case, $R_{10} = 0$. Form $B = R_{00} + R_{01}$. Using the table and Corollary 2 of Lemma 2, we may varify that B is a right ideal. Since $e \notin B$, we must then have $B = 0$. But then $R = R_{11}$, so that $e = 1$, contrary to assumption. Hence the case $A = 0$ cannot arise. The only open possibility is that $A = R$. Now from the table we see that $R_{10}^2 \subset R_{11} + R_{01}$, while $R_{10}^3 \subset (R_{11} + R_{01})R_{10} \subset R_{10} + R_{00}$, and $(R_{10} + R_{00})R_{10} \subset R_{11} + R_{01}$. Consequently, $R_{10}^{2n} \subset R_{11} + R_{01}$, and $R_{10}^{2n+1} \subset R_{10} + R_{00}$, for all positive integers n. Since the Peirce decomposition is direct and $A = R$, it must be that $R_{11} + R_{01} = \sum R_{10}^{2n} + R_{10}R_{01}$, and $R_{10} + R_{00} = \sum R_{10}^{2n-1}$. But note that by definition $R_{10}^{2n+2} = (R_{10}^{2n} R_{10})R_{10} \subset ([R_{11} + R_{01}]R_{10})R_{10} \subset R_{10}^2 + R_{00}R_{10}$ and so $\sum R_{10}^{2n} \subset R_{10}^2 + R_{00}R_{10}$. But then from two equations back it follows that $R_{11} + R_{01} \subset R_{10}^2 + R_{00}R_{10} + R_{10}R_{01}$. On the other hand it is a consequence of the table that $R_{10}^2 + R_{00}R_{10} + R_{10}R_{01} \subset R_{11} + R_{01}$, so that $R_{11} + R_{01} = R_{10}^2 + R_{00}R_{10} + R_{10}R_{01}$. This establishes part (i). To obtain (ii) from (i), simply reverse subscripts. By definition

$$R_{10}^{2n+1} \subset (R_{10}^{2n-1}R_{10})R_{10} \subset ([R_{10} + R_{00}]R_{10})R_{10} \subset R_{10}^3 + R_{01}R_{10} \subset (R_{11} + R_{01})R_{10} + R_{01}R_{10} \subset R_{10} + R_{01}R_{10} .$$

But then $R_{10} + R_{00} = \sum R_{10}^{2n-1} \subset R_{10} + R_{01}R_{10} \subset R_{10} + R_{00}$. But then $R_{10} + R_{01}R_{10} = R_{10} + R_{00}$. Using the directness of the Peirce decomposition we obtain that $R_{01}R_{10} = R_{00}$. This establishes part (iii). Part (iv) follows from part (iii) by reversing subscripts. This completes the proof of the lemma.

LEMMA 5. *For all* $a_{01} \in R_{01}$ *and* $x_{11}, y_{11} \in R_{11}$, $(a_{01}x_{11})\, y_{11} = a_{01}(y_{11}x_{11})$.

Proof. It follows from Lemma 4—(i) that $a_{01} \in R_{10}R_{01} + R_{10}^2 + R_{00}R_{10}$. Using (4) and the table we see that for

$$b_{10}, c_{10} \in R_{10}, (b_{10}c_{10})x_{11} = b_{10}(c_{10}x_{11} + x_{11}c_{10}) - (b_{10}x_{11})c_{10} = b_{10}(x_{11}c_{10}) .$$

By repeated use of this last equation, then $([b_{10}c_{10}]x_{11})y_{11} = (b_{10}[x_{11}c_{10}])y_{11} = b_{10}(y_{11}[x_{11}c_{10}])$. As previously observed, $(y_{11}, x_{11}, c_{10}) = -(y_{11}, c_{10}, x_{11}) = 0$. Thus, $b_{10}(y_{11}[x_{11}c_{10}]) = b_{10}([y_{11}x_{11}]c_{10}) = (b_{10}c_{10})(y_{11}x_{11})$ using the table, Lemma 2–Corollary 1, and the previous observation we made use of just before. Combining two previous equations, we see that $([b_{10}c_{10}]x_{11})y_{11} = (b_{10}c_{10})(y_{11}x_{11})$. Thus $b_{10}c_{10}$ has the desired property. Let $b_{00} \in R_{00}$. Then (4) and the table imply that $(b_{00}c_{10})x_{11} = b_{00}(c_{10}x_{11} + x_{11}c_{10}) - (b_{00}x_{11})c_{10} = b_{00}(x_{11}c_{10})$. Using the table, Lemma 2–Corollary 1, and the previous equation repeatedly, it follows that $([b_{00}c_{10}]x_{11})y_{11} = (b_{00}[x_{11}c_{10}])y_{11} = b_{00}(y_{11}[x_{11}c_{10}])$. As already noted, $(y_{11}, x_{11}, c_{10}) = 0$, so that $b_{00}(y_{11}[x_{11}c_{10}]) = b_{00}([y_{11}x_{11}]c_{10}) = (b_{00}c_{10})(y_{11}x_{11})$. Thus $b_{00}c_{10}$ has the desired property. Finally, if $z_{11} \in R_{11}$, then $(z_{11}x_{11})y_{11} - z_{11}(y_{11}x_{11}) \in R_{11}$ because of Lemma 2–Corollary 1. Hence, $(a_{01}x_{11})y_{11} - a_{01}(y_{11}x_{11}) \in R_{11}$, But from the table it follows that $(a_{01}x_{11})y_{11} - a_{01}(y_{11}x_{11}) \in R_{01}$. Since $R_{01} \cap R_{11} = 0$, it must be that $(a_{01}x_{11})y_{11} - a_{01}(y_{11}x_{11}) = 0$. This completes the proof of the lemma.

LEMMA 6. *R_{11} and R_{00} are commutative.*

Proof. Let $a_{01} \in R_{01}$, $b_{10} \in R_{10}$ and $x_{11}, y_{11} \in R_{11}$. As a result of (1), $(b_{10}a_{01}, x_{11}, y_{11}) + (b_{10}, a_{01}, (x_{11}, y_{11})) = b_{10}(a_{01}, x_{11}, y_{11}) + (b_{10}, x_{11}, y_{11})a_{01}$. Use of the table reveals that $(b_{10}, x_{11}, y_{11}) = 0$, since R_{11} is a subring. Moreover, Lemma 3 and the table imply that $(b_{10}a_{01}, x_{11}, y_{11}) = 0$. Thus only two terms survive in the first equation and we see that $(b_{10}, a_{01}, (x_{11}, y_{11})) = b_{10}(a_{01}, x_{11}, y_{11})$. Moreover,

$$(b_{10}, a_{01}, (x_{11}, y_{11})) = (b_{10}a_{01})(x_{11}y_{11} - y_{11}x_{11}) - b_{10}[a_{01}(x_{11}y_{11} - y_{11}x_{11})] ,$$

expanding the associator. But

$$\begin{aligned} - b_{10}[a_{01}(x_{11}y_{11} - y_{11}x_{11})] &= - b_{10}[(a_{01}y_{11})x_{11} - a_{01}(y_{11}x_{11})] \\ &= - b_{10}(a_{01}, y_{11}, x_{11}) = b_{10}(a_{01}, x_{11}, y_{11}) , \end{aligned}$$

using Lemma 5 and the right alternative identity. Now if we compare the last three equations we conclude that $(b_{10}a_{01})(x_{11}y_{11} - y_{11}x_{11}) = 0$. At this point Lemma 4–(iv) may be utilized to conclude that for every $z_{11} \in R_{11}$, $z_{11}(x_{11}y_{11} - y_{11}x_{11}) = 0$. In particular we may choose $z_{11} = e$. Then because of Lemma 2–Corollary 1, $x_{11}y_{11} - y_{11}x_{11} = 0$. Thus R_{11} is seen to be commutative. By reversing subscripts, it follows that R_{00} is also commutative. This completes the proof of the lemma.

LEMMA 7. $(R_{01}, R_{11}, R_{11}) = 0 = (R_{10}, R_{00}, R_{00})$.

Proof. Let $a_{01} \in R_{01}$ and $x_{11}, y_{11} \in R_{11}$. Then because of Lemmas 5 and 6, $(a_{01}x_{11})y_{11} = a_{01}(y_{11}x_{11}) = a_{01}(x_{11}y_{11})$, thus establishing $(a_{01}, x_{11}, y_{11}) = 0$. Hence, $(R_{01}, R_{11}, R_{11}) = 0$. By reversing subscripts, $(R_{10}, R_{00}, R_{00}) = 0$ follows. This completes the proof of the lemma.

DEFINITION. We define Q_{00} = nilpotent elements of R_{00}, and Q_{11} = nilpotent elements of R_{11}.

We note, since R_{11}, R_{00} are associative, commutative, subrings of R, that Q_{11} is an ideal of R_{11} and Q_{00} an ideal of R_{00}.

LEMMA 8. *If* $a_{11} \in R_{11}$, $b_{01} \in R_{01}$, $c_{10} \in R_{10}$, *then*

$$d_{11} = (a_{11}b_{01})c_{10} = (a_{11}, b_{01}, c_{10}) = -(a_{11}, c_{10}, b_{01})$$

satisfies $d_{11}^2 = 0$, *so that* $d_{11} \in Q_{11}$. *Similarly, if*

$$d_{00} = (a_{00}b_{10})c_{01} = (a_{00}, b_{10}, c_{01}) = -(a_{00}, c_{01}, b_{10}) ,$$

then $d_{00}^2 = 0$ *and* $d_{00} \in Q_{00}$. *Thus* $(R_{11}R_{01})R_{10} \subset Q_{11}$ *and* $(R_{00}R_{10})R_{01} \subset Q_{00}$.

Proof. As a result of (4), $(a_{11}b_{01})c_{10} = a_{11}(b_{01}c_{10} + c_{10}b_{01}) - (a_{11}c_{10})b_{01}$. But $a_{11}(b_{01}c_{10}) \in R_{11}R_{00} = 0$, from the table, while $a_{11}(c_{10}b_{01}) \in R_{11}^2 \subset R_{11}$, $-(a_{11}c_{10})b_{01} \in R_{10}R_{01} \subset R_{11}$, using the table and Lemma 2–Corollary. Hence $(a_{11}b_{01})c_{10} \in R_{11}$ and so $(R_{11}R_{01})R_{10} \subset R_{11}$. Let $f_{10} = a_{11}b_{01}$ and $d_{11} = (a_{11}b_{01})c_{10}$. Then $f_{10}c_{10} = d_{11}$, while $d_{11}^2 = (f_{10}c_{10})d_{11} = f_{10}(c_{10}d_{11} + d_{11}c_{10}) - (f_{10}d_{11})c_{10}$, using (4). Since $R_{10}R_{11} = 0$, follows from the table, two terms vanish in the last equation, so that $d_{11}^2 = f_{10}(d_{11}c_{10})$. But $f_{10}(d_{11}c_{10}) = f_{10}([f_{10}c_{10}]c_{10}) = f_{10}(f_{10}[c_{10}^2])$, because of the right alternative identity. Since

$$f_{10}(c_{10}^2) \in R_{10}R_{11} = 0 ,$$

as a result of the table, it follows that $d_{11}^2 = 0$, and so $d_{11} \in Q_{11}$. By interchanging subscripts we obtain the second part. This completes the proof of the lemma.

LEMMA 9. *Let*

$$Q = Q_{11} + Q_{11}R_{10} + R_{01}Q_{11} + R_{01}Q_{11}R_{10} + Q_{00} + Q_{00}R_{01} + R_{10}Q_{00} + R_{10}Q_{00}R_{01} + R_{11}R_{01} + (R_{11}R_{01})R_{01} + R_{00}R_{10} + (R_{00}R_{10})R_{10} .$$

Then Q is a right ideal of R.

Proof. Most of the calculations involved are routine, and (4) is an important tool. Unless the reasoning is complicated, we shall state the appropriate inclusions without comment. $(R_{11}R_{01})R_{11} \subset R_{10}R_{11} = 0$. $(R_{11}R_{01})R_{10} \subset Q_{11}$, because of Lemma 8. $(R_{11}R_{01})R_{01} \subset Q$. $(R_{11}R_{01})R_{00} \subset R_{11}(R_{01}R_{00} + R_{00}R_{01}) + (R_{11}R_{00})R_{01} \subset R_{11}(R_{00}R_{01}) \subset R_{11}R_{01}$, using (4) and the table.

$$([R_{11}R_{01}]R_{01})R_{11} \subset (R_{11}R_{01})(R_{01}R_{11} + R_{11}R_{01}) + ([R_{11}R_{01}]R_{11})R_{01} \subset (R_{11}R_{01})R_{01} + (R_{11}R_{01})R_{10} + ([R_{11}R_{01}]R_{11})R_{01} ,$$

using (4). Now $(R_{11}R_{01})R_{10} \subset Q_{11}$ because of Lemma 8, while we observed earlier in the proof that $(R_{11}R_{01})R_{11} = 0$. Consequently,

$$([R_{11}R_{01}]R_{01})R_{11} \subset (R_{11}R_{01})R_{01} + Q_{11} .$$
$$([R_{11}R_{01}]R_{01})R_{10} \subset (R_{11}R_{01})(R_{01}R_{10} + R_{10}R_{01}) + ([R_{11}R_{01}]R_{10})R_{01} \subset (R_{11}R_{01})R_{00} + (R_{11}R_{01})R_{11} + ([R_{11}R_{01}]R_{10})R_{01} ,$$

using (4). But we have already observed that $(R_{11}R_{01})R_{00} \subset R_{11}R_{01}$, and $(R_{11}R_{01})R_{11} = 0$, while Lemma 8 implies $(R_{11}R_{01})R_{10} \subset R_{11}$, so that

$$([R_{11}R_{01}]R_{10})R_{01} \subset R_{11}R_{01} .$$

Combining these observations it follows that $([R_{11}R_{01}]R_{01})R_{10} \subset R_{11}R_{01}$. $([R_{11}R_{01}]R_{01})R_{01} \subset (R_{10}R_{01})R_{01} \subset R_{11}R_{01}$.

$$([R_{11}R_{01}]R_{01})R_{00} \subset (R_{10}R_{01})R_{00} \subset R_{11}R_{00} = 0 .$$

As we have already observed, Q_{11} is an ideal of R_{11}, and so $Q_{11}R_{11} \subset Q_{11}$. $Q_{11}R_{10} \subset Q$. $Q_{11}R_{01} \subset R_{11}R_{01}$. $Q_{11}R_{00} = 0$. $(Q_{11}R_{10})R_{11} \subset R_{10}R_{11} = 0$. In order to obtain the desired inclusion for $(Q_{11}R_{10})R_{10}$, we observe first that $a_{10}^3 = 0$ follows from the table, hence $a_{10}^4 = 0$, so that $a_{10}^2 \in Q_{11}$. By linearization, then $a_{10}b_{10} + b_{10}a_{10} \in Q_{11}$. If $q_{11} \in Q_{11}$, then $q_{11}a_{10} \in R_{10}$, so that $(q_{11}a_{10})b_{10} + b_{10}(q_{11}a_{10}) = q'_{11} \in Q_{11}$. However, using (4), $(b_{10}a_{10})q_{11} = b_{10}(a_{10}q_{11} + q_{11}a_{10}) - (b_{10}q_{11})a_{10} = b_{10}(q_{11}a_{10})$, since $R_{10}R_{11} = 0$ follows from the table. Comparing the last two equations, $(q_{11}a_{10})b_{10} = q'_{11} - (b_{10}a_{10})q_{11}$. But

$$(b_{10}a_{10})q_{11} \subset (R_{10}^2)Q_{11} \subset (R_{11} + R_{01})Q_{11} \subset R_{11}Q_{11} + R_{01}Q_{11} \subset Q_{11} + R_{01}Q_{11} .$$

Hence $(q_{11}a_{10})b_{10} \in Q_{11} + R_{01}Q_{11}$, and thus $(Q_{11}R_{10})R_{10} \subset Q_{11} + R_{01}Q_{11}$. Also, $(Q_{11}R_{10})R_{01} \subset Q_{11}(R_{10}R_{01} + R_{01}R_{10}) + (Q_{11}R_{01})R_{10} \subset Q_{11}R_{11} + Q_{11}R_{00} + (R_{11}R_{01})R_{10}$, as a result of (4). But $Q_{11}R_{11} \subset Q_{11}$, $Q_{11}R_{00} \subset R_{11}R_{00} = 0$, and $(R_{11}R_{01})R_{10} \subset Q_{11}$ because of Lemma 8. Hence $(Q_{11}R_{10})R_{01} \subset Q_{11}$.

$$(Q_{11}R_{10})R_{00} \subset Q_{11}(R_{10}R_{00} + R_{00}R_{10}) + (Q_{11}R_{00})R_{10} \subset Q_{11}R_{10} + Q_{11}R_{01} \subset Q_{11}R_{10} + R_{11}R_{01} ,$$

using (4) and the table. Hence $(Q_{11}R_{10})R_{00} \subset Q_{11}R_{10} + R_{11}R_{01}$.

$$(R_{01}Q_{11})R_{11} \subset R_{01}(Q_{11}R_{11} + R_{11}Q_{11}) + (R_{01}R_{11})Q_{11} \subset R_{01}Q_{11} ,$$

using (4). $(R_{01}Q_{11})R_{10} \subset Q$. $(R_{01}Q_{11})R_{00} \subset R_{01}R_{00}$. To handle $(R_{01}Q_{11})R_{01}$, we recall from Lemma 4-(i) that $R_{01} \subset R_{11} + R_{10}^2 + R_{00}R_{10}$, so that

$$(5) \qquad (R_{01}Q_{11})R_{01} \subset (R_{11}Q_{11})R_{01} + (]R_{10}^2]Q_{11})R_{01} + ([R_{00}R_{10}]Q_{11})R_{01} .$$

Next we shall work on each of the three terms in the right hand side of (5). Thus $(R_{11}Q_{11})R_{01} \subset Q_{11}R_{01} \subset R_{11}R_{01}$, or

$$(6) \qquad (R_{11}Q_{11})R_{01} \subset R_{11}R_{01} .$$

As previously noted, $(R_{10}^2)Q_{11} = R_{10}(Q_{11}R_{10}) \subset (Q_{11}R_{10})R_{10} + Q_{11}$, since $a_{10}b_{10} + b_{10}a_{10} \in Q_{11}$. Thus $([R_{10}^2]Q_{11})R_{01} \subset ([Q_{11}R_{10}]R_{10})R_{01} + Q_{11}R_{01}$. But by use of (4), $([Q_{11}R_{10}]R_{10})R_{01} \subset (Q_{11}R_{10})(R_{10}R_{01} + R_{01}R_{10}) + ([Q_{11}R_{10}]R_{01})R_{10} \subset (Q_{11}R_{10})R_{11} + (Q_{11}R_{10})R_{00} + ([Q_{11}R_{10}]R_{01})R_{10}$. We saw previously in the lemma that $(Q_{11}R_{10})R_{11} = 0$, $(Q_{11}R_{10})R_{00} \subset Q_{11}R_{10} + R_{11}R_{01}$, and $([Q_{11}R_{10}]R_{01}) \subset Q_{11}$. Thus $([Q_{11}R_{10}]R_{10})R_{01} \subset Q_{11}R_{10} + R_{11}R_{01}$. Putting together the various inclusions we see that

$$(7) \qquad ([R_{10}^2]Q_{11})R_{01} \subset Q_{11}R_{10} + R_{11}R_{01} .$$

Using (4) it follows that

$$(R_{00}R_{10})Q_{11} \subset R_{00}(R_{10}Q_{11} + Q_{11}R_{10}) + (R_{00}Q_{11})R_{10} \subset R_{00}(Q_{11}R_{10}) ,$$

because of the table. But then

$$\begin{aligned} ([R_{00}R_{10}]Q_{11})R_{01} &\subset (R_{00}[Q_{11}R_{10}])R_{01} \subset R_{00}([Q_{11}R_{10}]R_{01} + R_{01}[Q_{11}R_{10}]) \\ &+ (R_{00}R_{01})(Q_{11}R_{10}) \subset R_{00}(R_{10}R_{01}) + R_{00}(R_{01}Q_{11}R_{10}) \\ &+ R_{01}Q_{11}R_{10} \subset R_{00}(R_{01}Q_{11}R_{10}) + R_{01}Q_{11}R_{10} , \end{aligned}$$

using (4) and the table. Observe that

$$(R_{00}, R_{01}, R_{10}) \subset -(R_{00}, R_{10}, R_{01}) \subset (R_{00}R_{10})R_{01} \subset Q_{00} ,$$

using the right alternative identity and Lemma 8. Hence

$$R_{00}(R_{01}Q_{11}R_{10}) \subset (R_{00}R_{01})(Q_{11}R_{10}) + Q_{00} \subset R_{01}Q_{11}R_{10} + Q_{00} .$$

Now piecing together various inclusions we see that

$$(8) \qquad ([R_{00}R_{10}]Q_{11})R_{01} \subset R_{01}Q_{11}R_{10} + Q_{00} .$$

By combining (5), (6), (7), and (8) we now see that $(R_{01}Q_{11})R_{01} \subset R_{11}R_{01} + Q_{11}R_{10} + R_{01}Q_{11}R_{10} + Q_{00}$. $(R_{01}Q_{11}R_{10})R_{11} \subset R_{00}R_{11} = 0$. $(R_{01}Q_{11}R_{10})R_{10} \subset R_{00}R_{10}$. $(R_{01}Q_{11}R_{10})R_{00} = R_{00}(R_{01}Q_{11}R_{10})$, as a result of Lemma 6. However, in the process of establishing (8) we observed that $R_{00}(R_{01}Q_{11}R_{10}) \subset Q_{00} + R_{01}Q_{11}R_{10}$. Therefore $(R_{01}Q_{11}R_{10})R_{00} \subset Q_{00} + R_{01}Q_{11}R_{10}$. Because of (4),

$$\begin{aligned} (R_{01}Q_{11}R_{10})R_{01} &\subset (R_{01}Q_{11})(R_{10}R_{01} + R_{01}R_{10}) \\ &+ ([R_{01}Q_{11}]R_{01})R_{10} \subset (R_{01}Q_{11})R_{11} + (R_{01}Q_{11})R_{00} + ([R_{01}Q_{11}]R_{01})R_{10} . \end{aligned}$$

We established earlier in the proof that $(R_{01}Q_{11})R_{11} \subset R_{01}Q_{11}$, $(R_{01}Q_{11})R_{00} = 0$, $(R_{01}Q_{11})R_{01} \subset R_{11}R_{01} + Q_{11}R_{10} + R_{01}Q_{11}R_{10} + Q_{00}$. Hence

$$\begin{aligned} ([R_{01}Q_{11}]R_{01})R_{10} &\subset (R_{11}R_{01})R_{10} + (Q_{11}R_{10})R_{10} \\ &+ (R_{01}Q_{11}R_{10})R_{10} + Q_{00}R_{10} \subset Q_{11} + R_{01}Q_{11} + R_{00}R_{10} , \end{aligned}$$

again utilizing inclusions previously established in the proof. Therefore $(R_{01}Q_{11}R_{10})R_{01} \subset Q_{11} + R_{01}Q_{11} + R_{00}R_{10}$. We have now established half of the necessary inclusions for proving that Q is a right ideal of R. The others all follow from reversing subscripts. This completes the proof of the lemma.

LEMMA 10. *If Q of Lemma 9 is zero, then the table becomes the same as that for alternative rings.*

Proof. $Q = 0$ implies $R_{11}R_{01} = 0 = R_{00}R_{10}$, as well as $a_{10}^2 = 0 = b_{01}^2$. Now define $A = R_{01} + R_{01}^2 + (R_{01}^2)R_{01} + R_{01}R_{10}$. We now proceed to establish that A is a right ideal of R. $R_{01}R_{11} \subset R_{01}$. $R_{01}R_{10} \subset A$. $R_{01}R_{01} \subset A$. $R_{01}R_{00} = 0$, follows from the table. Because of (4), $(R_{01}^2)R_{11} \subset R_{01}(R_{01}R_{11} + R_{11}R_{01}) + (R_{01}R_{11})R_{01} \subset R_{01}^2$, while $(R_{01}^2)R_{01} \subset A$. Again using (4),

$$(R_{01}^2)R_{10} \subset R_{01}(R_{01}R_{10} + R_{10}R_{01}) + (R_{01}R_{10})R_{01} \subset R_{01}R_{00} + R_{01}R_{11} + R_{00}R_{01} \subset R_{01} ,$$

as a result of the table. Again utilizing (4),

$$(R_{01}^2)R_{00} \subset R_{01}(R_{01}R_{00} + R_{00}R_{01}) + (R_{01}R_{00})R_{01} \subset R_{01}^2 ,$$

because of the table. Again because of (4) and the table

$$([R_{01}^2]R_{01})R_{11} \subset (R_{01}^2)(R_{01}R_{11} + R_{11}R_{01}) + ([R_{01}^2]R_{11})R_{01} \subset (R_{01}^2)R_{01} + ([R_{01}^2]R_{11})R_{01} .$$

But we just established that $(R_{01}^2)R_{11} \subset R_{01}^2$ so that

$$([R_{01}^2]R_{11})R_{01} \subset (R_{01}^2)R_{01} .$$

Thence $([R_{01}^2]R_{01})R_{11} \subset (R_{01}^2)R_{01}$. Again because of (4),

$$([R_{01}^2]R_{01})R_{10} \subset (R_{01}^2)(R_{01}R_{10} + R_{10}R_{01}) + ([R_{01}^2]R_{10})R_{01} \subset (R_{01}^2)R_{00} + (R_{01}^2)R_{11} + ([R_{01}^2]R_{10})R_{01} .$$

But we already know that $(R_{01}^2)R_{00} \subset R_{01}^2$, $(R_{01}^2)R_{11} \subset R_{01}^2$ and $(R_{01}^2R_{10}) \subset R_{01}$, so that $([R_{01}^2]R_{10})R_{01} \subset R_{01}^2$. Hence $([R_{01}^2]R_{01})R_{10} \subset R_{01}^2$. Since $R_{01}^2 \subset R_{00} + R_{10}$ follows from the table and $R_{11}R_{01} = 0$,

$$([R_{01}^2]R_{01})R_{01} \subset ([R_{10} + R_{00}]R_{01})R_{01} \subset (R_{11} + R_{01})R_{01} \subset R_{01}^2 .$$

$$([R_{01}^2]R_{01})R_{00} \subset (R_{11} + R_{01})R_{00} = 0 . \qquad (R_{01}R_{10})R_{11} \subset R_{00}R_{11} = 0 .$$

$$(R_{01}R_{10})R_{10} \subset R_{00}R_{10} = 0 .$$

$$(R_{01}R_{10})R_{01} \subset R_{00}R_{01} \subset R_{01} .$$

Because of (4), $(R_{01}R_{10})R_{00} \subset R_{01}(R_{10}R_{00} + R_{00}R_{10}) + (R_{01}R_{00})R_{10} \subset R_{01}R_{10}$. Thus we have proved that A is a right ideal of R. If $A = 0$, then $R_{01} = 0$. But then we may verify directly that $B = R_{11} + R_{10}$ is a right ideal, for $R_{11}^2 \subset R_{11}$, $R_{11}R_{10} \subset R_{10}$, $R_{11}R_{00} = 0$, $R_{10}R_{11} = 0$, $R_{10}R_{10} \subset R_{11}$, $R_{10}R_{00} \subset R_{10}$. As $1 - e \notin B$, then $B = 0$, and $R_{00} = R$. Since $e \notin R_{00}$, this leads to a contradiction. Hence we cannot have $A = 0$. But then $A = R$. We recall that $A = R_{01} + R_{01}^2 + (R_{01}^2)R_{01} + R_{01}R_{10}$, while $R_{01}^2 \subset R_{10} + R_{00}$,

$$(R_{01}^2)R_{01} \subset (R_{10} + R_{00})R_{01} \subset R_{11} + R_{01}, R_{01}R_{10} \subset R_{00} .$$

But because the Peirce decomposition is direct, we must have $R_{10} \subset R_{01}^2 + R_{00}$. And thus $R_{10}^2 \subset (R_{01}^2)R_{10} + R_{00}R_{10} \subset (R_{01}^2)R_{10}$. But as a result of (4), $(R_{01}^2)R_{10} \subset R_{01}(R_{01}R_{10} + R_{10}R_{01}) + (R_{01}R_{10})R_{01} \subset R_{01}R_{00} + R_{01}R_{11} + R_{00}R_{01} \subset R_{01}$. We have shown that $R_{10}^2 \subset R_{01}$. By reversing subscripts we also obtain $R_{01}^2 \subset R_{10}$. This completes the proof of the lemma.

LEMMA 11. *$R_{01}Q_{11}R_{10} \subset Q_{00}$ and $R_{10}Q_{00}R_{01} \subset Q_{11}$.*

Proof. Note that $(x_{01}q_{11}y_{10})^2 = -(x_{01}q_{11}y_{10}, x_{01}, q_{11}y_{10}) + ([x_{01}q_{11}y_{10}]x_{01})(q_{11}y_{10})$. Using (3) with $a = x_{01}, b = q_{11}y_{10}, c = x_{01}$, we see that $([x_{01}q_{11}y_{10}]x_{01})(q_{11}y_{10}) = x_{01}\{([q_{11}y_{10}]x_{01})(q_{11}y_{10})\}$. However, $([q_{11}y_{10}]x_{01})(q_{11}y_{10}) = (q_{11}, y_{10}, x_{01})(q_{11}y_{10}) + (q_{11}[y_{10}x_{01}])(q_{11}y_{10})$. Since $q_{11}[y_{10}x_{01}] \in R_{11}$, and $(R_{11}, R_{11}, R_{10}) = 0$, we use Lemma 6 to obtain $(q_{11}[y_{10}x_{01}])(q_{11}y_{10}) = (q_{11}^2[y_{10}x_{01}])y_{10}$. Going back to an earlier equation, thus

$$([q_{11}y_{10}]x_{01})(q_{11}y_{10}) = (q_{11}, y_{10}, x_{01})(q_{11}y_{10}) + (q_{11}^2[y_{10}x_{01}])y_{10} ,$$

and hence

$$\begin{aligned}(x_{01}q_{11}y_{10})^2 = &-(x_{01}q_{11}y_{10}, x_{01}, q_{11}y_{10}) \\ &+ x_{01}\{(q_{11}, y_{10}, x_{01})(q_{11}y_{10})\} \\ &+ x_{01}\{(q_{11}^2[y_{10}x_{01}])y_{10}\} .\end{aligned} \tag{9}$$

We shall now establish that $x_{01}q_{11}y_{10} \in Q_{00}$, by induction on the degree of nilpotency of q_{11}. Start off by assuming $q_{11}^2 = 0$. Then (1), with $a = b = q_{11}, c = y_{10}, d = x_{01}$ yields $(q_{11}^2, y_{10}, x_{01}) + (q_{11}, q_{11}, y_{10}x_{01} - x_{01}y_{10}) = q_{11}(q_{11}, y_{10}, x_{01}) + (q_{11}, y_{10}, x_{01})q_{11} = 2q_{11}(q_{11}, y_{10}, x_{01})$ as a result of Lemma 6. However, the left hand side of the equation is zero, since Lemma 3 implies $(R_{11}, R_{11}, R_{11}) = 0$, and the table implies that $(R_{11}, R_{11}, R_{00}) = 0$. But then $2q_{11}(q_{11}, y_{10}, x_{01}) = 0$, and so $q_{11}(q_{11}, y_{10}, x_{01}) = 0 = (q_{11}, y_{10}, x_{01})q_{11}$. Now in the light of this we go back to (9), which may now be rewritten as $(x_{01}q_{11}y_{10})^2 = -(x_{01}q_{11}y_{10}, x_{01}, q_{11}y_{10})$. But $-(x_{01}q_{11}y_{10}, x_{01}, q_{11}y_{10}) = (x_{01}q_{11}y_{10}, q_{11}y_{10}, x_{01}) = ([x_{01}q_{11}y_{10}][q_{11}y_{10}])x_{01}$ as a result of the right alternative identity and the table. Moreover, such an element belongs to $(R_{00}R_{10})R_{01} \subset Q_{00}$, as a result of Lemma 8. Thus $(x_{01}q_{11}y_{10})^2 \in Q_{00}$. But then it is obvious that $x_{01}q_{11}y_{10} \in Q_{00}$.

Assume inductively that $x_{01}q_{11}y_{10} \in Q_{00}$ whenever the degree of nilpotency of q_{11} is $k < n$ and let us then consider the case when q_{11} has degree of nilpotency n. As before, the proof that

$$-(x_{01}q_{11}y_{10}, x_{01}q_{11}, y_{10}) \in Q_{00}$$

goes over. Now $(q_{11}, y_{10}, x_{01}) \in R_{11}$ so that $(q_{11}, y_{10}, x_{01})q_{11} \in R_{11}$. Lemmas 3 and 6 imply that R_{11} is both associative and commutative so that $[(q_{11}, y_{10}, x_{01})q_{11}]^2 = (q_{11}, y_{10}, x_{01})^2 q_{11}^2$. However, as a result of Lemma 8, we have $(q_{11}, y_{10}, x_{01})^2 = 0$, so that $(q_{11}, y_{10}, x_{01})q_{11} \in Q_{11}$ and its degree of nilpotency is 2. But then by the previous calculation, or for that matter by the induction hypothesis, it becomes clear that

$$x_{01}\{[(q_{11}, y_{10}, x_{01})q_{11}]y_{10}\} \in Q_{00} .$$

Since $(R_{11}, R_{11}, R_{10}) = 0$ has already been established,

$$x_{01}\{(q_{11}, y_{10}, x_{01})(q_{11}y_{10})\} \in Q_{00} .$$

Finally, $[q_{11}^2(y_{10}x_{01})]^{[(n+1)/2]} = 0$, and $[(n+1)/2] < n$ in our situation, so that we may use the induction hypothesis to obtain that

$$x_{01}\{(q_{11}^2[y_{10}x_{01}])y_{10}\} \in Q_{00} .$$

Now going back to (9) we see that $(x_{01}q_{11}y_{10})^2 \in Q_{00}$, since Q_{00} is closed under addition. But then $x_{01}q_{11}y_{10} \in Q_{00}$, as before and the induction is completed. This proves $R_{01}Q_{11}R_{10} \subset Q_{00}$. By reversing subscripts we obtain the second part. This completes the proof of the lemma.

LEMMA 12. *$(Q_{11}R_{01})R_{01} \subset Q_{11}$ and $(Q_{00}R_{10})R_{10} \subset Q_{00}$.*

Proof. Let $q_{11} \in Q_{11}$ and $a_{01}, b_{01}, x_{01}, y_{01} \in R_{01}$. Then because of (4), $(q_{11}a_{01})b_{01} = q_{11}(a_{01}b_{01} + b_{01}a_{01}) - (q_{11}b_{01})a_{01}$. But $q_{11}(a_{01}b_{01} + b_{01}a_{01}) \in R_{11}R_{00} = 0$. Thus

$$(q_{11}a_{01})b_{01} = -(q_{11}b_{01})a_{01} . \tag{10}$$

Now $([q_{11}x_{01}]y_{01})^2 = -([q_{11}x_{01}]y_{01}, q_{11}x_{01}, y_{01}) + (\{[q_{11}x_{01}]y_{01}\}\{q_{11}x_{01}\})y_{01}$. But as a result of Lemma 8, $-([q_{11}x_{01}]y_{01}, q_{11}x_{01}, y_{01}) \in (R_{11}, R_{10}, R_{01}) \subset Q_{11}$. Hence let $-([q_{11}x_{01}]y_{01}, q_{11}x_{01}, y_{01}) = q'_{11}$. On the other hand (3) implies that $(\{[q_{11}x_{01}]y_{01}\}\{q_{11}x_{01}\})y_{01} = (q_{11}x_{01})([y_{01}(q_{11}x_{01})]y_{01})$. But then apply (10) with $a_{01} = x_{01}$, $b_{01} = [y_{01}(q_{11}x_{01})]y_{01}$. Thus

$$\begin{aligned}(q_{11}x_{01})([y_{01}(q_{11}x_{01})]y_{01}) &= -(q_{11}\{[y_{01}(q_{11}x_{01})]y_{01}\})x_{01} \\ = (q_{11}, y_{01}(q_{11}x_{01}), y_{01})x_{01} &= [(q_{11}, q_{11}x_{01}, y_{01})y_{01}]x_{01}\end{aligned}$$

using (2). Now $([q_{11}x_{01}]y_{01})^2 = q'_{11} + [(q_{11}, q_{11}x_{01}, y_{01})y_{01}]x_{01}$. Let

$$(q_{11}, q_{11}x_{01}, y_{01}) = t_{11} .$$

Then as a result of Lemma 8, we have $t_{11}^2 = 0$, and

$$([q_{11}x_{01}]y_{01})^2 = q'_{11} + (t_{11}y_{01})x_{01} . \tag{11}$$

In (11), replace q_{11} by t_{11}. Then

$$(12) \qquad ([t_{11}x_{01}]y_{01})^2 = S'_{11} + [(t_{11}, t_{11}x_{01}, y_{01})y_{01}]x_{01} .$$

In an arbitrary ring one may verify the Teichmüller identity:

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z .$$

Hence let $w = x = t_{11}$, $y = x_{01}$, and $z = y_{01}$. Then

$$(t_{11}^2, x_{01}, y_{01}) - (t_{11}, t_{11}x_{01}, y_{01}) + (t_{11}, t_{11}, x_{01}y_{01})$$
$$= t_{11}(t_{11}, x_{01}, y_{01}) + (t_{11}, t_{11}, x_{01})y_{01} .$$

Since $t_{11}^2 = 0$, the first term of the left hand side vanishes. Since $(t_{11}, t_{11}, x_{01}y_{01}) \subset (R_{11}, R_{11}, R_{00} + R_{10}) = 0$, the third term of the left hand side also vanishes. From (1) it follows that $(t_{11}^2, x_{01}, y_{01}) + (t_{11}, t_{11}, (x_{01}, y_{01})) = t_{11}(t_{11}, x_{01}, y_{01}) + (t_{11}, x_{01}, y_{01})t_{11}$. But $t_{11}^2 = 0$, while

$$(t_{11}, t_{11}, (x_{01}, y_{01})) \in (R_{11}, R_{11}, R_{00} + R_{10}) = 0 ,$$

so the left hand side of the last equation is zero. If we let $(t_{11}x_{01})y_{01} = a_{11}$, and $t_{11}(x_{01}y_{01}) = b_{10}$, then $t_{11}(a_{11} - b_{10}) + (a_{11} - b_{10})t_{11} = 0$. But $t_{11}b_{10} = t_{11}(t_{11}[x_{01}y_{01}]) = -(t_{11}, t_{11}, x_{01}y_{01}) \in (R_{11}, R_{11}, R_{00} + R_{10}) = 0$, while $b_{10}t_{11} = 0$, from the table. Thus $t_{11}a_{11} + a_{11}t_{11} = 0$. Then from Lemma 6 we have $2t_{11}a_{11} = 0$, so that $t_{11}a_{11} = 0$. But then $t_{11}(t_{11}, x_{01}, y_{01}) = t_{11}(a_{11} - b_{10}) = 0$. Thus what remains from the Teichmüller identity is $-(t_{11}, t_{11}x_{01}, y_{01}) = (t_{11}, t_{11}, x_{01})y_{01}$. Substituting this into (12) we see that $([z_{11}x_{01}]y_{01})^2 = S'_{11} - \{[(t_{11}, t_{11}, x_{01})y_{01}]y_{01}\}x_{01} = S'_{11} - \{(t_{11}, t_{11}, x_{01})y_{01}^2\}x_{01}$, as a result of the right alternative identity. But $y_{01}^2 \in Q_{00}$, as a result of the table, while $(t_{11}, t_{11}, x_{01}) = -(t_{11}, x_{01}, t_{11}) \in R_{10}$. Thus $-\{(t_{11}, t_{11}, x_{01})y_{01}^2\}x_{01} \in R_{10}Q_{00}R_{01}$. But as a result of Lemma 11, $R_{10}Q_{00}R_{01} \subset Q_{11}$. Thus $([t_{11}x_{01}]y_{01})^2 \in Q_{11}$. But then $(t_{11}x_{01})y_{01} \in Q_{11}$. Now we may go back to (11) and obtain $([q_{11}x_{01}]y_{01})^2 \in Q_{11}$ and so $(q_{11}x_{01})y_{01} \in Q_{11}$. We have shown that

$$(Q_{11}R_{01})R_{01} \subset Q_{11} .$$

By reversing subscripts we obtain $(Q_{00}R_{10})R_{10} \subset Q_{00}$.
This completes the proof of the lemma.

LEMMA 13. *$S = Q_{11} + R_{01}Q_{11} + Q_{11}R_{10} + Q_{11}R_{01} + Q_{00} + R_{10}Q_{00} + Q_{00}R_{01} + Q_{00}R_{10}$, is a right ideal of R.*

Proof. We observe that Q, as defined in Lemma 9, has six of the eight terms appearing in S. Indeed we can extract the following inclusions directly from the proof of Lemma 9.

$$Q_{11}R_{11} \subset Q_{11},\ Q_{11}R_{00} = 0,\ (R_{01}Q_{11})R_{11} \subset R_{01}Q_{11} ,$$
$$(R_{01}Q_{11})R_{01} \subset Q_{11}R_{01} + Q_{11}R_{10} + R_{01}Q_{11}R_{10} + Q_{00},\ (R_{01}Q_{11})R_{00} = 0 ,$$
$$(Q_{11}R_{10})R_{11} = 0,\ (Q_{11}R_{10})R_{10} \subset Q_{11} + R_{01}Q_{11} ,$$
$$(Q_{11}R_{10})R_{01} \subset Q_{11},\ (Q_{11}R_{10})R_{00} \subset Q_{11}R_{10} + Q_{11}R_{01} .$$

Also because of Lemma 11, $R_{01}Q_{11}R_{10} \subset Q_{00}$, so that in fact

$$(R_{01}Q_{11})R_{01} \subset Q_{11}R_{01} + Q_{11}R_{10} + Q_{00} \subset S .$$

Besides, $Q_{11}R_{10} \subset S$, and $Q_{11}R_{01} \subset S$. Thus we have proved that

$$(Q_{11} + R_{01}Q_{11} + Q_{11}R_{10})R \subset S .$$

Then $(Q_{11}R_{01})R_{11} \subset R_{10}R_{11} = 0$, because of the table. $(Q_{11}R_{01})R_{10} \subset Q_{11}$, as a consequence of Lemma 8. As a result of Lemma 12, $(Q_{11}R_{01})R_{01} \subset Q_{11}$. Using (4) and the table,

$$(Q_{11}R_{01})R_{00} \subset Q_{11}(R_{01}R_{00} + R_{00}R_{01}) + (Q_{11}R_{00})R_{01} \subset Q_{11}R_{01} \subset S .$$

This completes half of the required number of inclusions. The remaining ones follow by reversing subscripts. This completes the proof of the lemma.

COROLLARY. $S = 0$.

Proof. Assume $S \neq 0$. Then it follows from the lemma that $S = R$. But then from the directness of the Peirce decomposition we must have $Q_{11} = R_{11}$. Since $e \notin Q_{11}$, while $e \in R_{11}$, we have reached a contradiction. Hence, $S = 0$.

LEMMA 14. $Q = 0$.

Proof. Suppose $Q \neq 0$. Then as a result of Lemma 9, $Q = R$. Since the corollary to Lemma 13 gives us $S = 0$, looking at Lemma 9 we see that $R = R_{11}R_{01} + (R_{11}R_{01})R_{01} + R_{00}R_{10} + (R_{00}R_{10})R_{10}$. Since the Peirce decomposition is direct, then $R_{11}R_{01} = R_{10}$. But from this it follows that $R_{10}R_{10} \subset (R_{11}R_{01})R_{10} \subset Q_{11}$, as a result of Lemma 8. But $Q_{11} \subset S = 0$, hence $R_{10}R_{10} = 0$. At this point form $U = R_{11} + R_{10}$. Then it follows from the table that $R_{11}R_{11} \subset R_{11}, R_{11}R_{10} \subset R_{10}, R_{11}R_{01} \subset R_{10}$, $R_{11}R_{00} = 0, R_{10}R_{11} = 0, R_{10}R_{10} = 0, R_{10}R_{01} \subset R_{11}, R_{10}R_{00} \subset R_{10}$, so that U must be a right ideal. If $U = R$, then $R_{00} = 0$, so since $1 - e \in R_{00}$, we would have $e = 1$, contrary to assumption. On the other hand if $U = 0$, then $e = 0$, also a contradiction. The contradiction was brought about by supposing $Q \neq 0$. Hence, $Q = 0$. This completes the proof of the lemma.

We are now ready to state and prove our main result.

THEOREM. *Let R be a right alternative ring without proper right ideals, of characteristic not two. Suppose that e, $1 \in R$, where e is an idempotent other than 1, such that $(e, e, R) = 0$. Then R must be alternative, hence a Cayley vector matrix algebra of dimension eight over its center.*

Proof. Combining Lemmas 14 and 10, it follows that the table must be the same as that for an alternative ring and that R_{00} and R_{11} have no nilpotent elements. Then it follows from the main theorem of [4] that R must be alternative. However, the reader can get by with proving only Lemmas 14, 15, and 17 of that paper, since Lemma 16 coincides with our Lemma 7. Once R is alternative, the main result of [2] makes R either associative or a Cayley vector matrix algebra. But R cannot be associative, for having an identity element and no proper right ideals force R to be a division ring, which in turn could not have an idempotent $e \neq 1$. This completes the proof of the theorem.

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ON THE SUBRING STRUCTURE OF FINITE NILPOTENT RINGS

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This paper studies the nilpotent ring analogues of several well-known results on finite p-groups. We first prove an analogue for finite nilpotent p-rings [a ring is called a p-ring if its additive group is a p-group] of the Burnside Basis Theorem, and use this to obtain some information on the automorphism groups of these rings. Next we obtain Anzahl results, showing that the number of subrings, right ideals, and two-sided ideals of a given order in a finite nilpotent p-ring is congruent to 1 mod p. Finally, we characterize the class of nilpotent p-rings which have a unique subring of a given order.

The analogy between nilpotent groups and nilpotent rings which motivates the results of this paper is the replacement of group commutation by ring product. A nilpotent ring, of course, is itself a group under the circle composition $x \circ y = x + y + xy$ but the structure of this group implies little about the invariants to be studied here, as shown by the examples in the last section of the paper.

All rings considered here are associative. The reader may verify, however, that all results of §§ 1–3 hold without the assumption of associativity, with the exception of (3.3). The unqualified word "ideal" means two-sided ideal. The letter p always denotes a prime number. If $\mathfrak{R}$ is a ring, we denote the additive group of $\mathfrak{R}$ by $\mathfrak{R}^+$. The *order* of a ring $\mathfrak{R}$, denoted $|\mathfrak{R}|$, is the order of the group $\mathfrak{R}^+$; the *index* of a subring $\mathfrak{S}$ in a ring $\mathfrak{R}$, denoted $[\mathfrak{R}: \mathfrak{S}]$, is the index of $\mathfrak{S}^+$ in $\mathfrak{R}^+$. A ring is called *null* if all products are 0. A ring $\mathfrak{R}$ is called *nilpotent* of *exponent* e if all products of e elements from $\mathfrak{R}$ are 0, but not all products of $e - 1$ elements are 0. The *characteristic* of a finite ring is the maximum of the additive orders of its elements. The smallest ideal containing ideals $\mathfrak{S}$ and $\mathfrak{T}$ is denoted $\mathfrak{S} + \mathfrak{T}$.

We shall need the following elementary results:

(1.1) Let $\mathfrak{R}$ be a ring with periodic additive group. The primary decomposition of $\mathfrak{R}^+$ decomposes $\mathfrak{R}$ into a ring direct sum of p-rings.

Hence, in studying finite rings, it is sufficient to consider only p-rings.

(1.2.) Let $\mathfrak{J}$ be a maximal ideal of a nilpotent p-ring $\mathfrak{R}$. Then $[\mathfrak{R}: \mathfrak{J}] = p$, $\mathfrak{R}^2 \subseteqq \mathfrak{J}$, and $p\mathfrak{R} \subseteqq \mathfrak{J}$.

(1.3) Let $\mathfrak{S}$ be a proper subring of a finite nilpotent ring $\mathfrak{R}$. Then there is a maximal ideal of $\mathfrak{R}$ which contains $\mathfrak{S}$.

(1.4) If $\mathfrak{M}$ and $\mathfrak{N}$ are nonzero, nonempty subsets of a nilpotent ring, then $\mathfrak{M}$ is not contained in $\{\mu\nu \mid \mu \in \mathfrak{M}, \nu \in \mathfrak{N}\}$.

(1.5) A nilpotent ring of order p^n contains an ideal of every possible order $p^i, 0 \leqq i \leqq n$.

2. Burnside Basis Theorem. The *Frattini subring* $\Phi_{\mathfrak{R}}$ of a ring $\mathfrak{R}$ is defined to be the intersection of the maximal ideals of $\mathfrak{R}$, provided such exist. Otherwise, $\Phi_{\mathfrak{R}} = \mathfrak{R}$. A set of elements of a ring $\mathfrak{R}$ *generates* a subring $\mathfrak{S}$ if $\mathfrak{S}$ is the smallest subring of $\mathfrak{R}$ containing all the elements.

THEOREM 2.1. *Let $\mathfrak{R}$ be a finite nilpotent p-ring. Then $\mathfrak{A} = \mathfrak{R}/\Phi_{\mathfrak{R}}$ is a null ring, and $\mathfrak{A}^+$ is elementary abelian. Let $[\mathfrak{R}: \Phi_{\mathfrak{R}}] = p^d$. Then any set of elements of $\mathfrak{R}$ which generates $\mathfrak{R}$ contains a subset of d elements, $\{\theta_1, \cdots, \theta_d\}$, which generates $\mathfrak{R}$. In the canonical homomorphism of $\mathfrak{R}$ onto $\mathfrak{A}$ the elements $\theta_1, \cdots, \theta_d$ map onto a basis of $\mathfrak{A}^+$. If, conversely, $\theta_1 + \Phi_{\mathfrak{R}}, \cdots, \theta_d + \Phi_{\mathfrak{R}}$ form a basis of $\mathfrak{A}^+$, then $\theta_1, \cdots, \theta_d$ generate $\mathfrak{R}$.*

Proof. By (1.2) $\mathfrak{A}$ is a null ring and $\mathfrak{A}^+$ is elementary abelian. Thus the images under the canonical homomorphism $\mathfrak{R} \to \mathfrak{A}$ of any generating set for $\mathfrak{R}$ must contain a basis for $\mathfrak{A}^+$. Let $\{\theta_1, \cdots, \theta_d\}$ be a set of elements whose images form a basis for $\mathfrak{A}^+$. Suppose $\theta_1, \cdots, \theta_d$ generate a proper subring of $\mathfrak{R}$. By (1.3) this subring is contained in a maximal ideal $\mathfrak{J}$, which contains $\Phi_{\mathfrak{R}}$. Thus

$$\theta_1 + \Phi_{\mathfrak{R}}, \cdots, \theta_d + \Phi_{\mathfrak{R}}$$

are in $\mathfrak{J}/\Phi_{\mathfrak{R}}$, which is proper in $\mathfrak{A}$. This contradicts the assumption that the images of $\theta_1, \cdots, \theta_d$ form a basis of $\mathfrak{A}^+$. This completes the proof.

REMARK 1. Theorem 2.1 implies that a finite nilpotent p-ring contains a unique maximal subring (= ideal) if and only if it is generated by a single element. A ring [an associative algebra] generated by one element we call a *power ring* [*power algebra*], since the additive group of the ring [the underlying vector space of the algebra] is spanned by the generator and its powers. Whereas a group generated by one element is completely determined by its order, the same is not true for power rings. In fact, even specification of the additive group and the exponent of the ring are not generally sufficient to determine a nilpotent power ring up to isomorphism. There is, of course, only one nilpotent power algebra of a given dimension over any field. Note, finally, that all nilpotent power rings [algebras] are finite [finite-dimensional].

REMARK 2. It is frequently convenient to use the observation that $\Phi_{\mathfrak{R}} = \mathfrak{R}^2 + p\mathfrak{R}$ for every finite nilpotent p-ring $\mathfrak{R}$. To prove this, observe that (1.2) implies $\mathfrak{R}^2 + p\mathfrak{R} \subseteqq \Phi_{\mathfrak{R}}$, while $\mathfrak{J} = \mathfrak{R}/(\mathfrak{R}^2 + p\mathfrak{R})$ is a null ring and $\mathfrak{J}^+$ is elementary abelian, so the intersection of the maximal subrings of $\mathfrak{J}$ is 0, which means $\Phi_{\mathfrak{J}} = 0$, so $\Phi_{\mathfrak{R}} \subseteqq \mathfrak{R}^2 + p\mathfrak{R}$.

As an application of Theorem 2.1, we shall now derive some information about the group of automorphisms of a finite nilpotent p-ring.

THEOREM 2.2. *Let $\mathfrak{R}$ be a nilpotent ring of order p^n, and let $[\mathfrak{R}: \Phi_{\mathfrak{R}}] = p^d$. Then the order of the automorphism group of $\mathfrak{R}$ divides $p^{d(n-d)}\Theta(p^d)$, where*

$$\Theta(p^d) = (p^d - 1)(p^d - p) \cdots (p^d - p^{d-1}) .$$

The order of the group of automorphisms of $\mathfrak{R}$ which fix $\mathfrak{R}/\Phi_{\mathfrak{R}}$ elementwise divides $p^{d(n-d)}$.

Proof. This result, due to P. Hall for p-groups, follows in the same way as § 1.3 of [2].

If $\mathfrak{J}$ is an ideal of a ring $\mathfrak{R}$, we now define $\mathrm{Aut}(\mathfrak{R}; \mathfrak{J})$ to be the group of all automorphisms of $\mathfrak{R}$ which leave $\mathfrak{R}/\mathfrak{J}$ fixed elementwise. For $\mathfrak{R}$ a finite nilpotent p-ring we shall obtain a bound on the class of the p-group $\mathscr{P} = \mathrm{Aut}(\mathfrak{R}; \Phi_{\mathfrak{R}})$. These results are analogues of those obtained by H. Liebeck [4] for p-groups.

THEOREM 2.3. *Let $\mathfrak{R}$ be a finite p-ring, nilpotent of exponent e, for which $\Phi_{\mathfrak{R}} \neq 0$. Let $\mathfrak{R}^i/\mathfrak{R}^{i+1}$ have characteristic p^{m_i}, $i = 1, \cdots, e-1$. Then the class of $\mathscr{P} = \mathrm{Aut}(\mathfrak{R}; \Phi_{\mathfrak{R}})$ does not exceed*

$$\lambda(\mathfrak{R}) = \left(\sum_{i=1}^{e-1} m_i\right) - 1 .$$

This theorem will follow by induction from the next result.

THEOREM 2.4. *Let $\mathfrak{R}$ be a finite p-ring, nilpotent of exponent e, for which $\Phi_{\mathfrak{R}} \neq 0$. Let $\mathfrak{R}^{e-1}$ have characteristic p^m and let $\mathfrak{N} = p^{m-1}\mathfrak{R}^{e-1}$. Then*

(i) *the ideal $\mathfrak{N}$ is elementwise fixed by $\mathscr{P} = \mathrm{Aut}(\mathfrak{R}; \Phi_{\mathfrak{R}})$.*

(ii) $\mathscr{Z} = \mathrm{Aut}\,(\mathfrak{R}; \mathfrak{N})$ *is in the center of* $\mathscr{P}$.

(iii) $\mathscr{Z}$ *has order* p^{rd}, *where* p^r *is the order of* $\mathfrak{N}$, *and* $p^d = [\mathfrak{R}: \Phi_{\mathfrak{R}}]$.

(iv) $\mathscr{P}/\mathscr{Z}$ *is isomorphic to the subgroup* $\mathscr{Q}$ *of automorphisms from* $\mathrm{Aut}\,(\mathfrak{R}/\mathfrak{N}; \Phi_{\mathfrak{R}}/\mathfrak{N})$ *which can be extended to* $\mathfrak{R}$.

The proof of (2.4) requires three lemmas, the first of which is obvious.

LEMMA 2.5. *Let* $\mathfrak{R}$ *be a ring and* α *an automorphism of* $\mathfrak{R}$. *If* $\{\theta_1, \cdots, \theta_d\}$ *is a generating set for* $\mathfrak{R}$, *then* α *is completely determined by the values of* $\theta_i^\alpha, 1 \leqq i \leqq d$. *If* $\mathfrak{J}$ *is an ideal of* $\mathfrak{R}$ *then* $\alpha \in \mathrm{Aut}\,(\mathfrak{R}; \mathfrak{J})$ *if and only if* $\theta_i^\alpha - \theta_i \in \mathfrak{J}, 1 \leqq i \leqq d$.

LEMMA 2.6. *With* $\mathfrak{R}$ *as in* (2.4), *if* $\alpha \in \mathscr{P}$ *and* $\theta \in \mathfrak{R}^i$ *for some* i, $1 \leqq i \leqq e - 1$, *then* $\theta^\alpha - \theta \in p\mathfrak{R}^i + \mathfrak{R}^{i+1}$.

Proof. The lemma is true for $i = 1$, since $\Phi_{\mathfrak{R}} = p\mathfrak{R} + \mathfrak{R}^2$ and $\alpha \in \mathrm{Aut}\,(\mathfrak{R}; \Phi_{\mathfrak{R}})$. Assume the result for $i < j$. Let $\theta \in \mathfrak{R}^j$. Express θ as a sum of products

$$\theta = \sum_r \pi_r \rho_r$$

where the $\pi_r \in \mathfrak{R}^{j-1}$, $\rho_r \in \mathfrak{R}$. Then $\theta^\alpha = \sum \pi_r^\alpha \rho_r^\alpha = \sum (\pi_r + \sigma_r)(\rho_r + \tau_r)$, where, by induction hypothesis, $\sigma_r \in p\mathfrak{R}^{j-1} + \mathfrak{R}^j$ and $\tau_r \in p\mathfrak{R} + \mathfrak{R}^2$. Thus $\theta^\alpha - \theta = \sum (\sigma_r \rho_r + \pi_r \tau_r + \sigma_r \tau_r) \in p\mathfrak{R}^j + \mathfrak{R}^{j+1}$.

LEMMA 2.7. *With the notation of* (2.4), *every automorphism* $\zeta \in \mathscr{Z}$ *leaves* $\Phi_{\mathfrak{R}}$ *elementwise fixed.*

Proof. For any $\theta \in \mathfrak{R}$, $\theta^\zeta - \theta \in \mathfrak{N}$. Thus from $p\mathfrak{N} = 0$ follows $0 = p(\theta^\zeta - \theta) = (p\theta)^\zeta - p\theta$, so $\mathscr{Z}$ fixes $p\mathfrak{R}$ elementwise. Similarly, since $\mathfrak{N}\mathfrak{R} = \mathfrak{R}\mathfrak{N} = 0$, $\mathscr{Z}$ fixes $\mathfrak{R}^2$, hence $\Phi_{\mathfrak{R}} = \mathfrak{R}^2 + p\mathfrak{R}$, elementwise.

Proof of (2.4). (i) Suppose $\theta \in \mathfrak{N}$ and $\alpha \in \mathscr{P}$. Then $\theta = p^{m-1}\psi$ for some $\psi \in \mathfrak{R}^{e-1}$, and so, by (2.6),

$$\theta^\alpha - \theta = p^{m-1}(\psi^\alpha - \psi) \in p^{m-1}(p\mathfrak{R}^{e-1} + \mathfrak{R}^e) = 0\,.$$

(ii) Let $\theta \in \mathfrak{R}, \alpha \in \mathscr{P}$, and $\zeta \in \mathscr{Z}$. Then

$$\begin{aligned}(\theta^\alpha)^\zeta &= (\theta + \rho)^\zeta \text{ for some } \rho \in \Phi_{\mathfrak{R}} \\ &= \theta^\zeta + \rho \text{ by (2.7)} \\ &= \theta + \rho + \sigma \text{ for some } \sigma \in \mathfrak{N}\,,\end{aligned}$$

while

$$(\theta^{\zeta})^{\alpha} = (\theta + \sigma)^{\alpha} = \theta^{\alpha} + \sigma[\text{by (i)}] = \theta + \sigma + \rho\ .$$

Thus $\alpha\zeta = \zeta\alpha$.

(iii) Let $\theta_1, \cdots, \theta_d$ generate $\mathfrak{R}$, and $\psi_1, \cdots \psi_d$ be arbitrary elements of $\mathfrak{N}$. Then the mapping

$$(*) \qquad \theta_i \rightarrow \theta_i + \psi_i,\ i = 1, \cdots, d\ ,$$

defines an automorphim $\zeta \in \mathscr{X}$. But every automorphism $\zeta \in \mathscr{X}$ induces a mapping of the form (*). Thus $|\mathscr{X}| = p^{rd}$.

(iv) Let $\alpha \in \mathscr{P}$. Defining $(\theta + \mathfrak{N})^{\alpha} = \theta^{\alpha} + \mathfrak{N}$ for all $\theta \in \mathfrak{R}$ gives, by (i), a homomorphism f of $\mathscr{P}$ into $\mathscr{Q}$. If $(\theta + \mathfrak{N})^{\alpha} = \theta + \mathfrak{N}$, all θ, then $\theta^{\alpha} - \theta \in \mathfrak{N}$, so $\alpha \in \mathscr{X}$. Thus the kernel of f is $\mathscr{X}$. Now consider an automorphism $\beta' \in \mathscr{Q}$. Let β be an extension of β' to $\mathfrak{R}$, $\beta \in \mathscr{P}$. Then $f\beta \in \mathscr{Q}$, and for $\theta \in \mathfrak{R}$, $(\theta + \mathfrak{N})^{f\beta} = \theta^{\beta} + \mathfrak{N} = (\theta + \mathfrak{N})^{\beta'}$ so the homomorphism f is onto $\mathscr{Q}$. This completes the proof of (2.4).

Proof of (2.3). $\lambda(\mathfrak{R}) = 0$ implies $e = 2$ and $m_1 = 1$, hence $\Phi_{\mathfrak{R}} = 0$, contrary to assumption. If $\lambda(\mathfrak{R}) = 1$, then either $e = 2, m_1 = 2$, or else $e = 3, m_1 = m_2 = 1$. In either case, for $\mathfrak{N} = p^{m_{e-1}-1}\mathfrak{R}^{e-1}$, $\mathfrak{N} = \Phi_{\mathfrak{R}}$. Hence, by (ii) of (2.4), $\mathscr{P}$ is abelian. We now use induction on $\lambda(\mathfrak{R})$. Let $\mathfrak{N} = p^{m_{e-1}-1}\mathfrak{R}^{e-1}$. By (2.4) $\mathscr{X} = \mathrm{Aut}\,(\mathfrak{R}; \mathfrak{N})$ is central in $\mathscr{P}$ and $\mathscr{P}/\mathscr{X}$ is isomorphic to a subgroup $\mathscr{Q}$ of $\mathrm{Aut}\,(\mathfrak{R}/\mathfrak{N}; \Phi_{\mathfrak{R}}/\mathfrak{N})$. Since $\lambda(\mathfrak{R}/\mathfrak{N}) = \lambda(\mathfrak{R}) - 1 < \lambda(\mathfrak{R})$, by induction hypothesis the class of $\mathscr{Q}$ does not exceed $\lambda(\mathfrak{R}/\mathfrak{N})$. Thus the class of $\mathscr{P}$ does not exceed $\lambda(\mathfrak{R}) = \lambda(\mathfrak{R}/\mathfrak{N}) + 1$.

REMARK. The bounds given in Theorems 2.2 and 2.3 are attained by the free nilpotent rings of characteristic p with two or more generators.

3. Enumeration results. The results of this section depend upon the following lemma, which is essentially the enumeration principle of Philip Hall ([2], Th. 1.4).

LEMMA 3.1. *Let $\mathfrak{U}$ be a finite p-group, $\mathscr{M}$ the set of maximal subgroups of $\mathfrak{U}$ which contain a fixed subgroup $\mathfrak{B} \neq \mathfrak{U}$. Let $\mathscr{C}$ be any class whose members are subsets of $\mathfrak{U}$, and let each member of $\mathscr{C}$ be contained in at least one member of $\mathscr{M}$. Let $n(M)$ be the number of members of $\mathscr{C}$ which are contained in M for each $M \in \mathscr{M}$. Then the number of members of $\mathscr{C}$ is congruent to $\sum_{M \in \mathscr{M}} n(M)$* (mod p).

THEOREM 3.2. *Let $\mathfrak{R}$ be a nilpotent ring of order p^n. Let $\mathfrak{S}$ be a subring of $\mathfrak{R}$, of order p^s. Then for $s \leqq t \leqq n$, the number of* *subrings of* $\mathfrak{R}$ *of order* p^t *which contain* $\mathfrak{S}$ *is congruent to* 1 (mod p).

Proof. If $\mathfrak{S} = \mathfrak{R}$, the result is trivial. Suppose $\mathfrak{S} \neq \mathfrak{R}$. We proceed by induction on n. Let $\mathscr{M}$ be the set of maximal subgroups of $\mathfrak{R}^+$ which contain $\mathfrak{B}^+ = (\mathfrak{S} + \mathfrak{R}^2)^+$. By (1.2) and (1.3), $\mathscr{M}$ is non-empty. Letting $\mathscr{C} = \{\mathfrak{S}\}$ in (3.1), we see that the number of members of $\mathscr{M}$ is congruent to 1 (mod p). If $t = n$, the result is trivial. Suppose $t < n$. Let $\mathscr{C} = \{\mathfrak{T} \mid \mathfrak{S} \subseteqq \mathfrak{T}, |\mathfrak{T}| = p^t, \mathfrak{T}$ is a subring of $\mathfrak{R}\}$. By (1.3) and (1.2) each $\mathfrak{T} \in \mathscr{C}$ is contained in some $M \in \mathscr{M}$. Let $n(M)$ be the number of members of $\mathscr{C}$ contained in M, for each $M \in \mathscr{M}$. By induction, $n(M) \equiv 1$ (mod p). Hence, by (3.1), the number of members of $\mathscr{C}$ is congruent to 1 (mod p).

It is well known that the number of normal subgroups of a given order in a finite p-group is congruent to 1 (mod p). For rings there are several analogous results.

THEOREM 3.3. *Let* $\mathfrak{B}$ *be a right module of order* p^n *of a nilpotent ring* $\mathfrak{R}$. *The number of submodules of* $\mathfrak{B}$ *of order* p^k, $0 \leqq k \leqq n$, *is congruent to* 1 (mod p).

THEOREM 3.4. *Let* $\mathfrak{J}$ *be a right ideal of order* p^m *of a nilpotent ring* $\mathfrak{R}$ *of order* p^n. *The number of right ideals of* $\mathfrak{R}$ *of order* p^k *which contain* $\mathfrak{J}$ (*which are contained in* $\mathfrak{J}$), $m \leqq k \leqq n$ $(0 \leqq k \leqq m)$, *is congruent to* 1 mod p.

THEOREM 3.5. *Let* $\mathfrak{J}$ *be a two-sided ideal of order* p^m *of a nilpotent ring* $\mathfrak{R}$ *of order* p^n. *The number of two-sided ideals of* $\mathfrak{R}$ *of order* p^k *which contain* $\mathfrak{J}$ (*which are contained in* $\mathfrak{J}$), $m \leqq k \leqq n$ $(0 \leqq k \leqq m)$, *is congruent to* 1 mod p.

The proofs of these results are similar to that of (3.2).

REMARK 1. No analogue of the theorem of Kulakoff seems to hold for nilpotent rings. For example, the rings with basis α, β, such that $p^2\alpha = p^2\beta = 0$, $\alpha^2 = -\beta^2 = p\alpha$, and $\alpha\beta = \beta\alpha = 0$, have $3p + 1$ subrings of order p^2 if $p \neq 2$, and 5 if $p = 2$.

REMARK 2. Note that the Anzahl theorems fail to hold for non-nilpotent p-rings. For example, consider the ring $\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2$, where $\mathfrak{R}_1$ is generated by an element α of characteristic p with $\alpha^2 = \alpha$, and $\mathfrak{R}_2$ is generated by an element β of characteristic p with $\beta^2 = 0$. Then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are the only two subrings (and ideals) of order p, and $2 \not\equiv 1$ (mod p) for any prime p.

4. Nilpotent p-rings with only one subring of a given order. It is well-known that a finite p-group $\mathfrak{G}$ which contains only one subgroup $\mathfrak{S}$ of a given order, $1 \neq \mathfrak{S} \neq \mathfrak{G}$, must be cyclic, or else $|\mathfrak{S}| = 2$ and $\mathfrak{G}$ is generalized quaternion [1; 131–132]. This section obtains a characterization of nilpotent rings and (associative) algebras satisfying the analogous condition. Although the algebra result could be obtained as a corollary to the ring result, we shall give an independent proof to illustrate the general ideas used while avoiding much detail required for the ring proof. The result for algebras is

THEOREM 4.1. *A nilpotent algebra $\mathfrak{U}$ over a field $\mathfrak{F}$ contains only one subalgebra $\mathfrak{S}$ of some given finite dimension, $0 \neq \mathfrak{S} \neq \mathfrak{U}$, if and only if one of the following conditions holds:*

(1) $\dim \mathfrak{S} = \dim \mathfrak{U} - 1$ *and $\mathfrak{U}$ is a power algebra.*

(2) $\dim \mathfrak{S} = 1, \dim \mathfrak{U} \geqq 3, \mathfrak{U}^2 = \mathfrak{S}, \mathfrak{U}^3 = 0$, *and $\varphi \in \mathfrak{U}, \varphi^2 = 0$ implies $\varphi \in \mathfrak{S}$.*

REMARK 1. An algebra $\mathfrak{U}$ such that $\dim \mathfrak{U}^2 = 1, \mathfrak{U}^3 = 0$, and $\varphi \in \mathfrak{U}$, $\varphi^2 = 0$ implies $\varphi\mathfrak{U} = \mathfrak{U}\varphi = 0$, is called "almost-null." It may be of interest to note that a nil algebra has the property that every subalgebra is an ideal if and only if the algebra is almost-null (see Kruse [3]). Thus almost-null algebras seem in one way analogous to the quaternion group of order 8, which plays a key role both in the determination of p-groups with a unique subgroup of order p, and in the determination of groups in which all subgroups are normal.

REMARK 2. The classification of the finite-dimensional algebras $\mathfrak{U}$ over a field $\mathfrak{F}$ satisfying (2) of (4.1) is closely related to the study of quadratic forms over $\mathfrak{F}$. Let $\mathfrak{U}$ have a basis $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta\}$ with $\beta \in \mathfrak{U}^2$, and choose $a_{ij} \in \mathfrak{F}, 1 \leqq i, j \leqq n$, so that $\alpha_i\alpha_j = a_{ij}\beta$. Then condition (2) requires that the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j$$

have no nontrivial zero $(x_1, x_2, \cdots, x_n)$. Let us note that when $\mathfrak{F}$ is a finite field, then every quadratic form in three variables has a nontrivial zero, so if $\mathfrak{F}$ is finite then $\dim \mathfrak{U} = 3$. On the other hand, over each finite field there exists a quadratic form in two variables with no nontrivial zero, so algebras satisfying (2) always occur when $\mathfrak{F}$ is finite.

Finally, we note that an arbitrary almost-null algebra must either be null, or isomorphic to the direct sum of a null algebra and either a power algebra of dimension 2 or an algebra satisfying (2).

Proof of (4.1). It is easy to check that nilpotent algebras satisfying (1) or (2) have unique subalgebras of the dimensions indicated. For the converse we shall first establish two lemmas.

LEMMA 4.2. *If* $\mathfrak{A}$ *is a nilpotent algebra of dimension* 4, *over a field* $\mathfrak{F}$, *then* $\mathfrak{A}$ *has more than one subalgebra of dimension* 2.

Proof. If $\dim \mathfrak{A}^2 = 3$, then $\mathfrak{A}$ is a power algebra, generated by one element α. Then all subalgebras generated by $\alpha^2 + x\alpha^3$ for different $x \in \mathfrak{F}$ are distinct and all have dimension 2. If $\dim \mathfrak{A}^2 \leqq 1$ then $\mathfrak{A}/\mathfrak{A}^2$ is null, and $\mathfrak{A}$ contains more than one subalgebra of dimension 2. Thus we can suppose that $\dim \mathfrak{A}^2 = 2$, and $\mathfrak{A}^2$ is the only subalgebra of $\mathfrak{A}$ of dimension 2. It follows, for any $\varphi \in \mathfrak{A}, \varphi \notin \mathfrak{A}^2$, that $\varphi^3 \neq 0$. Since $\dim \mathfrak{A}^2 = 2$, there are elements $\alpha, \beta \in \mathfrak{A}$ which are linearly independent mod $\mathfrak{A}^2$, so $\{\alpha, \beta, \alpha^2, \alpha^3\}$ is a basis for $\mathfrak{A}$. Choose $x, y \in \mathfrak{F}$ so that $\alpha\beta = x\alpha^2 + y\alpha^3$ and let $\beta' = \beta - x\alpha - y\alpha^2$. Then $\beta' \notin \mathfrak{A}^2$, and $\alpha\beta' = 0$. Let $\beta'^2 = u\alpha^2 + v\alpha^3, u, v \in \mathfrak{F}$. Then $0 = (\alpha\beta')\beta' = u\alpha^3$ so $u = 0$. Then $\beta'^3 = 0$ and $\beta' \notin \mathfrak{A}^2$, a contradiction.

LEMMA 4.3. *Let* $\mathfrak{A}$ *be a nilpotent algebra with a unique subalgebra* $\mathfrak{S}$ *of dimension* 1, *and let* $\varphi \in \mathfrak{A}$. *If* $\varphi^2 = 0$ *then* $\varphi \in \mathfrak{S}$. *If* $\varphi \notin \mathfrak{S}$, *then* $0 \neq \varphi^2 \in \mathfrak{S}$.

Proof. If $\varphi^2 = 0$ then either $\varphi = 0$ or φ generates a subalgebra of dimension 1, which by hypothesis must be $\mathfrak{S}$. Thus $\varphi \in \mathfrak{S}$. Suppose $\varphi \notin \mathfrak{S}$, and let e be the natural number such that $\varphi^e = 0$ but $\varphi^{e-1} \neq 0$. By the above argument $e \geqq 3$. If $e \geqq 4$ then $(\varphi^{e-2})^2 = 0$ so $\varphi^{e-2} \in \mathfrak{S}$. But $\mathfrak{S}$ is an ideal, so nilpotence of $\mathfrak{A}$ implies $\mathfrak{A}\mathfrak{S} = 0$, so $\varphi\varphi^{e-2} = 0$, contradicting the definition of e. Thus $e = 3$. Then $(\varphi^2)^2 = 0$ so $0 \neq \varphi^2 \in \mathfrak{S}$.

Proof of 4.1, *continued.* Let $\mathfrak{A}$ be a nilpotent algebra with a unique subalgebra $\mathfrak{S}$ of a given dimension, $0 \neq \mathfrak{S} \neq \mathfrak{A}$. If $\dim \mathfrak{A} = \dim \mathfrak{S} + 1$, then $\mathfrak{A}$ is a power algebra, and condition (1) of the conclusion holds. If $\dim \mathfrak{A} \geqq \dim \mathfrak{S} + 2 \geqq 4$ then, by the algebra analogue of (1.5), $\mathfrak{A}$ contains a subalgebra $\mathfrak{B}$ with $\dim \mathfrak{B} = \dim \mathfrak{S} + 2$, and an ideal $\mathfrak{J}$ with $\dim \mathfrak{J} = \dim \mathfrak{S} - 2$. Then the algebra $\mathfrak{B}/\mathfrak{J}$ fails to satisfy (4.2). Hence we can suppose $\dim \mathfrak{S} = 1, \dim \mathfrak{A} \geqq 3$.

Next we show that $\mathfrak{A}^2 = \mathfrak{S}$. Choose $\varphi, \psi \in \mathfrak{A}$. By (4.3), φ^2, ψ^2, and $(\varphi + \psi)^2$ are in $\mathfrak{S}$. Thus $\varphi\psi + \psi\varphi \in \mathfrak{S}$. Hence $0 = \varphi(\varphi\psi + \psi\varphi) = \varphi\psi\varphi$, since $\varphi^2 \in \mathfrak{S}$ and $\mathfrak{S}\mathfrak{A} = 0$. Thus $(\varphi\psi)^2 = 0$, so by (4.3) $\varphi\psi \in \mathfrak{S}$. Thus $\mathfrak{A}^2 \subseteqq \mathfrak{S}$. $\mathfrak{A}^2 \neq 0$ is trivial, so $\mathfrak{A}^2 = \mathfrak{S}$. (4.3) now implies directly that $\mathfrak{A}$ satisfies (2) of (4.1). Thus the proof is complete.

We now turn to the analogous problem for rings. We shall establish the following:

THEOREM 4.4. *A nilpotent p-ring* $\mathfrak{R}$ *contains only one subring* $\mathfrak{S}$ *of a given order,* $0 \neq \mathfrak{S} \neq \mathfrak{R}$, *if and only if* $\mathfrak{R}$ *and* $\mathfrak{S}$ *satisfy one of the following conditions*:

(1) $\mathfrak{R}^+$ *is cyclic or quasi-cyclic.*

(2) $[\mathfrak{R}:\mathfrak{S}] = p$. $\mathfrak{R}$ *is a power ring.*

(3) $|\mathfrak{S}| = p$. *Let* $\mathfrak{U} = \{\varphi \in \mathfrak{R} \mid p\varphi = 0\}$. *Then* $\mathfrak{U}^+$ *has rank* 2 *or* 3, $\mathfrak{U}^2 = \mathfrak{S}$, *and* $\varphi \in \mathfrak{U}, \varphi^2 = 0$ *implies* $\varphi \in \mathfrak{S}$. *There is, moreover, an ideal* $\mathfrak{C}$ *of* $\mathfrak{R}$ *such that* $\mathfrak{R} = \mathfrak{C} + \mathfrak{U}, \mathfrak{C} \cap \mathfrak{U} = \mathfrak{S}$, *and* $\mathfrak{C}^+$ *is cyclic or quasi-cyclic.*

(4) $|\mathfrak{S}| = p^2$. $|\mathfrak{R}| = p^4, \mathfrak{R}^+$ *has type* (2,2), *and, if* $\varphi \in \mathfrak{R}$ *with* $p\varphi \neq 0$, *then* $p\varphi^2 = 0$ *and* φ^2 *is not a natural multiple of* φ.

REMARK. A description of the "exceptional" nilpotent p-ring $\mathfrak{R}$ satisfying (3) or (4) may be completed in terms of generators and relations as follows:

(3) Let $\mathfrak{S}$ be generated by an element σ. Then $p\sigma = 0$ and $\sigma\mathfrak{R} = \mathfrak{R}\sigma = 0$. If $\mathfrak{C}^+$ is quasi-cyclic, then $\mathfrak{C}\mathfrak{R} = \mathfrak{R}\mathfrak{C} = 0$. The subring $\mathfrak{U}$ satisfies one of the following conditions:

(a) $\mathfrak{U}^+$ has a basis σ, β, and $\beta^2 \neq 0$.

(b) $\mathfrak{U}^+$ has a basis σ, β_1, β_2. Let $\beta_i\beta_j = B_{ij}\sigma$ for suitable integers B_{ij}, $i, j = 1, 2$. Then $B_{11}X^2 + (B_{12} + B_{21})XY + B_{22}Y^2 \equiv 0 \pmod p$ for integers X and Y implies $X \equiv Y \equiv 0 \pmod p$.

(4) Let $\mathfrak{R}^+$ have a basis α_1 and α_2. Then $\alpha_i\alpha_j = A_{ij}p\alpha_1 + B_{ij}p\alpha_2$ for suitable integers A_{ij}, B_{ij}, $i, j = 1, 2$, and

$$B_{11}X^3 + (A_{11} + B_{12} + B_{21})X^2 + (B_{22} + A_{12} + A_{21})X + A_{22} \equiv 0 \pmod p$$

has no integer solution X.

Since, over the field of p elements, there are both quadratic forms in two variables which have no nontrivial zeroes, and irreducible cubic polynomials, rings satisfying (3) and (4) occur nontrivially for all primes p.

Proof of 4.4. It is easy to see that nilpotent p-rings satisfying (1)–(4) have unique subrings of the orders indicated. An infinite nilpotent p-ring which contains only one subring $\mathfrak{S}$ of a given (finite) order clearly satisfies one of (1)–(4) if and only if each of its finite subrings which properly contains $\mathfrak{S}$ also does. Thus for the converse we consider only finite rings. As a notational convenience, let $\mathscr{U}(n, s)$ denote the class of nilpotent rings of order p^n which contain only one subring, generically denoted $\mathfrak{S}$, of order p^s. If $\mathfrak{R} \in \mathscr{U}(n, n-1)$, then the basis theorem (2.1) implies $\mathfrak{R}$ is a power ring. The rings in $\mathscr{U}(n, 1)$

are studied in § 5. To characterize the rings in $\mathscr{U}(n, s), 1 < s < n-1$, we first determine those in $\mathscr{U}(4, 2)$, $\mathscr{U}(5, 2)$, and $\mathscr{U}(5, 3)$ and then proceed by induction. Several steps of the proof are separated as lemmas.

4.5 *Let $\mathfrak{R}$ be a nilpotent p-ring for which $\mathfrak{R}^+$ has type $(n, 1)$. Then, for $1 < i < n$, $\mathfrak{R}$ has exactly $p + 1$ ideals of order p^i.*

Proof. For $1 < i \leqq n + 1$, $\mathfrak{B}_i = \{\varphi \in \mathfrak{R} \mid p^{i-1}\varphi = 0\}$ is an ideal of $\mathfrak{R}$ of order p^i. For $1 \leqq i < n$, $\mathfrak{C}_i = \{p^{n-i}\varphi \mid \varphi \in \mathfrak{R}\}$ is an ideal of $\mathfrak{R}$ of order p^i. Hence $\mathfrak{R}$ has at least two, so by the Anzahl theorem (3.5) at least $p + 1$, ideals of order $p^i, 1 < i < n$. But these exhaust the subgroups of $\mathfrak{R}^+$ of order p^i.

4.6 *Suppose $\mathfrak{R}$ is a power ring, $\mathfrak{R} \in \mathscr{U}(n, s), 1 \leqq s < n - 1$. Then $\mathfrak{R}^+$ is cyclic.*

Proof. First suppose $s = 1$. Let $\mathfrak{R}$ be generated by an element α, and let $\Phi = p\mathfrak{R} + \mathfrak{R}^2$. Let $\mathfrak{M} = \{\varphi \in \Phi \mid p\varphi = 0, \varphi \notin \mathfrak{S}\}$. If $\mathfrak{M}$ is nonempty, then by (1.4) there is some $\delta \in \mathfrak{M}$ such that $\delta\alpha \in \mathfrak{S}$. From $\delta \in \Phi$ follows $\delta = p\psi + \alpha\xi$, some $\psi, \xi \in \mathfrak{R}$. Then $\delta^2 = \delta(p\psi + \alpha\xi) = 0$, so δ generates a second subring of order p. Thus $\mathfrak{M}$ is empty and Φ^+ is cyclic. By (2.1), $[\mathfrak{R}: \Phi] = p$. If $\mathfrak{R}^+$ had type $(n - 1, 1)$, then $\Phi = \{\varphi \in \mathfrak{R} \mid p^{n-2}\varphi = 0\}$, so Φ^+ would not be cyclic. Hence $\mathfrak{R}^+$ is cyclic, as desired.

We now proceed by induction on s. Suppose $s > 1$. Let $\mathfrak{J}$ be an ideal of order p of $\mathfrak{R}$. Applying the induction hypothesis to the power ring $\mathfrak{R}/\mathfrak{J}$ we find that $(\mathfrak{R}/\mathfrak{J})^+$ is cyclic. Hence either $\mathfrak{R}^+$ is cyclic or has type $(n - 1, 1)$. But type $(n - 1, 1)$ is excluded by (4.5).

4.7 *If $\mathfrak{R} \in \mathscr{U}(4, 2)$, then rank $\mathfrak{R}^+ \leqq 2$.*

Proof. $\mathfrak{R}^+$ cannot have rank 4 by Lemma 4.2, where $\mathfrak{F} = GF(p)$, the field of p elements. Suppose $\mathfrak{R}^+$ has rank 3, so $\mathfrak{R}^+$ has type $(2, 1, 1)$. Let $\mathfrak{T} = \{\varphi \in \mathfrak{R} \mid p\varphi = 0\}$. Since $|\mathfrak{T}| = p^3, \mathfrak{T}$ contains $\mathfrak{S}$, the unique subring of order p^2. It follows by (2.1) that $\mathfrak{T}$ is a power algebra over $GF(p)$, and so $\mathfrak{R}^+$ has a basis of the form $\{\varphi, \psi, \psi^2\}$ where φ has characteristic p^2, ψ and ψ^2 have characteristic p, and $\psi^3 = p\varphi$. Since $\psi^2 \in \mathfrak{R}^2$ and $\psi^3 \in \mathfrak{R}^2$, $|\mathfrak{R}^2| \geqq p^2$. By (4.6) $\mathfrak{R}$ is not a power ring, so $|\mathfrak{R}^2| \leqq p^2$. Thus $\mathfrak{R}^2 = \mathfrak{S}$, and $\mathfrak{S}^+$ has a basis $\{\psi^2, \psi^3\}$. Hence there are integers A, B such that $\varphi\psi = A\psi^2 + B\psi^3$. Let $\varphi' = \varphi - A\psi - B\psi^2$. Then $\varphi'\psi = 0$. Let $\varphi'^2 = C\psi^2 + D\psi^3$ for suitable integers C, D. Then $0 = \varphi'(\varphi'\psi) = \varphi'^2\psi = C\psi^3$ so $C \equiv 0 \pmod{p}$. Then $\varphi'^2 = Dp\varphi'$, so φ' generates a second subring of order p^2. Thus $\mathfrak{R}^+$ has rank at most 2.

4.8 *If* $\mathfrak{R} \in \mathscr{U}(5, s)$, $s = 2, 3$, *then* $\mathfrak{R}^+$ *is cyclic.*

Proof. Suppose rank $\mathfrak{R}^+ \geqq 3$. Then, for $s = 2$ (resp. for $s = 3$), we can find a subring $\mathfrak{U}$ of order p^4 (resp. an ideal $\mathfrak{J}$ of index p^4) with rank $\mathfrak{U}^+ \geqq 3$ (resp. rank $(\mathfrak{R}/\mathfrak{J})^+ \geqq 3$). This is impossible by (4.7), so rank $\mathfrak{R}^+ \leqq 2$.

Suppose rank $\mathfrak{R}^+ = 2$. By (4.5) $\mathfrak{R}^+$ has type $(3, 2)$. Let α and β, with $p^3\alpha = p^2\beta = 0$, be a basis for $\mathfrak{R}^+$. Since $p\alpha$ and $\{p^2\alpha, p\beta\}$ generate distinct subrings of order p^2, we have $s = 3$. Then $p\mathfrak{R} = \mathfrak{S}$. Moreover, $\mathfrak{R}^2 = \mathfrak{S}$, since otherwise $\mathfrak{R}$ is a power ring, which is excluded by (4.6). Let $\alpha^2 = Ap\alpha + Bp\beta, \beta^2 = Cp\alpha + Dp\beta, C \not\equiv 0 \pmod p$, $\alpha\beta = Ep\alpha + Fp\beta$. By replacing α by $\alpha' = \alpha - EC^{-1}\beta$, where $C^{-1}C \equiv 1$ (mod p^2), we may assume that $E = 0$. Then $(\alpha^2)\beta = BCp^2\alpha$, while $\alpha(\alpha\beta) = 0$. Thus $B \equiv 0 \pmod p$, and α generates a second subring of order p^3.

Proof of (4.4), *continued.* If $\mathfrak{R} \in \mathscr{U}(4, 2)$, then by (4.5) and (4.7) either $\mathfrak{R}^+$ is cyclic or has type $(2, 2)$. If $\mathfrak{R}^+$ has type $(2, 2)$ and, for some $\varphi \in \mathfrak{R}$, $p\varphi \neq 0$ and φ^2 is a multiple of φ, then both $p\mathfrak{R}$ and the subring generated by φ have order p^2. Thus (4) of (4.4) holds.

Suppose $\mathfrak{R} \in \mathscr{U}(n, s)$ with $n > 5, 1 < s < n - 1$. If $s = 2$ and rank $\mathfrak{R}^+ \geqq 2$, we can find a subring of order p^5 and rank $\geqq 2$, which contradicts (4.8). For $s > 2$ we proceed by induction on n. Let $\mathfrak{J}$ be an ideal of $\mathfrak{R}$ of order p. By induction hypothesis $(\mathfrak{R}/\mathfrak{J})^+$ is cyclic. $\mathfrak{R}^+$ cannot have type $(n - 1, 1)$ by (4.5), hence $\mathfrak{R}^+$ is cyclic.

5. Nilpotent p-rings with one subring of order p. In this section we shall show that a finite nilpotent p-ring $\mathfrak{R}$ which contains a unique subring $\mathfrak{S}$ of order p satisfies condition (3) of Theorem 4.4. Let $\mathfrak{S}$ be generated by an element σ. Then $p\sigma = 0$ and $\sigma\mathfrak{R} = \mathfrak{R}\sigma = 0$. Small Greek letters will denote elements of $\mathfrak{R}$. For ease of reference we restate the hypothesis that $\mathfrak{S}$ is the only subring of order p.

5.1 *If* $p\varphi = \varphi^2 = 0$, *then* $\varphi \in \mathfrak{S}$.

5.2 *Suppose* $p^a\varphi = 0$ *and* $p^{a-1}\varphi \notin \mathfrak{S}, a \geqq 1$. *Then* $a = 1$ *and* $\varphi^2 \in \mathfrak{S}$, $\varphi^2 \neq 0$.

Proof. By (5.1), $(p^{a-1}\varphi)^2 \neq 0$. This, with $p^a\varphi^2 = 0$, gives $2a - 2 < a$, so $a = 1$. Then φ together with $\mathfrak{S}$ generates an algebra over $GF(p)$, so (4.3) implies $\varphi^2 \in \mathfrak{S}, \varphi^2 \neq 0$.

LEMMA 5.3. *Let* a_1, a_2, a_3, *and* b *be elements of a ring such that* $pb = 0$ *and there are integers* $A_{ij}, 0 \leqq A_{ij} < p, i, j = 1, 2, 3$, *such that*

$a_i a_j = A_{ij} b$. *Then there exist integers* $0 \leqq X_i < p$, $i = 1, 2, 3$, *not all* 0, *such that*

$$(X_1 a_1 + X_2 a_2 + X_3 a_3)^2 = 0 .$$

Proof. This is equivalent to the well-known fact that a quadratic form in three variables over the field of p elements represents 0 nontrivially.

LEMMA 5.4. *Let* $\mathfrak{U} = \{\varphi \in \mathfrak{R} \mid p\varphi = 0\}$. *Then* $\mathfrak{U}^2 \subseteqq \mathfrak{S}$ *and one of the following conditions holds, according to the rank of* $\mathfrak{U}^+$:

(1) $\mathfrak{U} = \mathfrak{S}$.

(2) $\mathfrak{U}^+$ *has a basis* σ, β, *and* $\beta^2 \neq 0$.

(3) $\mathfrak{U}^+$ *has a basis* σ, β, γ, *and* $(X\beta + Y\gamma)^2 = 0$ *for integers* X *and* Y *implies* $X \equiv Y \equiv 0 \pmod{p}$.

Proof. Since $\mathfrak{S} \subseteqq \mathfrak{U}$, $\mathfrak{U}$ is an algebra over $GF(p)$ with a unique subalgebra of dimension 1. The result follows directly from (4.1) and (5.3).

In case $p\mathfrak{R} = 0$ we have $\mathfrak{R} = \mathfrak{U}$, and thus $\mathfrak{R}$ satisfies (3) of (4.4). If $p\mathfrak{R} \neq 0$, then, by (5.2), $\mathfrak{R}^+ = \mathfrak{C}^+ + \mathfrak{U}^+$ where $\mathfrak{C}^+$ is a cyclic p-group with $|\mathfrak{C}^+| > p$, and $\mathfrak{C}^+ \cap \mathfrak{U}^+ = \mathfrak{S}^+$. The rest of the proof is devoted to showing that $(\mathfrak{R}^2)^+ \subseteqq p\mathfrak{C}^+$. This implies that the set of elements of $\mathfrak{C}^+$ form a subring $\mathfrak{C}$, and thus (3) of (4.4) holds. Let α be a generator of $\mathfrak{C}^+$. The proof that $(\mathfrak{R}^2)^+ \subseteqq p\mathfrak{C}^+$ is divided into cases depending on the location of α^2 and on the rank of $\mathfrak{U}^+$, which of course equals the rank of $\mathfrak{R}^+$.

If $\mathfrak{U}^+$ has rank 1, then $\mathfrak{C} = \mathfrak{R}$, so (1) of (4.4) holds. Suppose $\mathfrak{U}^+$ has rank 2, with basis σ, β. If $\alpha^2 \notin p\mathfrak{C}^+$, then $(\mathfrak{R}^2)^+$ has rank 2 so (4.6) applies. Thus $\alpha^2 \in p\mathfrak{C}^+$. By (5.4) $\mathfrak{U}^2 \subseteqq \mathfrak{S}$, and $\mathfrak{S}^+ \subseteqq p\mathfrak{C}^+$. Finally, $\alpha\beta \in \mathfrak{S}$ and $\beta\alpha \in \mathfrak{S}$ by the nilpotence of α.

Thus we may assume that $\mathfrak{U}^+$ has rank 3, with basis σ, β, γ. If $(\mathfrak{R}^2)^+$ has rank 1, we are done. If $(\mathfrak{R}^2)^+$ has rank 3, then (4.6) applies. Thus assume $(\mathfrak{R}^2)^+$ has rank 2. Without loss of generality we may asssume $\beta \in \mathfrak{R}^2$. To complete the proof we make use of the following remark:

5.5 *Under the above assumptions, if* $\varphi \in \mathfrak{R}^2$, $p\varphi = 0$, *and* $\varphi\beta = 0$, *then* $\varphi \in \mathfrak{S}$.

Proof. Since $(\mathfrak{R}^2)^+$ has rank 2 and $\sigma, \beta \in \mathfrak{R}^2$, it follows that $\varphi =$

$X\sigma + Y\beta$, some integers X and Y. Thus $\varphi\beta = Y\beta^2$. Since $\beta^2 \neq 0$, $Y \equiv 0 \pmod p$. Thus $\varphi \in \mathfrak{S}$.

We now continue the proof. Since $\beta^2 \in \mathfrak{S}$, $0 = \alpha\beta^2 = (\alpha\beta)\beta$ so by (5.5) $\alpha\beta \in \mathfrak{S}$. Dually $\beta\alpha \in \mathfrak{S}$. By (5.4) $\mathfrak{U}^2 \subseteqq \mathfrak{S}$. Since $\gamma\beta \in \mathfrak{S}$, $0 = \alpha(\gamma\beta) = (\alpha\gamma)\beta$ so, by (5.5), $\alpha\gamma \in \mathfrak{S}$. Dually $\gamma\alpha \in \mathfrak{S}$. Since $\alpha\beta \in \mathfrak{S}$, $0 = \alpha(\alpha\beta) = \alpha^2\beta$. Thus $\mathfrak{R}^2\beta = 0$. Choose any $\varphi \in \mathfrak{R}^2$. Since $(\mathfrak{R}^2)^+$ has rank 2, and $\beta \in \mathfrak{R}^2$, $\mathfrak{S} \subseteqq \mathfrak{R}^2$, we can write $\varphi = pX_1\alpha + X_2\beta$ for some integers X_1 and X_2. Then $0 = \varphi\beta = (pX_1\alpha + X_2\beta)\beta = X_2\beta^2$. Since $\beta^2 \neq 0$, $X_2 \equiv 0 \pmod p$. Thus $\varphi \in p\mathfrak{C}^+$, so $(\mathfrak{R}^2)^+$ has rank 2.

6. Examples related to circle groups.[1] Every Jacobson radical ring is a group under the circle composition

$$x \circ y = x + y + xy\ ,$$

and every subring [two-sided ideal] of the ring is a subgroup [normal subgroup] of the circle group. In general, however, not all subgroups under circle are subrings, and normal subgroups, which may or may not be subrings, need not be ideals. In fact, a subgroup under circle is a subring if and only if it is also a subgroup under addition. We shall consider some examples which show that one cannot tell from the structure of the circle group alone which subgroups will or will not correspond to subrings.

6.1 *A fully invariant subgroup which is not a subring.* Let $\mathfrak{R}$ be the ring generated by an element φ of characteristic 8 with $\varphi^2 = 2\varphi$. The circle group $\mathfrak{C}$ of $\mathfrak{R}$ is abelian of order 8 and type (2, 1). The fully invariant subgroup of $\mathfrak{C}$ of elements of orders 1 and 2 in $\mathfrak{C}$ consists of $0, 3\varphi, 4\varphi$, and 7φ. These elements do not form a subring of $\mathfrak{R}$.

6.2 *Elementary abelian groups.* If $\mathfrak{R}$ is a radical ring whose additive and circle groups are elementary abelian p-groups, then all the additive and circle subgroups of a given order in $\mathfrak{R}$ are indistinguishable up to automorphisms of the groups. We shall, however, give examples of such rings in which the subring structure varies substantially.

Suppose $\mathfrak{R}$ is a radical ring such that $\mathfrak{R}^+$ is an elementary abelian p-group. Then the circle group $\mathfrak{C}$ of $\mathfrak{R}$ is elementary abelian if and only if $\mathfrak{R}$ is commutative and $\varphi^p = 0$ for all $\varphi \in R$. To prove this consider $\mathfrak{C}$ as the multiplicative group of elements $1 + \varphi$ where $\varphi \in \mathfrak{R}$ and 1 is an identity adjoined to $\mathfrak{R}$. Observe that $p\varphi = 0$ implies

[1] The authors wish to thank the referee and editor for encouraging the inclusion of this section.

$(1+\varphi)^p = 1 + \varphi^p$, all $\varphi \in \mathfrak{R}$.

A ring whose additive group is an elementary abelian p-group may be considered as an algebra over $GF(p)$. We now describe some examples of rings $\mathfrak{R}$ whose additive and circle groups are elementary abelian. We denote a basis for $\mathfrak{R}$ as an algebra over $GF(p)$ by $\{\varphi_1, \cdots, \varphi_n\}$.

(a) *Null algebra*, $\mathfrak{R} = \mathfrak{Z}_n$. Define $\varphi_i\varphi_j = 0$ for all $i, j = 1, \cdots, n$. Every subgroup (under $+$ or $\circ$) of $\mathfrak{R}$ is an ideal.

(b) *Power algebra*, $\mathfrak{R} = \mathfrak{P}_n$. Assume $n < p$. $\mathfrak{P}_n$ is the unique power algebra of dimension n over $GF(p)$. $\mathfrak{P}_n$ may be defined by $\varphi_1^k = \varphi_k, 1 \leqq k \leqq n, \varphi_1^{n+1} = 0$. By Theorem 2.1, $\mathfrak{P}_n$ contains only one subring of order p^{n-1}. For $1 \leqq k \leqq n-2, \mathfrak{P}_n$ contains only one ideal of order p^k, although more than one subring of order p^k.

(c) *Direct sum*. $\mathfrak{R} = \mathfrak{P}_m \oplus \mathfrak{Z}_{n-m}$, where $0 < m < n$ and $m < p$.

(d) *Almost-null algebras*. Assume $1 \leqq n \leqq 3$ and $p \neq 2$. $\mathfrak{R} = \mathfrak{U}$, where $\mathfrak{U}$ is a commutative ring whose structure is given in Lemma 5.4. 3 of 4.4. $\mathfrak{U}$ is called "almost-null," and its structure is typical both of nilpotent rings which have a unique subring of order p, and of nilpotent algebras in which all subalgebras are ideals.

6.3 *Remarks on commutative radical rings*. It is easy to find examples of commutative radical rings $\mathfrak{R}$ in which not every subring is an ideal. If $\mathfrak{C}$ is the circle group of $\mathfrak{R}$, then $\mathfrak{C}$ contains normal subgroups which correspond to subrings but not ideals. If, on the other hand, we start with the abelian group $\mathfrak{C}$, then the null ring whose additive group is $\mathfrak{C}$ also has circle group $\mathfrak{C}$, and every subgroup corresponds to an ideal. Every abelian group, moreover, appears as a circle group in this way.

In studying nilpotent rings one soon notices that the fruitful group analogy is between ring product and group commutation. Under this analogy an abelian group corresponds to a null ring, the center of a group to the annihilator of a ring, the lower central series of a group to the powers of a ring, etc.

6.4 *Three special rings*. We conclude by describing three nilpotent rings $\mathfrak{R}$ of order 16, each of which has an abelian circle group of type $(2, 1, 1)$, but which differ in several other properties.

$\mathfrak{R} = \mathfrak{A}$ is generated by elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, each of characteristic 2, such that $\alpha_1^2 = \alpha_2$ and $\alpha_i\alpha_j = 0$ if $i \neq 1$ or if $j \neq 1$.

$\mathfrak{R} = \mathfrak{B}$ is generated by elements $\beta_1, \beta_2, \beta_3$, such that char $\beta_1 = 4$, char $\beta_2 =$ char $\beta_3 = 2$, $\beta_1^2 = 2\beta_1$, $\beta_2^2 = \beta_3$, $\beta_2\beta_3 = \beta_3\beta_2 = 2\beta_1$, and $\beta_1\beta_2 = \beta_2\beta_1 = \beta_1\beta_3 = \beta_3\beta_1 = \beta_3^2 = 0$.

$\mathfrak{R} = \mathfrak{C}$ is generated by elements γ_1 and γ_2 of characteristic 4 such that $\gamma_1^2 = \gamma_2, \gamma_1\gamma_2 = \gamma_2\gamma_1 = 2\gamma_1$, and $\gamma_2^2 = 2\gamma_2$.

Invariant	Value for $\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$
Exponent of $\mathfrak{R}$: least integer e with $\mathfrak{R}^e = 0$	3	4	5
Number of generators required (see (2.1))	3	2	1
Number of subrings (ideals) of order 8	7	3	1
Number of subrings of order 4	11	3	3
Number of subrings of order 2	7	3	3
Number of ideals of order 4	11	3	1
Number of ideals of order 2	7	1	1
Order of $\mathfrak{R}^2$	2	4	8
Order of $\mathfrak{R}$ modulo its annihilator	2	8	8
Additive group type	(1, 1, 1, 1)	(2, 1, 1)	(2, 2)
Order of automorphism group of $\mathfrak{R}$	192	8	4

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SYMMETRIC POSITIVE DEFINITE MULTILINEAR FUNCTIONALS WITH A GIVEN AUTOMORPHISM

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Let V be an n-dimensional vector space over the real numbers R and let φ be a multilinear functional,

(1) $$\varphi: \overset{m}{\underset{1}{\times}} V \longrightarrow R$$

i.e., $\varphi(x_1, \cdots, x_m)$ is linear in each x_j separately, $j = 1, \cdots, m$. Let H be a subgroup of the symmetric group S_m. Then φ is said to be *symmetric* with respect to H if

(2) $$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \varphi(x_1, \cdots, x_m)$$

for all $\sigma \in H$ and all $x_j \in V, j = 1, \cdots, m$. (In general, the range of φ may be a subset of some vector space over R.) Let $T: V \rightarrow V$ be a linear transformation. Then T is an *automorphism* with respect to φ if

(3) $$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m)$$

for all $x_i \in V, i = 1, \cdots, m$. It is easy to verify that the set $\mathfrak{A}(H, T)$ of all φ which are symmetric with respect to H and which satisfy (3) constitutes a subspace of the space of all multilinear functionals symmetric with respect to H. We denote this latter set by $M_m(V, H, R)$.

We shall say that φ is *positive definite* if

(4) $$\varphi(x, \cdots, x) > 0$$

for all nonzero x in V, and we shall denote the set of all positive definite φ in $\mathfrak{A}(H, T)$ by $P(H, T)$. It can be readily verified that $P(H, T)$ is a convex cone in $\mathfrak{A}(H, T)$.

Our main results follow.

THEOREM 1. *If $P(H, T)$ is nonempty then*

(a) *m is even*

and

(b) *every eigenvalue of T has modulus* 1.

If, in addition, T has only real eigenvalues then

(c) *every elementary divisor of T is linear.*

Conversely if (a), (b) *and* (c) *hold then $P(H, T)$ is nonempty. Moreover, if $P(H, T)$ is nonempty then $\mathfrak{A}(H, T)$ is the linear closure of $P(H, T)$.*

In Theorem 2 we shall investigate the dimension of $\mathfrak{A}(H, T)$ in the event $P(H, T)$ is not empty. To do this we must introduce some combinatorial notation. Let $\Gamma_{m,n}$ denote the set of all sequences

$\omega = (\omega_1, \cdots, \omega_m)$ of length $m, 1 \leqq \omega_i \leqq n, i = 1, \cdots, m$. Introduce an equivalence relation $\sim$ in $\Gamma_{m,n}$ as follows: $\alpha \sim \beta$ if there exists a $\sigma \in H$ such that

$$\alpha^\sigma = \beta$$

where $\alpha^\sigma = (\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(m)})$. Let Δ be a system of distinct representatives for $\sim$, i.e., Δ is a set of sequences, one from each equivalence class with respect to $\sim$. We specify Δ uniquely by choosing each element $\alpha \in \Delta$ to be lowest in lexicographic order in the equivalence class in which α occurs.

THEOREM 2. *If $P(H, T)$ is nonempty and T has real eigenvalues $\gamma_1, \cdots, \gamma_n$ then $\gamma_i = \pm 1$, $i = 1, \cdots, n$. Suppose*

$$\gamma_{i_1} = \cdots = \gamma_{i_p} = 1 , \qquad \gamma_j = -1 , \qquad j \neq i_1, \cdots, i_p .$$

Let μ_p be the number of sequences ω in Δ such that the total number of occurrences of $i_1, \cdots, i_p$ in ω is even. Then

$$\dim \mathfrak{A}(H, T) = \mu_p . \tag{5}$$

COROLLARY. *If $H = S_m$ in Theorem 2 and T has p eigenvalues 1 and $n - p$ eigenvalues -1 then*

$$\dim \mathfrak{A}(H, T) = \sum_{k=0}^{m/2} \binom{p - 1 + 2k}{p - 1} \binom{n - p - 1 + m - 2k}{n - p - 1} .$$

In case $m = 2$, $H = S_2$, $\mathfrak{A}(H, T)$ is the totality of symmetric bilinear functionals φ for which

$$\varphi(Tx_1, Tx_2) = \varphi(x_1, x_2) , \qquad x_1, x_2 \in V ,$$

and $P(H, T)$ is just the cone of positive definite φ in $\mathfrak{A}(H, T)$ i.e.,

$$\varphi(x, x) \geqq 0$$

with equality only if $x = 0$. In this case we need not assume that T has real eigenvalues in order to analyze $\mathfrak{A}(H, T)$. We can easily prove the following result by our methods, most parts of which are known (see e.g. [1], Chapter 7).

THEOREM 3. *Assume that $m = 2$ and $H = S_2$. Then $P(H, T)$ is nonempty if and only if*

(a) *T has linear elementary divisors over the complex field,*

(b) *every eigenvalue of T has modulus 1.*

Suppose that T has distinct complex eigenvalues

$$\gamma_k = a_k + ib_k \qquad (\text{and } \bar{\gamma}_k = a_k - ib_k)$$

of multiplicity $e_k, k = 1, \cdots, p$ *and real eigenvalues*

$$\gamma_k = r_k\ , \qquad k = \sum_{j=1}^{p} 2e_j + 1, \cdots, n\ .$$

If $P(H, T)$ *is nonempty then the elementary divisors of* T *over the real field are*

$$\begin{aligned} \lambda^2 - 2\lambda a_k + 1\ , &\quad e_k \text{ times}, \quad k = 1, \cdots, p\ , \\ \lambda - 1\ , &\quad q \text{ times}, \\ \lambda + 1\ , &\quad l \text{ times}, \end{aligned}$$

where

$$\sum_{j=1}^{p} 2e_j + q + l = n\ .$$

Moreover, $\mathfrak{A}(H, T)$ *is the linear closure of* $P(H, T)$,

$$\dim \mathfrak{A}(H, T) = \frac{q(q+1)}{2} + \frac{l(l+1)}{2} + \sum_{j=1}^{p} e_j^2\ ,$$

and there exists a basis E *of* V *such that* $\mathfrak{A}(H, T)$ *consists of the set of all* φ *whose matrix representations with respect to* E, $[\varphi]_E^E$, *have the following form:*

$$[\varphi]_E^E = \sum_{k=1}^{p}{}^{\boldsymbol{\cdot}} (X_k \otimes I_2 + Y_k \otimes F) \dotplus H_q \dotplus H_l\ . \tag{6}$$

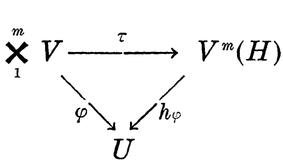
In (6), the dot indicates direct sum, $\otimes$ denotes the Kronecker product, $F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, X_k is e_k-square symmetric, Y_k is e_k-square skew-symmetric, H_q and H_l are q-square and l-square symmetric respectively.

2. **Proofs.** Let $V^m(H)$ denote the symmetry class of tensors associated with H [2]. That is, there exists a fixed multilinear function $\tau\colon \times_1^m V \rightarrow V^m(H)$ symmetric with respect to H, for which

(i) the linear closure of $\tau(\times_1^m V)$ is $V^m(H)$

(ii) the pair $(V^m(H), \tau)$ is universal: given any space U and any multilinear function $\varphi\colon \times_1^m V \rightarrow U$ symmetric with respect to H, there exists a (unique) linear $h_\varphi\colon V^m(H) \rightarrow U$ satisfying

$$h_\varphi \tau = \varphi. \tag{7}$$



We shall denote $\tau(x_1, \cdots, x_m)$ by $x_1 * \cdots * x_m$, and $x_1 * \cdots * x_m$ is called a decomposable tensor or a symmetric product of $x_1, \cdots, x_m$. If we take $\varphi(x_1, \cdots, x_m)$ to be $Tx_1 * \cdots * Tx_m$ in (7) then h_φ is denoted by $K(T)$ and is called the *induced transformation* on $V^m(H)$.

Before we embark on the proof of Theorem 1 we can define $\mathfrak{A}(H, T)$ in terms of $V^m(H)$. First observe that the mapping θ from the space of multilinear functionals φ symmetric with respect to H to the dual space of $V^m(H)$,

$$\theta \colon M_m(V, H, R) \longrightarrow (V^m(H))^* ,$$

defined by

$$\theta(\varphi) = h_\varphi ,$$

is one-to-one linear, and onto. That is, the correspondence $\varphi \leftrightarrow h_\varphi$ is linear bijective. Now, the subspace $\mathfrak{U}(H, T)$ of $M_m(V, H, R)$ is defined by

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m)$$

or in view of (7) by

$$h_\varphi(Tx_1 * \cdots * Tx_m) = h_\varphi(x_1 * \cdots * x_m) ,$$

for all $x_i \in V$, $i = 1, \cdots, m$. In other words, since the decomposable tensors span $V^m(H)$ (see (i) above), $\varphi \in \mathfrak{A}(H, T)$ if and only if $\theta(\varphi) = h_\varphi$ satisfies

$$h_\varphi K(T) = h_\varphi ,$$

or

$$h_\varphi(K(T) - I) = 0 \tag{8}$$

where I is the identity mapping on $V^m(H)$. We have proved the following.

LEMMA 1. *$\mathfrak{A}(H, T)$ is nonempty if and only if $K(T) - I$ is singular. Also,*

$$\dim \mathfrak{A}(H, T) = \eta(K(T) - I) \tag{9}$$

where η is the nullity of the indicated transformation.

LEMMA 2. *If $P(H, T)$ is nonempty then m is even and every eigenvalue of T has modulus 1. Moreover, corresponding to real eigenvalues, T has only linear elementary divisors.*

Proof. If $\varphi \in P(H, T)$ and $x \neq 0$ then

$$\varphi(-x, \cdots, -x) = (-1)^m \varphi(x, \cdots, x)$$

and hence $(-1)^m > 0$ and m is even. Suppose that γ is a real eigenvalue of T with corresponding eigenvector x. Then

$$\begin{aligned}\varphi(Tx, \cdots, Tx) &= \varphi(\gamma x, \cdots, \gamma x) \\ &= \gamma^m \varphi(x, \cdots, x) .\end{aligned}$$

Since $\varphi \in P(H, T)$, $\varphi(Tx, \cdots, Tx) = \varphi(x, \cdots, x) > 0$ and hence $\gamma^m = 1$ and $\gamma = \pm 1$. If γ were involved in an elementary divisor of degree greater than 1 then there would exist linearly independent vectors u_1 and u_2 such that $Tu_1 = \gamma u_1$, $Tu_2 = \gamma u_2 + u_1$ and hence

$$\varphi(Tu_1, \cdots, Tu_1, Tu_2) = \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2 + u_1) .$$

Now

$$\begin{aligned}\varphi(u_1, \cdots, u_1, u_2) &= \gamma^m \varphi(u_1, \cdots, u_1, u_2) \\ &= \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2)\end{aligned}$$

so that

$$\begin{aligned}0 &= \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2 + u_1) - \varphi(\gamma u_1, \cdots, \gamma u_1, \gamma u_2) \\ &= \varphi(\gamma u_1, \cdots, \gamma u_1, u_1) \\ &= \gamma^{m-1} \varphi(u_1, \cdots, u_1) ,\end{aligned}$$

a contradiction.

We now show that any complex eigenvalue of T has modulus 1. Since $\gamma = a + ib$ is now assumed not to be real there exists a pair of linearly independent vectors v_1 and v_2 in V such that

$$\begin{aligned}Tv_1 &= av_1 - bv_2 \\ Tv_2 &= bv_1 + av_2 .\end{aligned} \tag{10}$$

Let $\bar{V}$ be the extension of V to an n-dimensional space over the complex field. Now $\varphi \in \mathfrak{A}(H, T)$ means that

$$\varphi(Tx_1, \cdots, Tx_m) - \varphi(x_1, \cdots, x_m) = 0 \tag{11}$$

is an identity in $x_1, \cdots, x_m$. If we express the vectors in $\bar{V}$ in terms of a basis in V (using in general complex rather than real coefficients) the identity (11) continues to hold since it is a homogeneous polynomial of degree m in the components of $x_1, \cdots, x_m$, vanishing for all real values of these components. Of course it is not true that

$$\varphi(x, \cdots, x) > 0$$

continues to hold for nonzero $x \in \bar{V}$. Now define

$$
\begin{aligned}
e_1 &= v_1 + iv_2 \in \bar{V} \\
e_2 &= v_1 - iv_2 \in \bar{V}
\end{aligned} \tag{12}
$$

and observe that e_1 and e_2 are linearly independent in $\bar{V}$ and satisfy

$$
\begin{aligned}
Te_1 &= \gamma e_1 \\
Te_2 &= \bar{\gamma} e_2 \ .
\end{aligned}
$$

Let $\omega = (\omega_1, \cdots, \omega_m)$ be a sequence for which each ω_i is either 1 or 2, $i = 1, \cdots, m$:

$$\varphi(Te_{\omega_1}, \cdots, Te_{\omega_m}) = \gamma^k \bar{\gamma}^{m-k} \varphi(e_{\omega_1}, \cdots, e_{\omega_m}) \ ,$$

where k of the ω_i are 1 and $m - k$ are 2. But by the above remarks

$$\varphi(Te_{\omega_1}, \cdots, Te_{\omega_m}) = \varphi(e_{\omega_1}, \cdots, e_{\omega_m})$$

and taking absolute values we have

$$(|\gamma|^m - 1)\, |\varphi(e_{\omega_1}, \cdots, e_{\omega_m})| = 0 \ .$$

Thus if $|\gamma| \neq 1$ it follows that

$$\varphi(e_{\omega_1}, \cdots, e_{\omega_m}) = 0 \tag{13}$$

for all ω for which ω_i is 1 or 2 for $i = 1, \cdots, m$. From (12) we have $v_1 = (e_1 + e_2)/2$ and hence using (13) we see that

$$
\begin{aligned}
\varphi(v_1, \cdots, v_1) &= \varphi\left(\frac{e_1 + e_2}{2}, \cdots, \frac{e_1 + e_2}{2}\right) \\
&= 0 \ .
\end{aligned} \tag{14}
$$

However $v_1 \in V$ and $\varphi \in P(H, T)$ and therefore (14) is a contradiction. Thus $|\gamma| = 1$ and the proof of Lemma 2 is complete.

LEMMA 3. *If m is even, and T has real eigenvalues $r_1, \cdots, r_n$, and every elementary divisor of T is linear then $P(H, T)$ is nonempty.*

Proof. Since T has linear elementary divisors there exists a basis for V of eigenvectors $e_1, \cdots, e_n$. Let $g_1, \cdots, g_n$ be a dual basis in V^*. Let g_t^m denote the multilinear functional whose value for any $x_1, \cdots, x_m$ in V is

$$\prod_{j=1}^{m} g_t(x_j) \ .$$

Clearly $g_t^m \in M_m(V, H, R)$. Set

$$\varphi = \sum_{t=1}^{n} g_t^m .$$

Then if $x_j = \sum_{k=1}^{n} \xi_{jk} e_k$, $j = 1, \cdots, m$, and $Te_k = r_k e_k$, $k = 1, \cdots, n$,

$$\begin{aligned}
\varphi(Tx_1, \cdots, Tx_m) &= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t(Tx_j) \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t\Big(\sum_{k=1}^{n} \xi_{jk} Te_k\Big) \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} \xi_{jt} r_t \\
&= \sum_{t=1}^{n} r_t^m \prod_{j=1}^{m} \xi_{jt} \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} \xi_{jt} \\
&= \sum_{t=1}^{n} \prod_{j=1}^{m} g_t(x_j) \\
&= \varphi(x_1, \cdots, x_m) .
\end{aligned}$$

Hence $\varphi \in \mathfrak{A}(H, T)$. Moreover, if $x = \sum_{t=1}^{n} c_t e_t$ then

$$\begin{aligned}
\varphi(x, \cdots, x) &= \sum_{t=1}^{n} g_t(x)^m \\
&= \sum_{t=1}^{n} c_t^m .
\end{aligned}$$

But m is even and hence $\varphi \in P(H, T)$. To complete the proof of Theorem 1 we note that if $\varphi \in P(H, T)$ and if $e_1, \cdots, e_n$ is any basis of V then $\varphi(x, x, \cdots, x)$ is a homogeneous polynomial of degree m in $c_1, \cdots, c_n$. Hence, on the compact hypersphere S defined by $\sum_{t=1}^{n} c_t^2 = 1$ in V, φ must assume a positive minimum value m_φ. By a similar argument for any $\psi \in \mathfrak{A}(H, T)$, $|\psi|$ must assume a maximum M_ψ for $\sum_{t=1}^{n} c_t^2 = 1$. Now let ψ be an arbitrary element of $\mathfrak{A}(H, T)$ and choose a positive constant a such that $a > M_\psi / m_\varphi$. If $0 \neq x \in V$ and $\|x\|^2 = \sum_{t=1}^{n} c_t^2$ then $(x/\|x\|) \in S$ and

$$\begin{aligned}
a\varphi(x, \cdots, x) - \psi(x, \cdots, x) &= a\, \|x\|^m \varphi\Big(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}\Big) \\
&\quad - \|x\|^m \psi\Big(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}\Big) \\
&\geqq \|x\|^m (a m_\varphi - M_\psi) \\
&> 0 .
\end{aligned}$$

In other words,

$$a\varphi - \psi \in P(H, T)$$

so that ψ is a linear combination of elements in $P(H, T)$.

To proceed to the proof of Theorem 2 we use Theorem 1 to conclude immediately that since T has real eigenvalues the elementary divisors are all linear and thus there exists a basis of eigenvectors of T:

$$Te_k = \gamma_k e_k , \qquad k = 1, \cdots, n .$$

It is not difficult to show [2] that the decomposable tensors

$$e_\omega^* = e_{\omega_1} * \cdots * e_{\omega_m} , \qquad \omega \in \Delta ,$$

constitute a basis for $V^m(H)$.

We compute that

$$\begin{aligned} K(T)e_\omega^* &= Te_{\omega_1} * \cdots * Te_{\omega_m} \\ &= \gamma_{\omega_1} e_{\omega_1} * \cdots * \gamma_{\omega_m} e_{\omega_m} \\ &= \prod_{t=1}^{n} \gamma_t^{m_t(\omega)} e_\omega^* \end{aligned} \tag{15}$$

where $m_t(\omega)$ denotes the multiplicity of occurrence of t in ω, $t = 1, \cdots, n$. The formula (15) shows that $(K(T) - I)e_\omega^*$ is 0 or a nonzero multiple of e_ω^* according as

$$\prod_{t=1}^{n} \gamma_t^{m_t(\omega)}$$

is 1 or -1. Now we can assume without loss of generality that the eigenvalues $\gamma_1, \cdots, \gamma_n$ are so organized that $\gamma_1 = \cdots = \gamma_p = 1, \gamma_{p+1} = \cdots = \gamma_n = -1$. (This is of course merely a notational convenience.) Then

$$\begin{aligned} \prod_{t=1}^{n} \gamma_t^{m_t(\omega)} &= \prod_{t=p+1}^{n} (-1)^{m_t(\omega)} \\ &= (-1)^{m - \sum_{t=1}^{p} m_t(\omega)} \\ &= (-1)^{\sum_{t=1}^{p} m_t(\omega)} . \end{aligned}$$

Thus $\prod_{t=1}^{n} \gamma_t^{m_t(\omega)} = 1$ if and only if $\sum_{t=1}^{p} m_t(\omega)$ is even. This last statement just means that $1, \cdots, p$ (i.e., $i_1, \cdots, i_p$) occur altogether an even number of times in ω.

The proof of the corollary is completed by first noting that if $H = S_m$ then the set Δ is the totality of nondecreasing sequences of length m chosen from $1, \cdots, n$. Thus by Theorem 2 if $P(H, T)$ is nonempty and T has real eigenvalues $\gamma_1, \cdots, \gamma_n$ then these eigenvalues are ± 1 and we lose no generality in assuming that $\gamma_1 = \cdots = \gamma_p = 1$, $\gamma_{p+1} = \cdots = \gamma_n = -1$. We want to count the total number of ω in Δ for which

$$\sum_{t=1}^{p} m_t(\omega) \equiv 0 \pmod 2 . \tag{16}$$

Now, a sequence satisfying (16) may be constructed as follows. Suppose that k is a fixed integer, $0 \leqq 2k \leqq m$, and we count the number of sequences in Δ in which $\sum_{t=1}^{p} m_t(\omega) = 2k$. The total number of nondecreasing sequences of length $2k$ using the integers $1, \cdots, p$ is

$$\binom{p + 2k - 1}{2k} = \binom{p - 1 + 2k}{p - 1}$$

and any one of these can be completed to a nondecreasing sequence of length m by adjoining a nondecreasing sequence of length $m - 2k$ using the integers $p + 1, \cdots, n$. There are a total of

$$\binom{n - p + m - 2k - 1}{m - 2k} = \binom{n - p - 1 + m - 2k}{n - p - 1}$$

ways of doing this. This completes the proof of the corollary.

To proceed to the proof of Theorem 3 we remark that Theorem 1 cannot be directly applied because we are not assuming that the eigenvalues of T are real; in general this is not the case. However the statement (b) does follow from Theorem 1. If E is any basis of V, A is the matrix representation of T, and $C = [\varphi]_E^E$, then to say that $\varphi \in \mathfrak{A}(H, T)$ is equivalent to the assertion that

$$A^T C A = C . \tag{17}$$

If $\varphi \in P(H, T)$ then C is a positive definite symmetric matrix and can therefore be written $C = K^2$, where K is also positive definite symmetric. Then (17) is immediately equivalent to the statement that KAK^{-1} is a real orthogonal matrix and (a) is evident. Conversely if (a) and (b) obtain then there exists a real nonsingular matrix S such that $S^{-1}AS$ is a direct sum of a diagonal matrix with ± 1 along the main diagonal together with certain 2-square matrices of the form

$$\begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} . \tag{18}$$

Since $|\gamma_k| = 1$, $k = 1, \cdots, n$, the matrix (18) is orthogonal and hence $S^{-1}AS = U$ where U is an n-square real orthogonal matrix. If we set

$(S^T)^{-1}S^{-1} = C$ then C is a positive definite symmetric matrix and we compute that

$$\begin{aligned} A^T C A &= (S^{-1})^T U^T S^T (S^T)^{-1} S^{-1} S U S^{-1} \\ &= (S^{-1})^T S^{-1} \\ &= C \,. \end{aligned}$$

Thus if $[\varphi]_E^E = C$ then $\varphi \in P(H, T)$. The dimension of $\mathfrak{A}(H, T)$ can equally well be computed as in the general case by finding $\eta(K(T) - I)$ where $K(T)$ is the induced mapping on the complex space of 2-symmetric tensors over $\bar{V}$, i.e., $\bar{V}^2(S_2)$. The mapping $K(T)$ is just the 2nd Kronecker power of T restricted to the second symmetric space. This mapping is customarily denoted by $P_2(T)$[5]. Since T has a basis of eigenvectors $v_1, \cdots, v_n$, so does $P_2(T)$ and, for $1 \leqq i \leqq j \leqq n$,

$$P_2(T) v_i * v_j = \gamma_i \gamma_j v_i * v_j \,.$$

Thus $\dim \mathfrak{A}(H, T)$ is precisely the number of pairs of integers (i, j), $1 \leqq i \leqq j \leqq n$, for which

$$\gamma_i \gamma_j = 1 \,. \tag{19}$$

But T has the distinct eigenvalues $a_k + ib_k$ of multiplicity e_k, $k = 1, \cdots, p$. This yields a total of

$$\sum_{t=1}^{p} e_t^2$$

pairs (i, j) for which (19) is satisfied. Also, T has 1 as an eigenvalue q times and -1 as an eigenvalue l times and this yields an additional

$$\frac{q(q+1)}{2} + \frac{l(l+1)}{2}$$

pairs (i, j) for which (19) is satisfied. This proves that

$$\dim \mathfrak{A}(H, T) = \frac{q(q+1)}{2} + \frac{l(l+1)}{2} + \sum_{j=1}^{p} e_j^2 \,.$$

We now turn to the derivation of (6). First, we assert that since T has linear elementary divisors over the complex numbers [4] there exists a basis E of V such that the matrix representation of T has the following form:

$$A = \sum_{k=1}^{p}{}^{\cdot}\, I_{e_k} \otimes \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix} \dotplus I_q \dotplus -I_l \tag{20}$$

where I_s is the s-square identity matrix. We set $C = [\varphi]_E^E$ and partition C conformally with (20):

$$C = \left[\begin{array}{ccc|cc} C_{11} & \cdots & C_{1d} & & \\ \vdots & & \vdots & Z & \\ C_{d1} & \cdots & C_{dd} & & \\ \hline & & & C_q & C_r \\ & Z^T & & & \\ & & & C_r^T & C_l \end{array}\right],$$

C_{ij} is 2-square, $i, j = 1, \cdots, d = \sum_{j=1}^{p} e_j$, C_q is q-square symmetric and C_l is l-square symmetric. Set $L = \sum_{k=1}^{p} {}^{\cdot} I_{e_k} \otimes (a_k I_2 + b_k F)$ and observe that for (17) to be satisfied Z must satisfy

$$L^T Z(I_q \dotplus -I_l) = Z\,. \tag{21}$$

Now, $L^T \otimes (I_q \dotplus -I_l)$ has eigenvalues $\pm(a_k \pm ib_k)$ [3, p. 9] and none of these is equal to 1. Hence (21) has only the zero matrix as a solution. Similarly we see that $C_r = 0$. Next, consider a typical C_{ij}, $j > i$, call it K. Then K must satisfy an equation of the form

$$(a_s I_2 - b_s F)K(a_r I_2 + b_r F) = K\,. \tag{22}$$

The matrix

$$(a_s I_2 - b_s F) \otimes (a_r I_2 + b_r F)$$

has eigenvalues

$$(a_s \pm ib_s)(a_r \pm ib_r). \tag{23}$$

If $r \neq s$, (23) cannot be 1 and in this case $K = 0$. If $r = s$ then precisely two of the four complex numbers (23) are 1. Thus the nullity of the matrix

$$(a_s I_2 - b_s F) \otimes (a_s I_2 + b_s F) - I_4 \tag{24}$$

is 2. But $K = I_2$ and $K = F$ are two linearly independent solutions to (22) for $r = s$. Also note that since C is symmetric C_{ii} must be a multiple of I_2. It follows that the submatrix

$$\begin{bmatrix} C_{11} & \cdots & C_{1d} \\ \vdots & & \vdots \\ C_{d1} & \cdots & C_{dd} \end{bmatrix}$$

is itself a direct sum of $2e_k$-square matrices of the form

$$\left[\begin{array}{cc|cc|c|cc|c|cc|c|cc}
x_{11} & 0 & x_{12} & y_{12} & & & & & & & & & \\
0 & x_{11} & -y_{12} & x_{12} & & & & & & & & & \\
\hline
x_{12} & -y_{12} & x_{22} & 0 & & & & & & & & & \\
y_{12} & x_{12} & 0 & x_{22} & & & & & & & & & \\
\hline
 & & & & \ddots & & & & & & & & \\
\hline
 & & & & & x_{rr} & 0 & \cdots & x_{rs} & y_{rs} & & & \\
 & & & & & 0 & x_{rr} & \cdots & -y_{rs} & x_{rs} & & & \\
\hline
 & & & & & & & \ddots & \vdots & & & & \\
\hline
 & & & & & & & & x_{ss} & 0 & & & \\
 & & & & & & & & 0 & x_{ss} & & & \\
\hline
 & & & & & & & & & & \ddots & & \\
\hline
 & & & & & & & & & & & x_{e_k e_k} & 0 \\
 & & & & & & & & & & & 0 & x_{e_k e_k}
\end{array}\right]$$

This matrix is of the form $X_k \otimes I_2 + Y_k \otimes F$ where $X_k = (x_{ij})$ is e_k-square symmetric and $Y_k = (y_{ij})$ is e_k-square skew-symmetric. This completes the proof of Theorem 3.

3. Some examples. Let $m = 2p$ and let S'_p be the symmetric group of degree p on $p+1, \cdots, m$. In general if V is a Euclidean space with inner product (x, y) then $V^m(H)$ is also a Euclidean space [2] in which the inner product of two symmetric products $x_1 * \cdots * x_m$ and $y_1 * \cdots * y_m$ is given by

$$(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \frac{1}{m!} \sum_{\sigma \in H} \prod_{i=1}^{m} (x_i, y_{\sigma(i)}) \,. \tag{25}$$

Set $H = S_p \times S'_p$ (direct product) and define $\varphi \in M_m(V, H, R)$ by

$$\varphi(x_1, \cdots, x_p, x_{p+1}, \cdots, x_m) = (x_1 * \cdots * x_p, x_{p+1} * \cdots * x_m) \,. \tag{26}$$

Clearly φ is symmetric with respect to H and

$$\begin{aligned} \varphi(x, \cdots, x, x, \cdots, x) &= \| x * \cdots * x \|^2 \\ &\geqq 0 \,. \end{aligned}$$

Moreover $x * \cdots * x = 0$ if and only if $x = 0$ [2]. Hence φ is positive definite. Now suppose that $\varphi \in P(H, T)$ where $T: V \rightarrow V$. Then

$$\varphi(Tx_1, \cdots, Tx_p, Tx_{p+1}, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m)$$

and from (26) we have

(27) $(Tx_1 * \cdots * Tx_p, Tx_{p+1} * \cdots * Tx_m) = (x_1 * \cdots * x_p, x_{p+1} * \cdots * x_m)$.

It follows from (27) that

$$K(T^* T) = I \tag{28}$$

where T^* is the adjoint of T and $K(T)$ is the induced transformation in the symmetry class $V^p(S_p)$. It is not difficult to show [7] that (28) implies that $T^* T = \omega I_v$ where $|\omega| = 1$. However, since $T^* T$ is positive definite, $T^* T = I_V$, and hence T is orthogonal. It follows that T must have linear elementary divisors over the complex numbers.

In Theorem 1 we proved only that if $P(H, T)$ is nonempty then T has linear elementary divisors corresponding to real eigenvalues. We conjecture that in fact the preceding example is typical in the sense that T always has linear elementary divisors over the complex numbers if $P(H, T)$ is assumed to be nonempty.

We now give an example to show that if $\varphi \in \mathfrak{A}(H, T)$, but φ is not positive definite, then the elementary divisors of T over the complex numbers need not be linear. Let $H = S_2$ and let $\dim V = 4$. Choose T to have

$$(\lambda^2 + 1)^2$$

as its only elementary divisor. Then there exists a real basis $E = \{e_1, \cdots, e_4\}$ of V so that

$$[T]_E^E = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Let $A = [T]_E^E$. Then from (17) it suffices to determine a symmetric matrix C such that

$$A^T C A = C . \tag{29}$$

Define C as follows:

$$C = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix} .$$

Then C is symmetric (but not positive definite) and (29) is easily verified. This example also shows that $P(H, T)$ is empty. It is routine to verify that $\dim \mathfrak{A}(H, T) = 1$ in this case but the formula (5) produces the integer 4.

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PONTRYAGIN SQUARES IN THE THOM SPACE OF A BUNDLE

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The object of this note is to determine the action of the Pontryagin squares in the cohomology of the Thom space of a vector bundle. This computation is then applied to the case of the normal bundle of a manifold imbedded in Euclidean space to give simplified proofs of some theorems of Mahowald.

The first of Mahowald's theorems [3] was inspired by some 1940 results of Whitney [9], who showed that in certain cases the Euler class (with twisted integer coefficients) of the normal bundle of a nonorientable surface imbedded in Euclidean 4-space could be nonzero. This contrasts with the well-known theorem that the Euler class of the normal bundle of an orientable manifold in Euclidean space is always zero.

2. Notation and statement of results. For any space X, we will use integral cohomology $H^q(X, \mathbf{Z})$; cohomology with integers mod n as coefficients, $H^q(X, \mathbf{Z}_n)$; cohomology with twisted integer coefficients, $H^q(X, \mathscr{Z})$ cohomology with twisted integers mod n coefficients, $H^q(X, \mathscr{Z}_n)$; and rational cohomology, $H^q(X, Q)$. In the third and fourth cases the local system of groups which is used for coefficients will be determined by the Stiefel-Whitney class $w_1 \in H^1(X, \mathbf{Z}_2)$. Note that for the case $n=2$, we have

$$H^q(X, \mathscr{Z}_2) = H^q(X, Z_2) \ .$$

since a cyclic group of order 2 admits no nontrivial automorphisms.

Let (E, p, B, S^{n-1}) be an $(n-1)$-sphere bundle over the base space B with structure group $0(n)$. We will use the following notation for characteristic classes of such a bundle:

Stiefel-Whitney classes:

$$w_i \in H^i(B, Z_2)\ , \qquad 1 \leqq i \leqq n$$
$$W_i \in H^i(B, \mathscr{Z})\ , \qquad 1 \leqq i \leqq n,\ i \text{ odd}\ .$$

Pontrjagin classes:

$$p_i \in H^{4i}(B, Z)\ , \qquad 1 \leqq i \leqq n/2\ .$$

Euler class:

$$X_n \in H^n(B, \mathscr{Z})\ . \qquad (\text{If } n \text{ is odd, then } X_n = W_n.)$$

Let (A, π, B, D^n) be the associated n-dimensional disc bundle; we will call the pair (A, E) or the single space A/E the *Thom space* of the bundle. The *Thom class*, $U \in H^n(A, E, \mathscr{Z})$, has twisted integer coefficients; by taking cup products with U, we obtain the Thom isomorphism (see Thom [6]).

$$H^q(A, \mathscr{Z}) \approx H^{q+n}(A, E, Z) ,$$
$$H^q(A, Z) \approx H^{q+n}(A, E, \mathscr{Z}) ,$$
$$H^q(A, \mathscr{Z}_n) \approx H^{q+n}(A, E, Z_n) , \text{ etc.}$$

Recall also that the projection $\pi: A \rightarrow B$ is a deformation retraction, and hence induces isomorphisms of cohomology groups with any coefficients (even local coefficients!). For the sake of convenience, we will often identify the cohomology groups of A and B by means of this isomorphism; similarly we will identify the cohomology groups of the pair (A, E) and the space (A/E) (except in dimension 0) with ordinary coefficients (the local coefficient systems $\mathscr{Z}$ and $\mathscr{Z}_n$ do not exist in the space A/E).

The obvious epimorphism $\rho_n: Z \rightarrow Z_n$ and monomorphism $\theta: Z_2 \rightarrow Z_4$ induce homomorphisms of cohomology groups wich will be denoted as follows:

$$\rho_n: H^q(X, Z) \longrightarrow H^q(X, Z_n) ,$$
$$\tilde{\rho}_n: H^q(X, \mathscr{Z}) \longrightarrow H^q(X, \mathscr{Z}_n) ,$$
$$\theta: H^q(X, Z_2) \longrightarrow H^q(X, Z_4) ,$$
$$\tilde{\theta}: H^q(X, Z_2) \longrightarrow H^q(X, \mathscr{Z}_4) .$$

For convenience, we will let $U_2 = \tilde{\rho}_2(U)$, the Thom class reduced mod 2.

Our main concern will be the Pontryagin squaring operation,

$$\mathscr{P}: H^q(X, Z_2) \longrightarrow H^{2q}(X, Z_4) .$$

If q is odd, the Pontryagin square can be expressed in terms of simpler cohomology operations. (see formula (4.2) below); this is not true for q even. For a list of papers describing this operation, see the first paragraph of [7]. Our main result is the following, which describes the Pontryagin square of the mod 2 Thom class, U_2.

THEOREM I. *Let (E, p, B, S^{n-1}) be a (not necessarily orientable) $(n-1)$-sphere bundle with structure group* $0(n)$, *n even. Then*

$$\mathscr{P}(U_2) = [\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1 \cdot w_{n-1})] \cdot U .$$

As a corollary, we obtain the following result which was proved by Whitney [9] in 1940 for the case $n = 2$; the general case is due to Mahowald, [3, Th. I]:

COROLLARY 1. *Let M^n be a compact, connected, nonorientable n-manifold (n even) which is imbedded differentiably in R^{2n}. Then the twisted Euler class of the normal bundle, X_n, satisfies the following condition:*

$$\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1}) = 0 \ .$$

(Here $\bar{w}_i$ denotes the ith dual Stiefel-Whitney class of M^n.)

In particular, if $\bar{w}_1\bar{w}_{n-1} \neq 0$ (which can only happen if n is a power of 2, cf. [4]) then $X_n \neq 0$. Apparently this is the only general result known about the twisted Euler class of the normal bundle to a non-orientable manifold.

The corollary may be derived from the theorem as follows: Let (E, p, B, S^{n-1}) denote the normal sphere bundle of the imbedding, and (A, π, B, D^n) the associated disc bundle. It is well known that the top homology group of the Thom space,

$$H_{2n}(A/E, Z) = H_{2n}(A, E, Z) \ ,$$

is infinite cyclic, and the Hurewicz homomorphism

$$\pi_{2n}(A/E) \longrightarrow H_{2n}(A/E)$$

is an epimorphism. From this it follows that $\langle \mathscr{P}(U_2), x\rangle = 0$ for any $x \in H_{2n}(A/E, Z)$, and hence $\mathscr{P}(U_2) = 0$. Applying the formula for $\mathscr{P}((U_2)$ in Theorem I, we obtain the corollary.

Next, we give formulas for the Pontryagin square of an *arbitrary* mod 2 cohomology class of even degree in the Thom space of a vector bundle.

THEOREM II. *Let (E, p, B, S^{n-1}) be an $(n-1)$-sphere bundle with structure group $0(n)$, and let $x \in H^m(B, Z_2)$, $m + n$ even. Then if m and n are both even,*

$$\begin{aligned}\mathscr{P}(U_2x) = \{&\mathscr{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1})] \\ &+ \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\}\cdot U\end{aligned}$$

while if m and n are odd,

$$\begin{aligned}\mathscr{P}(U_2x) = \{&\mathscr{P}(x)[\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1w_{n-1} + w_1^2w_{n-2})] \\ &+ \tilde{\theta}[w_{n-1}xSq^1x + w_1w_nSq^{m-1}x]\}\cdot U \ .\end{aligned}$$

As a corollary, we derive a necessary condition due to Mahowald [3] for the imbeddability of an *orientable* manifold in Euclidean space of dimension $4k$ with codimension n.

COROLLARY 2. *Let M be a compact, connected, orientable manifold of dimension q which is differentiably imbedded in Euclidean space of dimension $q + n = 4k$. Then for any $x \in H^m(M, Z_2)$, where $m = 1/2(q - n)$, we must have*

$$\bar{w}_{n-1}xSq^1x = 0 .$$

Proof of corollary. One applies Theorem II with $B = M$ and (E, p, B, S^{n-1}) the normal bundle of the imbedding. Since M is assumed orientable, $\bar{w}_1 = 0$, $\bar{w}_n = 0$, $X_n = 0$, and $\bar{W}_n = 0$. Exactly as in the proof of the previous corollary we know that $\mathscr{P}(U_2 \cdot x) = 0$ in this case. Thus we conclude that

$$\theta(\bar{w}_{n-1}xSq^1x) = 0$$

for any $x \in H^m(M, Z_2)$. Since M is orientable, the homomorphism

$$\theta\colon H^q(M, Z_2) \longrightarrow H^q(M, Z_4)$$

is a monomorphism, and therefore we must have $\bar{w}_{n-1}xSq^1x = 0$, as desired.

Perhaps the neatest application of this corollary is to prove that q-dimensional real projective space does not imbed in R^{2q-2} for $q = 2^r + 1$. A discussion of the possibilities of using this theorem to prove non-imbedding results is given in § 5.

COROLLARY 3. *Let M be a compact, connected, nonorientable manifold of dimension q which is differentiably imbedded in Euclidean space of dimension $q + n = 4k$, q and n even. Then for any element $x \in H^m(M, Z_2)$, where $m = (1/2)(q - n)$, we must have*

$$\mathscr{P}(x) \cdot [\tilde{\rho}_4(X_n) + \tilde{\theta}(\bar{w}_1\bar{w}_{n-1})] + \tilde{\theta}(\bar{w}_{n-1}xSq^1x) = 0 .$$

This is a generalization of Corollary 1, and the proof is similar. Presumably this theorem would enable one to prove in certain cases that $\tilde{\rho}_4(X_n) \neq 0$, and hence $X_n \neq 0$, but the author knows of on examples to illustrate this possibility. Perhaps the most likely case in which this theorem could be applied is the case where $n = q - 4$, $m = 2$.

3. Proof of Theorem I. As is usual in such cases, one only need prove Theorem I in the case of the universal example, where $B = BO(n)$, n even. Then E has the same homotopy type as $BO(n - 1)$. Consider the following commutative diagram for this universal example:

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & H^*(A, E, Z_k) & \xrightarrow{j^*} & H^*(A, Z_k) & \xrightarrow{i^*} & H^*(E, Z_k) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow 1 & \nearrow & & & & & \\
 & & H^*(A, \mathscr{Z}_k) & & \uparrow \pi^* & & \updownarrow 2 & & \\
 & & \uparrow \pi^* & & & & & & \\
\cdots & \xrightarrow[\psi]{} & H^*(B, \mathscr{Z}_k) & \xrightarrow[\mu]{} & H^*(B, Z_k) & \xrightarrow[p^*]{} & H^*(E, Z_k) & \xrightarrow[\psi]{} & \cdots .
\end{array}
$$

The top line of this diagram is the mod k cohomology sequence of the pair (A, E) while the bottom line is the Gysin sequence of fibration. All vertical arrows are isomorphisms; arrow No. 1 denotes the Thom isomorphism, and arrow No. 2 is the identity. It is well known that in these exact sequences for the case $k=2$ (i.e., mod 2 cohomology), the following statements are true:

p^* and i^* are epimorphisms,

μ and j^* are monomorphisms, and

ψ and δ are zero.

We assert that these statements are also true in case $k = 4$. In order to prove this, it suffices to prove that j^* is a monomorphism, and for this purpose consider the following commutative diagram:

$$
\begin{array}{ccccc}
0 & \longrightarrow & H^{q-1}(A, E, Z_2) & \xrightarrow{j_2} & H^{q-1}(A, Z_2) \\
 & & \downarrow Sq^1 & & \downarrow Sq^1 \\
0 & \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) \\
 & & \downarrow \theta & & \downarrow \theta \\
\cdots & \longrightarrow & H^q(A, E, Z_4) & \xrightarrow{j_4} & H^q(A, Z_4) \\
 & & \downarrow \eta & & \downarrow \eta \\
0 & \longrightarrow & H^q(A, E, Z_2) & \xrightarrow{j_2} & H^q(A, Z_2) .
\end{array}
$$

The vertical lines are exact sequences corresponding to the following short exact sequence of coefficients:

$$0 \longrightarrow Z_2 \xrightarrow{\theta} Z_4 \xrightarrow{\eta} Z_2 \longrightarrow 0 .$$

Let $x \in H^q(A, E, Z_4)$ and assume that $j^*(x) \equiv j_4(x) = 0$. Therefore

$$j_2\eta(x) = \eta j_4(x) = 0$$

and since j_2 is a monomorphism, $\eta(x) = 0$. By exactness, there exists an element $y \in H^q(A, E, Z_2)$ such that

$$\theta(y) = x .$$

Since $\theta j_2(y) = 0$, there exists an element $z \in H^{q-1}(A, Z_2)$ such that

$$Sq^1(z) = j_2(y) .$$

We wish to show that z can be chosen so that $z \in$ image j_2. For this purpose, recall that we are identifying $H^*(A, Z_2)$ with $H^*(B, Z_2) = Z_2[w_1, w_2, \cdots, w_n]$; using this identification, the image of j^* is the ideal generated by w_n. We may split $H^*(A, Z_2)$ into the (vector space) direct sum of this ideal and a supplementary subspace as follows: one subspace is spanned by all monomials which have w_n as a factor, the other subspace is spanned by those monomials which do not have w_n as a factor. It is readily verified that the homomorphism

$$Sq^1 \colon H^*(A, Z_2) \longrightarrow H^*(A, Z_2)$$

maps each of these summands into itself (this depends on the fact that n is even). Since $j_2(y)$ belong to this ideal generated by w_n, we can choose z so it also belongs to this ideal. Therefore $z = j_2(u)$ for some element $u \in H^{q-1}(A, E, Z_2)$. It follows that

$$j_2(y - Sq^1u) = 0 .$$

Since j_2 is a monomorphism, $y = Sq^1u$, and

$$x = \theta(y) = \theta Sq^1u = 0$$

as asserted.

Next, let $X_n \in H^n(BO(n), \mathscr{Z})$ denote the Euler class (n even). We assert that

$$X_n^2 = p_{n/2} \in H^{2n}(BO(n), Z) .$$

To prove this, we make use of the fact that all torsion in $H^*(BO(n), Z)$ is of order 2 (cf. Borel and Hirzebruch, [2]). Hence it suffices to prove that the following two equations:

$$\rho_2(X_n^2) = \rho_2(p_{n/2}) \text{ and}$$
$$\rho_0(X_n^2) = \rho_0(p_{n/2}) ,$$

where ρ_0 is the homomorphism of cohomology groups induced by the coefficient map $Z \longrightarrow Q$.

As to the first equation, it is well known that $\rho_2(X_n) = w_n$ and $\rho_2(p_i) = w_{2i}^2$, hence

$$\rho_2(X_n^2) = w_n^2 = \rho_2(p_{n/2}) .$$

To prove the second equation, consider the following commutative diagram.

$$\begin{array}{ccc} H^{2n}(BO(n), Z) & \xrightarrow{f^*} & H^{2n}(BSO(n), Z) \\ \downarrow{\rho_0} & & \downarrow{\rho_0} \\ H^{2n}(BO(n), Q) & \xrightarrow{f^*} & H^{2n}(BSO(n), Q) . \end{array}$$

Here $f: BSO(n) \to BO(n)$ is the 2-fold covering induced by the inclusion of $SO(n)$ in $O(n)$. It is well known that $\rho_0 f^*(X_n^2) = \rho_0 f^*(p_{n/2})$ and that f^* is a monomorphism on rational cohomology (see Borel and Hirzebruch [2]). Hence $\rho_0(X_n^2) = \rho_0(p_{n/2})$ as required.

With these preliminaries out of the way, we will now prove Theorem I by consideration of the following commutative diagram:

$$\begin{array}{ccc} H^n(A, E, Z_2) & \xrightarrow{j_2} & H^n(A, Z_2) \\ \downarrow{\mathscr{P}} & & \downarrow{\mathscr{P}} \\ H^{2n}(A, E, Z_4) & \xrightarrow{j_4} & H^{2n}(A, Z_4) . \end{array}$$

It is well known that $j_2(U_2) = w_n$, and according to Thomas [8], Theorem C,

$$\mathscr{P}(w_n) = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n) .$$

Since j_4 is a monomorphism, it suffices to prove that

$$j_4\{\tilde{\rho}_4(X_n) + [\tilde{\theta}(w_1 w_{n-1})] \cdot U\} = \rho_4(p_{n/2}) + \theta(w_1 Sq^{n-1} w_n)$$

in order to complete the proof. Now

$$\tilde{\rho}_4(X_n) \cdot U = \rho_4(X_n \cdot U)$$

and

$$\begin{aligned} j_4\{\tilde{\rho}_4(X_n) \cdot U\} &= j_4 \rho_4(X_n \cdot U) = \rho_4 j(X_n \cdot U) \\ &= \rho_4(X_n^2) = \rho_4(p_{n/2}) \end{aligned}$$

since $j(U) = X_n$. Similarly,

$$[\tilde{\theta}(w_1 w_{n-1})] \cdot U = \theta(w_1 w_{n-1} \cdot U_2) = \theta(w_1 Sq^{n-1} U_2) ,$$

hence

$$\begin{aligned} j_4\{\tilde{\theta} w_1 w_{n-1}) \cdot U\} &= j_4 \theta(w_1 Sq^{n-1} U_2) \\ &= \theta j_2(w_1 Sq^{n-1} U_2) \\ &= \theta(w_1 Sq^{n-1} w_n) \end{aligned}$$

since $j_2(U_2) = w_n$. This completes the proof.

4. **Proof of Theorem 2.** The proof is a routine application of the following two formulas. For the first formula, assume that X is

a topological space, $u \in H^m(X, Z_2)$, $v \in H^n(X, Z_2)$, and $m \equiv n \bmod 2$; then the Pontryagin square of the cup product uv is given by the following formula:

$$\mathscr{P}(uv) = (\mathscr{P}u)(\mathscr{P}v) + \theta[(Sq^{m-1}u)vSq^1v + uSq^1u(Sq^{n-1}v)] \ . \tag{4.1}$$

For the case where m and n are both odd, this formula is given by Thomas [8], formula (10.5); in case m and n are even, the formula is given by Nakaoka [5], Theorem III. Our second formula expresses the Pontryagin square of an odd dimensional cohomology class in terms of more usual cohomology operations. Assume $u \in H^{2q+1}(X, Z_2)$; then

$$\mathscr{P}(u) = \rho_4 \beta Sq^{2q}u + \theta Sq^{2q}Sq^1u \ , \tag{4.2}$$

where β is the Bockstein coboundary operator associated with the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$. In particular, if we apply (4.2) to the computation of $\mathscr{P}(U_2)$ for an m-dimensional vector bundle, m odd, and make use of the formula $Sq^iU_2 = w_iU_2$, we obtain the formula

$$\mathscr{P}(U_2) = [\tilde{\rho}_4(W_m) + \tilde{\theta}(w_1w_{m-1} + w_1^2w_{m-2})] \cdot U \ . \tag{4.3}$$

The proof of Theorem II is now a direct application of formula (4.1); one also uses Theorem I in case m and n are even, and (4.3) in case m and n are odd.

5. Critique of corollary 2. We propose to discuss the following question: Under what conditions does Corollary 2 enable one to prove nonimbedding theorems not provable by more standard and/or elementary methods? We will assume, as in the statement of the corollary, that M is a compact, connected, orientable manifold of dimension q, that $\bar{w}_{n-1} \neq 0$, and

$$q + n \equiv 0 \bmod 4 \ .$$

We wish to prove that M can not be imbedded differentiably in Euclidean space of dimension $q + n$. We may as well assume that $\bar{w}_i = 0$ for all $i > n - 1$, otherwise the proof would be trivial.

We assert that *if n is even*, then for any $x \in H^m(M, Z_2)$, $m = (1/2)(q - n)$,

$$\bar{w}_{n-1}xSq^1x = 0$$

under the above hypotheses, and hence Corollary 2 can not be applied to prove nonimbedding results.

Proof of assertion. By Lemma 1 of Massey and Peterson [4],

$$\begin{aligned}\bar{w}_{n-1}xSq^1x &= Q^{n-1}(xSq^1x)\\ &= Q^{n-1}(xQ^1x)\\ &= \sum_{i+k=n-1}(Q^ix)(Q^jQ^1x)\ .\end{aligned}$$

But

$$Q^jQ^1 = \begin{cases}Q^{j+1} & \text{if } j \text{ is even },\\ 0 & \text{if } j \text{ is odd }.\end{cases}$$

Hence

$$\begin{aligned}\bar{w}_{n-1}xSq^1x &= \sum_{i+j=n-1}(Q^ix)(Q^{j+1}x)\\ &= \sum_{i+k=n}(Q^ix)(Q^kx)\ .\end{aligned}$$

where the summations are restricted to even values of j and odd values of k respectively.

If $n \equiv 0 \bmod 4$, then i must also be odd in this sum, and the nonzero terms occur in pairs which cancel. If $n \equiv 2 \bmod 4$, then all terms cancel in pairs except for the term where $i = k = n/2$, and one sees that in this case

$$\bar{w}_{n-1}xSq^1x = Q^n(x^2) = \bar{w}_n \cdot x^2\ .$$

But by our hypothesis, $\bar{w}_n = 0$; hence $\bar{w}_{n-1}xSq^1x = 0$ in this case also.

Thus this method is only of interest in case n and q are odd. Perhaps the first case of interest is the case where q is odd and $n = q - 2$. In this case $m = 1$, $x \in H^1(M, Z_2)$, $Sq^1x = x^2$, and

$$\bar{w}_{n-1}xSq^1x = Q^{n-1}(x^3) \in H^q(M, Z_2)\ .$$

The question is, for what values of n can $Q^{n-1}(x^3)$ be nonzero? Now it is easy to prove that for any 1-dimensional cohomology class x,

$$Q(x) = x + x^2 + x^4 + x^8 + \cdots + x^{2^k} + \cdots,$$

(see Atiyah and Hirzebruch [1], pp. 168–169), hence

$$\begin{aligned}Q(x^3) = (Qx)^3 = x^3 + (x^4 + x^5) + (x^8 + x^9)\\ + \cdots + (x^{2^k} + x^{2^k+1}) + \cdots\ .\end{aligned}$$

Therefore the only case for which $Q^{n-1}(x^3)$ can possibly be nonzero is the case $q = n + 2 = 2^k + 1$, and in this case

$$Q^{n-1}(x^3) = x^q\ .$$

Thus the example $M = q$-dimensional real projective space is typical for this situation.

The next case of interest would be the case q odd, $n = q - 6$, $m = 3$. The author knows no nontrivial examples to illustrate this case.

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PROOF OF A CONJECTURE OF WHITNEY

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Let M be a closed, connected, nonorientable surface of Euler characteristic χ which is smoothly embedded in Euclidean 4-space, R^4, with normal bundle ν. The Euler class of ν, denoted by $e(\nu)$, is an element of the cohomology group $H^2(M; \mathscr{Z})$ (the letter $\mathscr{Z}$ denotes twisted integer coefficients). Since the group $H^2(M; \mathscr{Z})$ is infinite cyclic, $e(\nu)$, is m times a generator for some integer m. In a paper presented to a Topology Conference held at the University of Michigan in 1940, H. Whitney studied the possible values that this integer m could take on for different embeddings of the given surface M. He gave examples to show that m can be nonzero (unlike the case for an orientable manifold embedded in Euclidean space) and proved that[1]

$$m \equiv 2\chi \pmod 4) .$$

Finally, he conjectured that m could only take on the following values:

$$2\chi - 4, 2\chi, 2\chi + 4, \cdots, 4 - 2\chi .$$

It is the purpose of the present paper to give a proof of this conjecture of Whitney. The proof depends on a corollary of the Atiyah-Singer index theorem.

This corollary is concerned with manifolds with an orientation preserving involution; an elementary proof of the corollary has recently been given by K. Jänich and E. Ossa, [5].

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The precise statement of the theorem which was conjectured by Whitney is contained in the next section. In order to remove the ambiguity in the sign of the integer m, it is necessary to give a rather thorough discussion of some basic notions regarding questions of orientation, local coefficient systems, etc. Although this material is more or less known, it is nowhere published in a form convenient for our purposes; hence it has been relegated to the appendix of this paper.

2. Precise statement of the theorem. We will assume that M is a closed, connected, nonorientable surface which is embedded smoothly

[1] This result of Whitney was generalized by M. Mahowald in 1964. For a proof of Mahowald's theorem, see a recent paper of the author entitled "Pontryagin squares in the Thom space of a bundle" (Pacific J. Math.).

in the 4-sphere, S^4 (the one point compactification of R^4), and that S^4 has been given a definite orientation. Let ν denote the normal bundle of this embedding; by the Whitney duality theorem, we have equality of Stiefel-Whitney classes,

$$w_1(\nu) = w_1(M) .$$

Let $\mathscr{Z}$ denote the local system of integers on M with twisting determined by $w_1(\nu) = w_1(M)$. The local systems of orientations $O(\nu)$ and $O(M)$ (see Appendix 1) are both isomorphic to $\mathscr{Z}$, and in each case the isomorphism may be chosen in two different ways. Note that if $\tau(M)$ denotes the tangent bundle to M, we have

$$\nu \oplus \tau(M) = \tau'$$

where τ' denotes the restriction of the tangent bundle of S^4 to M. Therefore we have a natural isomorphism

$$(*) \qquad O(\nu) \otimes O(\tau(M)) \approx O(\tau') .$$

The choosing of an orientation of S^4 determines an isomorphism of $O(\tau')$ with the group of integers, Z. Assume that one also chooses isomorphism

$$O(\nu) \approx \mathscr{Z} \quad \text{and} \quad O(\tau(M)) \approx \mathscr{Z} .$$

Then the equation $(*)$ becomes

$$(**) \qquad \mathscr{Z} \otimes \approx \mathscr{Z}\, Z .$$

We will consistently assume that the isomorphisms $O(\nu) \approx \mathscr{Z}$ and $O(\tau(M)) \approx \mathscr{Z}$ are chosen so that at each point of M the isomorphism of $(**)$ is that determined by ordinary multiplication of integers. This implies that the choice of the isomorphism $O(\nu) \approx \mathscr{Z}$ determines the choice of $O(\tau(M)) \approx \mathscr{Z}$ and conversely. It also implies that $e(\nu)[M]$ (the Euler class of ν evaluated on the fundamental class of M) is a positive or negative integer whose sign is determined by the orientation of S^4. With these conventions, we can state our main theorem;

THEOREM. *Let M be a closed, connected, nonorientable surface of Euler Characteristic χ which is smoothly embedded in the oriented 4-sphere, S^4. Then the integer $e(\nu)[M]$ has one of the following values:*

$$2\chi - 4, 2\chi, 2\chi + 4, \cdots, 4 - 2\chi .$$

Moreover, any of these possible values can be attained by an appropriate embedding of M in S^4.

REMARK. This theorem is actually true if S^4 is an oriented homology sphere; it is not necessary to assume that it is simply connected.

The rest of the paper is organized as follows: Section 3 contains an outline of the proof. The more tedious details are relegated to lemmas which are proved in §4 and 5. In §6 we prove the statement contained in the last sentence of the theorem; this part of the proof is completely independent of the rest.

3. Outline of the proof. We are assuming the surface M is smoothly embedded in the oriented 4-sphere, S^4. By the Alexander duality theorem,

$$H_1(S^4 - M; Z) \approx H^2(M, Z) = Z_2 .$$

Hence the space $S^4 - M$ has a unique 2-sheeted covering space, namely, that which corresponds to the commutator subgroup of the fundamental group $\pi_1(S^4 - M)$. This covering space can be "completed" to a branched covering space

$$p: S' \longrightarrow S^4$$

with M as the set of branch points (for the theory of branched covering spaces, see R. H. Fox, [2]. We orient S' so that its orientation agrees with that of S^4 under the map p. For the sake of convenience, we will identify M and $p^{-1}(M)$ by means of the map p. Note that S' is a 4-dimensional compact orientable manifold; we denote by

$$T: S' \longrightarrow S'$$

the obvious involution of S' which interchanges the two sheets of the covering. T is an orientation preserving smooth involution and its fixed point set is precisely the surface M.

We will denote by ν' the normal bundle of the imbedding of M in S'. Let $S(\nu)$ and $S(\nu')$ denote the associated 1-sphere bundles of the bundles ν and ν' respectively; $S(\nu)$ and $S(\nu')$ can be realized as the boundaries of smooth tubular neighborhoods of M in S^4 and S' respectively. The projection $p: S' \rightarrow S^4$ induces a fibre-preserving map $S(\nu') \rightarrow S(\nu)$ which has degree ± 2 on each fibre. We can now apply Lemma 1 (see §4) to this fibre preserving map and conclude that the Euler classes of the bundles ν and ν' are related by the following equation:

$$e(\nu) = \pm 2 \cdot e(\nu') . \tag{1}$$

Lemma 1 is applicable here, because the Euler class is the first obstruction to a cross section of a sphere bundle. We may assume that the local orientations, etc., are chosen so that $e(\nu) = 2\cdot e(\nu')$.

Next, we will apply equation (6.14) of Atiyah and Singer [1] to the involution T of the 4-manifold S'. The result is the following equation:

$$\text{Sign}\,(T, S') = \{\mathscr{L}(M)\cdot\mathscr{L}(\nu')^{-1}e(\nu')\}[M]\ . \tag{2}$$

In this equation, we have used the notation of Atiyah and Singer. Here Sign (T, S') denotes the signature of the involution T; for a simplified definition, see Hirzebruch, [4], or Jänich and Ossa, [5]. This simplified definition is repeated below. $\mathscr{L}(M)$ and $\mathscr{L}(\nu')$ are certain polynomials in the Pontrjagin classes of M and ν' respectively. Since M is a 2-dimensional manifold,

$$\mathscr{L}(M) = \mathscr{L}(\nu') = 1\ . \tag{3}$$

In view of (3) equation (2) simplifies to the following:

$$\text{Sign}\,(T, S') = e(\nu')[M]\ . \tag{4}$$

Thus to determine the possible values of the integer $e(\nu')[M]$ (and hence $e(\nu)[M]$, by equation (1)), we must determine the possible values of the signature, Sign (T, S').

We recall that Sign (T, S') may be defined as the signature of a quadratic form defined on the real cohomology group $H^2(S', R)$ as follows:

$$(x, y) = (x \cup T^*y)[S'],\ x, y \in H^2(S', R)\ .$$

Now by Lemma 2, $T^*(y) = -y$ for any $y \in H^2(S'\colon R)$, hence (x, y) is the negative of the usual quadratic form of the oriented 4-manifold S'; it follows that (x, y) is a nonsingular quadratic form. It is also proved in Lemma 2 than $H^2(S', R)$ has rank n, where $n = 2 - \chi$ is the (nonorientable) genus of the surface M (i.e., M is the connected sum of n projective planes). Therefore the possible values of Sign (T, S') are the following:

$$-n,\ -n+2,\ \cdots,\ n-2,\ n\ .$$

Whitney's conjecture now follows by making use of equation (1) and the equation $n = 2 - \chi$. To complete the proof, it remains to prove Lemmas 1 and 2; this is done in the following sections. We will also show that all the possible values of the integer $e(\nu)[M]$ can be attained by actual embeddings.

4. Statement of Lemma 1. Let B be a CW-complex, $p\colon E \to B$ and $p'\colon E' \to B$ locally trivial fibre spaces over B with fibres F and

F' respectively, and assume that $f: E \to E'$ is a fibre preserving map, i.e., the diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

is commutative. Finally, let us assume that the fibres F and F' are $(n-1)$-connected, $n \geqq 1$. Then the first obstructions to cross sections of these bundles are well-defined cohomology classes

$$c \in H^{n+1}(B, \pi_n(F)),\ c' \in H^{n+1}(B, \pi_n(F')) ,$$

(these are cohomology groups with local coefficients in general). The map f induces a coefficient homomorphism of cohomology groups,

$$f^{\#}: H^{n+1}(B, \pi_n(F)) \longrightarrow H^{n+1}(B, \pi_n(F'))$$

in an obvious way.

LEMMA 1. *Under the above hypotheses, the first obstructions satisfy the following naturality condition*:

$$f^{\#}(c) = c' .$$

The proof of this lemma is an easy consequence of the definition of obstructions. The details may be left to the reader.

5. Statement and proof of Lemma 2. In this section, we will use the same notation as in §2: $p: S' \to S^4$ is a 2-sheeted branched covering with the nonorientable surface M as the set of branch points, and $T: S' \to S'$ is the involution or covering transformation which interchanges the two sheets of the covering. The surface M is the connected sum of n projective planes, where $n = 2 - \chi$.

LEMMA 2. *The cohomology group $H^2(S', R)$ is a vector space over the reals of rank n and the homomorphism $T^*: H^2(S', R) \to H^2(S', R)$ induced by T satisfies the equation*

$$T^*(x) = -x,\ x \in H^2(S', R) .$$

The proof of this lemma involves several steps; as a first step, we will prove the following lemma which may be of independent interest:

LEMMA 3. *Let X be a finite, connected CW complex such that $H_1(X, Z)$ is cyclic of order 2, and let $\pi: \tilde{X} \to X$ denote the covering* *space corresponding to the commutator subgroup of* $\pi_1(X)$. *Then* $H_1(\tilde{X}, Z)$ *is a finite abelian group of odd order.*

Proof. Since $\pi: \tilde{X} \to X$ is a 2-sheeted covering space, it may be considered as a nonorientable 0-sphere bundle. Hence there is a Gysin sequence for this situation with Z_2-coefficients (see Thom [7]). We will make use of the following portion of this Gysin sequence:

$$H^{\circ}(X) \xrightarrow{\mu} H^1(X) \xrightarrow{\pi^*} H^1(\tilde{X}) \xrightarrow{\psi} H^1(X) \xrightarrow{\mu} H^2(X) .$$

Here the homomorphism $\mu: H^m(X, Z_2) \to H^{m+1}(X, Z_2)$ is cup product with the characteristic class, $w_1 \in H^1(X, Z_2)$. The hypothesis of the lemma implies that $H^1(X, Z_2)$ is cyclic of order 2; since $\tilde{X}$ is a nontrivial covering space, w_1 must be the unique nonzero element of $H^1(X, Z_2)$. From this it follows that $\mu: H^{\circ}(X) \to H^1(X, Z_2)$ is an isomorphism onto. We assert that $\mu: H^1(X) \to H^2(X)$ is a monomorphism; it then follows by exactness that $H^1(\tilde{X}, Z_2) = 0$. Since $H^1(\tilde{X}, Z_2) = \text{Hom}[H_1(\tilde{X}, Z), Z_2]$, and $H_1(\tilde{X}, Z)$ is a finitely generated abelian group, the conclusion of the lemma follows. It remains to prove the assertion. To do this, it suffices to prove that $\mu(w_1) \neq 0$. Now

$$\mu(w_1) = w_1 \cup w_1 = Sq^1(w_1) ,$$

and the homomorphism Sq^1 is well-known to be the composition of the Bockstein homomorphism (associated with the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$) and reduction mod 2. The hypothesis that $H_1(X, Z)$ is cyclic of order 2 enables one to prove that $Sq^1(w_1) \neq 0$; the details are left to the reader.

REMARK. Professor E. Schenkman has communicated to the author a purely group-theoretic proof of the following generalization of Lemma 3. Assume that X is a finite, connected CW-complex and $\pi; \tilde{X} \to X$ is the covering space corresponding to the commutator subgroup of $\pi_1(X)$, exactly as in the lemma. The generalization consists in assuming that $H_1(X, Z)$ is cyclic *of prime power order*. The conclusion is that $H_1(\tilde{X}, Z)$ is a finite abelian group, and the orders of $H_1(X, Z)$ and $H_1(\tilde{X}, Z)$ are relatively prime. Professor Schenkman also has an example to show that this conclusion does *not* necessarily hold if $H_1(X, Z)$ is a cyclic group of order 6.

We will now continue with the proof of lemma 2. Let A be a smooth closed tubular neighborhood of M in S^4, $C =$ closure of $S^4 - A$, and $E = A \cap C$. Then E is a closed, orientable 3-manifold which is the common boundary of A and C; also, E is a realization of the normal 1-sphere bundle $S(\nu)$. In general, we will denote the corresponding subsets of S' by means of primes, i.e.,

$$A' = p^{-1}(A)\,,$$
$$C' = p^{-1}(C), \text{ and}$$
$$E' = p^{-1}(E)\,.$$

Then A' is a closed tubular neighborhood of M in S', $A' \cup C' = S'$, and E' is the common boundary of A' and C'. Note that C is a deformation retract of $S^4 - M$, C' is a deformation retract of $S' - M$, and C' is a 2-fold (unbranched) covering of C. We can apply Lemma 3 with $X = C$, $\tilde{X} = C'$ to conclude that $H_1(C', Z)$ is a finite group of odd order. It follows immediately that

$$H^1(C', R) = 0\,. \tag{5}$$

Next, we wish to compute the real cohomology of the space E'. Since $E' = S(\nu')$ is a nonorientable 1-sphere bundle over M, we can use the Gysin sequence for this purpose:

$$\cdots \xrightarrow{\psi} H^{q-2}(M, \mathscr{R}) \xrightarrow{\mu} H^q(M, R) \xrightarrow{p^*} H^q(E', R)$$
$$\xrightarrow{\psi} H^{q-1}(M, \mathscr{R}) \xrightarrow{\mu} \cdots\,.$$

Here $H^m(M, \mathscr{R})$ means the m-dimensional cohomology group of M with local coefficient group the twisted real numbers. By the Poincaré duality theorem for nonorientable manifolds,

$$H^{q-2}(M, \mathscr{R}) \approx H_{4-q}(M, R)\,.$$

From this it follows readily that for any value of q, $H^{q-2}(M, \mathscr{R}) = 0$ or $H^q(M, R) = 0$. Therefore $\mu = 0$, and

$$\begin{aligned}\operatorname{rank} H^q(E', R) &= \operatorname{rank} H^q(M, R) + \operatorname{rank} H^{q-1}(M, \mathscr{R}) \\ &= \operatorname{rank} H^q(M, R) + \operatorname{rank} H_{3-q}(M, R)\,.\end{aligned}$$

From this we conclude that

$$\operatorname{rank} H^\circ(E', R) = \operatorname{rank} H^3(E', R) = 1\,, \tag{6}$$

$$\operatorname{rank} H^1(E', R) = \operatorname{rank} H^2(E', R) = n - 1\,. \tag{7}$$

Of course, (6) also follows from the fact that E' is a closed, connected, orientable 3-manifold.

Next, we consider the real cohomology sequence of the pair (C', E'). By making use of (5) and the fact that

$$\operatorname{rank} H^q(C', E', R) = \operatorname{rank} H^{4-q}(C', R)$$

(which is a consequence of the Lefschetz-Poincaré duality theorem for orientable manifolds with boundary) we conclude that

$$(8) \qquad H^3(C', R) = H^4(C', R) = 0 .$$

Next, since C' is a 2-sheeted covering of C, we have the following obvious relation between the Euler characteristics:

$$\chi(C') = 2 \cdot \chi(C) .$$

Now one readily computes that $\chi(C) = n$ (use the Alexander duality theorem). Hence $\chi(C') = 2n$; then making use of (5) and (8) we conclude that

$$(9) \qquad \operatorname{rank} H^2(C', R) = 2n - 1 .$$

Next, we will use the information we have already obtained about $H^*(C', R)$ and the real cohomology sequence of the pair (S', C') to determine $H^*(S', R)$. For this purpose, note that by the excision property,

$$H^q(S', C') \approx H^q(A', E') .$$

Now the pair (A', E') is the Thom space of the normal bundle ν'; therefore, we can apply the Thom isomorphism theorem for (non-orientable) vector bundles to conclude that

$$H^q(A', E', R) \approx H^{q-2}(M, \mathscr{R}) .$$

Also $H^{q-2}(M, \mathscr{R}) \approx H_{4-q}(M, R)$, as was noted above. Combining these isomorphisms, we see that

$$(10) \qquad \operatorname{rank} H^4(S', C', R) = 1 ,$$

$$(11) \qquad \operatorname{rank} H^3(S', C', R) = n - 1 ,$$

$$(12) \qquad H^q(S', C', R) = 0 \text{ for } q \neq 3 \text{ or } 4 .$$

If we incorporate all the information we have obtained about $H^*(S', C', R)$ and $H^*(C', R)$ together with the fact that

$$\operatorname{rank} H^3(S', R) = \operatorname{rank} H^1(S', R)$$

(which is a consequence of the Poincaré duality theorem) into the cohomology sequence of the pair (S', C'), we see that

$$\operatorname{rank} H^2(S', R) = n ,$$

as was to be proved. We note that it also follows that

$$H^1(S', R) = H^3(S', R) = 0 .$$

It remains to prove the last statement of Lemma 2. For this purpose, note that the projection $p: S' \to S^4$ induces a map of the real cohomology sequence of the pair (S^4, C) into that of the pair (S', C'); hence we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & H^2(C) & \xrightarrow{\delta} & H^3(S^4, C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_2^*} & & \\ 0 & \longrightarrow & H^2(S') & \xrightarrow{i^*} & H^2(C') & \xrightarrow{\delta'} & H^3(S', C') & \longrightarrow & 0 \end{array} \tag{13}$$

The involution T^* operates on each of the real vector spaces in the bottom line of this diagram, and the homomorphisms i^* and δ' commute with T^*. Each of these three vector spaces decomposes into the direct sum of two subspaces corresponding to the eigenvalues $+1$ and -1 respectively of the involution T^*. These subspaces are respectively the subspace of elements left fixed by T^*, and the subspace consisting of those elements x such that $T^*(x) = -x$. The homomorphisms i^* and δ' respect these direct sum decompositions. Furthermore, it is clear that the images of p_1^* and p_2^* are contained in the subspaces of elements left fixed by T^*.

Next, we assert that p_1^* is a monomorphism, and its image is the entire subspace of elements of $H^2(C')$ which are left fixed by T^*. To prove this, note that C' is a covering space of C; hence we can apply the results of appendix No. 2. By equation (V) we see that

$$H^2(C') = \text{image } p_1^* \oplus \text{kernel } t^* \tag{14}$$

and by (VI) the elements of kernel t^* satisfy the equation

$$x + T^*(x) = 0 ,$$

i.e., $T^*(x) = -x$. Thus the direct sum decomposition in (14) is the same as that corresponding to the eigenvalues of T^*.

Finally, we assert that p_2^* is an isomorphism. This follows from consideration of the following diagram:

$$\begin{array}{ccccc} H^3(S^4, C) & \xrightarrow{j} & H^3(A, E) & \xleftarrow{\varphi} & H^1(A, \mathscr{R}) \\ \downarrow{\scriptstyle p_2^*} & & \downarrow{\scriptstyle p_3^*} & & \downarrow{\scriptstyle p_4^*} \\ H^3(S', C') & \xrightarrow{j'} & H^3(A', E') & \xleftarrow{\varphi'} & H^1(A', \mathscr{R}) \end{array} \tag{15}$$

The left hand square of this diagram is commutative, and j and j' are isomorphisms by the excision property. (A, E) and (A', E') are the Thom spaces of the bundles ν and ν' respectively, and φ and φ' are the Thom isomorphisms defined by

$$\varphi(x) = x \cup U ,$$
$$\varphi'(y) = y \cup U' ,$$

where $U \in H^2(A, E; \mathscr{R})$ and $U' \in H^2(A', E', \mathscr{R})$ are the Thom classes (with twisted coefficients) of the bundles ν and ν' respectively. Note that $p_4: A' \to A$ is a homotopy equivalence, hence p_4^* is a isomorphism. The Thom classes are related by the following equation

$$p_3^*(U) = \pm 2U'$$

since the projection $p_3: (A', E') \to (A, E)$ is a fibre preserving map having degree ± 2 on each fibre (cf. Spanier [6], Chapter V, §7). Thus the right hand square of the diagram (15) is commutative up to a factor of ± 2. Putting all these facts together, we see that p_2^* is an isomorphism, as asserted.

It follows that every element of $H^3(S', C')$ is left fixed by T^*. Therefore every element of the subspace (kernel t^*) of $H^2(C')$ (i.e., those corresponding to the eigenvalue -1) is contained in kernel $\delta' =$ image i^*. But it is readily seen that

$$\text{rank (kernel } t^*) = n, \text{ and}$$
$$\text{rank (image } i^*) = n .$$

Therefore

$$\text{image } i^* = \text{kernel } t^* .$$

Since i^* is a monomorphism, it follows that on the vector space $H^2(S')$ the only eigenvalue of T^* is -1. This completes the proof of Lemma 2.

6. Proof that all possible values of the integer $e(\nu)[M]$ can actually be realized. It follows readily from our conventions that changing the orientation of the 4-sphere, S^4, changes the sign of the integer $e(\nu)[M]$. Alternatively, we could achieve the same result by keeping the orientation of S^4 fixed and replacing the given embedding $i: M \to S^4$ by the composite

$$M \xrightarrow{i} S^4 \xrightarrow{h} S^4$$

where h is an orientation reversing diffeomorphism of S^4.

If we are given two pairs (S_i^4, M_i), $i = 1, 2$, consisting of an oriented 4-sphere and a smoothly embedded nonorientable surface, we can form the connected sum

$$(S^4, M) = (S_1^4, M_1) \# (S_2^4, M_2)$$

as defined by Haefliger [3]. Denote the normal bundles of M, M_1, and M_2 by ν, ν_1, and ν_2 respectively. We then have the following equation:

$$e(\nu)[M] = e(\nu_1)[M_1] + e(\nu_2)[M_2] .$$

The proof of this equation is not difficult; we leave it to the reader.

Let P be a real projective plane imbedded smoothly in an oriented 4-sphere, S^4 with normal bundle ν. It is a consequence of the theorem proved so far that

$$e(\nu)[P] = \pm 2 ,$$

the sign depending on the orientation of S^4. Let us assume that the orientation is chosen so that

$$e(\nu)[P] = 2 .$$

If we now form the connected sum of i copies of the pair (S^4, P) and $(n - i)$ copies of the pair $(-S^4, P)$, we obtain a pair (S^4, M) such that

$$e(\nu_M)[M] = 4i - 2n ,$$

and $\chi(M) = 2 - n$. By choosing $i = 0, 1, 2, \cdots, n$ we obtain all possible values for the Euler class of the normal bundle of a surface M with $\chi(M) = 2 - n$.

Appendix 1. Generalities on orientations of vector bundles and local coefficients. If $E \to B$ is an n-dimensional real vector bundle over the space B, we will consistently use the notation $S(E) \to B$ and $D(E) \to B$ to denote the associated $(n - 1)$ -sphere bundle and the associated n-dimensional disc bundle respectively. For any point $b \in B$, the fibres of these bundle will be denoted by E_b, $S(E)_b$, and $D(E)_b$ respectively. Associated with the bundle $E \to B$ is a certain local system of groups $O(E)$, called "the local coefficient system of orientations of E". This local system of groups associates with each point $b \in B$ the group $H_n(D(E)_b, S(E)_b; Z)$ (or alternatively, the group $H_{n-1}(S(E)_b; Z)$ or $\pi_{n-1}(S(E)_b)$; these different groups are related by obvious canonical isomorphisms). The Euler class, $e(E)$, is an n-dimensional cohomology class with coefficients in $O(E)$. Note that the local system $O(E)$ is determined up to isomorphism by the first Stiefel-Whitney class, $w_1(E)$.

If M is a (possibly nonorientable) differentiable closed, connected n-manifold, the local coefficient system of orientations of M is, by definition, the local coefficient system of orientations of the tangent bundle of M; it is denoted by $O(M)$. The "fundamental homology class of M" is a uniquely defined homology class, $[M] \in H_n(M, O(M))$. If M is triangulated, it is represented by an n-cycle which assigns to each *oriented* n-simplex the corresponding "local orientation".

Let E and E' be vector bundles over B, and let $E \oplus E$ denote their Whitney sum. There is a natural isomorphism

$$O(E) \otimes O(E') \approx O(E \oplus E'))$$

which is determined at each point $b \in B$ by the natural isomorphism

$$H_n(D(E)_b, S(E)_b) \otimes H_{n'}(D(E')_b, S(E')_b)$$
$$\approx H_{n+n'}(D(E \oplus E')b, S(E \oplus E')_b) .$$

This natural isomorphism can also be looked on as a bilinear pairing

$$O(E) \times O(E') \longrightarrow O(E \oplus E') ,$$

which can be used to define cup products, cap products, etc.

Given the bundle E over a connected space B, the local system of groups $O(E)$ is isomorphic to a local system of groups in B which assigns to each point $b \in B$ the additive group of integers, Z, with the "twisting" of this local system of integers determined by $w_1(E)$. We will denote this local system of integers by $\mathscr{Z}$. As a matter of fact, there are actually two distinct isomorphisms between the local systems $O(E)$ and $\mathscr{Z}$; to choose one of them as a preferred isomorphism amounts to "orienting" the bundle E in some sense, even though the bundle E may be nonorientable in the usual sense.

Appendix 2. The transfer homomorphism in a covering space. Let X be an arcwise connected topological space and $p: \tilde{X} \rightarrow X$ a regular covering space of X with *finitely many* sheets. In this section, we shall consider some relations between the homology and cohomology groups of X and $\tilde{X}$. These relations are well known, but do not seem to have been published anywhere.

We will simultaneously consider the following two situations:

(a) X and $\tilde{X}$ are simplicial polyhedra, p is a simplicial map, and we use simplicial chains and cochains.

(b) X and $\tilde{X}$ are not assumed triangulated, and we use singular chains and cochains.

In either case, the projection p induces a chain transformation

$$p_\#: C_*(\tilde{X}) \longrightarrow C_*(X) .$$

Since the covering is assumed to have only finitely many sheets, the so called "transfer homomorphism" is defined in the opposite direction:

$$t: C_*(X) \longrightarrow C_*(\tilde{X}) .$$

The definition of t is as follows: for any n-simplex σ of X, $t(\sigma)$ is defined to be the sum of all n-simplexes σ' of $\tilde{X}$ such that $p_\#(\sigma') = \sigma$. It is readily verified that t is a chain transformation. One can also easily verify the following two relations:

(I) $$tp_\#(u) = \sum_{g \in G} g_\#(u),\ u \in C_*(\tilde{X}) ,$$

(II) $$p_\sharp t(v) = mv, \qquad v \in C_*(X) .$$

Here G denotes the group of covering transformations of $\tilde{X}$, and m denotes the number of sheets of the covering.

If we pass to cohomology with any coefficients, we have induced homomorphisms.

$$p^*: H^*(X) \longrightarrow H^*(\tilde{X})$$
$$t^*: H^*(\tilde{X}) \longrightarrow H^*(X)$$

and the relations (I) and (II) lead to the following relations:

(III) $$p^*t^*(x) = \sum_{g \in G} g^*(x),\ x \in H^*(\tilde{X})$$

(IV) $$t^*p^*(y) = my, \qquad y \in H^*(X) .$$

Let us assume that we use a field for coefficients *whose characteristic does not divide the number of sheets, m.* Then from (IV) we easily deduce that p^* is a monomorphism, t^* is an epimorphism, and $H^*(\tilde{X})$ breaks up into a direct sum,

(V) $$H^*(\tilde{X}) = \text{image } p^* \oplus \text{kernel } t^* .$$

The elements of the direct summand image p^* are obviously left fixed by the homomorphisms g^* for all $g \in G$. It follows from equation (III) that the elements of the direct summand kernel t^* satisfy the following condition:

(VI) $$\sum_{g \in G} g^*(x) = 0 .$$

Appendix 3 (*added in proof, August,* 1969). Glen Bredon has pointed out in a letter to the author (dated May 14, 1969) that the proof of Lemma 2 can be considerably shortened, as follows:

The last statement of Lemma 2 is an immediate consequence of the known fact that the homomorphism $p^* : H^*(S^4; R) \rightarrow H^*(S'; R)$ is an isomorphism onto the set of invariant elements of T^*; see Theorem 19.1 on page 85 of Bredon's book (*Sheaf Theory*, McGraw-Hill, 1967). The proof of this theorem depends on the fact that the notion of the transfer homomorphism (see Appendix 2) can be generalized to cover the case of an arbitrary action of a finite group on a Hausdorff space, provided, one uses a Čech type cohomology theory; see Bredon's book (*loc. cit.*) or *Annals of Mathematics Study No.* 46, *Seminar on Transformation Groups*, Chapter III, § 2 (by E. E. Floyd).

The first statement of Lemma 2 can be proved more directly by use of the exact sequences of P. A. Smith (see Bredon, *loc. cit.*, page 86, or Floyd, *loc. cit.*, Chapter III, § 4). In the case at hand this gives the following exact sequence (Z_2 coefficients):

$$\cdots \longrightarrow H^i(S^4, M) \longrightarrow H^i(S') \longrightarrow H^i(S^4, M) \oplus H^i(M) \longrightarrow H^{i+1}(S^4, M) \longrightarrow \cdots$$

Also, the part $H^i(M) \to H^{i+1}(S^4, M)$ of the connecting homomorphism is just the coboundary for the pair (S^4, M). From this it follows immediately that $H^1(S'; Z_2) = 0 = H^3(S'; Z_2)$ and the vector space $H^2(S'; Z_2)$ has rank n. Then one applies the universal coefficient theorem to conclude that $H^2(S'; R)$ is also a vector space of rank n.

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EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

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Denote by S a complex (nondegenerate) Banach space. Suppose that T is a transformation from a subset of S to S. A complex number λ is said to be in the resolvent of T if $(\lambda I - T)^{-1}$ exists, has domain S and is Fréchet differentiable (i.e., if p is in S there is a unique continuous linear transformation $F = [(\lambda I - T)^{-1}]'(p)$ from S to S so that

$$\lim_{q \to p} ||q - p||^{-1} ||(\lambda I - T)^{-1}q - (\lambda I - T)^{-1}p - F(q - p)|| = 0)$$

and locally Lipschitzean everywhere on S. A complex number is said to be in the spectrum of T if it is not in the resolvent of T.

Suppose in addition that the domain of T contains an open subset of S on which T is Lipschitzean.

THEOREM. *T has a (nonempty) spectrum.*

If T is a continuous linear transformation from S to S, then the notion of resolvent and spectrum given here coincides with the usual one ([1], p. 209, for example). Such a transformation T is, of course, Lipschitzean on all of S and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by C.

LEMMA. *Suppose that $d > 0$, p is in S, Q is a transformation from a subset of S to S, D is an open set containing p which is a subset of the domain Q, Q is Lipschitzean on D and $(I - cQ)^{-1}$ exists and has domain S if c is in C and $|c| < d$. Then,*

$$\lim_{c \to 0} (I - cQ)^{-1}p = p .$$

Proof. Denote by M a positive number so that $||Qr - Qs|| \leqq M||r - s||$ if r and s are in D. Suppose $\varepsilon > 0$. Denote by δ a number so that $0 < \delta < \min(\varepsilon, 1/2)$ and $\{q \in S: ||q - p|| \leqq \delta\}$ is a subset of D. Denote by δ' a positive number so that $\delta'(\max(M, ||Qp||)) < \delta/2$. Denote by c a member of C so that $|c| < \min(\delta', d)$. Denote $(I - cQ)^{-1}p$ by q, denote p by q_0 and $p + cQq_{n-1}$ by q_n, $n = 1, 2, \cdots$.

Then, $||q_1 - q_0|| = ||p + cQq_0 - q_0|| = |c|\,||Qq_0|| < \delta/2$. Suppose that k is a positive integer so that

$$||q_m - q_{m-1}|| < (\delta/2)^m, m = 1, 2, \cdots, k .$$

Then $\| q_m - p \| \leqq \sum_{j=0}^{m-1} \| q_{j+1} - q_j \| \leqq \sum_{j=0}^{m-1} (\delta/2)^{j+1} < \delta$, $m = 0, 1, \cdots, k$ and hence

$$\begin{aligned} \| q_{k+1} - q_k \| &= \| cQq_k - cQq_{k-1} \| \\ &\leqq | c | M \| q_k - q_{k-1} \| \\ &\leqq | c | M(\delta/2)^k \leqq (\delta/2)^{k+1} . \end{aligned}$$

Hence $\| q_n - q_{n-1} \| \leqq (\delta/2)^n$, $n = 1, 2, \cdots$ and therefore $q_1, q_2, \cdots$ converges to a point r of S. Note that $\| q_{n+1} - p \| \leqq \sum_{j=0}^{n} (\delta/2)^{j+1} < \delta$, $n = 1, 2, \cdots$ so that $\| r - p \| \leqq \delta$ and hence r is in D. But $\| r - (p + cQr) \| = \| (r - q_{n+1}) + (p + cQq_n) - (p + cQr) \| \leqq \| r - q_{n+1} \| + | c | \| Qq_n - Qr \| \leqq \| r - q_{n+1} \| + | c | M \| q_n - r \| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $r = p + cQr$, i.e., $(I - cQ)r = p$, i.e., $r = (I - cQ)^{-1}p = q$. Hence, $\| (I - cQ)^{-1}p - p \| \leqq \delta < \varepsilon$. This proves the lemma.

Proof of theorem. Suppose the statement of the theorem is false. Then T has an inverse since if not, 0 would be in the spectrum of T. Denote by D an open set on which T is defined and is Lipschitzean. Denote by p a point of D different from $-T(0)$.

Define $f(\lambda)$ to be $(\lambda I - T)^{-1}p$ for all λ in C. Suppose b is in C. If q is in S and different from p denote

$$(1/\| q - p \|)\{[bI - T)^{-1}q - (bI - T)^{-1}p] - [(bI - T)^{-1}]'(p)(q - p)\}$$

by $L(q)$. Denote by $L(p)$ the zero element of S and note that $\lim_{p \to p} L(q) = L(p)$ since $(bI - T)^{-1}$ is Fréchet differentiable at p. Denote $(bI - T)^{-1}$ by Q. If λ is in C, then

$$(\lambda I - T) = [I - (b - \lambda)(bI - T)^{-1}](bI - T)$$

and, since both $(\lambda I - T)^{-1}$ and $(bI - T)^{-1}$ exist and have domain S, it follows that $[I - (b - \lambda)(bI - T)^{-1}]^{-1} = [I - (b - \lambda)Q]^{-1}$ has the same properties and $(\lambda I - T)^{-1} = Q[I - (b - \lambda)Q]^{-1}$.

Hence, if λ is in C,

$$\begin{aligned} f(\lambda) - f(b) &= Q[I - (b - \lambda)Q]^{-1}p - Qp \\ &= Q'(p)[[I - (b - \lambda)Q]^{-1}p - p] \\ &\quad + \| [I - (b - \lambda)Q]^{-1}p - p \| L([I - (b - \lambda)Q]^{-1}p) . \end{aligned}$$

But $[I - (b - \lambda)Q]^{-1}p - p = (b - \lambda)Q[I - (b - \lambda)Q]^{-1}p$ so

$$\begin{aligned} &(\lambda - b)^{-1}[f(\lambda) - f(b)] \\ &\quad = -Q'(p)Q[I - (b - \lambda)Q]^{-1}p \\ &\qquad + (| b - \lambda |/(\lambda - b)) \| Q[I - (b - \lambda)Q]^{-1}p \| \\ &\qquad \times L([I - (b - \lambda)Q]^{-1}p) \longrightarrow -Q'(p)Qp \end{aligned}$$

as $\lambda \longrightarrow b$ since $\lim_{\lambda \to b}[I - (b - \lambda)Q]^{-1}p = p$. Hence,

$$f'(b) = -[(bI - T)^{-1}]'(p)(bI - T)^{-1}p .$$

Now $\lim_{c\to 0}(I - cT)^{-1}p = p$. Denote by δ a positive number so that if $|c| \leqq \delta$, then $\|(I - cT)^{-1}p\| \leqq \|p\| + 1$. Then if λ is in C and $|\lambda| \geqq 1/\delta$, $\|f(\lambda)\| = \|(\lambda I - T)^{-1}p\| = |1/\lambda|\ \|(I - (1/\lambda)T)^{-1}p\| \leqq \delta(\|p\| + 1)$. Hence f is bounded. So, by Liouville's theorem ([1], p. 129, for example), f is constant, i.e., there is a point q in S such that if λ is in C, $(\lambda I - T)^{-1}p = f(\lambda) = q$, and so $\lambda q = p + Tq$. Hence it must be that $q = 0$, i.e., $p = -T(0)$, a contradiction. This establishes the theorem.

The author considers it likely that the statement of the theorem is true if the condition (in the definition of resolvent) that $(\lambda I - T)^{-1}$ be locally Lipschitzean is dropped.

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MEASURE ALGEBRAS ON IDEMPOTENT SEMIGROUPS

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Taylor has shown that for every commutative convolution measure algebra M there is a compact topological semigroup S, called the structure semigroup of M, and an embedding $\mu \rightarrow \mu_S$ of M into $M(S)$ such that every complex homomorphism of M has the form $h_f(\mu) = \int_S f d\mu_s$ for some semicharacter f on S.

This paper deals with commutative convolution measure algebras whose structure semigroups are idempotent. The measure algebra on the interval $[0, 1]$, where the interval is given the semigroup operation of maximum multiplication, is an algebra of this type. These algebras are studied in this general setting in the hope of shedding new light on the known theory of measure algebras on locally compact idempotent semigroups and in the hope of extending attempts to classify a convolution measure algebra in terms of the algebraic nature of its structure semigroup.

An example is given of a measure algebra on a compact idempotent semigroup whose structure semigroup is not idempotent.

Our goal in this paper is to apply the structure theory for commutative convolution measure algebras developed by Taylor [5] to a special class of algebras which includes those studied by Hewitt and Zuckerman [3], Ross [4], and Baartz [1]. We will assume that each convolution measure algebra mentioned in this paper is commutative and, in addition, that each semigroup mentioned is commutative. We begin by giving the essential features of Taylor's structure theory.

A convolution measure algebra is roughly an ordered Banach space of measures with a multiplication which makes it a Banach algebra and which relates appropriately to the norm and the order. For a precise definition, see [5]. Examples include $L^1(G)$, the algebra of all absolutely continuous measures on a locally compact group G; $M(G)$, the measure algebra on G (all bounded regular Borel measures on G); and $M(S)$, the measure algebra on a locally compact semigroup S. Convolution is the multiplication operation in each of these examples. Both $L^1(G)$ and $M(G)$ are semisimple algebras as is $M(S)$ under certain not-too-restrictive conditions. We will therefore focus our attention only upon semisimple algebras.

Let M denote a semisimple convolution measure algebra. Taylor has shown in [5] that there is a compact topological semigroup S, called the structure semigroup of M, and an embedding $\mu \rightarrow \mu_S$ of M

into $M(S)$ with the following properties.

(A) $\mu \to \mu_S$ is an algebraic isomorphism and an order preserving isometry.

(B) The image M_S of the map $\mu \to \mu_S$ is weak* dense in $M(S)$; i.e., M_S separates points in $C(S)$.

(C) $C(S)$ is the closed linear span of $\hat{S}$; i.e., $\hat{S}$ separates points of S. ($\hat{S}$ is the collection of all continuous semicharacters on S),

(D) Each complex homomorphism of M has the form $h_f(\mu) = \int_S f d\mu_s$ for some f in $\hat{S}$.

The reader will recall that a semicharacter is a nonzero, bounded, complex valued function f defined on the semigroup S which satisfies $f(x \cdot y) = f(x)f(y)$ for all x and y in S. As a result of (D), the set $\hat{S}$ of semicharacters with the weak* topology induced by M can be considered the maximal ideal space of M. We will regard a semicharacter f in S as both a continuous function on S and a complex homomorphism of M via the indentification given by (D), Thus, we write $f(\mu)$ in place of $h_f(\mu)$.

We are now in a position to define the type of algebra which will be our object of study.

DEFINITION 1. A semisimple convolution measure algebra M will be called a P-algebra provided $f(\mu) \geqq 0$ for every positive measure μ in M and every complex homomorphism f of M.

Examples of P-algebras are the measure algebra $M(T)$, under convolution, of the compact semigroup $T = [a, b]$ with multiplication $x \cdot y = \max\{x, y\}$ [3], and more generally, the measure algebra $M(T)$ of a finite product T of locally compact, totally ordered spaces with co-ordinatewise maximum multiplication [1]. In both examples, each complex homomorphism of $M(T)$ has the form

$$h_A(\mu) = \int_T \chi_A d\mu \qquad \mu \in M(T) ,$$

for some subsemigroup A of T whose complement $T \backslash A$ is a (prime) ideal of T (Definition 1.5, [1]). Consequently, $M(T)$ is a P-algebra. In § 4 we will give an alternate proof the $M(T)$ is a P-algebra, based on the results of § 3.

We pause to define several terms with which the reader may not be familiar. The reader is referred to Taylor [5] for terms not defined here.

Let M be a convolution measure algebra.

DEFINITION 2. A closed subspace (subalgebra, ideal) N of M is called an L-subspace (subalgebra, ideal) if whenever $\mu \in N$, then $\nu \in N$ for all $\nu < \mu(\nu$ absolutely continuous with respect to μ).

DEFINITION 3. An L-ideal N of M is called a prime L-ideal if $N^{\perp} = \{\mu \in \mu \mid \mu \perp \nu$ (μ and ν are mutually singular) for all $\nu \in N\}$ is a subalgebra of M.

2. Some characterizations of P-algebras. Our first theorem gives six equivalent conditions for a semisimple convolution measure algebra M to be a P-algebra. The identity e mentioned in statements (5) and (6) of the theorem is the identity in M if M has an identity and is the identity adjoined to M in the usual manner if M does not have an identity. Similarly, the inversion mentioned in statement (6) takes place in the algebra M if M has an identity, and in the algebra "M with identity adjoined" if M does not have an identity.

THEOREM 1. *Let M be a semisimple convolution measure algebra. Then the following statements are equivalent.*

(1) *M is a P-algebra.*

(2) *$\hat{S}$ is an idempotent semigroup.*

(3) *S is an idempotent semigroup.*

(4) *For each $f \in \hat{S}$, $M = N_f + N_f^{\perp}$ where N_f is a prime L-ideal such that if $\mu = \mu_1 + \mu_2(\mu_1 \in N_f, \mu_2 \in N_f^{\perp})$, then $f(\mu) = (\mu_2)_S(S)$.*

(5) *The spectral radius of $\mu - e$ is less than or equal to one for every positive measure μ of norm one.*

(6) *$\mu + e$ is invertible for every positive measure μ of norm one.*

Proof. The order of proof will be $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (1)$ and $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)$. $(1) \Rightarrow (2)$. Since the integral of each semicharacter in $\hat{S}$ with respect to any positive measure in M is nonnegative, each semicharacter in S is a nonnegative function by (B). Let f be in $\hat{S}$. Then f is nonnegative and hence for fixed z, f^z is in $\hat{S}$ if $\operatorname{Re} z > 0$. Now $g_s(z) = f^z(s)$ is analytic in $\operatorname{Re} z > 0$ for fixed $s \in S$. But then $g_s(z)$ is a nonnegative analytic function and is therefore constant. If we evaluate $g_s(z)$ at $z = 1$, we obtain $f^z(s) = f(s)$ for all $\operatorname{Re} z > 0$. If we let $z = 2$ in the above equality, we obtain $f^2(s) = f(s)$ for each $s \in S$ and hence $f^2 = f$. Therefore $\hat{S}$ is an idempotent semigroup.

(2) $\Leftrightarrow$ (3). Let $f \in \hat{S}$ and $s \in S$. We conclude that $f(s) = f^2(s) = f(s)f(s) = f(s \cdot s)$ since $f^2 = f$. Thus $s \cdot s = s$ by (C), and hence S is idempotent. Obviously, $(3) \Rightarrow (2)$.

(3) $\Rightarrow$ (4). If S is idempotent, so is $\hat{S}$. Given $f \in \hat{S}$, let $J = \{s \in S \mid f(s) = 0\}$, We note that J is a prime ideal in S. If we let $N_f = \{\mu \in M \mid \mu_S$ is concentrated on $J\}$, then N_f is a prime L-ideal

(Theorem 3.2, [5]) with orthogonal complement $N_f^{\perp}$. Thus if $\mu = \mu_1 + \mu_2$ where $\mu_1 \in N_f$ and $\mu_2 \in N_f^{\perp}$, then

$$f(\mu) = \int_S f d\mu_s = \int_S fd(\mu_1 + \mu_2)_s = \int_J fd(\mu_1)_s + \int_S fd(u_2)_s = (\mu_2)_s(S) .$$

(4) ⇒ (1). Obvious.

(1) ⇒ (5). Recall that the spectral radius of an element χ is a Banach algebra (written $\|\chi\|_{sp}$) is given by $\|\chi\|_{sp} = \lim_{n\to\infty} \|\chi^n\|^{1/n}$. If the algebra is commutative, then the spectral radius of χ is also the supremum norm of the Gelfand transform of χ. If (1) holds, then, by definition, each complex homomorphism takes positive measures to nonnegative numbers. Let μ be a positive measure in M of norm one. Then $0 \leqq f(\mu) = \hat{\mu}(f) \leqq 1$ for every $f \in \hat{S}$. Therefore,

$$1 \geqq \sup_{f \in \hat{S}} |(\mu - e)^\wedge(f)| = \|(\mu - e)^\wedge\|_\infty = \|\mu - e\|_{sp} .$$

(5) ⇒ (6). Let μ be a positive measure in M. Then $\mu/\|\mu\|$ is a positive measure of norm one and hence

$$\|(\mu/\|\mu\| - e)^\wedge\|_\infty = \|\mu/\|\mu\| - e\|_{sp} \leqq 1 .$$

Clearly $\hat{\mu}$ can never assume the value -1. Thus $-1 \notin \sigma(\mu)$, and hence $\mu + e$ is invertible.

(6) ⇒ (1). Let μ be a positive measure in M. If $\lambda > 0$, then $\mu/\lambda + e$ is invertible. Hence $\mu + \lambda e$ is invertible and $-\lambda \notin \sigma(\mu)$. Therefore, the spectrum of any positive measure in M contains no negative members. We claim that this fact ensures us that every positive measure will have real, nonnegative spectrum. For suppose μ is a positive measure whose spectrum is not real. Then there is an $f \in \hat{S}$ such that $\hat{\mu}(f) = \lambda = \lambda_1 + i\lambda_2$ where λ_1 and λ_2 are real and $\lambda_2 \neq 0$. We can choose a number $t > 0$ such that $\exp(t\lambda) < 0$. Thus $\exp(t\mu)$ is a positive measure with a negative number in its spectrum, a contradiction. The proof of the theorem is complete.

An L-subalgebra of a convolution measure algebra is again a convolution measure algebra. Since the spectral radius of a measure depends only upon the norms of the measure and its powers, statement (5) together with the above observation yields the following corollary.

COROLLARY. *Every L-subalgebra of a P-algebra is a P-algebra.*

3. A sufficient condition.

Our next theorem gives a sufficient condition for an algebra to be a P-algebra. We suspect that the condition is also necessary but have not been able to prove it.

THEOREM 2. *Let M be a semisimple convolution measure algebra* *with structure semigroup S. If for every positive (nonzero) measure μ in M there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ of positive measures in M such that*

(1) $\mu_n \to \mu$,

(2) $\mu_n * \nu' = \mu_n(S)\nu'$ *for all* $\nu' < \nu_n$ *and each* n,

(3) $\nu_n < \sum_{m=0}^{\infty} \mu^m/2^m$ *for all* n

then M is a P-algebra.

Proof. Throughout this proof, M is considered a subalgebra of $M(S)$. Let λ be a positive measure in M, let f be in $\hat{S}$, and let $J = \{s \in S \mid f(s) = 0\}$. Then J is a prime ideal in S, and hence

$$M_J = \{\nu \in M \mid \nu \text{ is concentrated on } J\}$$

is a prime L-ideal of M with orthogonal complement $M_J^\perp$. Define μ, μ' by $\mu'(E) = \lambda(E \cap J)$, for Borel sets E, and $\mu = \lambda - \mu'$. Then $\mu \in M_J$ and $\mu' \in M_J^\perp$.

If $\mu = 0$, then $f(\lambda) = 0$. If $\mu \neq 0$, then choose sequences $\{\mu_n\}$ and $\{\nu_n\}$ guaranteed by the hypothesis of the theorem. We claim that there is a measure $\nu_n' < \nu_n$ such that $f(\nu_n') \neq 0$ for each n. If not, then for some n, $f(\nu_n') = \int f d\nu_n' = 0$ for all $\nu_n' < \nu_n$ and $f = 0$ a.e. $[\nu_n]$. Thus there is a Borel set $E \subset S$ of ν_n-measure zero such that $f = 0$ on E. Hence $\nu_n \in M_J$. But since $\mu \in M_J^\perp$ and $M_J^\perp$ is an L-subalgebra, $\sum_{m=0}^{\infty} \mu^m/2^m$ is in $M_J^\perp$. Since $\nu_n < \sum_{m=0}^{\infty} \mu^m/2^m$, ν_n is in $M_J^\perp$. Therefore, $\nu_n \perp \nu_n$ and so $\nu_n = 0$, a contradiction. This establishes our claim.

Choose measures $\nu_n' < \nu_n$ such that $f(\nu_n') \neq 0$. Note that since $\mu_n * \nu_n' = \mu_n(S)\nu_n'$ and $f(\nu_n') \neq 0$, then, $f(\mu_n) = \mu_n(S) \geqq 0$. But $\mu_n \to \mu$; thus, $f(\mu_n) \to f(\mu)$. Therefore, $f(\mu) = \mu(S) \geqq 0$ and $f(\lambda) = f(\mu + \mu') = f(\mu) + f(\mu') = f(\mu) \geqq 0$. Hence M is a P-algebra.

4. An application of Theorem 2. If T is an idempotent semigroup we can introduce a partial ordering "$\leqq$" in T by defining $x \leqq y$ if and only if $x \cdot y = y$ for all x any y in T. A totally ordered idempotent semigroup is one in which the above partial ordering is a total ordering. Our goal in this section is to show that the measure algebra on a finite product of totally ordered, locally compact, idempotent semigroups is a P-algebra. This result follows trivially from a theorem of Baartz (Theorem 3.5, [1]); however, we shall give an independent development using Theorem 2. We will need the three lemmas that follow.

LEMMA 1. *Let T be a locally compact idempotent semigroup and let μ and ν be in $M(T)$. Suppose* supp $\mu \leqq$ supp ν *in the sense that* *for any* $s \in \operatorname{supp} \mu$ *and* $t \in \operatorname{supp} \nu$, $s \leqq t (s \cdot t = t)$. *Then* $\mu * \nu = \mu(T)\nu$.

Proof. Supp μ denotes the support of the measure μ. Let $A = \operatorname{supp} \mu$, $B = \operatorname{supp} \nu$, and E be a Borel subset of T. Then

$$\mu * \nu(E) = \iint \chi_E(x \cdot y) d\mu(x) d\nu(y) = \int \mu(E_y) d\nu(y) \text{ where } E_y = \{x \in T \mid x \cdot y \in E\} .$$

But $\int_T \mu(E_y) d\nu(y) = \int_B \mu(E_y) d\nu(y)$ since ν is concentrated on B. Furthermore,

$$\mu(E_y) = \begin{cases} 0 & y \in B \backslash E \\ \mu(T) & y \in B \cap E \end{cases}$$

since for $y \in B \cap E$, $A \subset E_y$ and for $y \in B \backslash E$, $A \cap E_y = \phi$. Thus

$$\int_B \mu(E_y) d\nu(y) = \mu(T)\nu(E)$$

and hence $\mu * \nu = \mu(T)\nu$.

LEMMA 2. *Let* T *be a totally ordered, locally compact, idempotent semigroup and let* μ *be a positive measure in* $M(T)$. *Then given* $\varepsilon > 0$, *there is an* $x \in T$ *such that* $\mu(\{y \in T \mid y \geqq x\}) > 0$ *and* $\mu(\{y \in T \mid y > x\}) < \varepsilon$.

Proof. Since μ is a bounded regular measure, there is a compact set $K \subset \operatorname{supp} \mu$ such that $\mu(T \backslash K) < \varepsilon$. Let $z = \sup \{y \mid y \in K\}$. If $\mu(\{y \mid y \geqq z\}) > 0$, then the choice of $x = z$ completes the proof. If $\mu(\{y \mid y \geqq z\}) = 0$, we again apply the regularity of μ to obtain an $x < z$ such that $\mu(\{y \mid y > x\}) < \varepsilon$. The choice of z forces $\mu(\{y \mid y \geqq x\}) > 0$. The proof of the lemma is complete.

LEMMA 3. *Let* $T = P_{i=1}^m T_i$ *be a finite product of totally ordered, locally compact, idempotent semigroups* T_i. *Let* μ *be a positive measure in* $M(T)$. *Then given* $\varepsilon > 0$, *there is an* $x = (x_1, \cdots x_m)$ *in* T *such that* $\mu^m(\{y \in T \mid y \geqq x\}) > 0$ *and* $\mu(T \backslash \{y \in T \mid y \leqq x\}) < \varepsilon$.

Proof. Let π_i be the projection map of T onto T_i for

$$i = 1, 2, \cdots m .$$

The measure $\mu_i = \mu \circ \pi_i^{-1}$ is a positive measure in $M(T_i)$, Therefore, by Lemma 2, there is an x_i in T_i such that $\mu_i(\{y \in T_i \mid y > x_i\}) < \varepsilon/m$ and $\mu_i(\{y \in T_i \mid y \geqq x_i\}) > 0$. For notational convenience, let

$$J_i = \{y \in T_i \mid y > x_i\} \quad \text{and} \quad K_i = \{y \in T_i \mid y \geqq x_i\} .$$

Then the above statement becomes $\mu_i(J_i) < \varepsilon/m$ and $\mu_i(K_i) > 0$.

Let $x = (x_1, x_2, \cdots x_m)$. We first note that

$$T\backslash\{y \in T \mid y \leqq x = \bigcup_{i=1}^{m} \pi_i^{-1}(J_i) .$$

Thus $\mu(T\backslash\{y \in T \mid y < x\}) = \mu(\bigcup_{i=1}^{m} \pi_i^{-1}(J_i)) \leqq \sum_{i=1}^{m} \mu \circ \pi_i^{-1}(J_i) < \sum_{i=1}^{m} \varepsilon/m = \varepsilon$. We next note that

$$\pi_1^{-1}(K_1)\cdot\pi_2^{-1}(K_2) \cdots \pi_m^{-1}(K_m) = \bigcap_{i=1}^{m} \pi_i^{-1}(K_i) = \{y \in T \mid y \geqq x\} .$$

Since μ has mass on each of the sets $\pi_i^{-1}(K_i)$, μ^m has mass on

$$\pi_1^{-1}(K_1)\cdot\pi_2^{-1}(K_2) \cdots \pi_m^{-1}(K_m); \text{ i.e., } \mu^m(\{y \in T \mid y \geqq x\}) > 0 .$$

This establishes the lemma.

THEOREM 3. *Let $T = P_{i=1}^{m} T_i$ be a finite product of totally ordered, locally compact, idempotent semigroups T_i. Then the measure algebra $M(T)$ is a P-algebra.*

Proof. Let μ be a positive measure in $M(T)$. Lemma 3 guarantees the existence of a sequence $\{x_n\}_{n=1}^{\infty}$ in T such that

$$\mu(T\backslash\{y \in T \mid y \leqq x_n\} < 1/n$$

and $\mu^m(\{y \in T \mid y \geqq x_n\} > 0$.

Let $\mu_n = \mu \mid \{y \in T \mid y \leqq x_n\}$ and let $\nu_n = \mu^m \mid \{y \in T \mid y \geqq x_n\}$. Here we denote the measure μ restricted to a set A by $\mu \mid A(\mu \mid A(E) = \mu(A \cap E)$ for any Borel set E). We claim that the sequences $\{\mu_n\}$ and $\{\nu_n\}$ satisfy the hypothesis of Theorem 2. Clearly, $\mu_n \to \mu$. Since $\operatorname{supp} \mu_n \leqq \operatorname{supp} \nu_n$, and since $\operatorname{supp} \nu'_n \subset \operatorname{supp} \nu_n$ for any $\nu'_n < \nu_n$, Lemma 1 assures us that $\mu_n * \nu'_n = \mu_n(T)\nu'_n$. Finally, ν_n is a nonzero measure such that $\nu_n < \mu^m < \sum_{k=0}^{\infty} \mu^k/2^k$. Since $M(T)$ is semisimple [1], $M(T)$ is a P-algebra by Theorem 2.

5. **A counterexample.** Theorem 1 shows that any P-algebra may be considered as an L-subalgebra of the measure algebra on a compact idempotent semigroup. Each of the examples given in § 1 is a measure algebra on a locally compact idempotent semigroup (not the structure semigroup). It is therefore natural to ask whether or not the measure algebra on *any* locally compact, idempotent semigroup is a P-algebra. The answer to this question is "no" as the counterexample of this section will show. We first make the following definition.

DEFINITION 4. A subset Q of an idempotent semigroup S will be called independent if whenever $x_1 \cdot x_2 \cdots x_n = y_1 \cdot y_2 \cdots y_m$ for $\{x_i\}_{i=1}^n \cup \{y_j\}_{j=1}^m \subset Q$ and $m < n$, then $x_i = y_j$ for some i and j ($1 \leqq i \leqq n$ and $1 \leqq j \leqq m$).

Let C denote the Cantor set on the interval $[0, 1]$. Let S denote the collection of all finite subsets of C and let union be the semigroup operation in S. Note that the one point sets form an independent subset of S in the sense of Definition 4. For an open-compact subset $U \subset C$, $X \in S$, define

$$\chi_U(X) = \begin{cases} 1 & \text{if } X \subset U \\ 0 & \text{if } X \not\subset U . \end{cases}$$

Give S the weak topology generated by the functions $\{\chi_U\}$ (U open-compact). Observe that each χ_U is a continuous semicharacter on S. Let V be a countable open-compact base for C and let $\tilde{V} = \{\tilde{U} \mid U \in V\}$. Finally, let $\{U_i\}_{i=1}^\infty = V \cup \tilde{V}$ and note that the family $\{\chi_{U_i}\}_{i=1}^\infty$ separates points in S.

Let T be the countable topological product of the two-point semigroup $\{0, 1\}$, under multiplication. Thus T is a compact idempotent semigroup. We now define a map $\alpha: S \to T$ by $[\alpha(X)]_i = \chi_{U_i}(X)$ for any $X \in S$. Note that α is a continuous one-to-one homomorphism from S into T. We further observe that C is embedded in an obvious way in S, and hence in T, as an independent set.

The concluding argument is similar to the one given in the Hewitt-Kakutani paper on $M(G)$ [2]. There is a positive continuous measure μ of norm one concentrated on C. Using Fubini's theorem and the fact that C is independent, it can be shown that μ and all its powers are mutually singular [2]. Now let $\sigma = \delta_e - \mu$ (e is the identity in T). Then $\| \sigma^n \| = \| \sum_{k=0}^n C_{n,k}(-1)^k \mu^k \| = \sum_{k=0}^n C_{n,k} = 2^n$. Hence

$$\| \hat{\sigma} \|_\infty = \lim_{n \to \infty} \| \sigma^n \|^{1/n} = 2 .$$

Thus there is a complex homomorphism h of $M(T)$ such that $| h(\sigma) | = 2$. This forces $h(\mu) = -1$. Therefore, $M(T)$ is not a P-algebra.

The countable product of the two point semigroup, with the operation of coordinatewise multiplication, is a sub-semigroup of the countable product of unit intervals, with the operation of coordinatewise minimum. Thus although the measure algebra on a finite product of intervals with coordinatewise maximum multiplication is a P-algebra, this is not the case for an infinite product of intervals. We are therefore led to conjecture that a measure algebra $M(T)$ is a P-algebra if and only if T is an idempotent semigroup which satisfies a certain "finite dimensionality" condition.

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MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES

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One of the important investigations in the theory of summability is that of finding necessary and sufficient conditions on an infinite matrix in order that the matrix should transform one (complex) sequence space into the same or another sequence space. In this note some such theorems are given.

Let

C_0 = the space of null sequences;

C = the space of convergent sequences;

Γ = the space of sequences $x = \{x_p\}$ such that $|x_p|^{1/p} \to 0$, as $p \to \infty$. The space Γ can be regarded as the space of all integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$;

Γ^* = the space of sequences $s = \{s_p\}$ such that the sequence $\{|s_p|^{1/p}\}$ is bounded. Γ^* may also be considered as the space conjugate to Γ regarded as the space of integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$. Each continuous linear functional $U \in \Gamma^*$ is of the form $U(f) = \sum_{p=1}^{\infty} s_p z_p$.

Let $A = (a_{np})$, $(n, p = 1, 2, \cdots)$, be an infinite matrix of complex elements. The A transform of $x = \{x_p\}$, $y = \{y_n\}$ is the sequence defined by the equations

$$(1) \qquad y_n = \sum_{p=1}^{\infty} a_{np} x_p, \ (n = 1, 2, \cdots) .$$

Here $y = \{y_n\}$ and $x = \{x_p\}$ are complex sequences. Similarly, the A transform of $s = \{s_p\}$, $t = \{t_n\}$ is the sequence defined by the equations

$$(2) \qquad t_n = \sum_{p=1}^{\infty} a_{np} s_p, \ (n = 1, 2, \cdots) .$$

Here also $t = \{t_n\}$ and $s = \{s_p\}$ are both complex sequences.

The following theorems are true:

THEOREM I. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* Γ *whenever* $\{x_p\}$ *belongs to* C_0, *it is necessary and sufficient that*

(I, 1) *the sequence* $\{\theta_n\}$ *is a null sequence, where*

$$(3) \qquad \theta_n = \left(\sum_{p=1}^{\infty} |a_{np}|\right)^{1/n}, \ (n = 1, 2, \cdots) .$$

Theorem I holds even if C_0 is replaced by C.

THEOREM II. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* Γ^* *whenever* $\{x_p\}$ *belongs to* C, *it is necessary and sufficient that*

(II, 1) *the sequence* $\{\theta_n\}$ *is bounded, where* θ_n, $(n = 1, 2, \cdots)$, *are given by* (3).

THEOREM III. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* C *whenever* $\{x_p\}$ *belongs to* Γ, *it is necessary and sufficient that*

(III, 1) $|a_{np}|^{1/p} \leqq M$ *independently of* n, p;

(III, 2) $\lim_{n\to\infty} a_{np} = a_p$ *exists for each fixed* p.

THEOREM IV. *Let* (2) *hold. In order that* $\{t_n\}$ *should belong to* C *whenever* $\{s_p\}$ *belongs to* Γ^*, *it is necessary and sufficient that*

(IV, 1) *the sequence* $\{f_n(z)\}$ *of integral functions*

$$(4) \qquad f_n(z) = \sum_{p=1}^{\infty} a_{np} z^p, \; (n = 1, 2, \cdots) ,$$

is uniformly bounded on every compact set (*of the complex plane*);

(IV, 2) = (III, 2) $\lim_{n\to\infty} a_{np} = a_p$ *exists for each fixed* p.

THEOREM V. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* Γ^* *whenever* $\{x_p\}$ *belongs to* Γ, *it is necessary and sufficient that*

(V, 1) $|a_{np}|^{1/(n+p)} \leqq M$ *independently of* n, p.

THEOREM VI. *Let* (2) *hold. In order that* $\{t_n\}$ *should belong to* Γ *whenever* $\{s_p\}$ *belongs to* Γ^*, *it is necessary and sufficient that*

(VI, 1) $|f_n(z)|^{1/n} \to 0$, *as* $n \to \infty$, *uniformly on every compact set* (*of the complex plane*), *where* $\{f_n(z)\}$ *is the sequence of integral functions* $f_n(z)$ *given by* (4).

THEOREM VII. *Let* (1) *hold with* $a_{ij} = 0$ *for* $i > j$. *In order that* $\{y_n\}$ *should belong to* Γ *whenever* $\{x_p\}$ *belongs to* Γ, *it is necessary and sufficient that*

(VII, 1) $|a_{np}|^{1/p} \leqq M$ *independently of* n, p.

The matrix transformation of Γ^* into Γ^* was studied by Heller [6].

The sufficiency, in each case is a straightforward calculation. The necessity of any of the above conditions is proved by taking special sequences, and constructing sequences to contradict the given condition, or by using Functional Analysis. Indeed, to prove the necessity of (III, 1), let $U_n(x) = y_n = \sum_{p=1}^{\infty} a_{np} x_p$, $(n = 1, 2, \cdots)$, for each fixed $x = \{x_p\} \in \Gamma$. Then $\{U_n(x)\}$ represents a sequence of continuous linear functionals on Γ ([4], Th. 4). Here $\{|a_{np}|^{1/p}\}$ is bounded for each fixed n. Since $\{y_n\} \in C$, it follows that $\overline{\lim}_{n\to\infty} |U_n(x)| < \infty$ for each fixed $x \in \Gamma$. Define for each $x \in \Gamma$, $|x| =$ upper bound $(|x_p|^{1/p}, p \geqq 1)$.

Then for $x, x' \in \Gamma$, $|x - x'|$ defines a metric or distance in Γ. With the metric, Γ is a complete metric space. Therefore, by Theorem 11 of ([1], p.19), there is a closed sphere S and a fixed number M such that

$$(5) \qquad |U_n(x)| \leqq M \text{ for } x \in S \text{ and all } n \geqq 1 .$$

Take the sphere S as $|x| \leqq d$. Set $x_p = (d/2)^p$ and $x_j = 0$ for all $j \neq p$ so that $|x| \leqq d/2$ and hence $x = \{x_p\} \in S \subset \Gamma$. Then, by (5), it at once follows that

$$|U_n(x)| = |a_{np}(d/2)^p| \leqq M .$$

That is, $|a_{np}|^{1/p} \leqq M^{1/p}(2/d) < 2m(M)/d$ where $m(M) = \max(1, M)$. This proves the necessity of (III, 1).

A similar proof applies to condition (VII, 1).

Finally, I thank Professor V. Ganapathy Iyer for his help and guidance. I also thank the referee for drawing my attention to the papers of Sheffer [10] and Zeller [14], and other useful comments. Conditions of Theorems V, VI and VII neither include nor are included in Sheffer's conditions. However, Γ and Γ^* are included in the spaces considered by Sheffer. Sheffer [10] and Zeller [14] also dealt with the spaces of all power series with a certain minimal radius of convergence.

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SOME THEOREMS IN FOURIER ANALYSIS ON SYMMETRIC SETS

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Let R be the real line and $A = A(R)$ the space of continuous functions on R which are the Fourier transforms of functions in $L^1(R)$. $A(R)$ is a Banach Algebra when it is given the $L^1(R)$ norm. For a closed $F \subsetneqq R$ one defines $A(F)$ as the restrictions of $f \in A$ to F with the norm of $g \in A(F)$ the infimum of the norms of elements of A whose restrictions are g. Let $F_r \subsetneqq R$ be of the form

$$F_r = \{\textstyle\sum_1^\infty \varepsilon_j r(j) : \varepsilon_j \text{ either } 0 \text{ or } 1\}.$$

This paper shows that if

$$\sum (r(j+1)/r(j))^2 < \infty \quad \text{and} \quad \sum (s(j+1)/s(j))^2 < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. We also show that, in some sense square summability is the best possible criterion. In the course of the proof we show that F_r is a set of synthesis and uniqueness if $\sum (r(j+1)/r(j))^2 < \infty$. This is almost a converse to a theorem of Salem.

We shall also consider sets $E_m \subsetneqq \prod_1^\infty Z_{m(j)}$ of the form

$$E_m = \{x : j^{\text{th}} \text{ coordinate is } 0 \text{ or } 1\}\,.$$

The E_m will have analogous properties to the F_r that will depend on the $m(j)$.

The original work on isomorphisms of the algebras was done in [2] where Beurling and Helson show that any automorphism of A must arise from a map φ by $f \circ \varphi$ where $\varphi(x) = ax + b$. For restriction algebra the situation is more complex. In [5] it is shown that an isomorphism between $A(F_1)$ and $A(F_2)$ of norm one must be given by $f \to f \circ \varphi$ where $\varphi: F_2 \to F_1$ is continuous and $e^{i\varphi}$ is a restriction to F_2 of a character of the discrete reals. Further if F_2 is thick in some appropriate sense the character is continuous. However, McGehee [11] gives examples of F_1 and F_2 for which the restriction algebras $A(F_1)$ and $A(F_2)$ are isomorphic under an isomorphism induced by a discontinuous character. Meyer [12] has shown that if

$$\sum r(j+1)/r(j) < \infty \quad \text{and} \quad \sum s(j+1)/s(j) < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. For appropriate $r(j)$ this is an example of an isomorphism induced by a φ with $e^{i\varphi}$ not even a discontinuous character. He also showed that under these hypothesis F_r was a set of synthesis and uniqueness.

DEFINITIONS AND NOTATIONS. For background material and notation not defined here we refer the reader to [7] and [15].

In this paper G will always be a locally compact abelian group with dual group Γ. If g and γ are elements of G and Γ respectively, the value of the character γ at the point g will be denoted by (γ, g).

When we have a sequence of compact abelian groups G_j, we shall denote their *direct product* (complete direct sum [15]) by ΠG_j. If Γ_j is the dual of G_j, then the direct sum [15] $\Sigma\Gamma_j$ is the dual of ΠG_j. The j^{th} *coordinate* of elements g of ΠG_j or γ of $\Sigma\Gamma_j$ will be denoted by g_j and γ_j. One has:

$$(\gamma, g) = \Pi(\gamma_j, g_j)$$

where all but a finite number of elements in the product are 1.

We shall be dealing with the following basic groups:

(i) The *multiplicative circle group* will be denoted by T. T shall be identified with the unit interval by $x \in [0, 1) \rightarrow \exp(x)$ where $\exp(x) = e^{2\pi i x}$. The additive group of integers Z is the dual group of T. If $x \in [0, 1)$ represents an element of T and $n \in Z$ then $(n, x) = \exp(nx)$.

(ii) R will denote the *additive group* of *reals*. R is isomorphic to its dual under the pairing given by

$$(y, x) = \exp(xy),$$

$x, y \in R$.

(iii) Z_n for $n \geqq 2$ will denote the *additive group* of *integers mod n*. Z_n is also isomorphic to its dual under the pairing given by

$$(r, s) = \exp(rs/n),$$

$r, s \in Z_n$.

Any nonzero regular translation invariant measure on a locally compact abelian group G is called a Haar measure. If μ_G and μ_Γ are Haar measures on G and its dual group Γ respectively, the Fourier transform $\hat{f}$ of f in $L^1(\Gamma, \mu_\Gamma)$ is defined by

$$\hat{f}(g) = \int_\Gamma f(\gamma)(\gamma, g)d\mu_\Gamma$$

for $g \in G$. The inversion theorem gives

$$\int_G \hat{f}(g)(\gamma, -g)d\mu_G = Cf(\gamma) .$$

We shall normalize μ_G and μ_Γ so that $C = 1$. If G is compact we can place $\mu_G(G) = 1$ and if Γ is discrete $\mu_\Gamma(\gamma) = 1$ for $\gamma \in \Gamma$. $L^1(G)$ will denote $L^1(G, \mu_G)$ for a normalized Haar measure.

For $f, h \in L^1(\Gamma)$ define the convolution $f_* h$ by

$$f*h(\gamma) = \int_{\lambda \in \Gamma} f(\gamma - \lambda)h(\lambda)d\mu_\Gamma .$$

In [15] it is shown that $L^1(\Gamma)$ is a commutative Banach algebra under convolution and for $g \in G$

$$\widehat{f*h}(g) = \hat{f}(g)\hat{h}(g) .$$

We denote by $M(G)$ the space of all regular, complex valued Borel measures on G of finite total variation. In [15] the Fourier transform $\hat{\mu}$ of $\mu \in M(G)$ and the convolution $\mu * \nu$ of measures in $M(G)$ are defined. It is shown that $M(G)$ is a Commutative Banach Algebra under convolution and

$$\widehat{\mu * \nu}(\gamma) = \hat{\mu}(\gamma) \cdot \hat{\nu}(\gamma)$$

for $\gamma \in \Gamma$.

Let $A = A(G)$ be defined by

$$A(G) = \{\hat{f}: f \in L^1(\Gamma)\} .$$

$A(G)$ is a Banach algebra under pointwise multiplication and with norm $\|\cdot\|_A$ defined by $\|\hat{f}\|_A = \|f\|_{L^1(\Gamma)}$ and is isomorphic to $L^1(\Gamma)$ under $*$. For a closed set $E \subseteq G$ define the restriction algebra

$$A(E) = \{\hat{f}/E: f \in L^1(\Gamma)\}$$

with norm $\|\cdot\|_{A(E)}$ defined by

$$\|h\|_{A(E)} = \inf\{\|\hat{f}\|_A: \hat{f}/E = h\} .$$

$A(E)$ is again a Banach algebra under pointwise multiplication. Set

$$I(E) = \{\hat{f}: \hat{f}/E = 0 \quad \text{and} \quad f \in L^1(\Gamma)\}$$

$A(E)$ can be identified with the quotient algebra $A(G)/I(E)$.

The dual space of $A(G)$ is denoted by PM (or $PM(G)$). Its elements are called *pseudomeasures*. Each $S \in PM$ can be identified with a function $\hat{S} \in L^\infty(\Gamma)$ as follows. The action of $S \in PM$ as a linear functional on $\hat{f} \in A(G)$ is given by

$$(S, \hat{f}) = \int_\Gamma f(\gamma)\overline{\hat{S}(\gamma)}d\mu_\Gamma .$$

We shall denote by $\|S\|_{PM}$ the $L^\infty(\Gamma)$ norm of $\hat{S}$. Thus PM under $\|\cdot\|_{PM}$ is identical with $L^\infty(\Gamma)$ under the sup norm.

Since $A(E)$ is the quotient of $A(G)$ by $I(E)$, the dual of $A(E)$ consists of those $S \in PM$ which annihilate every function in $I(E)$.

We shall denote this dual of $A(E)$ by $N(E)$. If $N(E)$ is the set of all $S \in PM$ with supp $S \subseteqq E$ [7, p. 161], then E is said to be a set of synthesis. The set of all $\mu \in M(G)$ with support in E we denote by $M(E)$. $M(E)$ can be considered a subspace of $N(E)$ with $(\mu, \hat{f}) = \int \hat{f}\, d\bar{\mu}$. The two definitions for $\hat{\mu}$ coincide.

If G_1 and G_2 are locally compact abelian groups and E_1 and E_2 are closed subsets of G_1 and G_2 respectively we say that $\Phi\colon A(E_1) \rightarrow A(E_2)$ is an isomorphism into if and only if it is an injective algebraic homomorphism and is continuous. If the range of Φ is dense in $A(E_2)$ there exists a continuous $\varphi\colon E_2 \rightarrow E_1$ with $\Phi f = f \circ \varphi$ [9]. We always denote the adjoint of Φ taking $N(E_2)$ into $N(E_1)$ by Φ^*.

Symmetric sets in R are defined as follows. For any sequence $r = \{r(j)\colon j = 1, \cdots\}$ of positive reals with the property

$$\sum_{k}^{\infty} r(j) < r(k-1)$$

we define the subset F_r of R by

$$F_r = \left\{\sum_{1}^{\infty} \varepsilon_j r(j)\colon \varepsilon_j \text{ either } 0 \text{ or } 1\right\}.$$

The representation of the elements of F_r as an infinite sum is unique. For each positive integer k, the subset F_r^k or F_r is defined by

$$F_r^k = \left\{\sum_{1}^{k} \varepsilon_j r(j)\colon \varepsilon_j \text{ either } 0 \text{ or } 1\right\}.$$

We define the subspace $N_1(F_r)$ of $N(F_r)$ by

$$N_1(F_r) = \bigcup_{k=1}^{\infty} M(F_r^k).$$

For any given sequence $m = \{m(j)\colon j = 1, 2, \cdots\}$ of positive integers we define the subset E_m of $\Pi_j Z_{m(j)}$ by

$$E_m = \{x\colon x \in \Pi Z_{m(j)};\ x_j \text{ either } 0 \text{ or } 1\}.$$

For each positive integer k the subset E_m^k of E_m is defined by

$$E_m^k = \{x\colon x \in E_m;\ x_j = 0 \text{ if } j > k\}.$$

Define the subspace $N_1(E_m)$ of $N(E_m)$ by

$$N_1(E_m) = \bigcup_{k=1}^{\infty} M(E_m^k).$$

For r and m as above there is a standard homeomorphism $\varphi\colon E_m \rightarrow F_r$ which takes $x \rightarrow \Sigma x_j r(j)$. Let the inverse of φ be called ψ.

We shall frequently write E for E_m, E^k for E^k_m, F for F_r, and F^k for F^k_r when the respective sequences are clear.

Throughout this work ε_j will always denote a quantity that may take on the values 0 or 1.

1. The symbols r and m shall always denote $\{r(j): j = 1, 2, \cdots\}$ and $\{m(j): j = 1, 2, \cdots\}$ respectively. F_r and E_m will then represent the previously defined sets with $\varphi: E_m \to F_r$ and $\psi: F_r \to E_m$ the standard homeomorphisms. The maps φ and ψ induce maps between $N_1(E_m)$ and $N_1(F_r)$ which we shall again denote by φ and ψ. The maps have the form

$$\varphi(\mu)(\{\varphi(x)\}) = \mu(\{x\})$$

for $\mu \in N_1(E)$, and

$$\psi(\mu)(\{\psi(x)\}) = \mu(\{x\})$$

for $\mu \in N_1(F)$.

If $x = \langle \varepsilon_1, \cdots \varepsilon_k, 0, \cdots \rangle$ is an element of E^k_m and $\mu \in M(E^k)$ set

$$a(\varepsilon_1, \cdots, \varepsilon_k) = \mu(\{x\}) .$$

If $y = \sum_1^k \varepsilon_j r(j)$ is an element of F^k and $\nu \in M(F^k)$ set

$$b(\varepsilon_1, \cdots, \varepsilon_k) = \nu(\{y\}) .$$

We see that

$$\| \mu \|_{PM} = \sup_{\xi_1, \cdots, \xi_k} | \sum a(\varepsilon_1, \cdots, \varepsilon_k) \xi_1^{\varepsilon_1} \cdots \xi_k^{\varepsilon_k} |$$

where ξ_j is an arbitrary $m(j)$ root of unity and the sum is taken over all combinations with ε_j being 0 or 1. Similarly

$$\| \nu \|_{PM} = \sup_x \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left(x \sum_1^k \varepsilon_j r(j) \right) \right|$$

where $x \in R$.

For any $\mu \in N_1(E)$ we define

$$\| \mu \|_{\mathrm{MAX}} = \sup_{\theta_1, \cdots \theta_k} \left| \sum a(\varepsilon_1 \cdots, \varepsilon_k) \exp (\sum \varepsilon_j \theta_j) \right|$$

where $\theta_j \in R$. Define $\| \nu \|_{\mathrm{MAX}}$ for $\nu \in N_1(F)$ by

$$\| \nu \|_{\mathrm{MAX}} = \sup_{\theta_1, \cdots \theta_k} \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp (\sum \varepsilon_j \theta_j) \right| .$$

It is clear that $\|\mu\|_{PM} \leqq \|\mu\|_{\mathrm{MAX}}$ and $\|\nu\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}$. For any standard homeomorphism φ we have

$$\|\varphi\mu\|_{PM}/\|\mu\|_{PM} \leqq \|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} .$$

Similarly

$$\|\psi\nu\|_{PM}/\|\nu\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}/\|\nu\|_{PM} .$$

One should note that if r is a sequence of reals independent mod 1 over the rationals, Kronecher's Theorem [4, p. 99] implies that $\|\nu\|_{\mathrm{MAX}} = \|\nu\|_{PM}$ for $\nu \in N_1(E_r)$.

In order to achieve isomorphisms between certain quotient algebras we shall first study the ratios $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM}$ and $\|\nu\|_{\mathrm{MAX}}/\|\nu\|_{PM}$.

LEMMA 1.1. *If* $\sum (1/m(j))^2 < \infty$ *then there is a* C *depending only on* m *so that* $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} \leqq C$ *for all nonzero* $\mu \in N_1(E_m)$.

Proof. For each k, since $M(E^k)$ is finite dimentional, there is a smallest constant $A(k)$ so that $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} \leqq A(k)$ for all nonzero $\mu \in M(E^k)$. We shall show that there are constants C_k with $\Pi C_k < \infty$ so that $A(k)/A(k-1) \leqq C_k$.

The quotient $\|\mu\|_{PM}/\|\mu\|_{\mathrm{MAX}}$ is equal to

$$\frac{\sup_{\xi_j}\left|\sum_{\varepsilon_j} [(a(\varepsilon_1, \cdots, \varepsilon_{k-1}, 0) + a(\varepsilon_1, \cdots \varepsilon_{k-1}, 1)\xi_k)(\xi_1^{\varepsilon_1} \cdots \xi_{k-1}^{\varepsilon_{k-1}})]\right|}{\sup_{Z_j}\left|\sum_{\varepsilon_j} [(a(\varepsilon_1, \cdots, \varepsilon_{k-1}, 0) + a(\varepsilon_1, \cdots, \varepsilon_{k-1} 1)Z_k)(Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}})]\right|} \tag{1.2}$$

where ξ_j are $m(j)$ roots of unity and Z_j are complex numbers of modulus 1. By a division and multiplication $\|\mu\|_{PM}/\|\mu\|_{\mathrm{MAX}}$ becomes

$$\begin{aligned} &\frac{\sup_{\xi_j}\left|\sum [(a(\cdots, 0) + a(\cdots, 1)\xi_k)\xi_1^{\varepsilon_1} \cdots \xi_{k-1}^{\varepsilon_{k-1}}]\right|}{\sup_{\xi_k, Z}\left|\sum [(a(\cdots, 0) + a(\cdots, 1)\xi_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|} \\ &\times \frac{\sup_{\xi_k, Z_j}\left|\sum [(a(\cdots, 0) + a(\cdots, 1)\xi_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|}{\sup_{Z_j}\left|\sum [(a(\cdots, 0) + a(\cdots, 1)Z_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|} . \end{aligned} \tag{1.3}$$

The factor used in division and multiplication in (1.3) is nonzero. If it were zero $\|\mu\|_{PM}$ would be zero and hence μ would be zero. The fraction on the left of (1.3) is greater than or equal to $1/A(k-1)$. Choose $z_j = y_j$ so that the maximum of the denominator in (1.2) is achieved. The fraction on the right in (1.3) is greater than or equal to

$$(1.4)\qquad \left|1+\frac{\sum[a(\cdots,1)(\xi_k-y_k)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}]}{\sum[(a(\cdots,0)+a(\cdots,1)y_k)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}]}\right|.$$

If $\sum a(\cdots,1)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}$ is zero (1.4) is equal to one. Otherwise set $e^{ix}=\xi_k/y_k$ and (1.4) is equal to

$$(1.5)\qquad \left|1+\frac{e^{ix}-1}{\left[\dfrac{\sum[a(\cdots,0)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}]}{y_k\sum[a(\cdots,1)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}]}\right]+1}\right|.$$

However, in order that the choice $z_j=y_j$ give $\|\mu\|_{\text{MAX}}$, the quotient

$$\frac{\sum a(\cdots,0)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}}{y_k\sum a(\cdots,1)y_1^{\varepsilon_1}\cdots y_{k-1}^{\varepsilon_{k-1}}}$$

must be a real positive real number. Call that number s and (1.5) becomes

$$\left|1+\frac{(\cos x-1)+i\sin x}{s+1}\right|$$

which is greater than or equal to

$$1-x^2/2\ .$$

For an appropriate ξ_k, $|x|$ is less than or equal to $2\pi/m(k)$.

From the above calculation we get

$$\|\mu\|_{PM}/\|\mu\|_{\text{MAX}}\geqq\frac{(1-2\pi^2/(m(k))^2)}{A(k-1)}$$

and therefore

$$A(k)\leqq A(k-1)\cdot\left(1+\frac{C^1}{(m(k))^2}\right)$$

for some absolute constant C^1 and for all $m(k)$ sufficiently large. Since $\sum(1/m(j))^2<\infty$ the theorem is proven.

For the symmetric sets F_r we shall need the following lemma similar to Lemma 1.1.

LEMMA 1.6. *Suppose that* $\sum(r(j+1)/r(j))^2<1/24$. *Choose a real number* x_0 *and define the interval* I *to be*

$$\left\{x\colon|x-x_0|<2\left(\sum_1^k 1/r(j)\right)\right\}.$$

There is then a constant C_1 *independent of* k *and* x_0, *so that*

$$\|\nu\|_{\text{MAX}}/\sup|\hat{\nu}(x)|<C_1,\ \textit{for all nonzero}\ \nu\in M(F_r^k)\ .$$

Proof. Fix k and choose a nonzero $\nu \in M(F_r^k)$. There exists real numbers $\theta_1, \cdots, \theta_k$ less than or equal to one, for which

$$\|\nu\|_{\mathrm{MAX}} = |\sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp(\sum \varepsilon_j \theta_j)| .$$

Define the functions $\hat{\nu}_k, \cdots, \hat{\nu}_2, \hat{\nu}_1 = \hat{\nu}$ on R by

$$\hat{\nu}_j(\mathrm{x}) = \sum\left[b(\varepsilon_1, \cdots, \varepsilon_k) \exp\left(\sum_1^{j-1} \varepsilon_j \theta_j\right) \exp x\left(\sum_j^k \varepsilon_j r(j)\right)\right] .$$

Let us estimate $\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/\|\nu\|_{\mathrm{MAX}}$ where

$$I_1 = \left\{x \colon |x - x_0| \leqq \sum_{k-1}^{k} (2/r(j))\right\} .$$

There is an x_0' within $(1/r(k))$ of x_0 for which $x_0' \cdot r(k) = \theta_k \pmod 1$. Pick x_1 within $1/r(k-1)$ of x_0' so that $x_1 \cdot r(k-1) = \theta_{k-1} \pmod 1$. Then

$$\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/\|\nu\|_{\mathrm{MAX}} \geqq |\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} .$$

As a function of $x, \hat{\nu}_k(x)$ is the Fourier Stieltjes transform of a measure ν_k having support in $[0, r(k)]$. Now,

$$\begin{aligned} |\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} &= |\hat{\nu}_k(x_1)|/|\hat{\nu}_k(x_0')| \\ &= \left|1 + \frac{\hat{\nu}_k'(x_0')}{\hat{\nu}_k(x_0')}(x_1 - x_0') + \frac{\hat{\nu}_k''(x_0')}{\hat{\nu}_k(x_0')}\frac{(x_1 - x_0')^2}{2} + \cdots\right| \end{aligned}$$

$|\hat{\nu}_k|^2$ has a maximum at x_0'. Therefore, if $\hat{\nu}_k = f + ig$, with f and g real, $f \cdot f' + g \cdot g' = 0$ at x_0'. But, at x_0',

$$\begin{aligned} \hat{\nu}_k'/\hat{\nu}_k &= f' + ig'/f + ig \\ &= (ff' + gg' + i(fg' - f'g))/f^2 + g^2 , \end{aligned}$$

which is purely imaginary. Therefore,

$$|\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} \geqq 1 - \left|\frac{\hat{\nu}_k''(x_0')}{\hat{\nu}_k(x_0')}\frac{(x_1 - x_0')^2}{2} + \cdots\right| .$$

If a measure μ has support in $[0, \delta]$ a theorem of Bernstein [1, p. 138] shows that for all x

$$[\hat{\mu}'(x)| \leqq \delta \|\mu\|_{PM}$$

and hence its *nth* derivative $\hat{\mu}^{(n)}$ has

$$|\hat{\mu}^{(n)}(x)| \leqq \delta^n \|\mu\|_{PM} .$$

Since ν_k has support in $[0, r(k)]$ we obtain

$$|\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} \geqq 1 - (r(k)^2/r(k-1)^2) .$$

In effect, we have just shown that there is an $x_1 \in I_1$ for which

$$\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-1}(x_1)| \leqq 1 + 2(r(k)/r(k-1))^2 .$$

Assume that for some $j < k-1$ there is an

$$x_j \in I_j = \left\{x: |x - x_0| \leqq \sum_{k-j}^{k} (2/r(l))\right\}$$

for which

$$\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-j}(x_j)| \leqq \prod_{l=k-j}^{\infty} (1 + 24(r(l+1)/r(l))^2) .$$

We shall show there is then an $x_{j+1} \in I_{j+1}$ for which

$$\begin{aligned} &\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-(j+1)}(x_{j+1})| \\ &\leqq \prod_{l=k-j-1}^{\infty} (1 + 24(r(l+1)/r(l))^2) . \end{aligned} \tag{1.7}$$

Consider $S = \{x: |x - x_j| \leqq 1/r(k-(j+1))\}$. If $|\hat{\nu}_{k-j}|$ does not have a relative maximum in S greater than or equal to $|\hat{\nu}_{k-j}(x_j)|$, then $|\hat{\nu}_{k-j}|$ would be greater than or equal to $|\hat{\nu}_{k-j}(x_j)|$ on some interval in S of length equal to $1/r(k-(j+1))$. However there would be an x_{j+1} in the interval for which $x_{j+1}\cdot r(k-(j+1)) = \theta_{k-(j+1)} \pmod 1$ and hence $\hat{\nu}_{k-(j+1)}(x_{j+1}) = \hat{\nu}_{k-j}(x_{j+1})$, which implies the induction step. Let us assume therefore that there is an x'_j where

$$|x'_j - x_0| \leqq (1/r(k-(j+1)) + \sum_{k-j}^{k} 2/r(l)),$$

$|\hat{\nu}_{k-j}(x'_j)| \geqq |\hat{\nu}_{k-j}(x_j)|$ and at which $|\hat{\nu}_{k-j}|$ has a relative maximum. As before, choosing x_{j+1} within $1/r(k-j+1)$ of x'_j and satisfying $x_{j+1}\cdot r(k-(j+1)) = \theta_{k-(j+1)}$ gives

$$\begin{aligned} &|\hat{\nu}_{k-(j+1)}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| = |\hat{\nu}_{k-j}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| \\ &\geqq \sum 1 - \left| \frac{\hat{\nu}''_{k-j}(x'_j)}{\hat{\nu}_{k-j}(x'_j)} \cdot \frac{(x_{j+1} - x'_j)^2}{2} + \cdots \right| . \end{aligned} \tag{1.8}$$

$\hat{\nu}_{k-j}$ as a function of x is the Fourier Stieltjes of a measure ν_{k-j} having support in $[0, 2r(k-j)]$. Since $\|\nu_{k-j}\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}$, the previously stated theorem of Bernstein gives

$$|\hat{\nu}^{(n)}_{k-j}(x')| \leqq (2r(k-j))^n \|\nu\|_{\mathrm{MAX}} .$$

However

$$\begin{aligned} \|\nu\|_{\mathrm{MAX}} &\leqq \left[\prod_{l=k-j}^{\infty} (1 + 24(r(l+1)/r(l))^2)\right] \times |\hat{\nu}_{k-j}(x'_j)| \\ &\leqq e^{24\Sigma(r(l+1)/r(l))^2}\cdot |\hat{\nu}_{k-j}(x'_j)| \\ &\leqq 3\,|\hat{\nu}_{k-j}(x'_j)| \end{aligned}$$

Since $\Sigma(r(l+1)/r(l))^2 \leqq (1/24)$. Therefore in (1.8),

$$|\hat{\nu}_{(k-j+1)}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| \geqq 1 - 12(r(k-j)/r(k-(i+1))^2$$

and hence (1.7) is true, finishing the induction.

Lemma 1.6 in its present form is an adaptation and extension of a lemma of Meyer [12]. Previously we had much more stringent conditions on the r, to arrive at a similar conclusion to Lemma 1.6.

To utilize the Lemmas 1.1 and 1.6 to obtain isomorphisms of restriction algebras we shall introduce some functional analysis.

Let V represent a Banach Space and V^* its dual. For $r > 0$ let $B_r = \{t: t \in V^*, ||t|| \leqq r\}$. A set $O \subseteqq V^*$ is said to be open in the *bounded topology* on V^* if and only if $O \cap B_r$ is open in the relative weak* topology of B_r for all r > 0. For a distribution of the bounded topology the reader should consult [6, p. 427].

LEMMA 1.10. *Let V, W be Banach spaces with duals V^* and W^*. Let $K \subset V^*$ be a weak* dense subspace of V^*. Suppose that $T: K \to W$ is linear and continuous when K has the topology induced by the bounded topology on V^* and W^* has the weak topology. Then there exists a bounded linear transformation S: $W \to V$ for which $T = S^*/K$.*

Proof. For each $w \in W$, define the linear functional T_w on K by

$$T_w(t) = Tt(w) .$$

Each T_w is continuous in the topology induced by the bounded topology of V^* which is a locally convex topology by Corollary 5, page 428 of [6]. Hence by the Hahn-Banach theorem there exists an extension $\tilde{T}_w$ of T_w to all of V^*, continuous in the bounded topology of V^*.

By Theorem 6, page 428 of [6], $\tilde{T}_w$ is continuous in the weak* topology on V^*. Hence there exists an element $v \in V$ such that $T_w(t) = t(v)$ for all $t \in K$. Since K is assumed weak* dense in V^*, the element v is determined by w. Define S: $W \to V$ by $S(w) = v$. S is linear. Since K is weak* dense S is closed. Therefore by the Closed Graph Theorem S is bouned. If $t \in K$, $w \in W$

$$S^*t(w) = t(S(w)) = Tt(w) ,$$

which completes the proof.

It is clear that $N_1(E_m)$ and $N_1(F_r)$ are weak* dense in $N(E_m)$ and $N(F_r)$, respectively. By studying the continuity of the standard maps between $N_1(E_m)$ and $N_1(F_r)$, we shall be able to use Lemma 1.10 to obtain isomorphisms between $A(E_m)$ and $A(F_r)$ for certain classes of sequences m and r.

Choose $\mu \in N_1(E)$. For each k we define an approximating measure μ_k in $M(E^k)$ by

$$\mu_k(\{x\}) = \sum_{y \in D} \mu(\{y\})$$

where $x \in E^k$ and $D = \{y : y \in E \text{ and } y_j = x_j \text{ for } j \leqq k\}$. Let

$$\Gamma^k = \{\gamma : \gamma \in \Sigma Z(m(j)) \quad \text{and} \quad \gamma_j = 0 \quad \text{if} \quad j > k\}.$$

If $\gamma \in \Gamma^k \hat{\mu}_k(\gamma) = \hat{\mu}(\gamma)$. It is easy to see that

$$|| \mu_k ||_{PM} = \sup_{\gamma \in \Gamma} {}_k | \hat{\mu}_k(\gamma) | .$$

To each $\lambda \in M(E^k)$ we associate the measure λ' in $M(E^k)$ defined by

$$\lambda'(\{x\}) = \begin{cases} 0 & \text{if} \quad x_k = 0 \\ \lambda(\{x\}) & \text{if} \quad x_k = 1 \end{cases} .$$

It is not hard to see that

$$|| \lambda' ||_{PM} \leqq 2 \, || \lambda ||_{PM} .$$

Choose $\nu \in N_1(F)$. For each k define an approximating measure ν_k in $M(F^k)$ by

$$\nu_k(\{x\}) = \sum_{y \in D} \nu(\{y\})$$

where $x = \sum_1^k x_j r(j)$ and $D = \{y : y = \Sigma \varepsilon_j r(j) \text{ and } \varepsilon_j = x_j \text{ for } j \leqq k\}$.

To each $\beta \in M(F^k)$ we associate the measure β' in $M(F^k)$ defined by

$$\beta'(\{x\}) = \begin{cases} 0 & \text{if} \quad x = \sum_1^k \varepsilon_j r(j) \quad \text{and} \quad \varepsilon_k = 0 \\ 1 & \text{if} \quad x = \sum_1^k e_j r(j) \quad \text{and} \quad \varepsilon_k = 1 \end{cases} .$$

We are now ready to prove the following theorem.

THEOREM 1.11. *If $\Sigma(1/m(j))^2 < \infty$ and $\Sigma(r(j+1)/r(j))^2 < \infty$ then $A(E_m)$ is isomorphic to $A(F_r)$.*

We shall break the proof into two lemmas.

LEMMA A. *Let F_r be any symmetric set. Let $\Sigma(1/m(j))^2 < \infty$ $\varphi \colon E_m \to F_r$ the standard homeomorphism. Then there is an iso-*

morphism into $\Phi: A(F_r) \to A(E_m)$ *given by*

$$\Phi(f) = f \circ \varphi, \qquad f \in A(F_r) .$$

Proof. We shall study the continuity properties of

$$\varphi: N_1(E) \to N_1(F) .$$

For $f \in A(F)$ define

$$U_{\varepsilon,f} = \{\nu: \nu \in N_1(F) \quad \text{and} \quad |(\nu, f)| < \varepsilon\} .$$

To establish that φ is continuous from the bounded weak* topology of $N_1(E)$ to the weak* topology of $N_1(F)$ it is sufficient to prove that the zero element of $N_1(E)$ is an interior point of $\varphi^{-1}(U_{\varepsilon,f})$ (i.e., that φ is continuous at 0). This follows at once if we prove that given a and ε, there exists δ, k such that if for $\mu \in N_1(E)$

$$\begin{aligned} &\|\mu\|_{PM} \leqq a \quad \text{and} \quad |\hat{\mu}(\gamma)| < \delta \quad \text{for} \quad \gamma \in \Gamma^k \\ &\varphi(\mu) \text{ is an element of } U_{\varepsilon,f}. \end{aligned} \tag{1.12}$$

In view of Lemma 1.1 (1.12) follows if we can show that given a, ε, and M then there exists δ, k such that for $\mu \in N_1(E)$,

$$\begin{gathered} \|\mu\|_{PM} \leqq a \quad \text{and} \quad \hat{\mu}(\gamma) < \delta \quad \text{for} \quad \gamma \in \Gamma^k \\ \text{then} \\ |\widehat{\varphi(\mu)}(x)| < \varepsilon \quad \text{for} \quad |x| \leqq M . \end{gathered} \tag{1-13}$$

We first estimate $|\widehat{\varphi(\mu)} - \widehat{\varphi(\mu_k)}|$ for $\mu \in M(E^s)$.

$$\begin{aligned} |\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| &\leqq \sum_k^{s-1} |\widehat{\varphi(\mu_{j+1})}(x) - \widehat{\varphi(\mu_j)}(x)| \\ &\leqq \sum_k^{s-1} |\exp(-xr(j+1)) - 1| \cdot \|\varphi(\mu'_{j+1})\|_{PM} . \end{aligned}$$

By Lemma 1.1, for any s

$$|\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| \leqq 4\pi C |x| \, \|\mu\|_{PM} \cdot \sum_{k+1}^{\infty} r(j) .$$

For μ with $\|\mu\|_{PM} \leqq a$, pick $\delta < \varepsilon/2C$ where C is the constant of Lemma 1.1 and choose k so that $4\pi CMa \sum_{k+1}^{\infty} r(j) < \varepsilon/2$. If $|\hat{\mu}(\gamma)| < \delta$ for $\gamma \in \Gamma^k$, then $\|\mu_k\|_{PM} < \delta$ and by Lemma 1.1 $\|\varphi(\mu_k)\|_{PM} < \varepsilon/2$. If $|x| \leqq M$, then $|\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| < \varepsilon/2$ so

$$|\widehat{\varphi(\mu)}(x)| < \varepsilon, \quad \text{for} \quad |x| \leqq M .$$

The conditions of Lemma 1.10 are satisfied so $\varphi = \Phi^*$ for some linear $\Phi: A(F) \to A(E)$. For $\mu \in N_1(E)$ and $f \in A(F)$

$$(\Phi f, \mu) = (f, \varphi(\mu)) .$$

Therefore if $x \in \bigcup_1^\infty E^s$

$$\Phi f(x) = f(\varphi(x)) .$$

Since φ, f and Φf are continuous, Φ is the linear map wanted.

LEMMA B. *Let F_r be a symmetric set with $\Sigma(r(j+1)/r(j))^2 < \infty$. Let $\psi: F_r \to E_m$ be the standard homeomorphism of F_r with some E_m. Then there is an isomorphism into $\bar{\Psi}: A(E_m) \to A(F_r)$ given by*

$$\bar{\Psi}(f) = f \circ \psi, \qquad f \in A(E_m) .$$

Proof. There is an l so that $\sum_{l+1}^\infty (r(j+1)/r(j))^2 < 1/24$. F is a union of 2^l sets which are translations of the set $F' = \{x: x = \sum_{l+1}^\infty \varepsilon_j r(j)\}$. It is therefore sufficient to prove the theorem for F'. For convenience, assume F_r has the property $\sum_1^\infty (r(j+1)/r(j))^2 < 1/24$. We shall show as in Lemma A that $\psi: N_1(F_r) \to N_1(E_m)$ has the required continuity properties to be the adjoint of a continuous linear map $\bar{\Psi}: A(E_m) \to A(F_r)$ satisfying $\bar{\Psi}(f) = f \circ \psi$.

Using Lemmas 1.6 and 1.10 as in Lemma A, it is enough to show that if a, ε, M are given, then there exists $\delta, x_1, \cdots, x_t$ so that the following holds.

If $\nu \in N_1(F)$, $\|\nu\|_{PM} \leqq a$ and $\hat{\nu}(x_j) < \delta$ for $j = 1, \cdots, t$, then $|\widehat{\psi(\nu)}(\gamma)| < \varepsilon$ for $\gamma \in \Gamma^M$.

Choosing $\nu \in N_1(F)$ with $\|\nu\|_{PM} \leqq a$ and estimating $|\hat{\nu} - \hat{\nu}_k|$ gives

$$\begin{aligned} |\hat{\nu}(x) - \hat{\nu}_k(x)| &\leqq \sum_k^S |\hat{\nu}_{j+1}(x) - \hat{\nu}_j(x)| \\ &\leqq \sum_k^\infty |\exp(-xr(j+1)) - 1| \, \|\nu'_{j+1}\|_{PM} . \end{aligned}$$

Lemma 1.1 and 1.6 show that the PM norm on $N_1(F_r)$ and $N_1(E_{m'})$ are equivalent when $\sum (1/m'(j))^2 < \infty$. Hence

$$\begin{aligned} |\hat{\nu}(x) - \hat{\nu}_k(x)| &\leqq 4\pi x C_1 C \, \|\nu\|_{PM} \sum_{k+1}^\infty r(j) \\ &\leqq 8\pi |x| C_1 C a \cdot r(k+1) . \end{aligned}$$

An easy consequence of the condition $\Sigma(r(j+1)/r(j))^2 < 1/24$ is that

$$\lim_{k\to\infty} 8\pi C_1.C.a. \left(\sum_1^k 2/r(j)\right) \cdot r(k+1) = 0 .$$

Pick $k \geqq M$ large enough so that

$$8\pi C_1 Ca\left(\sum_1^k 2/r(j)\right)r(k+1) < \varepsilon/4C_1 .$$

Then

$$(1.14) \qquad |\hat{\nu}(x) - \hat{\nu}_k(x)| < \varepsilon/4C_1$$

for $|x| < \sum_1^k (2/r(j))$. By Lemma 1.6 there is an x_0 with

$$|x_0| < \sum_1^k (2/r(j)$$

so that for $\nu_k \in M(F^k)$

$$||\nu_k||_{\mathrm{MAX}}/||\hat{\nu}_k(x_0)|| < C_1 .$$

By a theorem of Bernstein [1, p. 138]

$$|\hat{\nu}_k(x_0) - \hat{\nu}_k(x_*)| \leqq C_1 |\hat{\nu}_k(x_0)| \left(\sum_1^\infty r(j)\right) |x_* - x_0| .$$

Therefore, if $|x_* - x_0| < 1/2(\sum r(j))\cdot C_1$

$$(1.15) \qquad ||\nu_k||_{\mathrm{MAX}}/|\hat{\nu}_k(x_*)| \leqq 2C_1 .$$

Choose for $i = 1, \cdots, t; x_i$ with $|x_i| \leqq \sum_1^k (2/r(j))$ so that for every x with $|x| \leqq \sum_1^k (2/r(j))$ there is an x_j with $|x - x_j| < 1/2(\Sigma r(j))\cdot C_1$. If $|\hat{\nu}(x_j)| < \varepsilon/4C_1$ for $x_j, j = 1, \cdots, t$, then $|\hat{\nu}_k(x_j)| < \varepsilon/2C_1$ by (1.14), and by (1.15) $||\nu_k||_{\mathrm{MAX}} < \varepsilon$. Consequently, $||\psi(\nu_k)||_{PM} < \varepsilon$. Since $k > M$ we see that $|\widehat{\psi(\nu)}(\gamma)| < \varepsilon$ for $\gamma \in \Gamma^M$.

As in Lemma A, the continuity conditions of Lemma 1.10 are satisfied and

$$\Psi(f) = f \circ \psi .$$

Theorem 1.11 is an immediate consequence of Lemmas A and B. Meyer [12] has proven that if $\Sigma(r(j+1)/r(j)) < \infty$ and

$$\Sigma(s(j+1)/s(j)) < \infty$$

then $A(F_r) \cong A(F_s)$. Lemmas 1.6 was an analogue and improvement on his main lemma which allowed us to obtain the theorem with square summability.

If $r_0(j) = \{e^{-j}\cdot 2^{-j^2}\}$ then every $A(F_r)$ and $A(E_m)$ with

$$\Sigma(r(j+1)/r(j))^2 < \infty \quad \text{and} \quad \Sigma(1/m(j))^2 < \infty$$

is isomorphic to $A(F_{r_0})$. The isomorphisms are given by

$$f \to f \circ \varphi$$

where f is in an appropriate restriction algebra and φ one of the standard homeomorphisms. We shall call an isomorphism between any two restriction algebras induced in this manner a *standard isomorphism*. If $A(F_r)$ or $A(E_m)$ is isomorphic to $A(F_{r_0})$ by standard isomorphisms, F_r or E_m will then be said to belong to the *class* M_y. One should note that for $\mu \in N_1(F_{r_0})$, $\|\mu\|_{PM} = \|\mu\|_{\text{MAX}}$.

Define sets of multiplicity and uniqueness as in [7, p. 52]. In [7, p. 100] it is shown that if $\alpha \in [0, 1/2)$ one can construct sets F_r of multiplicity with $r(j+1)/r(j) = 0(j^{-\alpha})$. The next theorem shows, in particular, that if $r(j+1)/r(j) = 0(j^{-\alpha})$ with $\alpha \in (1/2, \infty)$ then F_r is a set of uniqueness.

THEOREM 1.16. *Suppose that $\Sigma(r(j+1)/r(j))^2 < \infty$. Then F_r is a set of synthesis and there is a constant B so that for all $S \in N(F_r)$*

$$\|S\|_{PM} \leqq B \overline{\lim} |\hat{S}(x)| .$$

Hence F_r is a set of uniqueness.

Proof. Choose l so that $\sum_{l+1}^{\infty}(r(j+1)r(j))^2 < 1/24$. Then F is a union of 2^l disjoint sets of the form $a(\varepsilon) + F(l)$ where $\varepsilon = \langle\varepsilon_1, \cdots, \varepsilon_l\rangle$ and $F(l) = \{x: x = \sum_{l-1}^{\infty} \varepsilon_j r(j)\}$. We can find 2^l functions φ_ε in $A(R)$ where $\varphi_\varepsilon = 1$ on $a(\varepsilon) + F(l)$ and 0 on the other sets. Let $S \in PM$ with support in F_r. $S = \Sigma_\varepsilon \varphi_\varepsilon S$ and hence if $\varphi_\varepsilon S \in N(a(\varepsilon) + F(l))$ for each ε, $S \in N(F_r)$. Moreover, for some ε the inequality

$$\|\varphi_\varepsilon S\|_{PM} \geqq 2^{-l} \|S\|_{PM}$$

must hold. If $\|S\|_{PM} > B \lim |\hat{S}(x)|$ we see that

$$\|\varphi_\varepsilon S\|_{PM} \geqq \frac{2^{-l}B}{\|\varphi_\varepsilon\|_A} \overline{\lim} |\widehat{\varphi_\varepsilon S}(x)| .$$

We may therefore assume that $\Sigma(r(j+1)/r(j))^2 < 1/24$.

Lemma 1.6 and [12, Proposition 2.2.3] imply that there is a natural isomorphism T from $A(F_r^k \times [-2r(k+1), 2r(k+1)])$ in $A(R \times R)$ to $A(F_r^k + [-2r(k+1), 2r(k+1)]$ with norm

$$T \leqq (1 - \alpha 4r(k+1) \cdot (\Sigma_1^k 1/r(j)))^{-1}$$

and $\|T^{-1}\| = 1$, where $\alpha \leqq 1$ and is independent of k. For large enough k the norm is smaller than some constant B_1. For each $x \in R$ consider the funtion $f_x \in A(F_r^k + [-2r(k+1), 2r(k+1)])$

$$f_x(y) = \exp(xy) - \exp(x \cdot \Sigma_1^k \varepsilon_j r(j)) \quad \text{for} \quad |y - \Sigma_1^k \varepsilon_j r(j)| \leqq 2r(k+1) .$$

Its image in $A(F_r^k \times [-2r(k+1), 2r(k+1)]$ is

$$\tilde{f}_x(t, y) = \exp(xt)\cdot(\exp(xy) - 1) .$$

Then

$$\| f_x \|_{A(F_r^k + [\cdot])} \leqq B_1 \| \tilde{f}_x \|_{A(F_r^k \times [\cdot])} \leqq B_2 | x | r(k+1) .$$

Define $v_k \in M(F_r^k)$ by

$$v_k(\{\Sigma \varepsilon_j r(j)\}) = (\widehat{S |_{\Sigma_1^k \varepsilon_j r(j) + [\cdot]}})(0) .$$

where S is a given element of PM with support in F_r. Then for sufficiently large k

$$| \hat{S}(x) - \hat{v}_k(x) | = | (S, f_x) | \leqq B_2 \cdot | x | \cdot \| S \|_{PM} \cdot r(k+1) .$$

By Lemma 1.6 we have that

$$\hat{v}_k(x) \longrightarrow \hat{S}(x) \forall x \in R; \lim \| v_k \|_{PM} \leqq C \| S \|_{PM}$$

and hence $S \in N(F_r)$ and F_r is a set of synthesis.

For convenience assume that $\| S \|_{PM} = 1$ and $| \hat{S}(0) | > 1/2$. Suppose that $| \hat{S}(x) | < \varepsilon$ for $x > x_0$. Pick a constant k_0 so that

$$(x_0 + 4 \cdot \Sigma_1^k r(j)) B_2 \| S \|_{PM} \cdot r(k+1) < \varepsilon$$

for $k > k_0$. Then if $k > k_0$

$$| \hat{v}_k(x) | < 2\varepsilon$$

for all x satisfying $| x - x_* | \leqq \Sigma_1^k (2/r(j))$ where x_* is the center of the interval $[x_0, x_0 + 4\Sigma_1^k(1/r(j))]$. Since $| \hat{v}_k(0) | > 1/2$ Lemma 1.6 shows that

$$\varepsilon > 1/4C_1 .$$

Theorem 1.16 is essentially methods of McGehee and Meyer utilizing Lemma 1.6.

We next examine the sets E_m. By [15, p. 166] they are sets of synthesis. If $m(j) = 2$ for all but a finite number of j, E_m has positive measure and there is an $S \in N(E_m)$ with $\inf_T \sup_{\gamma \in \sim T} | \hat{S}(\gamma) | = 0$. The following is a converse.

THEOREM 1.17. *Let $m(j)$ be a sequence of integers with infinitely many $m(j) \geqq 3$. Then there is a constant C so that for all $S \in N(E_m)$*

$$\| S \|_{PM} \leqq C \inf_T \sup_{\gamma \in \sim T} | \hat{S}(\gamma) |$$

where T is any finite set in $\Sigma Z_{m(j)}$.

Proof. Let $S \in N(E)$ and assume for simplicity that $\|S\|_{PM} = 1$ and $\hat{S}(0) > 3/4$. Let $\{\mu_k\}$ be the measure defined by

$$\mu_k\{x\} = \left(\widehat{S\,|_{x+\prod_{j=k+1}^{\infty} Z_{m(j)}}}\right)(0)$$

where $x = \langle \varepsilon_1, \cdots, \varepsilon_k, 0, 0, \cdots\rangle$. Let $\gamma^s \varepsilon \sum \Gamma_{m(j)}$ be that element with

$$\gamma_j^s = \begin{cases} 0 & \text{if } j \neq s \\ 1 & \text{if } j = s \end{cases}.$$

Then for $1 \leqq s \leqq k$

$$\hat{\mu}_k(\gamma^s) = \sum_{\varepsilon(s)=0} a(\varepsilon(1), \cdots, \varepsilon(k)) + \sum_{\varepsilon(s)=1} a(\varepsilon(1), \cdots, \varepsilon(k)) \exp(1/m(s)) .$$

If we call $\sum_{\varepsilon(s)=0} a(\varepsilon(1), \cdots, \varepsilon(k)) = \alpha$

$$\sum_{\varepsilon(s)=1} a(\varepsilon(1), \cdots, \varepsilon(k)) = \beta \quad \text{then} \quad \hat{\mu}_k(0) = \alpha + \beta .$$

It is easy to see that $\alpha \leqq 1$ and $\beta \leqq 2$. Therefore

$$|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(0)| \leqq 2\,|\exp(1/m(s) - 1)| \leqq 4\pi/m(s) .$$

Therefore, if $m(s) > 8\pi$

$$|\hat{\mu}_k(\gamma^s)| > 1/4 .$$

Let $\tilde{\gamma}^s \in \Sigma\Gamma_{m(j)}$ be the element with

$$\tilde{\gamma}_j^s = \begin{cases} 0 & \text{if } j \neq s \\ m(s) - 1 & \text{if } j = s \end{cases}.$$

Then

$$\hat{\mu}_k(\tilde{\gamma}^s) = \alpha + \beta \exp(-1/m(s))$$

and hence

$$|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s)| = 2\beta \sin(2\pi/m(s)) .$$

If $3 \leqq m(s) < 8\pi$ and $|\hat{\mu}_k(\gamma^s)| < (1/100)$ then $\beta > (1/3)$ and

$$|\hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s)| > 1/50$$

and hence $|\hat{\mu}_k(\tilde{\gamma}^s)| > 1/50$. Therefore we may conclude that for all k either $|\hat{\mu}_k(\gamma^s)|$ or $|\hat{\mu}_k(\tilde{\gamma}^s)|$ is greater than 1/100 provided $m(s) \geqq 3$.

On Γ^k, $\hat{\mu}_k$ and $\hat{S}$ are identical. Suppose there is a t so that

$$|\hat{S}(\gamma)| < 1/200 \tag{1.19}$$

for $\gamma \notin \Gamma^t$. Pick a $k > t$ so that there is an s with $k > s > t$ for which $m(s) \geqq 3$. Then either $|\hat{\mu}_k(\tilde{\gamma}^s)|$ or $|\hat{\mu}_k(\tilde{\gamma}^s)|$ is greater than 1/100. Hence $|\hat{S}(\gamma^s)|$ or $|\hat{S}(\tilde{\gamma}^s)|$ is greater than 1/100 contradicting (1.19).

2. In this section we shall exhibit sets E_m, F_r that do not have $A(E_m)$ or $A(F_r)$ isomorphic to $A(F_{r_0})$ by standard isomorphisms. They are then not in the class M_y.

The first theorem is a converse to Lemma A.

THEOREM 2.1. *If $\Sigma(1/m(j))^2 = \infty$, then E_m is not an element of the class M_y.*

Proof. It is sufficient to show that

$$\sup_{\mu \in N(E)} ||\mu||_{\mathrm{MAX}}/||\mu||_{PM} = \infty$$

since for $\nu \in N_1(F_{r_0})$ $||\nu||_{PM} = ||\nu||_{\mathrm{MAX}}$. For each integer s, let $x^s \in \Pi Z_{m(j)}$ be that element with $x^s_j = \delta^s_j$. Let α_s be the two point measure

$$\alpha_s\{x^s\} = \exp(1/3m(s)) .$$

For each k, define an element μ_k of $M(E^k)$ by

$$\mu_k = \alpha_1 * \cdots * \alpha_k .$$

we see that

$$||\mu_k||_{\mathrm{MAX}} = 2^k$$

while

$$||\mu_k||_{PM} = \sup_{\xi_s}\left|\prod_{s=1}^{k}(1 + \exp(1/(3m(s)))\cdot\xi_s\right|,$$

where the ξ_s are $m(s)$ roots of unity. Since

$$|1 + \exp(1/3m(s))| \geqq |1 + \exp(1/3m(s))\xi_s|$$

for ξ_s any $m(s)$ root of unity, and since $\cos(\theta) < 1 - \theta^2/4$ for $\theta < 1$

$$\begin{aligned} ||\mu_k||_{PM} &= 2^k \prod_{s=1}^{k} \cos(\pi/3m(s)) \\ &\leqq 2^k \prod_{s=1}^{k} (1 - (1/3m(s))^2) . \end{aligned}$$

Therefore

$$||\mu_k||_{\mathrm{MAX}}/||\mu_k||_{PM} \geqq 1/\prod_{s=1}^{k}(1 - (1/3m(s))^2)$$

and since $\Sigma(1/m(s))^2 = \infty$, $\|\mu_k\|_{\text{MAX}}/\|\mu_k\|_{PM} \to \infty$ as $k \to \infty$.

We have actually shown more than claimed in Theorem 2.1. The proof shows that if $\{r(j)\}$ is any independent sequence and $\Sigma(1/m(j))^2 = \infty$, then $A(E_m)$ is not isomorphic to $A(F_r)$ by a standard isomorphism.

The next theorem will imply that no condition on the convergence of $(r(j+1)/r(j))$ weaker than

$$\Sigma(r(j+1)/r(j))^2 < \infty ,$$

is sufficient for a set F_r to be a member of the class M_y.

THEOREM 2.2. *Suppose that n_j is an increasing sequence of integers. Let $b \geqq 2$ be an integer and put $r(j) = b^{-n_j}$. If*

$$\Sigma(r(j+1)/r(j))^2 = \infty$$

then F_r is not an element of the class M_y.

Proof. Let us assume for convenience that $\Sigma_1^\infty(r(2j)/r(2j-1))^2 = \infty$ and $b = 10$. We can also assume our set F to be on the circle. For any integer j define the two point measure γ_j by

$$\gamma_j\{0\} = 1$$
$$\gamma_j\{r(j)\} = \exp\left(-\frac{1}{2}\right).$$

For each k, define an element ν_k of $M(F^k)$ by

$$\nu_k = \gamma_1 * \cdots * \gamma_k .$$

Then for any integer s

$$|\hat{\nu}_{2k}(s)| = 2^{2k}\left|\prod_1^{2k}\cos\left(\pi\left(s\cdot 10^{-n_j} - \frac{1}{2}\right)\right)\right| .$$

In this product, consider terms $\delta_j(s)$ of the form

$$\left|\cos\left(\pi\left(s\cdot 10^{-n_{2j-1}} - \frac{1}{2}\right)\right)\cdot\cos\left(\pi\left(s\cdot 10^{-n_{2j}} - \frac{1}{2}\right)\right)\right| .$$

If

$$\left|s\cdot 10^{-n_{2j-1}} - \frac{1}{2}\right| < 1/10 \bmod 1 ,$$

then

$$\left|s\cdot 10^{-n_{2j}} - \frac{1}{2}\right| \geqq \frac{1}{10}\cdot(10^{n_{2j-1}}/10^{n_{2j}}) \bmod 1.$$

Then

$$|\hat{\nu}_{2k}(s)| = 2^{2k} \prod_{j=1}^{k} |\delta_j(s)|$$
$$\leqq 2^{2k} \prod_{j=1}^{k} (1 - D\cdot(10^{n_{2j-1}}/10^{n_{2j}})^2) ,$$

where D is an absolute constant. Therefore

$$\|\nu_{2k}\|_{PM} \leqq 2^{2k} \prod_{j=1}^{k} (1 - D(r(2j)/r(2j-1))^2) .$$

However, $\|\nu_{2k}\|_{\mathrm{MAX}} = 2^{2k}$, so

$$\|\nu_{2k}\|_{\mathrm{MAX}}/\|\nu_{2k}\|_{PM} \geqq \left| / \prod_{j=1}^{k} (1 - D(r(2j)/r(2j-1))^2) \right| .$$

Therefore $\|\nu_{2k}\|_{\mathrm{MAX}}/\|\nu_{2k}\|_{PM} \to \infty$ as $k \to \infty$. Hence F_r is not a member of the class M_y. The proof with $b \neq 10$ is completely analogous to the proof with $b = 10$.

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CENTRALIZERS OF ABELIAN, NORMAL SUBGROUPS OF HYPERCYCLIC GROUPS

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J. L. Alperin proved the following theorem about finite p-groups G: if E is maximal among the abelian, normal subgroups of G of exponent dividing p^n, then $\Omega_n \mathfrak{C}_G(E) = E$, provided that $p^n \neq 2$. It turns out that the restriction to p-groups and also to finite groups in Alperin's proof is not essential. In fact a similar theorem holds in a large class of hypercyclic groups (Theorem 2.2). By the same method also a modified version (Theorem 2.8) will be obtained, the word "normal" in the assumptions about E being replaced by "characteristic", here G is supposed to be hypercentral; the modification results in enlarging E to a characteristic subgroup $\mathfrak{X}_G(E)$ of class 2 in a very definite way before taking its centralizer.

The proofs of both theorems rely on a fairly general version (not used in its full strength) of the lemma used by Alperin on p-automorphisms of abelian p-groups that centralize all elements of order p. The first paragraph is devoted to this generalization (Theorem 1.11) and may be of independent interest.

TERMINOLOGY. We denote by $\mathscr{F}$ the set of all functions from the set of all primes to the set of all rational integers extended by the symbol ∞. Addition and subtraction are defined on $\mathscr{F}$ by $(f \pm q)(p) = f(p) \pm g(p)$, where ∞ is handled in the usual manner; also $f \leqq g$ for $f, g \in \mathscr{F}$ if and only if $f(p) \leqq g(p)$ for all primes p. A function $f \in \mathscr{F}$ is called *finite*, if $f(p) < \infty$ for every prime p. The constant functions in $\mathscr{F}$ will be denoted by their single value. The function $d \in \mathscr{F}$ which is 2 at 2 and 1 elsewhere will play a particular role in our discussion.

Let f be a function in $\mathscr{F}$. The nonegative part f^+ of f is defined by $f^+(p) = f(p)$ if $f(p) \geqq 0$, and $f^+(p) = 0$, if $f(p) \leqq 0$. With every torsion element x of a group X there is associated a function $e_x \in \mathscr{F}$ such that $\Pi\{p^{e_x(p)} \mid p \text{ prime}\}$ is the order of x. We say that an element x is *restrained* by f, if x is a torsion element and $e_x \leqq f^+$. The elements of X restrained by f generate a (characteristic) subgroup $\Omega_f(X)$ of X. We say that X is *restrained by* f, if every element of X is restrained by f.

For every torsion element x of a group X and every prime p there is a uniquely determined power x_p of x such that the order of x_p is a power of p and the order of $x_{p'} = x_p^{-1}x$ is prime to p. X_p is the set of all p-elements of X.

A group G is called *hypercyclic*, if every epimorphic image, not 1, of G has a cyclic, normal subgroup, not 1. This implies that every normal subgroup, not 1, of an epimorphic image H of G contains a cyclic, normal subgroup, not 1, of H. A group G is called *hypercentral*, if every epimorphic image, not 1, of G has a nontrivial center. This implies that every normal subgroup, not 1, of an epimorphic image H of G intersects the center of H nontrivially.

We use the notation $(a, b) = a \circ b = a^{-1}a^b$ where b is an endomorphism or a group element and $a^b = b^{-1}ab$. If b operates on A, then $A \circ b$ is the set of all (a, b) with $a \in A$. $(a, b, c) = ((a, b), c)$. An element or automorphism x of X *centralizes* the x-invariant factor B/A, if x fixes every element in B/A. $\mathfrak{N}_X(A) =$ normalizer of A in X, $\mathfrak{C}_X(A) =$ centralizer of A in X, $\mathfrak{Z}(X) =$ center of X, $X' =$ commutator subgroup of X, $A_X =$ largest normal subgroup of X contained in A, $\langle S \rangle =$ subgroup generated by the set S, $p' =$ set of all primes different from p.

1. After a few lemmas of a general nature this paragraph will be concerned with torsion automorphisms that centralize $\Omega_d(G)$.

LEMMA 1.1. *For a group G, an endomorphism x of G and an element b of G define the elements b_i, $i \geqq 0$, by*

$$b_0 = b,\ b_i^x = b_i b_{i+1}\ ,$$

and assume $b_i b_{i+1} = b_{i+1} b_i$ for $i > 1$. Then for every integer $s \geqq 0$,

$$b^{x^s} = b_0^{\binom{s}{0}} b_0^{\binom{s}{1}} \cdots b_s^{\binom{s}{s}}\ .$$

Proof. Since $\binom{s}{0} = 1$ for all $s > 0$, the statement is true for $s = 0$. We proceed by induction on s to get

$$\begin{aligned} b^{x^{s+1}} &= [b_0^{\binom{s}{0}}]^x [b_1^{\binom{s}{1}}]^x \cdots [b_s^{\binom{s}{s}}]^x \\ &= [b_0 b_1]^{\binom{s}{0}} [b_1 b_2]^{\binom{s}{1}} \cdots [b_s b_{s+1}]^{\binom{s}{s}} \\ &= b_0^{\binom{s}{0}} [b_1^{\binom{s}{0}} b_1^{\binom{s}{1}}] \cdots [b_s^{\binom{s}{s-1}} b_s^{\binom{s}{s}}] b_{s+1}^{\binom{s}{s}} \\ &= b_1^{\binom{s+1}{0}} b_1^{\binom{s+1}{1}} \cdots b_s^{\binom{s+1}{s}} b_{s+1}^{\binom{s+1}{s+1}} \end{aligned}$$

where we use the formulas $\binom{s}{i} + \binom{s}{i+1} = \binom{s+1}{i+1}$ and $\binom{s}{0} = \binom{s}{s} = 1$. This proves the lemma.

LEMMA 1.2. *If x is an automorphism of the group G that centralizes the subgroup U of G, then U and $\langle \mathfrak{N}_G(U) \circ x \rangle$ centralize each other.*

Proof. Let $u \in U$ and $g \in \mathfrak{N}_G(U)$. Then $u^g \in U$, hence $u^g = (u^g)^x = (u^x)^{g^x} = u^{g^x}$. Therefore $g^{-1}g^x$ centralizes U which proves the lemma.

LEMMA 1.3. *Let U be a subgroup of a group G and x an automorphism of G that centralizes U and satisfies $\langle G \circ x \rangle \subseteqq U$. Then*

(1) $\langle G \circ x \rangle \subseteqq \mathfrak{C}_U(U_G)$;

(2) *if the set $G \circ x$ has finite exponent n, then x is a torsion automorphism of order dividing n;*

(3) *if x is a torsion automorphism, then for every $g \in G$ the order of $g \circ x$ divides the order of x.*

Proof. (1) follows from Lemma 1.2 applied to U_G. Let $g \in G$ and put $h = g^{-1}g^x \in G \circ x \subseteqq U$. Then $g^x = gh$ and, by induction, $g^{x^r} = gh^r$. In particular, under the assumption of (2), $g^{x^n} = gh^n = g$. Hence $x^n = 1$ proving (2). On the other hand, if x is a torsion automorphism of order m, then $g = g^{x^m} = gh^m$, hence $h^m = 1$ proving (3).

LEMMA 1.4. [4, p. 49, 1.5 Hilfssatz.] *Every automorphism of a finite p-group P that centralizes $\Omega_d(P)$ has order a power of p.*

Proof. Let P be a counterexample of minimal order and $x \neq 1$ a p'-automorphism of P centralizing $\Omega_d(P)$. Assume by way of contradiction that $\langle P \circ x \rangle$ is a proper subgroup of P. P being a minimal counterexample and $\langle P \circ x \rangle$ being x-admissible, $\langle P \circ x \rangle$ must be centralized by x. So Lemma 1.3 (2) implies $x = 1$, a contradiction. Therefore $P = \langle P \circ x \rangle$. Since P is solvable, P' is a proper, x-admissible subgroup of P. Hence P' is centralized by x. By Lemma 1.2, $P' \subseteqq \mathfrak{C}_P(\langle P \circ x \rangle) = \mathfrak{Z}(P)$ and P has class 2. Let t be minimal such that $P = \Omega_{t+1}(P)$, hence $t \geqq 1$, and let $g \in P$ be an element of order dividing p^{t+1}. Then $g^{p^t} \in \Omega_1(P)$ is centralized by x, hence

$$(g^{1}g^{x})^{p^t} = (g^{-1})^{p^t}(g^{x})^{p^t}(g^{x}, g^{-1})^{\binom{p^t}{2}} = (g^{x}, g^{-1})^{\binom{p^t}{2}} ,$$

see [9, p. 8., (10)]. If $p \neq 2$, then p^t divides $\binom{p^t}{2}$; put $s = t$. If $p = 2$, then p^{t-1} divides $\binom{p^t}{2}$; put $s = t - 1$. Now

$$(g^{x}, g^{-1})^{p^s} = (g^{x}, g^{-p^s}) ,$$

see [9, p. 80, (9)], and $g^{p^s} \in \Omega_d(P) \subseteqq \mathfrak{Z}(P)$ by Lemma 1.2. Therefore

$$(g^{x}, g^{-p^s}) = 1, (g^{x}, g^{-1})^{\binom{p^t}{2}} = 1, \quad \text{and} \quad (g^{-1}g^{x})^{p^t} = 1$$

showing that

$$P = \langle P \circ x \rangle = \langle \Omega_{t+1}(P) \circ x \rangle \subseteqq \Omega_t(P) \subset P ,$$

a contradiction. No counterexamples exist.

COROLLARY 1.5. *Let E be a subgroup of a finite group G such that $\mathfrak{C}_G(E)$ has a p-Sylow subgroup S which satisfies $\Omega_d(S) \subseteqq E$. Then $\mathfrak{C}_G(E)$ has a normal p-complement.*

Proof. Suppose U is a subgroup of S and x is a p'-element of $\mathfrak{N}_C(U)$, $C = \mathfrak{C}_G(E)$. Then $\Omega_d(U) \subseteqq \Omega_d(S) \subseteqq E$ is centralized by x. Lemma 1.4 implies that U is centralized by x. By a well-known theorem of Frobenius [3], C has a normal p-complement.

PROPOSITION 1.6. *If A is a locally finite, normal subgroup of the group B, then any torsion automorphism of B that leaves A invariant and centralizes B/A and $\Omega_d(A)$ has order divisible by primes that are orders of elements in A only.*

Proof. Such an automorphism x is the product of its primary components x_q, q prime, x_q being a power of x. Put $y = x_q$ and assume that q is not the order of an element in A. Pick $a \in A$. Being finitely generated $F = \langle a^{\langle y \rangle} \rangle$ is a finite, y-admissible subgroup of A. For any prime p the number of p-Sylow subgroups of F is prime to q and y normalizes at least one p-Sylow subgroup P of F. By Lemma 1.4, P is centralized by y. So $F = \langle P \mid p \text{ prime} \rangle$ is centralized by y and, in particular, a is centralized by y.

Pick $b \in B$. A being a torsion group, $b^{-1}b^y \in A$ has finite order prime to q. By Lemma 1.3 (3) this order divides the order of y, $b^{-1}b^y = 1$ and $x_q = y = 1$.

We shall only need the following special case of Proposition 1.6.

COROLLARY 1.7. *Let A be a normal torsion subgroup of a group B, assume $\Omega_i(A)/\Omega_{i-d}(A) \subseteqq \mathfrak{Z}(\Omega_{i+1}(A)/\Omega_{i-d}(A))$ for all functions $i \in \mathscr{F}$ with $i \geqq d$, let x be a torsion automorphism of B that leaves A invariant and centralizes B/A and $\Omega_d(A)$. If the order of x is prime to the order of every element in A, then $x = 1$.*

Proof. A is locally finite, since it is the union of solvable (hence locally finite) torsion subgroups. Exploiting the structure of A one may prove Corollary 1.7 also without reference to Lemma 1.4.

REMARK 1.8. The quaternion group of order 8 shows that $\Omega_d(P)$ in Lemma 1.4 may not be replaced by $\Omega_1(P)$. However, if P is abelian, this may be done. Similarly in Proposition 1.6 and Corollary 1.7 $\Omega_d(A)$ may be replaced by $\Omega_1(A)$ provided that finite 2-subgroups of A are abelian.

LEMMA 1.9. *Let A be a torsion group with abelian factors $\Omega_i(A)/\Omega_{i-1}(A)$ for all finite functions $i \in \mathscr{F}$. Let $0 \leqq k \in \mathscr{F}$ satisfy $k(2) \geqq 2$ and $\Omega_k(A) \subseteqq \mathfrak{Z}(A)$, assume that only finitely many primes are orders of elements in A. Then any automorphism of A that centralizes $\Omega_k(A)$ also centralizes all the factors $\Omega_n(A)/\Omega_{n-k}(A)$ for finite functions $n \in \mathscr{F}$.*

Proof. Clearly by the structure of A, $\Omega_r(A)$ is restrained by r for $r \in \mathscr{F}$. If there are counterexamples, then there are also counterexamples of finite exponent, since only finitely many primes are orders of elements in A. Let A be a counterexample of minimal (finite) exponent $\Pi p^{m(p)}$. Then automorphisms of A that centralize $\Omega_k(A)$ also centralize

$$\Omega_j(A)/\Omega_{j-k}(A)$$

for $j < m$. In particular, A centralizes $\Omega_j(A)/\Omega_{j-k}(A)$ for $j < m$.

Let x be an automorphism of A that centralizes $\Omega_k(A)$, pick

$$a \in \Omega_m(A) = A \ ,$$

and let p, a prime, be the order of an element in A. Let g_t have the value t at p and 0 elsewhere. Then (a^x, a^{-1}) lies in $\Omega_{m-g_1}(A)$, since $\Omega_m(A)/\Omega_{m-g_1}(A)$ is abelian, and this commutator commutes with a^x and $a^{-1} \bmod \Omega_{m-g_1-k}(A)$, since A centralizes $\Omega_{m-g_1}(A)/\Omega_{m-g_1-k}(A)$. Therefore

$$(a^{-1}a^x)^{p^t} \equiv (a^{-1})^{p^t}(a^x)^{p^t}(a^x, a^{-1})^{\binom{p^t}{2}} \bmod \Omega_{m-g_1-k}(A)$$

for natural numbers t, cf. [9, p. 81, (10)]. Since x centralizes

$$\Omega_{m-g_1}(A)/\Omega_{m-g_1-k}(A)$$

by the minimality of the exponent of A, we have

$$(a^{-1})^{p^t}(a^x)^{p^t} = (a^{p^t})^{-1}(a^{p^t})^x \equiv 1 \bmod \Omega_{m-g_1-k}(A)$$

and

$$(1) \qquad (a^{-1}a^x)^{p^t} \equiv (a^x, a^{-1})^{\binom{p^t}{2}} \bmod \Omega_{m-g_1-k}(A) \ .$$

Using a well-known formula, cf. [9, p. 80, (9)], and remembering that A centralizes $\Omega_{m-g_1}(A)/\Omega_{m-g_1-k}(A)$ we get

$$(2) \qquad (a^x, a^{-1})^p \equiv ((a^x)^p, a^{-1}) \equiv 1 \bmod \Omega_{m-g_1-k}(A).$$

Assume first that $p \neq 2$ and choose $t = 1$. Then p divides $\binom{p}{2}$. Hence by (1) and (2), $(a^{-1}a^x)^p \equiv 1 \bmod \Omega_{m-g_1-k}(A)$, $a^{-1}a^x \in \Omega_{m-k}(A)$, and x centralizes $\Omega_m(A)/\Omega_{m-k}(A)$.

If $p = 2$, choose $t = 2$. Then p divides $\binom{p^t}{2}$ and it follows (2) that $(a^x, a^{-1})^{\binom{p^t}{2}} \equiv 1 \bmod \Omega_{m-g_1-k}(A)$. So (1) implies

$$(a^{-1}a^x)^{p^2} \equiv 1 \bmod \Omega_{m-g_1-k}(A)$$

and $a^{-1}a^x \in \Omega_{m-g_1-k+g_2}(A) \subseteqq \Omega_{m-g_1}(A)$, since $k(2) \geqq 2$ implies $k \geqq g_2$. Therefore

$$(a^x, a^{-1}) = (a(a^{-1}a^x), a^{-1}) = (a^{-1}a^x, a^{-1})$$

lies in $\Omega_{m-g_1-k}(A)$, and $t = 1$ in (1) implies

$$(a^{-1}a^x)^p \equiv 1 \bmod \Omega_{m-g_1-k}(A) .$$

Consequently $a^{-1}a^x \in \Omega_{m-k}(A)$, and x centralizes $\Omega_m(A)/\Omega_{m-k}(A)$.

LEMMA 1.10. *Let $0 \leqq k \in \mathscr{f}$ satisfy $k(2) \geqq 2$, let A be a torsion group with $\Omega_i(A)/\Omega_{i-k}(A) \subseteqq \mathfrak{Z}(\Omega_{i+1}(A)/\Omega_{i-k}(A))$ for all finite functions $i \in \mathscr{f}$, and assume that only finitely many primes are orders of elements in A. Then any automorphism of A that centralizes $\Omega_k(A)$ also centralizes all the factors $\Omega_n(A)/\Omega_{n-k}(A)$ for finite functions $n \in \mathscr{f}$.*

Proof. Let A be a counterexample of minimal (finite) exponent $\Pi p^{m(p)}$. Then $\Omega_k(A) \neq 1$ and there exists a prime p with $k(p) > 0$ and $\Omega_g(A) \neq 1$ where $g(p) = 1$ and 0 elsewhere. By assumption $\bar{A} = \Omega_m(A)/\Omega_{m-g-k}(A)$ satisfies $\Omega_k(\bar{A}) \subseteqq \mathfrak{Z}(\bar{A})$, since $\Omega_k(\bar{A}) = \Omega_{m-g}(A)/\Omega_{m-g-k}(A)$. By the minimality of the exponent of A, an automorphism x of A that centralizes $\Omega_k(A)$ also centralizes $\Omega_{m-g}(A)/\Omega_{m-g-k}(A) = \Omega_k(\bar{A})$. Therefore by Lemma 1.9, x centralizes $\Omega_{k+g}(\bar{A})/\Omega_g(\bar{A})$, i.e., $\Omega_m(A)/\Omega_{m-k}(A)$.

THEOREM 1.11. *Let A be a normal torsion subgroup of a group B, assume $\Omega_i(A)/\Omega_{i-d}(A) \subseteqq \mathfrak{Z}(\Omega_{i+1}(A)/\Omega_{i-d}(A))$ for all finite functions $i \in \mathscr{f}$, let x be a torsion automorphism of B that leaves A invariant and centralizes B/A and $\Omega_d(A)$, and let $f \geqq 0$ be a function in $\mathscr{f}$. Then x centralizes $B/\Omega_f(A)$, if and only if x is restrained by f.*

Proof. (1) To prove the if-part of the theorem we shall assume without loss of generality that f is finite and assumes only finitely many positive values, because x is a torsion automorphism.

Assume first that the statement in question is false for some group A of finite exponent. Then there are counterexamples A of minimal finite exponent $\Pi p^{j(p)}$ of A. Choose one where also f is minimal with respect to the partial ordering $\leqq$. It follows $j \neq 0, f \neq 0$, and $x \neq 1$. There exists a prime q such that both $A^q \neq 1$ and q divides the order of x, since otherwise Corollary 1.7 would imply $x = 1$. Define $g \in \mathscr{f}$ to be 1 at q and 0 elsewhere. $A/\Omega_g(A)$ is restrained by $j - g < j$ and has the required structure. x induces in $B/\Omega_g(A)$ an automorphism that leaves $A/\Omega_g(A)$ invariant and centralizes $[B/\Omega_g(A)]/[A/\Omega_g(A)]$ and $\Omega_d(A/\Omega_g(A))$; this last fact follows from Lemma 1.10. So the minimality of j yields that x centralizes $[B/\Omega_g(A)]/\Omega_f(A/\Omega_g(A))$, i.e., $B/\Omega_{f+g}(A)$.

Again by our hypothesis and Lemma 1.10, x centralizes

$$\Omega_f(A)/\Omega_{f-d}(A) \quad \text{and} \quad \Omega_{f+g}(A)/\Omega_{f+g-d}(A) .$$

So we may apply Lemma 1.1 for any $b \in B$ to get

$$b^{x^q} \equiv bb_1^q b_2^{\binom{q}{2}} \bmod \Omega_{f-d}(A)$$

with $b_1 \in \Omega_{f+g}(A)$, $b_2 \in \Omega_{f+g-d}(A) \subseteqq \Omega_f(A)$, $b_3 \in \Omega_{f-d}(A)$, since

$$b_1b_2 \equiv b_2b_1 \bmod \Omega_{f-d}(A)$$

by the structure of A. If $q \neq 2$, then $b_2^{\binom{q}{2}} \in \Omega_{f-g}(A)$. If $q = 2$, then even $b_2 \in \Omega_{f-g}(A)$. Hence

$$b^{x^q} \equiv bb_1^q \bmod \Omega_{f-g}(A) .$$

On the other hand by the minimality of f, x^q being restrained by $f - g \geqq 0$ centralizes $B/\Omega_{f-g}(A)$ so that

$$b \equiv b^{x^q} \bmod \Omega_{f-g}(A) .$$

Hence $b_1^q \in \Omega_{f-g}(A)$ and $b_1 \in \Omega_f(A)$. This signifies that x centralizes $B/\Omega_f(A)$ contradicting our assumption and proving the statement for groups A of finite exponent.

Now consider the general case and let $b \in B$. Since A is a torsion group, there exists a function $j \geqq f$ in $\mathscr{F}$ such that $\Omega_j(A)$ has finite exponent and contains $b^{-1}b^x$. Hence x leaving $\Omega_j(A)$ invariant centralizes $\langle b\rangle\Omega_j(A)/\Omega_j(A)$. By what we have already proved above we may conclude that x centralizes $\langle b\rangle\Omega_j(A)/\Omega_f(A)$, whence $b^x \equiv b \bmod \Omega_f(A)$. This shows that x centralizes $B/\Omega_f(A)$.

(2) Conversely, if x is a torsion automorphism of B centralizing $B/\Omega_f(A)$, we may assume $x \neq 1$. Let p be a prime that divides the order of x and define $g \in \mathscr{F}$ to be 1 at p and 0 elsewhere. There exists $h_b \leqq f$ depending on $b \in B$ such that $\Omega_{h_b}(A)$ has finite exponent and contains $b_1 = b^{-1}b^x$. By Lemma 1.10 applied to $\Omega_{h_b}(A)$, $b_2 = b_1^{-1}b_1^x$ is contained in $\Omega_{h_b-d}(A) \subseteqq \Omega_{f-g}(A)$. Therefore

$$b^{x^p} \equiv bb_1^p \bmod \Omega_{f-g}(A)$$

by Lemma 1.1. But $b_1^p \in \Omega_{f-g}(A)$, hence x^p centralizes $B/\Omega_{f-g}(A)$.

By induction on the order of x, x^p is restrained by $f - g$. So x is restrained by f.

REMARK 1.12. H. Leptin [6, p. 101] proved that in the case of a reduced abelian p-group A with $p \geqq 5$ the conclusion of Theorem 1.11 remains valid under the weaker hypothesis that x only centralizes certain factors of $\Omega_1(A)$ instead of $\Omega_1(A)$ as a whole.

REMARK 1.13. Let A be an abelian 2-group of exponent $\geqq 8$ and let x be the automorphism of A that maps every element onto its inverse. Then x centralizes $\Omega_1(A)$ and has order 2, but does not centralize $A/\Omega_1(A)$.

2. Let E be a normal subgroup of the p-group G and denote by $\mathfrak{S}^k(E)$ the subgroup of G formed by all the elements of G that centralize all the factors $\Omega_i(E)/\Omega_{i-k}(E)$, $i \geqq k$. The following proposition may be generalized to the case where E satisfies $E \subseteqq \mathfrak{S}^k(E)$ instead of being abelian ($k = \infty$). But if $x \in \mathfrak{S}^k(E)$ satisfies $x^p \in E$ and $(x, g) \subseteqq E$ then it does not follow that the subgroup W generated by E and x satisfies $W \subseteqq \mathfrak{S}^k(W)$, since $\Omega_1(W) \subseteqq \mathfrak{Z}(W)$ may be violated. Hence no application of the proposition in this case which would be similar to the proof of Theorem 2.2 or Theorem 2.8 below is to be expected. Consequently we shall restrict our attention to the abelian case and follow Alperin's argument.

PROPOSITION 2.1. *Let G be a group, E an abelian subgroup of G, E_1 a subgroup of G that contains E, and $f \geqq 0$ a function in $\mathscr{f}$ such that*

(1) $f(2) \neq 1$,

(2) *if h is a function in $\mathscr{f}$ with $0 \leqq h \leqq f$ and if $\Omega_h\mathfrak{C}_G(E_1)$ is restrained by f, then $\Omega_h\mathfrak{C}_G(E_1) \subseteqq E$,*

(3) *there exists an abelian subgroup A of G and a subgroup A_1 of G such that*

(a) *$\Omega_f\mathfrak{C}_G(E_1)$ normalizes A_1,*

(b) $A_1 \supseteqq A \supseteqq \Omega_f(A) = E$, $E \trianglelefteqq A_1$,

(c) $\Omega_f\mathfrak{C}_G(E_1)' \cap \mathfrak{C}_G(A_1) \subseteqq A$,

(d) *if the element x of $\mathfrak{C}_G(E_1)$ is restrained by f, then x centralizes $A_1/\Omega_f(A)$.*

Then $\Omega_f\mathfrak{C}_G(E_1) \subseteqq E$.

Proof. Assume that the proposition is false and choose $h \in \mathscr{f}$ minimal with respect to $0 \leqq h \leqq f$ and $\Omega_h\mathfrak{C}_G(E_1) \not\subseteqq E$. So by (2), $\Omega_h\mathfrak{C}_G(E_1)$ is not restrained by f.

Aiming at a contrary statement pick $x \neq 1$ and y in $\mathfrak{C}_G(E_1)$ where x is restrained by h and y is restrained by f. We shall examine $\langle x, y\rangle$.

By (3d), x and y centralize $A_1/\Omega_f(A)$. Hence $\langle x, y\rangle$ induces an abelian group of automorphisms in A_1, whence

$$(x, y) \in \Omega_f\mathfrak{C}_G(E_1)' \cap \mathfrak{C}_G(A_1) \subseteqq A$$

by (3c). But again x and y centralize A/E, so (x, y, x) and (x, y, y) are in E, and $\langle x, y\rangle$ has class of nilpotency at most 3. By Lemma 1.1,

$$y = y^{x^e} = y(y, x)^e(y, x, x)^{\binom{e}{2}},$$

hence

$$(*1) \qquad 1 = (y, x)^e (y, x, x)^{\binom{e}{2}}$$

for every natural number e that is divisible by the order of x. Let q be a prime that divides the order of x. Let $g \in \mathscr{F}$ be 1 at q and 0 elsewhere and let k be the smallest function $\geqq 0$ in $\mathscr{F}$ that restrains x. Then x^q is restrained by $k - g$, hence $x^q \in \Omega_{k-g}\mathfrak{C}_G(E_1)$. We have still $k - g \geqq 0$, but $k - g < k \leqq h$, since k is finite. So the minimality of h yields $\Omega_{k-g}\mathfrak{C}_G(E_1) \subseteqq E \subseteqq A$, hence

$$(*2) \qquad 1 = (y, x, x^q) = (y, x, x)^q ,$$

cf. [9, p. 80, (9)].

Assume first that $h(2) \neq 1$ and let y be restrained even by h. Choose e to be the least common multiple of $q^{h(q)}$ and the orders of x and y. Note that $h(q) \geqq d(q)$, since q divides the order of x. Therefore q divides $\binom{e}{2}$, entailing $(y, x, x)^{\binom{e}{2}} = 1$ because of (*2) and $(y, x)^e = 1$ because of (*1). By the choice of e this proves that (y, x) is contained in $\Omega_h(A) \subseteqq E$. Therefore (cf. [9, p. 80, (9)]), $(y,x)^e = (y, x^q) = 1$, since $x^q \in \Omega_{k-g}\mathfrak{C}_G(E_1) \subseteqq E$, and (cf. [9, p. 81, (10)])

$$(xy)^e = x^e y^e (y, x)^{\binom{e}{2}} = 1 .$$

This proves that $\Omega_h\mathfrak{C}_G(E_1)$ is restrained by $h \leqq f$; a contradiction.

Assume now that $h(2) = 1$ and let $h' \in \mathscr{F}$ have the value 2 at 2 and coincide with h elsewhere. Then $h \leqq h' \leqq f$, since $1 = h(2) \leqq f(2) \neq 1$. Suppose that y is restrained by h' and choose e' to be the least common multiple of $4 = 2^{h'(2)}$ and the orders of x and y. Again q divides $\binom{e'}{2}$, (y, x) is contained in $\Omega_{h'}(A) \subseteqq E$, and $(xy)^{e'} = 1$. This proves that xy is restrained by h' and hence that $\Omega_h\mathfrak{C}_G(E_1)$ is restrained by $h' \leqq f$, if $h(2) = 1$; again a contradiction.

THEOREM 2.2. *If f is a function in $\mathscr{F}$, if G is a hypercylic group, and if E is maximal among its abelian, normal subgroups restrained by f, then $\Omega_f\mathfrak{C}_G(E) = E$, provided that $f(2) \neq 1$ and that*

(*) *there exists an abelian, normal torsion subgroup $A \supseteqq E$ of G such that $\Omega_f\mathfrak{C}_G(E)' \cap \mathfrak{C}_G(A) \subseteqq A$ and $\Omega_d(A) = \Omega_d(E)$.*

Proof. We have to establish the hypotheses of Proposition 2.1. Without loss of generality $f \geqq 0$. Let $E_1 = E$. Assume by way of contradiction that (2) is not satisfied, so that $E\Omega_h\mathfrak{C}_G(E) \supset E$ for some h in $\mathscr{F}$ with $0 \leqq h \leqq f$ such that $\Omega_h\mathfrak{C}_G(E)$ is restrained by f. Then by hypercylicity, $E\Omega_h\mathfrak{C}_G(E)/E$ contains a cyclic, normal subgroup $H/E \neq 1$ of G/E. But $E \subseteqq \mathfrak{Z}(H)$, hence $H = \mathfrak{Z}(H)$. Furthermore $E\Omega_h\mathfrak{C}_G(E)$ is restrained by f, and so is H. This contradicts the maximality of E. Hence (2) is satisfied. Put $A_1 = A$. By maximality of E, $\Omega_f(A) = E$. So (3a, b, c) are satisfied. Now let $x \in \mathfrak{C}_G\Omega_f(A)$ be restrained by f. Then x centralizes E and $\Omega_d(E) = \Omega_d(A)$, so that Theorem 1.11 is applicable, x centralizes $A/\Omega_f(A)$. By Proposition 2.1, $\Omega_f\mathfrak{C}_G(E) = E$.

REMARK 2.3. The condition (*) in Theorem 2.2 may not be dropped as shown by the following example of [2, p. 19, Example 1]. Let A be a torsionfree, abelian group and x the automorphism of A that sends every element onto its inverse. Then $G = A\langle x\rangle$ is hypercylic and every element in the coset Ax is of order 2. Let $f = \infty$. Then $E = 1$ is a maximal abelian, normal subgroup of G restrained by f, but $\Omega_f\mathfrak{C}_G(E) = G \neq E$.

REMARK 2.4. As already indicated by J. L. Alperin the condition $f(2) \neq 1$ in Theorem 2.2 may not be dropped. If G is a dihedral group of order $2^{n+1} \geqq 16$, then G has no elementary abelian, normal subgroups of index 2, since G contains elements of order 8. Therefore it follows from [8, Lemma 1] that a maximal elementary abelian, normal subgroup E of G has order 2 and as such lies in the center of G. But at least half of the elements of G have order 2. This proves $\Omega_1\mathfrak{C}_G(E) \supset E$. This question has been investigated further by G. Tani Corsi [7].

COROLLARY 2.5. *If $f \geqq 0$ is a function in $\mathscr{F}$, if G is a hypercylic group, and if E is maximal among its abelian, normal subgroups restrained by f, then $\Omega_f\mathfrak{C}_G(E) = E$ provided that one of the following holds*:

(1) *$f(2) \neq 1$ and $\Omega_f(G)$ is restrained by $\bar{f}$ where $\bar{f}(p) = 0$ if $f(p) \leqq 0$ and $\bar{f}(p) = \infty$ if $f(p) > 0$.*

(2) *$f \geqq d$ and $\Omega_f(G)$ is a torsion group.*

(3) *There exists a prime q such that $f(p) = 0$ for $p \leqq q$ and $f(p) \geqq 1$ for $p > q$.*

(4) *$f(2) \neq 1$ and the set of p-elements of G is a subgroup for every prime p.*

(5) *$f(2) \neq 1$ and G is hypercentral.*

Proof. Assume (1). It will suffice to show that condition (*) of Theorem 2.2 is satisfied. Let A be maximal among the abelian, normal subgroups of G containing E and restrained by $\bar{f}$; such a subgroup exists by the maximum principle of set theory. $F = \Omega_f(G) \cap \mathfrak{C}_G(A)$ is restrained by $\bar{f}$. Therefore, since G is hypercylic, a similar argument as used in the proof of Theorem 2.2 shows that F is contained in A. In particular, $\Omega_f(G)' \cap \mathfrak{C}_G(A) \subseteqq F \subseteqq A$. If for some prime p the component A_p of A is not 1, then $\bar{f}(p) = \infty$ and $f(p) > 0$, hence

$f(p) \geqq d(p)$. Therefore, $\Omega_d(A) \subseteqq \Omega_f(A)$. By the maximality of E, $\Omega_f(A) = E$. This implies $\Omega_d(A) = \Omega_d(E)$, and Theorem 2.2 is applicable.

(2) is a special case of (1).

In any hypercylic group G the torsion elements of an order divisible by primes $p > q$ only form a subgroup $G(q)$; cf. [2, p. 21, Proposition 1]. Therefore (3) is also a special case of (1).

Clearly (4) implies (1).

Every hypercentral group is locally nilpotent [5, p. 223] and every locally nilpotent group has a unique p-Sylow subgroup for every prime p [5, p. 229]. Hence (5) is a special case of (4).

COROLLARY 2.6. *For a p-Sylow subgroup P of a finite group G let E be maximal among the abelian, normal subgroups of P of exponent dividing p^n, $n \geqq d(p)$. Then $\mathfrak{C}_G(E)$ has a normal p-complement and E is the set of all elements in $\mathfrak{C}_G(E)$ of order dividing p^n.*

Proof. Since P normalizes $C = \mathfrak{C}_G(E)$, $S = \mathfrak{C}_P(E)$ is a p-Sylow subgroup of C. Moreover, $\Omega_d(S) = \Omega_d\mathfrak{C}_P(E) \subseteqq E$ by Corollary 2.5 (5), hence Corollary 1.5 yields the existence of a normal p-complement in C. An arbitrary p-Sylow subgroup S_0 of C is conjugate to S in $C = \mathfrak{C}_G(E)$. Therefore $E = \Omega_n(S) = \Omega_n(S_0)$. This completes the proof.

DEFINITION 2.7. (a) For $f \in \mathscr{F}$ define $f' \in \mathscr{F}$ by $f'(p) = 0$ if $f(p) \leqq 0$ and $f'(p) = 1$ if $f(p) > 0$.

(b) For an abelian normal subgroup E of a group G such that E is restrained by $f \in \mathscr{F}$ define $\mathfrak{X}_G^f(E)$ by

$$\mathfrak{X}_G^f(E)/E = \Omega_{f'}[\Omega_f\mathfrak{C}_G(E)/E \cap \mathfrak{Z}(G/E)] .$$

THEOREM 2.8. *If f is a function in $\mathscr{F}$ with $f(2) \neq 1$, if G is a hypercentral group, and if E is maximal among its abelian characteristic subgroups restrained by f, then $\Omega_f\mathfrak{C}_G\mathfrak{X}_G^f(E) = E$.*

Proof. (1) Let U be a normal subgroup of G contained in $\mathfrak{C}_G\mathfrak{X}_G^f(E)$ and restrained by f, and assume by way of contradiction that $U \not\subseteqq E$. Then $UE/E \neq 1$, whence $UE/E \cap \mathfrak{Z}(G/E) \neq 1$ by hypercentrality. Since UE is restrained by f we see that $\Omega_1(S) = \Omega_{f'}(S)$ for every subgroup S of UE/E, in particular

$$1 \neq \Omega_{f'}[UE/E \cap \mathfrak{Z}(G/E)] \subseteqq [UE \cap \mathfrak{X}_G^f(E)]/E .$$

But $UE \cap \mathfrak{X}_G^f(E) = [U \cap \mathfrak{X}_G^f(E)]E$, where $U \cap \mathfrak{X}_G^f(E) \subseteqq \Omega_f\mathfrak{Z}\mathfrak{X}_G^f(E) \subseteqq E$ by the maximality of E. This contradiction shows that $U \subseteqq E$.

(2) Consider first the case where f assumes only the values 0 and ∞. Since G is hypercentral, $\Omega_f(G)$ is restrained by f (cf. proof

of Corollary 2.5 (5)) and so is $U = \Omega_f \mathfrak{C}_G \mathfrak{X}_G^f(E)$. Therefore (1) implies $\Omega_f \mathfrak{C}_G \mathfrak{X}_G^f(E) \subseteqq E$. Clearly $E \subseteqq \Omega_f \mathfrak{C}_G \mathfrak{X}_G^f(E)$, and the theorem is proved in this case.

(3) Now consider the general case and put $E_1 = \mathfrak{X}_G^f(E)$. Then condition (2) of Proposition 2.1 follows from (1) above. Let A be maximal among the abelian, characteristic subgroups of G containing E and restrained by $\bar{f}$, where $\bar{f}$ is defined as in Corollary 2.5 (1). Put $A_1 = \mathfrak{X}_G^{\bar{f}}(A)$. Then $\Omega_f(A) = E$ by the maximality of E and $\Omega_{\bar{f}} \mathfrak{C}_G \mathfrak{X}_G^{\bar{f}}(A) = A$ by (2) above. In particular since $\Omega_f \mathfrak{C}_G(E_1)' \subseteqq \Omega_{\bar{f}}(G)$ is restrained by $\bar{f}$,

$$\Omega_f \mathfrak{C}_G(E_1)' \cap \mathfrak{C}_G(A_1) \subseteqq \Omega_{\bar{f}} \mathfrak{C}_G(A_1) = A$$

proving (3c) of Proposition 2.1. Clearly A_1/A is centralized by every element in G. As in the proof of Corollary 2.5 (1), $\Omega_d(A) = \Omega_d(E)$. Therefore (3d) follows from Theorem 1.11, and Proposition 2.1 yields $\Omega_f \mathfrak{C}_G(E_1) = E$.

COROLLARY 2.9. *For a p-Sylow subgroup P of a finite group G let E be maximal among the abelian, characteristic subgroups of P of exponent dividing p^n, $n \geqq d(p)$. Then $\mathfrak{C}_G \mathfrak{X}_P^n(E)$ has a normal p-complement and E is the set of all elements in $\mathfrak{C}_G \mathfrak{X}_P^n(E)$ of order dividing p^n.*

Proof. Use Theorem 2.8 in the proof of Corollary 2.6.

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G-SPACES, H-SPACES AND W-SPACES

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The notions of G-space, W-space, H-space, and higher order Whitehead product are differentiated through example.

In [3], [4] and [5] D. H. Gottlieb introduces certain subgroups, $G_n(X, x_0)$, of the homotopy groups of a space. These groups are related to the problem of sectioning fibrations with fibre X. Related to the groups $G_n(X, x_0)$ is the notion of a G-space. A G-space is a space with $G_n(X, x_0) = \pi_n(X, x_0)$ for all n. It is a simple matter to show that every H-space is a G-space (see below). However, till recently the status of the converse remained undecided. Recently, Gottlieb produced an example of a two-stage Postnikov system that is a G-space but not an H-space (unpublished). The purpose of this note is to clarify the situation further. We produce a 3-dimensional manifold that is a G-space but not an H-space. Incidently, the theory of G-spaces tells us that our example is also a W-space, that is, a space whose Whitehead products all vanish.

Finally we would like to resolve a question of G. Porter [6]. Namely, our example is also an example of a space whose higher order Whitehead products all vanish but, again, is not an H-space.

We would like to acknowledge the priority of D. H. Gottlieb's example mentioned above and thank him for his help in the preparation of this paper.

1. Preliminaries.

In this section we review the elementary theory of G-spaces presented in [4] and [5].

NOTATION 1.1. We assume all our spaces X are path connected C. W. complexes with base point x_0. We let X^X be the space of maps X to X. We let $M(X)$ be the component of the identity map $1: X \to X$ in X^X. Consider the evaluation map $e: M(X) \to X$ given by $e(f) = f(x_0)$. This map gives a fibration with fibre $M(X)_0$, the space of maps in $M(X)$ with $f(x_0) = x_0$.

DEFINITION 1.2. We define

$$G_n(X, x_0) = e_*(\pi_n(M(X), 1)) \subseteq \pi_n(X, x_0) .$$

THEOREM 1.3. *The groups $G_n(X, x_0)$ are invariant with respect to base point and homotopy type but not natural with respect to maps.*

Proof. [5].

THEOREM. 1.4.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \cdot F: X \times S^n \longrightarrow X \text{ with } F/X \vee S^n = 1 \vee f\} .$$

Proof. [5].

THEOREM 1.5.

$$G_n(X, x_0) = \{[f] \mid \cdot \exists \text{ a fibration } X \subseteqq E \xrightarrow{p} S^{n+1} \text{ with } [f] = \partial_*[1]\} ,$$

where $1: S^{n+1} \to S^{n+1}$ *is the identity map.*

Proof. [4].

DEFINITION 1.6. $P_n(X, x_0)$ is the subgroup of elements $[f]$ in $\pi_n(X, x_0)$ with $[[f], [g]] = 0$ (Whitehead product) for all m and all $[g] \in \pi_m(X, x_0)$.

THEOREM 1.7. $G_n(X, x_0) \subseteqq P_n(X, x_0)$.

Proof. [5] (see 1.4 above).

REMARK. Ganea [1] has shown that in general $G_n(X, x_0) \neq P_n(X, x_0)$. (see 3.4 below).

DEFINITION 1.8. (a) A *G-space* is a space X with $G_n(X, x_0) = \pi_n(X, x_0)$, all n.

(b) A *W-space* is a space X with $P_n(X, x_0) = \pi_n(X, x_0)$ for all n.

THEOREM 1.9. (a) *Every H-space is a G-space.*
(b) *Every G-space is a W-space.*

Proof. [5]. (a) Follows from 1.4.
(b) Follows from 1.7.

2. A G-space that is not an H-space. As mentioned in 1.3 the groups $G_n(X, x_0)$ are not natural with respect to maps. However, we can prove the following.

LEMMA 2.1. *Suppose we are given a map* $F: Y \times X \to Y$ *with* $F/Y \vee X = 1 \vee f$ *then* $f_*: \pi_n(X, x_0) \to G_n(Y, y_0)$.

Proof. For $g: S^n \to X$ consider the composition

$$Y \times S^n \xrightarrow{1 \times g} Y \times X \xrightarrow{F} Y .$$

Now apply 1.4.

EXAMPLE 2.2. Let H be a closed subgroup of a Lie group G. Let $(H\backslash G)$ be the left coset space. Let $p\colon G \to (H\backslash G)$ be the projection. We have the usual pairing $(H/G) \times G \xrightarrow{F} (H/G)$ with $F/(H/G) \vee G = 1 \vee p$.

THEOREM 2.3. *In the situation of 2.2 assume $i_*\colon \pi_n(H, e) \to \pi_n(G, e)$ is an inclusion for all n, then $(H\backslash G)$ is a G-space hence a W-space.*

Proof. Since i_* is an inclusion $p_*\colon \pi_n(G, e) \to \pi_n(H\backslash G, [e])$ is an epimorphism. On the other hand, by 2.1 $p_*\pi_n(G, e) \subseteqq G_n(H\backslash G, [e])$ hence $G_n(H\backslash G, [e]) = \pi_n(H\backslash G, [e])$ or $H\backslash G$ is a G-space.

We are now prepared to produce our example. We represent S^1 by the complex numbers $e^{i\theta}$ $0 \leqq \theta \leqq 2\pi$. In $SO(3)$ we let the symbol (θ) denote the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

2.4. *Example of a G-space that is not an H-space.* Embed $S^1 \subseteqq SO(3) \times S^1$ as a subgroup by the following map $i(e^{i\theta}) = (2\theta) \times e^{i3\theta}$. We let

$$T \overset{\text{def}}{\equiv} i(S^1)\backslash SO(3) \times S^1 .$$

LEMMA 2.5. *T is a G-space, hence a W-space.*

Proof. By 2.3 we need only check

$$\begin{array}{ccc} i_*\colon \pi_1(S^1) & \longrightarrow & \pi_1(SO(3) \times S^1) \\ \| & & \| \\ Z & & Z_2 \oplus Z \end{array}$$

is an inclusion, but it is easy to check $i_*(1) = 0 \oplus 3$. Note this implies $\pi_1(T) = Z_2 \oplus Z_3$.

LEMMA 2.6. *T is not an H-space.*

Proof. (a) T is a 3-dimensional manifold hence $H^n(T, Z_3) = 0$,

$n > 3$.

(b) $H'(T, Z_3) = Z_3$, generated say by α. This is by remarks at the end of 2.5.

(c) From the universal coefficient theorem we know there is $\beta \neq 0$ in $H^2(T, Z_3)$ β indecomposable ($\alpha^2 = 0$).

(a), (b) and (c) implies that $H^*(T, Z_3)$ does not support a Hopf algebra structure, hence, T is not an H space. In particular if $T \times T \xrightarrow{h} T$ is a Hopf map.

$$0 = h^*(\beta^2) = (1 \otimes \beta + \beta \otimes 1 + r(\alpha \otimes \alpha))^2 = 2\beta \otimes \beta + \cdots \neq 0 .$$

We could also note that $T = Z_3 \backslash SO(3)$ where Z_3 is the group (0), $(2/3\pi)$, $(4/3\pi)$. Then, using the spectral sequence of a covering we have

$$H^n(T, Z_3) = \begin{cases} Z_3 & n = 0, 1, 2, 3 \\ 0 & n > 3 . \end{cases}$$

This does not support a Hopf algebra structure.

3. Higher order Whitehead products. The purpose of this section is to point out that our example also answers a question of G. Porter [6].

DEFINITION 3.1. A space X is said to have *trivial higher order Whitehead products.* If given any set of homotopy elements

$$[f_i] \in \pi_{p_i}(X, x_0) \quad 1 \leqq i \leqq n .$$

The map $\bigvee_{i=1}^{n} f_i \colon \bigvee S^{p_i} \to X$ extends to some $f \colon \bigtimes_{i=1}^{n} S^{p_i} \to X$. (see [6]).

THEOREM 3.2. *Any G-space has trivial spherical Whitehead products.*

Proof.

LEMMA. *Given any* $n-1$ *elements* $[f_i] \in \pi_{p_i}(X, x_0)$ $1 \leqq i \leqq n-1$ *we can find a map*

$$h \colon \left(\bigtimes_{i=1}^{n-1} S^{p_i} \right) \times X \longrightarrow X \qquad \textit{with}$$

$$h \Big/ \left(\bigvee_{i=1}^{n-1} S^{p_i} \right) \vee X = \left(\bigvee_{i=1}^{n-1} f_i \right) \vee 1 .$$

This is proved by induction. For $n = 2$ this is 1.4. Suppose we have a map

$$\bar{h}\colon \left(\mathop{\times}_{i=1}^{n-2} S^{p_i}\right) \times X \longrightarrow X$$

with the required property.

Consider $\tilde{h}\colon S^{p_{n-1}} \times X \to X$ an extension of $f_{n-1} \vee 1$ (1.4). Finally, consider the composition

$$S^{p_{n-1}} \times \left(\left(\mathop{\times}_{i=1}^{n-2} S^{p_i}\right) \times X\right) \xrightarrow{1 \times \bar{h}} S^{p_{n-1}} \times X \xrightarrow{\tilde{h}} X .$$

Set $h = \tilde{h}(1 \times \bar{h})$.

We now finish the proof by noting that the composition

$$\left(\mathop{\times}_{i=1}^{n-1} S^{p_i}\right) \times S^{p_n} \xrightarrow{1 \times f_n} \left(\mathop{\times}_{i=1}^{n-1} S^{p_i}\right) \times X \xrightarrow{h} X$$

is the required extension of $\bigvee_{i=1}^{n} f_i$.

THEOREM 3.3. *There exists finite dimensional spaces with trivial higher order Whitehead products that are not H-spaces.*

Proof. The space T of 2.4 is such an example.

FINAL REMARKS 3.4. Ganea [2] has constructed an infinite dimensional example of a *W*-space that is not a *G*-space. G. Lang (unpublished) points out that using recent results of Gottlieb [5] one can show that $CP(3)$ is a finite dimensional example of such a space. In [1] it is shown that $CP(3)$ is a *W*-space, but in [5] it is shown that every finite dimensional *G*-space has Euler-Poinare characteristic 0 hence $CP(3)$ is not a *G*-space.

Porter [7] shows that $CP(3)$ has nontrivial higher order Whitehead products. It would be interesting to have examples of spaces with vanishing higher order Whitehead products that are not *G*-spaces.

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COHESIVE SETS AND RECURSIVELY ENUMERABLE DEDEKIND CUTS

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In this paper the methods of recursive function theory are applied to certain classes of real numbers as determined by their Dedekind cuts or by their binary expansions. Instead of considering recursive real numbers as in constructive analysis, we examine real numbers whose lower Dedekind cut is a recursively enumerable (r.e.) set of rationals, since the r.e. sets constitute the most elementary nontrivial class which includes nonrecursive sets. The principal result is that the sets A of natural numbers which "determine" such real numbers α (in the sense that the characteristic function of A corresponds to the binary expansion of α) may be very far from being r.e., and may even be cohesive. This contrasts to the case of recursive real numbers, where A is recursive if and only if the corresponding lower Dedekind cut is recursive.

With each subset A of the set of natural numbers N, there is naturally associated a real number in the interval $[0, 2]$, namely $\Phi(A) = \sum_{n \in A} 2^{-n}$, and $\Phi(\varnothing) = 0$. Fix a one-one effective map from N onto Q, the set of rationals in the interval $[0, 2]$, and denote the image under this map of an element n by the **bold** face $\mathbf{n}$. Identifying each natural number n with its rational image $\mathbf{n}$, the (lower) *Dedekind cut associated with* A is simply

$$L(A) = \{n \mid \mathbf{n} \leqq \Phi(A)\} .$$

It is well known in recursive analysis [4] that A is recursive if and only if $L(A)$ is recursive, and in this case $\Phi(A)$ is said to be a *recursive real number*.

From the point of view of recursion theory, however, it is more natural to consider certain wider classes of Dedekind cuts, especially those which are recursively enumerable (r.e.). The most interesting results in recursion theory concern these sets. In going from recursive to recursively enumerable Dedekind cuts, we find that: A r.e. implies $L(A)$ r.e.; but not conversely. (C.G. Jockusch has observed the following simple counter-example to the converse. If A is any r.e. set and if $B = A$ join $\bar{A} = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in \bar{A}\}$, then $L(B)$ is r.e., but B is not r.e. unless A is recursive.) It is now natural to ask just how "sparse" the set A can be so that $L(A)$ remains r.e. At the end of §3 in [8] we indicated how to construct a hyperimmune set H such that $L(H)$ is r.e. We now consider two notions (dominant and hyper-

hyperimmune) *which are natural extensions* (as explained in §2) *of the* two equivalent properties used to define a hyperimmune set. We will prove that:

(1) There is a set A such that: (i) A is dominant (i.e. the principal function of A dominates every recursive function); (ii) $L(A)$ is r.e.; and (iii) A contains an infinite retraceable subset, and is not hyperhyperimmune.

(2) There is a cohesive (and hence hyperhyperimmune) set C such that $L(C)$ is r.e.

In addition to illustrating the wide range of sets A which can yield r.e. Dedekind cuts, $L(A)$, these results suggest another method of classifying r.e. Dedekind cuts. Recursively enumerable Dedekind cuts appear to defy classification by the usual division of the r.e. *sets* into such categories as creative or simple, because the dense linear ordering imposed by the rationals prevents any Dedekind cut from being simple or creative (see [8]). We have suggested in [8] a partial classification of r.e. Dedekind cuts using certain classes of fixed point free recursive maps which preserve them. The construction of the dominant set now suggests the notion of an r.e. Dedekind cut being *stably recursively enumerable*, a requirement which is strictly intermediate between requiring that A be r.e., and requiring merely that $L(A)$ be r.e.

Background material may be found in the references listed at the end of the paper, especially [6] and [7]. We used the standard enumeration of the r.e. sets, $W_0, W_1, \cdots$, that is obtained by setting $W_e = \{x \mid (\exists y) T_1(e, x, y)\}$ for each e; and we set $W_e^z = \{x \mid (\exists y)_{<z} T_1(e, x, y)\}$ for each e and z. For natural numbers $x < y$, $I[x, y]$ will denote the finite set $\{x, x+1, x+2, \cdots, y\}$. We will also used the standard effective indexing of the finite sets, $\{D_x\}$. Namely, if $x_1, x_2, \cdots, x_n$ are distinct natural numbers, and $x = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$, then D_x denotes $\{x_1, x_2, \cdots, x_n\}$, and D_0 denotes the empty set, $\emptyset$. We use the standard pairing function, $j(x, y) = x + (1/2)(x + y)(x + y + 1)$, and following Rogers [6] we will let $\langle x, y \rangle$ denote the image $j(x, y)$. If $P(x)$ is a predicate, then $\sim P(x)$ denotes the negation of $P(x)$, and $!xP(x)$ denotes "the unique x such that $P(x)$ holds". For any set $A \subseteq N$, $\bar{A}$ denotes $N - A$, card A denotes "cardinality of A", and $\Phi(A)$ denotes the real number $\sum_{n \in A} 2^{-n}$, while $\Phi(\emptyset) = 0$. Finally, we write $A \subset^* B$ if $B - A$ is finite.

1. Stably recursively enumerable Dedekind cuts. Before defining the notion of a *stably* r.e. Dedekind cut, it will be convenient to have the following characterization of a r.e. Dedekind cut. (From now on "cut" will always mean Dedekind cut.) A sequence of finite sets, $\{A^s\}$, is said to be *canonically* r.e. if there is a recursive function

f such that $A^s = D_{f(s)}$ for all s.

LEMMA 1.1. *For any set A, the cut $L(A)$ is* r.e. *if and only if there is a canonically* r.e. *sequence of finite sets, $\{A^s\}$, such that*

$$(s)[\Phi(A^{s+1}) \geqq \Phi(A^s)], \text{ and} \tag{1.1}$$

$$A = \lim_s A^s \text{ (i.e. } (n)(\exists s)(m)_{\leqq n}(t)_{\geqq s}[m \in A \Leftrightarrow m \in A^t]) \tag{1.2}$$

Proof. If A is recursive the lemma is clear, so we may assume that A is nonrecursive and thus $\Phi(A)$ is nonrational. Now assume that $\{A^s\}$ is canonically r.e. and satisfies (1.1) and (1.2). For each s, define the rational $\mathbf{x}_s = \Phi(A^s)$. Then $\lim_s \mathbf{x}_s = \Phi(A)$, and $L(A)$ is r.e. because $\bigcup \{x_s\}$ is r.e., and because $y \in L(A) \Leftrightarrow (\exists s)[\mathbf{y} \leqq \mathbf{x}_s]$, since $\Phi(A)$ is nonrational.

Conversely, assume $L(A)$ is r.e., say $L(A) = W_e$. For every s such that $W_e^s \neq \emptyset$, define $\mathbf{x}_s = \max\{\mathbf{y} \mid y \in W_e^s\}$, and let B^s be the recursive set such that $\Phi(B^s) = \mathbf{x}_s$. Let $A^s = B^s \cap I[0, s]$. Note that B^s is recursive since $\mathbf{x}_s$ is rational, and B^s is unique if whenever a rational has two distinct binary expansions, we always favor the expansion ... 1000 ... instead of ... 0111 ... (Since for each x, $\mathbf{x}$ is effectively presented as a quotient of natural numbers, we can effectively recognize this case.) Clearly, the sequence $\{A^s\}$ satisfies (1.1) and (1.2).

In general there is no further restriction upon these sets A^s, so that in particular an element n may appear and disappear in subsequent sets many times (at most 2^{n+1}) as long as

$$(s)[n \in A^s - A^{s+1} \Rightarrow (\exists y)[y \in A^{s+1} - A^s \,\&\, y < n]$$

so that $\Phi(A^{s+1}) \geqq \Phi(A^s)$ holds.

In view of this we define an r.e. cut $L(A)$ to be *stably recursively enumerably* (s.r.e.) if there is a canonically r.e. sequence of finite sets $\{A^s\}$ satisfying (1.1) and (1.2) as well as

$$(n)(s)(t)_{>s}[n \in A^s - A^{s+1} \Rightarrow n \notin A^t] . \tag{1.3}$$

If the set A itself is r.e., say $A = W_e$, then $L(A)$ is clearly s.r.e. because we may take $A^s = W_e^s$ so that the antecedent in (1.3) never holds. The converse, however, is false by Jockusch's example $L(B)$ given earlier which is easily seen to be s.r.e. but B is not necessarily r.e.

Furthermore, Theorems 1.2 and 3.1 together will imply that not every r.e. cut is s.r.e., and hence that the requirement that $L(A)$ be stably r.e. is strictly intermediate between requiring that A be r.e., and requiring merely that $L(A)$ be r.e. Theorem 1.2 proves that if A is infinite and $L(A)$ is s.r.e. then A contains an infinite retraceable subset. Theorem 3.1 proves that there is a cohesive set C such that

$L(C)$ is r.e. Since no cohesive set contains an infinite retraceable subset (Rogers [6], Exercise 12–48), $L(C)$ cannot be s.r.e.)

Dekker and Myhill [1] define a set R to be *retraceable* if there is a partial recursive function f such that $f(r_0) = r_0$, and $f(r_{n+1}) = r_n$ for all n, where $r_0, r_1, \cdots$, are the elements of R in ascending order. Given such an f, for each x in the domain of f, we adopt the convention that $f^0(x) = x$, and define the set,

$$\hat{f}(x) = \{y \mid (\exists n)[f^n(x) = y]\} .$$

THEOREM 1.2. *If A is infinite, and $L(A)$ is stably* r.e., *then A contains an infinite retraceable subset B, which is retraceable by a finite-one, partial recursive fuction f.*

Proof. Assume that A is infinite and that $\{A^s\}$ is a canonically r.e. sequence of finite sets satisfying (1.1), (1.2) and (1.3). At each stage s, the partial recursive retracing fuction f will be defined on at most a finite number of elements. Let $a_0 = \mu x[x \in A]$, and $s_0 = \mu s[a_0 \in A^s]$. Our construction begins at stage $s = s_0$.

Stage $s = s_0$. Let $a_1^{s_0}, a_2^{s_0}, \cdots$, be the elements of A^{s_0} which are greater than a_0, listed in ascending order. Define $f(a_0) = a_0$, and $f(a_{i+1}) = a_i$ for all i.

Stage $s > s_0$. Let $a_1^s, a_2^s, \cdots$ be the elements of $A^s - \bigcup_{t<s} A^t$ listed in ascending order. Define

$$\begin{aligned} f(a_1^s) = {} & !\, x[x < a_1^s \;\&\; \hat{f}(x) \subseteqq A^s \\ & \&\; (y)[y < a_1^s \;\&\; \hat{f}(y) \subseteqq A^s \Rightarrow \Phi(\hat{f}(x)) \geqq \Phi(\hat{f}(y))]] \\ f(a_{i+1}^s) = {} & a_i^s, \text{ for all } i > 1 . \end{aligned}$$

Clearly f is partial recursive and finite-one because of our conditions on the sequence $\{A^s\}$.

We now exhibit an infinite subset of A, namely $B = \{b_0, b_1, \cdots\}$, which is retraced by f. Define $b_0 = a_0$,

$$b_{n+1} = \mu x[f(x) = b_n \;\&\; x > a_0] .$$

Clearly, B is retraced by f, and B is infinite since A is infinite. To show $B \subseteqq A$ we first define $s(m) = \mu s[b_m \in A^s \;\&\; s \geqq s_0]$. We prove simultaneously by induction on m that,

$$(1.4) \qquad (m)[b_m \in A]$$

$$(1.5) \qquad (t)_{\geqq s(m)}(n)_{\leqq b_m}[n \in A \Leftrightarrow n \in A^t] .$$

These are clearly true for $m = 0$. Assume true for all $m \leqq p$. Now $s(p+1) \geqq s(p)$ because $f(b_{p+1}) = b_p$. Suppose $n \in A^{t+1} - A^t$ for some

$n < b_{p+1}$ and some $t \geqq s(p+1)$. Let n' be the least such n, and t' the least corresponding t. By inductive hypothesis $b_p < n'$, but by stability of $\{A^s\}$, $n' \notin \bigcup_{u \leqq t'} A^u$. Thus at stage $t'+1$ we must define $f(n') = b_p$, contradicting the definition of b_{p+1}. Hence (1.5) holds for $m = p+1$. But then $b_{p+1} \in A$ by (1.5) and (1.1) since $b_{p+1} \in A^{s(p+1)}$.

2. A dominant set with recursively enumerably lower cut. Following Martin [3], we say that a function f *dominates* a function g, if for all but finitely many n, $f(n) \geqq g(n)$. The *principal function* of an infinite set A is that function which enumerates the members of A in order of magnitude without repetition. A function f dominates an infinite set A if f dominates the principal function of A.

We define an infinite set A to be *dominant* if the principal function of A dominates every recursive function. It is easily seen that A is dominant if and only if the principal function of A dominates every infinite r.e. set, and we will use this property in the proof of Theorem 2.1. (Martin [3] used no name for a dominant set, but called a set A *dense* if $\bar{A}$ is either finite or dominant.)

A set H is said to be *hyperimmune* if there is no recursive function f such that for all x and y,

$$D_{f(x)} \cap H \neq \varnothing \; \& \; [x \neq y \Rightarrow D_{f(x)} \cap D_{f(y)} = \varnothing] \, ,$$

or equivalently if no recursive function dominates the principal function of H (Rice [4]). A set H is *hyperhyperimmune* if there is no recursive function f such that for all x and y,

$$W_{f(x)} \cap H \neq \varnothing \; \& \; W_{f(x)} \text{ is finite } \& \; [x \neq y \Rightarrow W_{f(x)} \cap W_{f(y)} = \varnothing] \, .$$

The notions of hyperhyperimmune and dominant represent respectively the strengthenings of the two equivalent conditions of hyperimmunity. Since it is possible [8] to construct a hyperimmune set H such that $L(H)$ is r.e., it is natural to attempt to obtain the same conclusion for these two "sparser" types. We construct below a dominant set A such that $L(A)$ is stably r.e. By Theorem 1.2, A contains an infinite subset B retraced by a finite-one, partial recursive retracing function, and hence A is not hyperhyperimmune (by the same proof as in Rogers [6], Exercise 12-48 (a)). (Martin [2], p. 275 constructs a co-r.e. set A which is dominant but not hyperhyperimmune. Of course, our set A cannot be co-r.e. since $L(A)$ would be recursive.)

For each s and e, we define the partial recursive function $h(s, e, n)$ to be that function which enumerates the members of W_e^s in ascending order and is undefined for $n \geqq$ cardinality of W_e^s (denoted card W_e^s). (Since the first element of W_e^s is given by $h(s, e, o)$, the function will be defined only for $n <$ card W_e^s.) Now $\lim_s h(s, e, n)$ clearly exists for each e and $n < \text{card } W_e^s$, and will be denoted by the partial function $h(e, n)$, which is the principal function of W_e if W_e is infinite. Note also that,

$$(2.1) \qquad (s)(e)(n)[h(s, e, n) \geqq h(s+1, e, n) \geqq h(e, n)]$$

whenever the functions are defined.

THEOREM 2.1. *There is a dominant set A such that $L(A)$ is stably recursively enumerable.*

(Intuitively, one may think of the following proof as an attempt to satisfy an infinite number of "requirements", where requirement $\langle e, i\rangle$, denoted $R_{\langle e,i\rangle}$, states that

$$(n)[\langle e, i\rangle < n \leqq \langle e, i+1\rangle \Rightarrow a(n) \geqq h(e, n)] ,$$

where $a(n)$ is the principal function of A. We say that requirement $R_{\langle e,i\rangle}$ has *higher priority* than requirement $R_{\langle x,y\rangle}$ if $\langle e, i\rangle < \langle x, y\rangle$. In Lemma 2.4 we will prove that if W_e is infinite, then for every i, $R_{\langle e,i\rangle}$ is satisfied, and thus $a(n)$ dominates $h(e, n)$. To convert our proof into a "movable markers" argument as in Rogers [6] one need merely imagine that a "marker" $\Lambda_{\langle e,i\rangle}$ is uniquely associated with $R_{\langle e,i\rangle}$ for each $\langle e, i\rangle$, and that $v(s, e, i)$ *denotes the integer occupied by marker $\Lambda_{\langle e,i\rangle}$ at stage s.* Then for example, (2.2) states that the markers are always arranged in order according to the priority of $R_{\langle e,i\rangle}$, and the definition of $v(s+1, e, i)$ may be viewed as a description of how the markers move.)

Proof. We will construct by stages a canonically r.e. sequence of finite sets, $\{A^s\}$, which satisfies (1.1), (1.2) and (1.3), and such that if $a(n)$ is the principal function of the set $A = \lim_s A^s$, then $a(n)$ dominates $\lambda n h(e, n)$ whenever W_e is infinite. Simultaneously, we will define by stages a recursive function $v(s, e, i)$ such that for all s, e, i, x and y,

$$(2.2) \qquad v(s, e, i) < v(s, x, y) \Leftrightarrow \langle e, i\rangle < \langle x, y\rangle$$

$$(2.3) \qquad v(s, e, i) \leqq v(s+1, e, i) .$$

Define $A^0 = \emptyset$, and $v(0, e, i) = \langle e, i\rangle$ for all e and i.

Stage $s \geqq 0$. We say that the integer $\langle e, i\rangle$ is *eligible at stage s* if $v(s, e, i) \notin A^s$ and $\text{card } W_e^s > \langle e, i+1\rangle$. If no integer is eligible at stage s then set $A^{s+1} = A^s$ and $v(s+1, e, i) = v(s, e, i)$ for all e and i, and go to stage $s+1$. Otherwise, let $\langle e_s, i_s\rangle$ denote the least integer eligible at stage s, and define,

$$A^{s+1} = (A^s \cap I[0, v(s, e_s, i_s)]) \cup \{v(s, e_s, i_s)\} .$$

Note that in either case,

$$\Phi(A^{s+1}) \geqq \Phi(A^s) \tag{2.4}$$

because in the second case $\langle e_s, i_s \rangle$ eligible at stage s implies that $v(s, e_s, i_s) \notin A^s$.

In order to insure stability of A as well as (2.2) and (2.3) we define a predicate $V(t+1, e, i, n)$ which specifies certain integers n which are *available as values for* $v(t+1, e, i)$. (It will be clear that the function $v(t, e, i)$ is recursive by recursion first upon t and then upon $\langle e, i \rangle$ because $v(t+1, e, i)$ is uniformly recursive in $V(t+1, e, i, n)$ which itself is uniformly recursive in $v(t, e, i)$ and $v(t+1, x, y)$ for $\langle x, y \rangle < \langle e, i \rangle$.)

$$\begin{aligned} V(t+1, e, i, n) \equiv (u)_{\leqq t}[&n \notin A^u \,\&\, n \geqq v(t, e, i) \\ &\&\, (x)(y)[\langle x, y \rangle < \langle e, i \rangle \Rightarrow v(t+1, x, y) < n]] . \end{aligned}$$

We now complete our construction by defining at stage s,

$$v(s+1, e, i) = \begin{cases} v(s, e, i) & \text{if } \langle e, i \rangle \leqq \langle e_s, i_s \rangle \\ \mu n[n \geqq h(s, e_s, \langle e, i \rangle) & \text{if } \langle e_s, i_s \rangle < \langle e, i \rangle \leqq \langle e_s, i_s + 1 \rangle \\ \quad \& V(s+1, e, i, n)] & \\ \mu n V(s+1, e, i, n) & \text{if } \langle e_s, i_s + 1 \rangle < \langle e, i \rangle . \end{cases}$$

Note that the second and third clauses of V guarantee that $v(s, e, i)$ satisfies (2.3) and (2.2) respectively. (Notice how by the second clause in the definition of v we attempt to satisfy requirement $R_{\langle e_s, i_s \rangle}$ at stage s.) Furthermore, we have for all s, e, i and n,

$$n \in A^{s+1} - A^s \Leftrightarrow n = v(s, e_s, i_s) \tag{2.5}$$

$$v(s, e, i) < v(s+1, e, i) \Rightarrow (\exists m)[m < v(s, e, i) \& m \in A^{s+1} - A^s] \tag{2.6}$$

$$n \in A^{s+1} - A^s \Rightarrow (t)_{>s}[n = v(t, e, i) \Rightarrow \langle e, i \rangle = \langle e_s, i_s \rangle] \tag{2.7}$$

$$(t)_{>s}[v(t, e_t, i_t) < v(s, e_s, i_s) \Rightarrow v(s, e_s, i_s) < v(t+1, e_s, i_s)] , \tag{2.8}$$

where (2.8) is considered vacuous unless $\langle e_s, i_s \rangle$ and $\langle e_t, i_t \rangle$ are defined. Clearly (2.6) follows from the definition of $v(s+1, e, i)$ and in fact $m = v(s, e_s, i_s)$ by (2.5). To prove (2.7) fix s and suppose for some n that $n \in A^{s+1} - A^s$. Then $n = v(s, e_s, i_s)$. But $n \in A^{s+1}$ implies $(t)_{>s} \sim V(t, e, i, n)$. Thus, if $n = v(t, e, i)$ for some e and i, and some $t > s$, it can only be through the first clause in the definition of $v(t, e, i)$. It follows by an easy induction on t that $\langle e, i \rangle = \langle e_s, i_s \rangle$, thus establishing (2.7). In (2.8) fix s and $t > s$, and assume that

$\langle e_s, i_s\rangle$ and $\langle e_t, i_t\rangle$ are defined, and that the antecedent holds. Now $v(s, e_s, i_s) \leqq v(t, e_s, i_s)$ by (2.3), and thus $\langle e_t, i_t\rangle < \langle e_s, i_s\rangle$ by (2.2). If $n = v(s, e_s, i_s)$, then $n \in A^{s+1} - A^s$ implies $\sim V(t+1, e_s, i_s, n)$ because $t > s$. Hence, by the definition of v, $v(t+1, e_s, i_s) \neq n$. Thus by (2.3), $n = v(s, e_s, i_s) < v(t+1, e_s, i_s)$.

By (2.4) we know that $\Phi(A^{s+1}) \geqq \Phi(A^s)$ for all s. Hence, $\lim_s A^s$ must exist and will be denoted by A. That A is infinite will follow by Lemma 2.3.

LEMMA 2.2. *$L(A)$ is stably recursively enumerable.*

Proof. By Lemma 1.1 and the above $L(A)$ is r.e. because the sequence $\{A^s\}$ satisfies (1.1) and (1.2). To prove that $L(A)$ is stably r.e. fix $n \in A$, and suppose that $n \in A^{s+1} - A^s$. Then $n = v(s, e_s, i_s)$ by (2.5). Now suppose for some $t > s$ that $n \in A^t - A^{t+1}$, and that t' is the least such t. Necessarily $n > v(t', e_{t'}, i_{t'})$. Now by (2.8) and (2.3), $(u)_{>t'}[n \neq v(u, e_s, i_s)]$. But then by (2.7), $(u)_{>t'}(e)(i)[n \neq v(u, e, i)]$, and thus $(u)_{>t'}[n \notin A^u]$.

LEMMA 2.3. *For all e and i, $\lim_s v(s, e, i)$ exists (and is denoted by $v(e, i)$), and $A = \{v(e, i) \mid \text{card } W_e > \langle e, i+1\rangle\}$.*

Proof. We prove both parts simultaneously by induction on $\langle e, i\rangle$. If $\langle e, i\rangle = 0$, then $e = i = 0$, and $v(s, 0, 0) = v(0, 0)$ for all s. Furthermore, clearly

$$v(0, 0) \in A \Leftrightarrow (\exists s)[\text{card } W_0^s > \langle 0, 1\rangle] .$$

Fix e and i, and assume by induction that the lemma holds for all x and y such that $\langle x, y\rangle < \langle e, i\rangle$. Now define,

$$\begin{aligned} s' = \mu s(t)_{\geqq s}(x)(y)[&\langle x, y\rangle < \langle e, i\rangle \\ &\Rightarrow [v(t, x, y) = v(x, y)] \;\&\; [v(x, y) \in A \Leftrightarrow v(x, y) \in A^t]] . \end{aligned}$$

Then $v(s', e, i) = v(e, i)$ because if $v(s+1, e, i) > v(s, e, i)$ for some $s \geqq s'$, then by (2.6), $n \in A^{s+1} - A^s$ for some $n < v(s, e, i)$. But by (2.5), $n = v(s, e_s, i_s)$, and by (2.2), $\langle e_s, i_s\rangle < \langle e, i\rangle$ contradicting the definition of s'.

Before proving the second half of the lemma note that for all s and n,

$$(2.9) \qquad [n \in A \,\&\, n \in A^{s+1} - A^s] \Rightarrow n = v(s, e_s, i_s) = v(e_s, i_s)$$

because $n = v(s, e_s, i_s)$ by (2.5), but if $v(s, e_s, i_s) < v(t, e_s, i_s)$ for some $t > s$, then $(u)_{\geqq t}[n \notin A^u]$ by the proof of Lemma 2.2.

Now suppose $v(e, i) \in A$, say $v(e, i) \in A^{s+1} - A^s$. Then $v(e, i) =$

$v(s, e, i) = v(s, e_s, i_s)$ by (2.9). Hence, $\langle e, i\rangle = \langle e_s, i_s\rangle$, and card $W_e^s > \langle e, i+1\rangle$ by the eligibility of $\langle e, i\rangle$ at s.

Conversely let $t' = (\mu t)_{>s'}[\text{card } W_e^t > \langle e, i+1\rangle]$, where s' is defined as above. If $v(t', e, i) \notin A^{t'}$ already, then $\langle e, i\rangle$ is eligible at t', and is the least eligible at t' by the definition of s'. Hence, $v(t', e, i) \in A^{t'+1}$, and $v(t', e, i) = v(e, i)$ by (2.6) since $t' \geqq s'$. Finally, $v(e, i) \in A$ because if $v(e, i) \in A^t - A^{t+1}$ for some $t > t'$, then $v(t, x, y) \in A^{t+1} - A^t$ for some $\langle x, y\rangle < \langle e, i\rangle$ contradicting the definition of s'.

Before proceding to Lemma 2.4, we note that by (2.2),

$$(2.10) \qquad (e)(i)(x)(y)[v(e, i) < v(x, y) \Leftrightarrow \langle e, i\rangle < \langle x, y\rangle] .$$

Now from (2.10) and the second part of Lemma 2.3,

$$(2.11) \qquad (x)(y)[a(\langle x, y\rangle) \geqq v(x, y)]$$

where $a(n)$ is the principal function of A.

LEMMA 2.4. *For all e, if W_e is infinite, then*

$$(n)[\langle e, 0\rangle < n \Rightarrow a(n) \geqq h(e, n)] .$$

Proof. If false, let e, i, and n be such that W_e is infinite, and $\langle e, i\rangle < n \leqq \langle e, i+1\rangle$, and $a(n) < h(e, n)$. Now $v(e, i) \in A$ by Lemma 2.3 since W_e is infinite. Let $v(e, i) \in A^{s+1} - A^s$. Then by (2.9), $v(e, i) = v(s, e, i) = v(s, e_s, i_s)$, and thus $\langle e, i\rangle = \langle e_s, i_s\rangle$. Let $n = \langle x, y\rangle$. Since $\langle e, i\rangle < \langle x, y\rangle \leqq \langle e, i+1\rangle$, we have by the second clause in the definition of v,

$$(2.12) \qquad v(s+1, x, y) \geqq h(s, e, \langle x, y\rangle) .$$

Now by (2.11) and (2.3) respectively,

$$(2.13) \qquad a(\langle x, y\rangle) \geqq v(x, y) \geqq v(s+1, x, y), \text{ and}$$

$$(2.14) \qquad h(s, e, \langle x, y\rangle) \geqq h(e, \langle x, y\rangle), \text{ by (2.1)} .$$

Arranging in order the inequalities of (2.13), (2.12) and (2.14) respectively, we conclude that $a(\langle x, y\rangle) \geqq h(e, \langle x, y\rangle)$, that is $a(n) \geqq h(e, n)$, contrary to hypothesis.

3. A cohesive set with recursively enumerable lower cut. An infinite set C is *cohesive* if there is no r.e. set W_e such that $W_e \cap C$ and $\bar{W}_e \cap C$ are both infinite. An r.e. set M is *maximal* if $\bar{M}$ is cohesive. Although the construction of a maximal set requires a priority argument, it is easy to give a *noneffective* construction of a cohesive set (which is not co-r.e.). (The following in substance is the construction of Dekker and Myhill which appears in Rogers [6], p. 232.) Define a sequence of indices, $e_0, e_1 \cdots$, as follows:

$$e_0 = \mu e[W_e \text{ is infinite}]$$
$$e_{i+1} = (\mu e)_{>e_i}[W_e \cap S_i \text{ is infinite}], \text{ where } S_i = \cap \{W_{e_j} \mid j \leqq i\} .$$

Now define $C = \bigcup_i \{x_i\}$ where x_i is some element of S_i, then C is clearly cohesive since

$$(e)[W_e \cap C \text{ infinite} \Rightarrow C \subset^* W_e] .$$

(Recall that $A \subset {}^*B$ denotes that $B - A$ is finite.)

This procedure is so noneffective, however, that it has rarely been used in an *effective* construction of some r.e. set. (For instance, the usual co-maximal cohesive sets C given by the Yates construction (see Rogers [6]) do not satisfy the property that $C \subset {}^*S_i$ for every i.) We will construct a cohesive set A such that $L(A)$ is r.e., and such that for every i, $A \subset {}^*S_i$. The latter property guarantees that A is cohesive because if $A \cap W_e$ is infinite, then $e = e_i$ for some i, but then $A \subset {}^*S_i$, and hence $A \subset {}^*W_{e_i}$. (Throughout the proof we will refer to the indices $\{e_i\}$ and the sets $\{S_i\}$ defined above.)

THEOREM 3.1.[1] *There is an infinite set A such that $L(A)$ is* r.e., *and $A \subset {}^*S_i$ for every i (and hence A is cohesive).*

(Again our proof will be an attempt to satisfy certain "requirements". Requirement x, denoted R_x, states that,

$$A \subset {}^* \cap \{W_j \mid j \in D_x\} .$$

Naturally, it will be impossible to simultaneously satisfy all requirements, but we will prove (Lemma 3.8) that if $D_x = \{e_0, e_1, \cdots, e_i\}$ for some i, then R_x is satisfied, i.e., that

$$A \subset {}^* \cap \{W_j \mid j \in D_x\} = S_i .$$

We say R_x has *higher priority* than R_y just if $\Phi(D_x) > \Phi(D_y)$. To aid intuition one may imagine that a "marker" Λ_x corresponds to R_x for every x, and that $v(s, x)$ *denotes the integer occupied by Λ_x at stage s.* Ideally, we would like to reflect the priority of requirements as in (2.2) by defining $v(s, x)$ so that for all s, $v(s, x) < v(s, y) \Leftrightarrow \Phi(D_x) > \Phi(D_y)$, because the leftmost markers (i.e., markers occupying smaller integers) will exercise greatest control over elements eventually admitted to A. Naturally, this is impossible since markers would have an infinite number of predecessors. We must therefore begin more modestly with

[1]This question was suggested to us by T.G. McLaughlin.

a recursive well ordering of type ω, $W(x, y)$, and then allow markers to change their relative positions so as to more closely approximate the priority ordering when desirable in order to attempt to satisfy a certain requirement.)

Proof. From now on we adopt the convention that max D_x denotes $\max[n \mid n \in D_x]$, and $\max \varnothing = 0$. Define the recursive predicate,

$$W(x, y) \equiv \{\max D_x < \max D_y\} \vee [\max D_x = \max D_x \& \Phi(D_x) > \Phi(D_y)] .$$

We define a canonically r.e. sequence of finite sets, $\{A^s\}$, and a recursive function $v(s, x)$ as follows. Set $A^0 = \varnothing$, $v(0, 0) = 0$, and for $x > 1$, define

$$v(0, x) = \mu n(y)[W(y, x) \Rightarrow v(0, y) < n] .$$

Stage $s \geqq 0$. Define the function f,

$$f(s, x) = \max\{\cup D_y \mid \text{all } y \text{ such that } v(s, y) \leqq v(s, x)\} .$$

(That f is recursive will follow because v will be recursive and because $\lambda y\, v(s, y)$ will be a one-one function.) We define x to be *eligible at stage* s, denoted $E(s, x)$, as follows:

$$E(s, x) \equiv \text{card}\{n \mid n > v(s, x) \& n \in \cap \{W_i^s \mid i \in D_x\}\} > 2^{f(s,x)+2} .$$

Case 1. There is no eligible x at stage s. Then set $A^{s+1} = A^s$, and $v(s + 1, x) = v(s, x)$ for all x, and go to stage $s + 1$. (Note that $(\exists x)E(s, x)$ is decidable given $\lambda x\, v(s, x)$ since one need only examine those x such that $v(s, x) < s$, because $(j)(z)_{\geqq s}[z \notin W_j^s]$ by the Gödel numbering.)

Case 2. Otherwise. Let x_s be the unique eligible x which satisfies the predicate

$$L(s, x) \equiv E(s, x) \& \sim (\exists y)[E(s, y) \& v(s, y) < v(s, x)] .$$

(That is, x_s is the unique eligible x whose marker Λ_x is *leftmost* among all the markers Λ_y such that y is eligible at s.)

Now let $m_s = f(s, x_s) + 1$, and define the sets,

$$X_1^s = \{x \mid v(s, x) < v(s, x_s)\}$$
$$X_2^s = \{x \mid v(s, x) \geqq v(s, x_s) \& D_x \subseteqq I[0, m_s]\}$$
$$X_3^s = \{x \mid v(s, x) \geqq v(s, x_s) \& D_x \nsubseteq I[0, m_s]\} .$$

Note that card $X_2^s \leqq 2^{f(s,x_s)+2}$. (Viewing the following definition of $v(s + 1, x)$ as a description of how the markers move, notice that only the markers Λ_x for $x \in X_2^s$ are allowed to change their relative order, and they move only so as to more closely approximate our priority ranking. Furthermore, since the elements $v(s+1, x)$ are potential elements of A^t for some $t > s+1$, the first conjunct of the case $x \in X_2^s$ attempts to partially satisfy requirement R_{x_s}). Define,

$$A^{s+1} = [A^s \cap I[0, v(s, x_s)] \cup \{v(s, x_s)\}, \text{ and}$$

$$v(s+1, x) = \begin{cases} v(s, x) & \text{if } x \in X_1^s \\ \mu n[n \in \cap \{W_i^s \mid i \in D_{x_s}\} \,\&\, n > v(s, x) & \text{if } x \in X_2^s \\ \quad \&(y)[[y \in X_2^s \,\&\, \Phi(D_y) > \Phi(D_x)] \Rightarrow v(s+1, y) < n]] & \\ \mu n(y)[[y \in X_2^s \vee [y \in X_3^s \,\&\, v(s, y) < v(s, x)]] & \text{if } x \in X_3^s \\ \qquad \Rightarrow v(s+1, y) < n] . & \end{cases}$$

(It is clear by recursion on s that the function $v(s, x)$ is recursive since $\lambda x\, v(s+1, x)$ is uniformly recursive in $\lambda x\, f(s, x)$, $E(s, x)$, and X_i^s, $1 \leqq i \leqq 3$, which in turn are uniformly recursive in $\lambda x\, v(s, x)$.)

By the definition of $v(s+1, x)$ we have for all s, x, y and z,

$$(3.1) \qquad v(s, x) \neq v(s, y) \Rightarrow x \neq y$$

$$(3.2) \qquad x \in X_1^s \,\&\, y \in X_2^s \,\&\, z \in X_3^s \Rightarrow v(s+1, x) < v(s+1, y) < v(s+1, z)$$

$$(3.3) \qquad v(s, x) < v(s, y) \,\&\, v(s+1, x) > v(s+1, y) \Rightarrow x \in X_2^s \,\&\, y \in X_2^s$$

$$(3.4) \qquad x \in X_2^s \Rightarrow v(s+1, x) \in \cap \{W_i^s \mid i \in D_{x_s}\} .$$

To see (3.2), suppose $x \in X_1^s$ and $y \in X_2^s$, then $v(s+1, y) > v(s, y) \geqq v(s, x_s) > v(s, x) = v(s+1, x)$. The rest of (3.2) is clear, while (3.3) follows from (3.2) and the fact that if $x, y \in X_1^s$ or $x, y \in X_3^s$ then $v(s, x) < v(s, y)$ if and only if $v(s+1, x) < v(s+1, y)$. Finally, (3.4) follows by the definition of $v(s+1, x)$.

By the definitions of v and f, we have for all s that if $(\exists x)L(s, x)$, i.e., if x_s is defined, then

$$(3.5) \qquad f(s, x_s) < f(s+1, x_s)$$

because if $D_y = D_{x_s} \cup \{m_s\}$ then $y \in X_2^s$, and so by the second clause in the definition of v, $v(s+1, y) < v(s+1, x_s)$ because $\Phi(D_y) > \Phi(D_{x_s})$. But then $f(s+1, x_s) \geqq m_s = 1 + f(s, x_s)$.

Furthermore, it is clear that for all x, n and s,

$$(3.6) \qquad n \in A^{s+1} - A^s \Leftrightarrow n = v(s, x_s)$$

$$(3.7) \qquad v(s, x) \neq v(s+1, x) \Rightarrow (\exists m)[m < v(s, x) \,\&\, m \in A^{s+1} - A^s]$$

$$(3.8) \qquad n \in A^s - A^{s+1} \Rightarrow (\exists m)[m < n \,\&\, m \in A^{s+1} - A^s] .$$

Using (3.6) and the fact that $v(s+1, x_s) > v(s, x_s)$ (because $x_s \in X_2^s$), it is easily seen by induction on s that

$$(s)(x)[v(s, x) \notin A^s] . \tag{3.9}$$

Now by (3.9) and the definition of A^{s+1}, we have $\Phi(A^{s+1}) \geqq \Phi(A^s)$ for all s. Thus $\lim_s A^s$ must exist, and will be denoted by A. Since the canonically r.e. sequence, $\{A^s\}$, of finite sets satisfies (1.1) and (1.2), we have proved.

LEMMA 3.2. *$L(A)$ is* r.e.

(Of course, unlike the sequence in Theorem 2.1, we know that $\{A^s\}$ cannot satisfy (1.3) because no cohesive set may contain an infinite retraceable subset.)

For future reference we will define the nonrecursive function s,

$$s(n) = \mu t(m)_{<n+1}[m \in A \Leftrightarrow m \in A^{t+1}] . \tag{3.10}$$

By (3.8) and the definition of $s(n)$ we have,

$$(t)_{>s(n)}(m)_{<n+1}[m \in A \Leftrightarrow m \in A^t] . \tag{3.11}$$

Finally, by the minimality of $s(n)$ we see that if $n \in A$, then $n \in A^{s(n)+1} - A^{s(n)}$, so that by (3.6),

$$(n)[n \in A \Leftrightarrow n = v(s(n), x_{s(n)})] . \tag{3.12}$$

LEMMA 3.3. *A is an infinite set.*

Proof. If A is finite, let $m \equiv \max\{n \mid n \in A\}$. Then by (3.11) and (3.7),

$$(x)(t)_{>s(m)}[A = A^t \,\&\, v(t, x) = v(s(m) + 1, x)] .$$

But since there are an infinite number of x such that $\cap\{W_i \mid i \in D_x\}$ is infinite, there must exist some $t > s(m)$ and some x such that x is eligible at stage t. But then $v(t, x_t) \in A^{t+1} - A^t$, contradicting $A^{t+1} = A = A^t$ for $t > s(m)$.

LEMMA 3.4. *For all $x \neq 0$, if $\cap\{W_i \mid i \in D_x\}$ is finite, then $\{s \mid (\exists y)[D_y \supseteqq D_x \,\&\, L(s, y)]\}$ is finite also.*

Proof. Fix $x \neq 0$. Let $m = \max\{n \mid n \in \{W_i \mid i \in D_x\}\}$. (Recall that $\max \varnothing = 0$.) Then

$$(y)(t)_{>s(m)}[D_y \supseteqq D_x \Rightarrow \sim L(t, y)] ,$$

because if $v(t, y) \leqq m$ and $L(t, y)$ then $v(t, y) \in A^{t+1} - A^t$ contradicting

(3.11). But if $v(t, y) > m$, then $\sim L(t, y)$ because $\sim E(t, y)$ since $\cap \{W_i^t \mid i \in D_y\} \subseteqq I[0, m]$.

Now define a (nonrecursive) function d as follows:

$$D_{d(i)} = \{e_0, e_1, \cdots, e_i\}$$

where $e_0, e_1, \cdots$ is the sequence of indices defined in the beginning of §3. (Note that $S_i = \cap \{W_j \mid j \in D_{d(i)}\}$.)

LEMMA 3.5. $(n)[n \in A \Rightarrow n \leqq v(s(n), d(i))]$.

Proof. Suppose that $n > v(s(n), d(i))$. Now by (3.7) and (3.11), $v(t, d(i)) = v(s(n), d(i))$ for all $t > s(n)$. Now since $\cap \{W_j \mid j \in D_{d(i)}\}$ is infinite, there must be some $t > s(n)$ such that $d(i)$ is eligible at stage t. But then $L(t, y)$ holds for some y such that $v(t, y) \leqq v(t, d(i))$, and hence $m \in A^{t+1} - A^t$ for some $m < v(s(n), d(i))$, contradicting (3.11).

LEMMA 3.6. *For all* s, x, *and* y,

$$v(s, x) < v(s, y) \Rightarrow [\max D_x < \max D_y \vee \Phi(D_x) > \Phi(D_y)] .$$

Proof. This is clearly true for $s = 0$ by definition of $\lambda x\, v(0, x)$. Assume true for some fixed s, and suppose $v(s + 1, x) < v(s + 1, y)$. Now if $v(s, x) < v(s, y)$ then the conclusion follows by inductive hypothesis. But by (3.3) if $v(s, x) > v(s, y)$, then $x, y \in X_2^s$, and thus $v(s + 1, x) < v(s + 1, y)$ only if $\Phi(D_x) > \Phi(D_y)$.

LEMMA 3.7. *For every* i, *there exists* t_i *such that*

$$(s)_{>t_i}(x)[L(s, x) \;\&\; v(s, x) \leqq v(s, d(i)) \Rightarrow D_x \supseteqq D_{d(i)}] .$$

Proof. The proof is by induction on i.

Case $i = 0$. Define

$$t_0 = \max \{t \mid (\exists j)(\exists y)[j < e_0 \;\&\; \{j\} \subseteqq D_y \;\&\; L(t, y)]\} ,$$

which is at most a finite set by Lemma 3.4. Now by Lemma 3.6, for all s and x,

$$v(s, x) < v(s, d(0)) \Rightarrow [\max D_x < d(0) \vee \Phi(D_x) > \Phi(D_{d(0)})]$$
$$\therefore (s)_{>t_0}(x)[L(s, x) \;\&\; v(s, x) \leqq v(s, d(0)) \Rightarrow D_x \supseteqq D_{d(0)}] .$$

Case $i + 1$. By induction, assume that for all $j \leqq i$, t_j is defined so that the above statement holds. Define

(3.13) $\quad w = \max \{s \mid (\exists j)(\exists y)[e_i < j < e_{i+1} \;\&\; D_y \supseteqq D_{d(i)} \cup \{j\} \;\&\; L(t, y)]\}$

which is at most a finite set by Lemma 3.4, and the definition of e_{i+1}. Define $r = \max\{t_i, w\}$. Thus,

$$(3.14) \qquad (s)_{>r}(x)[[L(s, x) \,\&\, v(s, x) < v(s, d(i))] \Rightarrow D_x \supseteqq D_{d(i+1)}]$$

because by inductive hypothesis and (3.1), $D_x \not\supseteqq D_{d(i)}$, and by (3.13), $D_x \not\supseteqq D_{d(i)} \cup \{j\}$ for any $j < e_{i+1}$. (That $D_x \not\supseteqq D_{d(i)} \cup \{j\}$ for $j < e_i$ and $j \notin D_{d(i)}$ follows of course by inductive hypothesis.)

Subcase 1. $(\exists s)_{>r}[v(s, d(i+1)) < v(s, d(i))]$. If u is the least such s, then a second induction on s for $s \geqq u$ proves simultaneously that,

$$(3.15) \quad (s)_{\geqq u}[v(s+1, d(i+1)) < v(s+1, d(i))], \text{ and}$$

$$(3.16) \quad (s)_{\geqq u}(x)[L(s, x) \,\&\, v(s, x) \leqq v(s, d(i+1)) \Rightarrow D_x \supseteqq D_{d(i+1)}] .$$

By the definition of u, we have $v(u, d(i+1)) < v(u, d(i))$. Choose $t \geqq u$, and assume (3.15) and (3.16) for all s such that $u \leqq s < t$. We may assume that,

$$(\exists x)[L(t, x) \,\&\, v(t, x) \leqq v(t, d(i+1))]$$

because otherwise $v(t+1, y) = v(t, y)$ for all y, such that $v(t, y) \leqq v[t, d(i))$, and (3.15) and (3.16) hold trivially for $s = t$. Now by (3.14), $D_x \supseteqq D_{d(i+1)}$ thus establishing (3.16) for $s = t$.

To prove (3.15) for $s = t$, note that $f(t, x_t) \geqq e_{i+1}$ by the definition of f since $D_{x_t} \supseteqq D_{d(i+1)}$. But then $d(i), d(i+1) \in X_2^t$ because $D_{d(i+1)} \subseteqq I[0, f(t, x_t) + 1]$, and $v(t, x_t) \leqq v(t, d(i+1)) < v(t, d(i))$. Hence, $v(t+1, d(i+1)) < v(t+1, d(i))$ by the second clause in the definition of v because $\Phi(D_{d(i+1)}) > \Phi(D_{d(i)})$.

Subcase 2. $(s)_{>r}[v(s, d(i)) < v(s, d(i+1))]$. This assumption will lead to a contradiction. Define

$$u(1) = (\mu s)_{>r}(\exists y)[L(s, y) \,\&\, v(s, y) \leqq v(s, d(i))] .$$

(Such an s exists by Lemma 3.5 and (3.12) since A is infinite and A^r is finite.) Now by (3.1) and (3.14), $D_{x_{u(1)}} \supseteqq D_{d(i+1)}$ or $x_{u(1)} = d(i)$. But if the former then $d(i), d(i+1) \in X_2^{u(1)}$, because

$$v(u(1), x_{u(1)}) \leqq v(u(1), d(i)) < v(u(1), d(i+1)) .$$

But then since $\Phi(D_{d(i+1)}) > \Phi(D_{d(i)})$ we have by the definition of v that

$$v(u(1)+1, d(i+1)) < v(u(1)+1, d(i))$$

contrary to the hypothesis.

We conclude that $x_{u(1)} = d(i)$. But then by (3.5),

$$f(u(1)+1, d(i)) > f(u(1), d(i)) \geqq e_i .$$

Now define,

$$u(2) = (\mu s)_{>u(1)}(\exists y)[L(s, y) \;\&\; v(s, y) \leqq v(s, d(i))] \;.$$

By the same argument as above, $x_{u(2)} = d(i)$, and

$$f(u(2) + 1, d(i)) > f(u(2), d(i)) > f(u(1), d(i)) \geqq e_i \;.$$

Continuing in this manner, after at most $k = e_{i+1} - e_i$ steps, we must have $f(u(k), d(i)) \geqq e_{i+1} - 1$. But then $D_{d(i+1)} \subseteqq I[0, m_{u(k)}]$ so that $d(i), d(i + 1) \in X_2^{u(k)}$. Thus by the definition of v,

$$v(u(k) + 1, d(i + 1)) < v(u(k) + 1, d(i)) \;,$$

contradicting the assumption of subcase 2.

Thus if we define

$$t_{i+1} = (\mu s)_{>r}[v(s, d(i + 1)) < v(s, d(i))]$$

then Lemma 3.7 follows.

LEMMA 3.8. *For every i, $A \subset {}^*S_i$.*

Proof. Fix i, and let t_i be defined as in Lemma 3.7. Let $n = \mu m[m \in A - A^{t_i}]$. By (3.12), $n = v(s(n), x_{s(n)})$, and $s(n) > t_i$ since $n \notin A^{t_i}$. By Lemma 3.5, $n < v(s(n), d(i))$. Now $v(s(n), x_{s(n)}) < v(s(n), d(i))$ implies by Lemma 3.7 that $D_{x_{s(n)}} \supseteqq D_{d(i)}$. Hence, $d(i) \in X_2^{s(n)}$. But then by (3.2) and the definition of v we have for all y,

$$n < v(s(n) + 1, y) \leqq v(s(n) + 1, d(i)) \Rightarrow y \in X_2^{s(n)} \;.$$

Thus, by the second clause in the definition of v, we have for all y and for $t = s(n) + 1$,

$$n < v(t, y) \leqq v(t, d(i)) \Rightarrow v(t, y) \in \cap \{W_j \mid j \in D_{x_{s(n)}}\} \;.$$

But since $D_{x_s} \supseteqq D_{d(i)}$, we have for all y, and for $t = s(n) + 1$,

$$n < v(t, y) \leqq v(t, d(i)) \Rightarrow v(t, y) \in S_i \;. \tag{3.17}$$

Now we will prove by induction on t that (3.17) holds for all $t \geqq s(n) + 1$. This will prove that

$$(m)_{>n}[m \in A \Rightarrow m \in S_i]$$

because if $m \in A$ then by (3.12) $m = v(s(m), x_{s(m)})$. But $m > n$ implies $s(m) \geqq s(n) + 1$. Now by Lemma 3.5,

$$v(s(m), x_{s(m)}) = m < v(s(m), d(i)) \;.$$

Hence, $v(s(m), x_{s(m)}) \in S_i$ by (3.17).

It remains to prove (3.17) by induction on $t \geqq s(n) + 1$. Since

(3.17) clearly holds for $t = s(n) + 1$, choose $u \geqq s(n) + 1$ and assume by induction that (3.17) holds for all $t \leqq u$. Now (3.17) follows trivially for $t = u + 1$ by inductive hypothesis and the definition of v unless,

$$(\exists y)[L(u, y) \,\&\, v(u, y) \leqq v(u, d(i))] .$$

In this case by Lemma 3.7, $D_{x_u} \supseteqq D_{d(i)}$ since $u > s(n) > t_i$. But $D_{x_u} \supseteqq D_{d(i)}$ implies $d(i) \in X_2^u$. Thus by (3.2) for all y,

$$v(u + 1, y) < v(u + 1, d(i)) \Rightarrow [y \in X_1^u \vee y \in X_2^u] .$$

Now if $y \in X_1^u$, then $v(u + 1, y) = v(u, y)$ and so if $n < v(u, y)$, then $v(u + 1, y) \in S_i$ by inductive hypothesis. But $y \in X_2^u$ implies $v(u + 1, y) \in \cap \{W_j \mid j \in D_{x_u}\}$ by (3.4). Hence, since $D_{x_u} \supseteqq D_{d(i)}$, $v(u + 1, y) \in S_i$.

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ISOMETRIES OF CERTAIN FUNCTION SPACES

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Let X be a discrete symmetric Banach function space with absolutely continuous norm. We prove by the method of generalized hermitian operator that an operator U on X is an onto isometry if and only if it is of the form:

$$Uf(.)=u(.)f(T.) \quad \text{all} \quad f \in X ,$$

where u is a unimodular function and T is a set isomorphism of the underlying measure space. That other types of isometries occur if the symmetry condition is not present is illustrated by an example. We completely describe the isometries of a reflexive Orlicz space $L_{M\Phi}(\neq L_2)$ provided the atoms have equal mass (the atom-free case has been treated by G. Lumer); similarly for the case that no Hilbert subspace occurs.

We shall reproduce some definitions and results from [4] which will be needed in the sequel.

DEFINITION. Let X be a vector space. A semi-inner-product on X is a mapping $[\,,\,]$ of $X \times X$ into the field of numbers (real or complex) such that

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ \lambda[x, z] &= [\lambda x, z] \text{ for all } x, y, z \in X \text{ and } \lambda \text{ scaler .} \\ [x, x] &> 0 \text{ for all } x \neq 0 \\ |\,[x, y]\,|^2 &\leqq [x, x][y, y] \,. \end{aligned}$$

We call X a semi-inner-product space (in short, s.i.p.s.). If X is a s.i.p.s., one shows easily that $[x, x]^{1/2}$ is a norm on X. On the other hand, let X be a normed space and X^* its dual. For each $x \in X$, there exists by the Hahn-Banach theorem, at least one (and we shall choose one) functional $Wx \in X^*$ such that $\langle x, Wx\rangle = \|\,x\,\|^2$. Given any such mapping W from X into X^* (ank in general, there are infinitely many such mappings), it is at once verified that $[x, y] = \langle x, Wy\rangle$ defines a semi-inner-product (s.i.p.).

DEFINITION. Given a linear transformation T on a s.i.p.s., we call the set $W(T) = \{[Tx, x]: [x, x] = 1\}$ the numerical range of T.

An important fact concerning the notion of numerical range is the following [4, Th. 14]:

Let X be a complex Banach space, and T an operator on X.

Although there may be many different s.i.p. consistent with the original norm of X, in the sense that $[x, x] = \|x\|^2$, nonetheless, the convex hulls of numerical range of T relative to all such s.i.p. are equal. It has real numerical range with respect to one s.i.p., then it has real numerical range with respect to any other s.i.p. inducing the same norm.

DEFINITION. Let T be an operator on a complex Banach space X, then T is called hermitian if its numerical range is real, relative to any s.i.p. consistent with the norm.

1. A general setting. We shall call an algebra A over the complex field C a $*$-algebra if there is a mapping $*$ defined on A satisfying:

(i) $a \in A$ implies $a^* \in A$.

(ii) $(a + b)^* = a^* + b^*$ and $(\lambda a)^* = \bar{\lambda} a^*$.

(iii) $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for all $a, b \in A$ and $\lambda \in C$. An element a such that $a^* = a$ is said to be self-adjoint (s.a.). Every element a of a $*$-algebra can be written in a unique way: $a = u + iv$ where u and v are s.a. A $*$-algebra-isomorphism ρ is an algebra isomorphism on a $*$-algebra A with the condition that $(\rho(a))^* = \rho(a^*)$ for all a in A.

Let X be a complex s.i.p.s. and A be a $*$-algebra with a topology. Assume that X is a two-sided module over A. Suppose that there is a net $\{e_\alpha\}$ in A such that $\lim_\alpha f e_\alpha = f$ for all f in X. For a $*$-subalgebra A_0 of A such that A_0 is a subset of X, and $\{e_\alpha\}$ is contained in A_0, the following holds:

THEOREM 1. *Suppose that for any* s.a. h *in* A, $H_h f = hf$ *for all* f *in* X *defines a bounded hermitian operator on* X; *and that conversely every bounded hermitian operator is of this form. Then any onto isometry* U *of* X *when restricted to* A_0 *is given by*

$$Uf = \lim_{\alpha} \rho(f) U e_\alpha$$

where ρ *is a* $*$*-algebra-isomorphism on* A.

Proof. Let h in A be s.a., then H_h is a bounded hermitian operator on X. On the other hand, let a s.i.p. $[,]$ on X be given, then $[f, g]' = [U^{-1}f, U^{-1}g]$ defines another s.i.p. on X inducing the same norm. It follows that

$$[UH_h U^{-1}f, f]' = [H_h U^{-1}f, U^{-1}f] \text{ is real for all } f\,.$$

Thus $UH_h U^{-1}$ is another hermitian operator on X, and by hypothesis there is a s.a. $\hat{h}$ in A such that

$$UH_hU^{-1}f = H_{\hat{h}}f \quad \text{for all } f \text{ in } X .$$

Clearly the mapping $h \to \hat{h}$ is linear. If $\hat{h} = 0$, then for all $f \in X$, $UH_hU^{-1}f = 0$; in particular $UH_hU^{-1}Ue_\alpha = U(he_\alpha) = 0$. Since U is one to one, $he_\alpha = 0$ and $\lim_\alpha he_\alpha = h = 0$. Hence this mapping is one to one. We shall set $\rho(h) = \hat{h}$. With s.a. h and h' in A,

$$H_{\rho(hh')} = UH_{hh'}U^{-1} = UH_hU^{-1}UH_{h'}U^{-1} = H_{\rho(h)}H_{\rho(h')} .$$

Thus $\rho(hh') = \rho(h)\rho(h')$. Extending ρ on A trivially by letting

$$\rho(h + ih') = \rho(h) + i\rho(h') ,$$

it can easily be shown that ρ is a *-algebra-isomorphism on A. For all f in A_0, $U(fe_\alpha) = UH_fU^{-1}Ue_\alpha = \rho(f)Ue_\alpha$, so that

$$Uf = \lim_\alpha \rho(f)Ue_\alpha .$$

2. Function spaces. Let X be a Banach function space with absolutely continuous norm [6] over a σ-finite measure space (Ω, Σ, μ).

LEMMA 1. *Assume that ω is a measurable subset of Ω and let P be the projection of X onto the subspace E of functions in X vanishing outside ω. Then for any hermitian operator H on X, PHP is a hermitian operator on E.*

Proof. Since X has absolutely continuous norm $X^* = X'$, the associated space of X. Let W be a mapping as before. Then a consistent s.i.p. on X is given by: with each $g \in X$,

$$[f, g] = \langle f, Wg \rangle = \int fWg \quad \text{for all } f \in X .$$

Without loss of generality we can take Wg to be χWg if $g \in E$ where χ is the characteristic function of ω. Then for all $g \in E$ such that $\| g \| = 1$, we obtain

$$[Hg, g] = \int Hg\chi Wg = \int \chi Hg\chi Wg = [(PHP)g, g]$$

which is real valued. Thus PHP is hermitian on E.

LEMMA 2. [5, Lemma 7]. *If $h \in L_\infty$ is a real function, the operator H_h, defined by $H_hf = hf$ for all $f \in X$, is a bounded hermitian operator on X; and $\| H_h \| = \| h \|_\infty$.*

We shall use the following fact several times later.

LEMMA 3. *For α, β, γ complex numbers such that $e^{i\theta}\alpha + e^{-i\theta}\beta + \gamma$ is real for all $0 \leqq \theta < 2\pi$, then $\alpha = \bar{\beta}$ and γ is real.*

Let E be a two-dimensional Banach space. Denote the element f of E as a function defined on the set $\Omega = \{x, y\}$. We shall assume that the norm in E has the following properties:

(1) $\| f \| = \| | f | \|$.

(2) $| f | \leqq | g |$ implies that $\| f \| \leqq \| g \|$ with all $f, g \in E$.

The real functions in E can be considered as points in the two-dimensional Euclidean plane; let γ be the convex curve of the boundary of its real unit ball. At each point $p \in \gamma$ there is a supporting hyperplane, and suppose that the normal vector at p to the hyperplane is given by (α, β). We shall define sgn g as the function

$$\operatorname{sgn} g = \begin{cases} 0 & \text{if } g = 0 \\ \dfrac{| g |}{g} & \text{otherwise}. \end{cases}$$

LEMMA 4. *For any nonzero $g \in E$*

$$[f, g] = \| g \| A(g)\{f(x) \operatorname{sgn} g(x)\alpha(g) + f(y) \operatorname{sgn} g(y)\beta(g)\}$$

where

$$A(g) = \left\{\frac{| g(x) |}{\| g \|}\alpha(g) + \frac{| g(y) |}{\| g \|}\beta(g)\right\}^{-1} \quad and \quad (\alpha(g), \beta(g))$$

is a normal vector at $(| g(x) |/\| g \|, | g(y) |/\| g \|)$ for all $f \in E$, defines a consistent s.i.p. on E.

Proof. Clearly it is linear in f and $[g, g] = \| g \|^2$. First we assume that f and g are real valued. The fact that $\| g \| = \| | g | \|$ implies that the curve γ is symmetric with respect to both axes. The function $A(g)\{s\alpha(g) + t\beta(g)\}$ has absolute value no greater than one on the region between the two lines L_1 and L_2 where they are two chosen supporting hyperplanes at $(| g(x) |/\| g \|, | g(y) |/\| g \|)$ and $(-| g(x) |/\| g \|, -| g(y) |/\| g \|)$ with normal vectors $(\alpha(g), \beta(g))$ and $(-\alpha(g), -\beta(g))$ respectively. So that $A(g) \| g \| \{| s\alpha(g) | + | t\beta(g) |\} \leqq \| g \|$ for all $(s, t) \in \gamma$. For all nonzero $f \in E$, $(| f(x) |/\| f \|, | f(y) |/\| f \|) \in \gamma$, we obtain

$$A(g) \| g \| \{| f(x) \operatorname{sgn} g(x)\alpha(g) | + | f(y) \operatorname{sgn} g(y)\beta(g) |\} \leqq \| f \| \| g \| .$$

Now in the above inequality, only the absolute values are involved, it holds for all complex functions f and g as well.

Let X_n be a n-dimensional real Banach space ($n \geqq 2$) and S its unit ball. We shall fix a basis for X_n and denote every element x as a point in the n-dimensional Euclidean space E_n. Define a function F on E_n as $F(x_1, x_2, \cdots, x_n) = \|(x_1, x_2, \cdots, x_n)\| - 1$. For each $i = 1, 2, \cdots n$, let $e^i = (0, \cdots, 0, 1, 0, \cdots 0)$ (1 at the i-th position).

LEMMA 5. *Let S' be an open set of E_n consisting of smooth points of X_n, then the function F has continuous first partial derivatives at every point of S'.*

Proof. If x is a point of S', then the norm function is Gateaux differentiable at x [7]. Therefore with $i = 1, 2, \cdots, n$

$$\lim_{t\to 0} \frac{\|x + te^i\| - \|x\|}{t} = \frac{\partial F}{\partial x_i}(x) .$$

Suppose W is as before, then from [5, Lemma 1]

$$\|x\| \frac{\partial F}{\partial x_i}(x) = \langle e^i, Wx\rangle = [e^i, x] .$$

Since the norm topology of X_n and that of E_n are equivalent, the weak star compactness of the unit ball of X_n^* and the smoothness of S' implies that this mapping W is weak star continuous on S'. Thus $\partial F/\partial x_i(x)$ is continuous on S'.

LEMMA 6. *Let H be a hermitian operator on E and*

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Then either $b = c = 0$ or else $b/\bar{c} > 0$ and E is a Hilbert space; in either case a and d are real numbers.

Proof. We shall start by proving that the set $S' = \{(s, t): s \neq 0 \neq t\}$ consists of smooth points if b and c are not both zero. For $0 \leqq \theta \leqq 2\pi$, let $f = (e^{i\theta}s, t)$ be such that $(s, t) \in S'$ and $\|f\| = 1$, then by Lemma 4 $[Hf, f] = A(f)(as\alpha + dt\beta + e^{-i\theta}bt\alpha + e^{i\theta}cs\beta)$ is real, where (α, β) is the normal vector of a supporting hyperplane to the real unit ball S at (s, t). We have by Lemma 3 that

$$bt\alpha - \bar{c}s\beta = 0 .$$

We assume that $c \neq 0$. If $b = 0$ then $\beta = 0$ for all such f and γ is a rectangle. As $\beta = 0$ cannot occur on all four sides of a rectangle, b and c are not zero. (α, β) is uniquely determined up to a scaler multiple. Therefore the hyperplane is unique and every point of S'

is smooth. Now for b and c being nonzero, the function $F(s, t) = \|(s, t)\| - 1$ is differentiable at $(s, t) \in S'$. The hyperplane is thus given by the tangent plane. So that for all $g \in E$ such that $g(x) \neq 0 \neq g(y)$, the linear functional in Lemma 4 can be replaced by

$$[f, g] = A(g)\|g\|\left\{f(x)\operatorname{sgn} g(x)\frac{\partial F}{\partial s} + f(y)\operatorname{sgn} g(y)\frac{\partial F}{\partial t}\right\}$$

and we obtain the equation

$$\frac{bt}{\bar{c}}\frac{\partial F}{\partial s} - s\frac{\partial F}{\partial t} = 0 .$$

Now $(b/\bar{c})t^2 + s^2$ satisfies the partial differential equation. By the uniqueness of solution, the curve γ is given by the equation $s^2 + (b/\bar{c})t^2 = K$. Since the unit ball is bounded, $b/\bar{c}$ and K must be positive. Then an inner-product on E can be defined by

$$(f, g) = \frac{f(x)\overline{g(x)}}{K} + \frac{bf(y)\overline{g(y)}}{\bar{c}K} .$$

Thus E is a Hilbert space.

For nonzero $g \in E$ such that $g(y) = 0$, by Lemma 4 $[f, g] = \|g\|^2 f(x)/g(x)$ for all f in E. As $[Hg, g] = a\|g\|^2$ is real, a is real; similarly d is real.

3. Discrete symmetric Banach function spaces. Let X be a Banach function space with absolutely continuous norm and the measure is purely atomic; so that X is a sequence space. Assume that X is symmetric, i.e., if f in X and ϕ is an isomorphism of the atoms, then $\|f\| = \|f(\phi)\|$. Choose the set of all characteristic functions of atoms to be a fixed basis for X. Let H be a hermitian operator on X and be represented as an infinite matrix (a_{ij}), then Lemmas 1 and 6 imply that $a_{ij} = \bar{a}_{ji}$.

LEMMA 7. *If there is a hermitian operator H on X such that its matrix representation is not diagonal, then there is a hermitian operator H' on X with all nonzero off diagonal entries.*

Proof. We write $H = (a_{ij})$. Assume that without loss of generality that $a_{12} \neq 0$; then $\bar{a}_{21} \neq 0$. Suppose that i_1 is the smallest positive integer such that $a_{1i_1} = 0$. Define U_1 on X as operator obtained from the identity I by interchanging its 2nd and i_1-th row. Then U_1 is isometric and $H_1 = U_1HU_1$ is hermitian. Choose $\alpha_1 > 0$ such that $\|\alpha_1 H_1\| \leqq 1/2$ and the matrix entries of a_{1j} of $H + \alpha_1 H_1$ are nonzero for all $2 \leqq j \leqq i_1$. Assume that this has been done for i_n steps and let i_{n+1} be the smallest integer greater than i_n such that $a_{1i_{n+1}} = 0$. Again let $H_{n+1} = U_{n+1}H_{n+1}U_{n+1}$ where U_{n+1} is the isometric operator obtained from I by interchanging the 2nd and i_{n+1}-th row. Take $\alpha_{n+1} > 0$ with $\|\alpha_{n+1}H_{n+1}\| \leqq 1/2^{n+1}$ and the matrix entries a_{ij} of $H + \sum_{1\leqq k\leqq n+1} \alpha_k H_k$ are not zero for $j = 2, \cdots, i_{n+1}$. Then the operator $G_1 = H + \sum_{k\geqq 1} \alpha_k H_k$ is a bounded hermitian on X. Its entries $a_{1j} \neq 0$ for all $j \geqq 2$. With $i = 2, 3, \cdots$ let V_i be the operator by interchanging first and i-th row of I. Then $G_i = V_i G_1 V_i$ is hermitian and its entries $a_{ij} \neq 0$ for $j = 1, 2, \cdots, i-1, i+1, \cdots$. Choose a sequence $\{\beta_i\}$ of positive numbers such that $\sum_{\beta_i} < \infty$ and for each $k = 2, 3, \cdots$ the first k rows of $\sum_{1\leqq j\leqq k} \beta_j G_j$ are not zero except may be at the (j, j) position. Then $H' = \sum \beta_j G_j$ is the required hermitian operator.

Let $X_n = \{f \in X: f(k) = 0 \text{ all } k > n\}$. Suppose that S is the real unit ball in X_n as represented in the n-dimensional Euclidean space E_n and γ its boundary. For $\alpha \in \gamma$ there exists at least one supporting hyperplane to S at α with a normal vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

LEMMA 8. *For nonzero* $g \in X_n$,

$$[f, g] = A(g)\,\|g\|\Big\{\sum_{1\leqq j\leqq n} f(j)\operatorname{sgn} g(j)\alpha_j\Big\} \qquad \text{all } f \in X_n\,,$$

where $A(g) = \{\sum_{j=1}^{n} |g(j)|/\|g\|\,\alpha_j\}^{-1}$ *and* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *is the normal vector to a hyperplane at* $\left(\frac{|g(1)|}{\|g\|}, \frac{|g(2)|}{\|g\|}, \cdots, \frac{|g(n)|}{\|g\|}\right)$, *defines a consistent* s.i.p. *on* X_n.

The proof is similar to that as in Lemma 4.

LEMMA 9. *If there is a* H' *as in Lemma* 7, *then the set* $S' = \{f \in X_n: f(j) \neq 0 \text{ all } j\}$ *consists of smooth points.*

Proof. Let $(x_1, x_2, \cdots, x_n) \in S'$ and $k = 1, 2, \cdots, n-1$, $g_k = (x_1, x_2, \cdots, e^{i\theta}x_k, \cdots, x_n)$ in X_n is of unit norm where $0 \leqq \theta < 2\pi$. The restriction of H' to X_n, $H_n = (a_{ij})_{i,j=1,2,\cdots n}$ is hermitian by Lemma 1 and

$$\begin{aligned}[H_n g_k, g_k] &= A(g_k)\{(a_{11}x_1 + \cdots + e^{i\theta}a_{1k}x_k + \cdots + a_{1n}x_n)\alpha_1 + \cdots \\ &\quad + e^{-i\theta}(a_{k1}x_1 + \cdots + e^{i\theta}a_{kk}x_k + \cdots + a_{kn}x_n)\alpha_k + \cdots \\ &\quad + (a_{n1}x_1 + \cdots + e^{i\theta}a_{nk}x_k + \cdots + a_{nn}x_n)\alpha_n\} \\ &= A(g_k)\{e^{i\theta}(a_{1k}x_k\alpha_1 + \cdots + a_{k-1k}x_k\alpha_k + a_{k+1k}x_k\alpha_{k+1} + \cdots \\ &\quad + a_{nk}x_k\alpha_n) + e^{-i\theta}(a_{k1}x_1 + \cdots + a_{kk-1}x_{k-1} \\ &\quad + a_{kk+1}x_{k+1} + \cdots + a_{kn}x_n)\alpha_k + \cdots\}\end{aligned}$$

is real valued. By Lemma 3 we obtain the system of equations:

$$(1) \qquad \left(\sum_{\substack{j=1\\ \neq k}}^{n} a_{kj}x_j\right)\alpha_k - \sum_{\substack{j=1\\ \neq k}}^{n} \bar{a}_{jk}x_k\alpha_j = 0$$

for $k = 1, 2, \cdots, n-1$.

For every real number β, let U be a diagonal matrix whose first diagonal element is $e^{-i\beta}$ and the rest is one. In place of H' we substitute $UH'U^{-1}$. Then the resulting matrix elements are changed only for the first row and first column; and the subsequent form of equations (1) are:

$$\left(\sum_{j=2}^{n} a_{1j}e^{-i\beta}x_j\right)\alpha_1 - \sum_{j=2}^{n} \bar{a}_{j1}e^{-i\beta}x_1\alpha_j = 0$$

$$\left(a_{k1}e^{i\beta}x_1 + \sum_{\substack{j=2\\ \neq k}}^{n} a_{kj}x_j\right)\alpha_k - \left(\bar{a}_{1k}e^{i\beta}x_k\alpha_1 + \sum_{\substack{j=2\\ \neq k}}^{n} \bar{a}_{jk}x_k\alpha_j\right) = 0$$

for $k = 2, 3, \cdots, n-1$. With any fixed $(x_1, x_2, \cdots, x_n)$ where $x_j \neq 0, j = 1, 2, \cdots, n$, we shall show that this system is linearly independent for some β; equivalently we show that the following matrix is rank $n-1$:

$$\begin{bmatrix} \sum_{j=2}^{n} a_{1j}e^{-i\beta}x_j & -\bar{a}_{21}e^{-i\beta}x_1 & \cdots & -\bar{a}_{k1}e^{-i\beta}x_1 & & -\bar{a}_{n1}e^{-i\beta}x_1 \\ -\bar{a}_{12}e^{i\beta}x_2 & a_{21}e^{i\beta}x_1 + \sum_{j=3}^{n} a_{2j}x_j & & \cdot & & -\bar{a}_{n2}x_2 \\ \vdots & & \cdot & & & \vdots \\ -\bar{a}_{1k}e^{i\beta}x_k & \cdot & & a_{k1}e^{i\beta}x_1 + \sum_{\substack{j=2\\ \neq k}}^{n} a_{kj}x_j & \cdots & -{}_n\bar{a}_k x_k \\ \vdots & \cdot & & & & \vdots \\ -\bar{a}_{1n-1}e^{i\beta}x_{n-1} & -\bar{a}_{2n-1}x_{n-1} & \cdot & \cdot & \cdot & -\bar{a}_{nn-1}x_{n-1} \end{bmatrix}.$$

If we take the first $n-1$ columns, we obtain a square matrix and its determinant is a polynomial $P(e^{i\beta})$ of degree $n-2$. The coefficient of the $e^{i(n-2)\beta}$ term is obtained by finding the determinant of the following matrix:

$$\begin{bmatrix} \sum_{j=2}^{n} a_{1j}x_j & -\bar{a}_{21}x_1 & \cdots & -\bar{a}_{k1}x_1 & \cdots & -\bar{a}_{n-11}x_1 \\ -\bar{a}_{12}e^{i\beta}x_2 & a_{21}e^{i\beta}x_1 & & & \mathbf{0} & \\ \vdots & & & & & \\ -\bar{a}_{1k}e^{i\beta}x_k & & & a_{k1}e^{i\beta}x_1 & & \\ \vdots & \mathbf{0} & & & & \\ -\bar{a}_{1n-1}x_{n-1}e^{i\beta} & & & & & a_{n-1}e^{i\beta}x_1 \end{bmatrix}.$$

For $k = 2, 3, \cdots, n-1$, we add $\bar{a}_{k1}e^{-i\beta}/a_{k1}$ multiple of k-th row to the first row. We obtain by the condition that $a_{1k} = \bar{a}_{k1}$ a matrix of nonzero diagonal elements and whose entries above diagonal are zero. Thus the polynomial P is not identically zero and the original matrix has rank $n-1$ for some β. Thus we may assume that the system (1) is linearly independent. This implies that the normal vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is uniquely determined up to a multiple of constant. The proof is complete.

THEOREM 2. *Suppose that H is a hermitian operator on X, then either there is real valued function $h \in l_\infty$ such that*

$$Hf = hf \qquad \textit{for all } f \in X$$

and $\|H\| = \|h\|_\infty$ or else X is a Hilbert space. Conversely for every real valued function $h \in l_\infty$ the above formula defines a hermitian operator on X.

Proof. The converse is the content of Lemma 2. Assume that there is a hermitian operator H on X which is not diagonal, then Lemmas 7 and 9 imply that the function F defined on E_n, given by $F(x_1, x_2, \cdots, x_n) = \|(x_1, x_2, \cdots, x_n)\| - 1$, is differentiable at points of S'. So that the supporting hyperplane at $g \in S'$ is given by tangent place and the system (1) can be replaced by

$$\sum_{\substack{j=1\\ \neq k}}^{n} a_{kj}x_j \frac{\partial F}{\partial x_k} - \sum_{\substack{j=1\\ \neq k}}^{n} \bar{a}_{jk}x_k \frac{\partial F}{\partial x_j} = 0$$

$k = 1, 2, \cdots, n-1$. Observe that the function $\sum_{i=1}^n x_i^2$ satisfies this system. Let $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ be a point on the unit ball and $\sum_{i=1}^n (x_i^0)^2 = K$ for some $K > 0$. For all other $x \in S'$ which is on this sphere we have

$$F(x) = F(x^0) + \int_\Gamma \text{grad } F = \int_\Gamma \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{dx_i}{ds} ds$$

where $T = (dx_1/ds, dx_2/ds, \cdots, dx_n/ds)$ is the unit tangent vector. If $F(x) \neq 0$, since grad $F. T$ is continous, then there is a s_0 such that $x(s_0) \in \Gamma$ and grad $F(s_0).T(s_0) \neq 0$. But $T(s_0)$ at $x(s_0)$ is on the tangent plane to the sphere at $x(s_0)$ and grad $F(s_0)$ is normal to this plane, this is a contradiction. Therefore $F(x) = 0$ and all $x \in S'$ such that $\sum_{i=1}^n x_i^2 = K$ are on the real unit ball. As the surface γ is continuous, this equation gives the set of points on γ.

This will suffice to imply that X_n is a Hilbert space, since an inner-product on it can be found to give the original norm. The absolute continuity of the norm thus implies that X is a Hilbert space.

If X is not a Hilbert space, then every H on X is real diagonal and the rest is clear.

THEOREM 3. *Suppose U is an isometry from X onto itself and assume that X is not a Hilbert space. Then there is a fixed unimodular function u and an isomorphism T of atoms such that*

$$Uf(.) = u(.)f(T.) \qquad \textit{for all } f \in X .$$

Conversely such a transformation always defines as isometry on X.

Proof. The line of argument follows that of Theorem 5 below. u is unimodular because of the symmetry condition on X.

4. Reflexive Orlicz spaces. Let $L_{M\Phi}$ be a reflexive Orlicz space defined by the convex function Φ. We assume that Φ is everywhere finite. Suppose that the measure is finite.

LEMMA 10. [5, Lemma 6]. *Let H be a bounded hermitian operator on $L_{M\Phi}$. If Ω', Ω'' are* a.e. *disjoint, i.e., $\mu(\Omega' \cap \Omega'') = 0$, let χ' and χ'' be their characteristic functions; then $\int_{\Omega'} H\chi'' = 0$ if and only if $\int_{\Omega''} H\chi' = 0$.*

LEMMA 11. [5, Th. 9]. *Suppose H is as above, and μ is purely nonatomic, then either there exists a real valued function $h \in l_\infty$ such that $Hf = hf$ for all $f \in L_M$ and $\| H \| = \| h \|_\infty$ or else $L_{M\Phi} = L_2$.*

Let (Ω, Σ, μ) be a general measure space and decompose $L_{M\Phi} = L'_{M\Phi} + l_{M\Phi}$ where $L'_{M\Phi}$ are functions on nonatomic part and $l_{M\Phi}$ are functions on purely atomic part.

LEMMA 12. *Suppose H is as above, then either $L_{M\Phi}$ is L_2 or else $L'_{M\Phi}$ and $l_{M\Phi}$ are both invariant under H.*

Proof. Assume that $L_{M\Phi}$ is not a L_2 space. Let Ω' be a nonzero atom and χ' its characteristic function. Suppose that $H\chi'$ is not zero on a nonatomic set Ω'', and $\int_{\Omega''} H\chi' \neq 0$. Take χ'' to be the characteristic function of Ω''. Then for $\alpha \geqq 0$ we obtain the equality as in the proof of Lemma 11 [see 5]:

$$\Psi\left(\frac{\alpha}{\| \alpha\chi'' + \chi' \|}\right) = \alpha\Psi\left(\frac{1}{\| \alpha\chi'' + \chi' \|}\right)$$

where $\Psi = 1/2(\Phi^+ + \Phi^-)$ and Φ^+, Φ^- are the right and left hand derivatives of Φ respectively. Since Ω'' is nonatomic, we may replace

Ω'' by subset of Ω'' with arbitrarily small measure, so that

$$\Psi\left(\frac{\alpha}{\|\chi'\|}\right) = \alpha\Psi\left(\frac{1}{\|\chi'\|}\right) \qquad \alpha \geqq 0 .$$

Then $\Phi(t) = ct^2$ and $L_{M\Phi}$ is actually a L_2 space. This contradict our hypothesis, $H\chi' \in l_{M\Phi}$.

Conversely, if Ω'' is nonatomic and χ'' its characteristic function, then by Lemma 10, $\int_{\Omega'} H\chi'' = 0$ if and only if $\int_{\Omega''} H\chi' = 0$ where Ω' is any atom. The previous result shows that $H\chi' \in l_{M\Phi}$ for every atom Ω'. Hence $\int_{\Omega'} H\chi'' = 0$. Therefore $H\chi'' \in L'_{M\Phi}$. Since the step functions are dense in their respective subspaces, both $L'_{M\Phi}$ and $l_{M\Phi}$ are invariant under H.

THEOREM 4. *Suppose H is a bounded hermitian operator on $L_{M\Phi}$ which is not a L_2 space, then one of the following three cases holds:*

(1) *$l_{M\Phi}$ is a Hilbert space.*

(2) *$l_{M\Phi}$ contains a two-dimensional Hilbert space but is not a Hilbert space.*

(3) *There is a fixed real valued function $h \in l_\infty$ such that $Hf = hf$ for all $f \in L_{M\Phi}$ and $\|H\| = \|h\|_\infty$.*

Proof. By Lemma 12 and Lemma 11 it is enough to consider the restriction H' of H on $l_{M\Phi}$. If $l_{M\Phi}$ does not have a two-dimensional Hilbert subspace, the H' is real diagonal by Lemma 6 and case (3) follows.

REMARK. Let μ be a σ-finite measure and $\Omega = \bigcup_{n=1}^\infty \Omega_n$ where $\{\Omega_n\}$ is a fixed increasing sequence of measurable sets with finite mass. Suppose that for each n, P_n is the projection onto the subspace X_n of functions restricted to Ω_n. $H_n = P_nHP_n$ is hermitian. As $L_{M\Phi}$ has absolutely continuous norm, we have for $g \in L_{M\Phi}$ $\|Hg - HP_ng\| \to 0$ as $n \to \infty$, and $\|Hg - H_ng\| \leqq \|Hg - HP_ng\| + \|HP_ng - H_ng\|$, so that $Hg = \lim_n H_ng$. Thus we show that Theorem 4 holds for σ-finite measure as well.

Let $L^b_{M\Phi}$ be the set of all $f \in L_{M\Phi} \cap L_\infty$. L_∞ forms a *-algebra under the ordinary conjugation with the set of elements $\{\chi_n : \chi_n$ characteristic functions of $\Omega_n\}$ satisfying $\lim_n f\chi_n = f$ for all $f \in L_{M\Phi}$. $L^b_{M\Phi}$ contains this sequence. Suppose that $l_{M\Phi}$ is not a Hilbert space and contains no Hilbert subspace. Then the following is true.

LEMMA 13. *Suppose that U is an isometry of $L_{M\Phi}$. Then there is a *-isomorphism ρ on L_∞ such that $Ug = u\rho(g)$ for all $g \in L^b_{M\Phi}$, where $u \neq 0$* a.e.

Proof. By Theorem 1 and Theorem 4 we have for all $g \in L^b_{M\Phi}$, $\lim_n \rho(g) U\chi_n = Ug$. It is enough to show that $U\chi_n$ converges a.e. to a nonzero function u. Since ρ is an isomorphism, it sends characteristic functions onto themselves. Define $T\omega = \omega'$, where $\rho(\chi_\omega) = \chi_{\omega'}$. For every $n \geqq 1$, $UH_{\chi_n}U^{-1} = H_{\rho(\chi_n)} = H_{\chi_{T\Omega_n}}$, so that $U(\chi_n) = \chi_{T\Omega_n} U\chi_n$. That is $U\chi_n = 0$ on $\Omega - T\Omega_n$. Similarly $U(\chi_{\Omega_n - \Omega_m})$ vanishes on $\Omega - T(\Omega_n - \Omega_m) = \Omega - (T\Omega_n - T\Omega_m)$ for $1 \leqq m \leqq n$; therefore $U\chi_n = U\chi_m$ on $T\Omega_n$ and $\lim_n U\chi_n = u$ exists a.e.

Assume that ω is a measurable subset of $T\Omega$ such that $0 < \mu(\omega) < \infty$ and $u = 0$ on ω. For every $h \in L^b_{M\Phi}$, $Uh = u\rho(h) = 0$ on ω. $L^b_{M\Phi}$ is dense in $L_{M\Phi}$, so that with every $f \in L_{M\Phi}$, there is a sequence $\{f_n\}$ in $L^b_{M\Phi}$ such that $f_n \to f$ as $n \to \infty$. Since the norm is absolutely continuous, there is a subsequence $\{f_{n_k}\}$ such that $Uf_{n_k} \to Uf$ a.e. Thus $Uf = 0$ on ω. But U is onto and χ_ω is in the range of U. Hence u is nonzero a.e.

DEFINITION. A regular set isomorphism of a measure space (Ω, Σ, μ) will mean a mapping S of Σ into Σ defined modulo set of measure zero, satisfying: (i) $S(\Omega - \omega) = S\Omega - S\omega$. (ii) $S(\bigcup_{n=1}^\infty \omega_n) = \bigcup_{n=1}^\infty S\omega_n$ for disjoint sets $\{\omega_n\}$. (iii) $\mu(\omega) = 0$ if and only if $\mu(S\omega) = 0$.

LEMMA 14. *T, defined as in the proof above, is a regular set isomorphism of the underlying measure space; and it induces a linear transformation on $L_{M\Phi}(f(.) \to f(T^{-1}.))$.*

Proof. It is routine to show that T is regular. Let $f \in L_{M\Phi}$ be $a \leqq f < b$ on a measurable set ω and zero elsewhere. Assume that $\{f_n\}$ is a sequence of step functions whose values lying between a and b on ω and zero elsewhere, such that $f_n \to f$ as $n \to \infty$. Then $Uf_n = u\rho(f_n)$ converges to $Uf = u\rho(f)$ as $n \to \infty$. There is a subsequence $u\rho(f_{n_k})$ converging to $u\rho(f)$ a.e. Since $u \neq 0$, $\rho(f_{n_k}) \to \rho(f)$ a.e. We denote the step function $\rho(f_{n_k})$ as $f_{n_k}(T^{-1}.)$. Then $a \leqq f_{n_k}(T^{-1}.) < b$ on $T\omega$; $\rho(f)$, the a.e. limit of $f_{n_k}(T^{-1}.)$, has the same property. We shall let this function be g. For any nonnegative function f of $L_{M\Phi}$, let $\omega_n = \{x : n \leqq f(x) < n+1\}$ and f_n be the restriction of f to ω_n. Then g_n is $n \leqq g_n < n+1$ on $T\omega_n$ and zero elsewhere. Since T is regular, we can compose these functions to be a function g; and denote it by $f(T^{-1}.)$. Extend this definition to negative and then complex functions. The mapping so defined is clearly linear.

Combining the results, we obtain the following isometry theorem:

THEOREM 5. *Let U be an isometry from a reflexive Orlicz space $L_{M\Phi} = L'_{M\Phi} + l_{M\Phi}$ onto itself. Suppose that $L_{M\Phi} \neq L_2$, then U can be* *decomposed into* $U_1 + U_2$ *where* U_1 *and* U_2 *are isometric on* $L'_{M\Phi}$ *and* $l_{M\Phi}$ *respectively. Moreover one of the following three cases holds.*

(1) $l_{M\Phi}$ *is a Hilbert space.*

(2) $l_{M\Phi}$ *is not a Hilbert space but contains a two-dimensional Hilbert subspace.*

(3) *There is a regular set isomorphism* T *of the underlying measre space and a fixed* a.e. *nonzero function* u *such that*

$$Uf(.) = u(.)f(T^{-1}.) \qquad \text{for all } f \in L_{M\Phi}\,.$$

Proof. We first show that U decomposes. For all real function $h \in L_\infty$, $U^{-1}H_h UL'_{M\Phi} \subseteqq L'_{M\Phi}$ by Lemma 12. Hence $H_h UL'_{M\Phi} \subseteqq UL'_{M\Phi}$. If $UL'_{M\Phi} \nsubseteq L'_{M\Phi}$, then there is a characteristic function χ of some atom $\{a\}$ such that $Ug = \chi$ with some g in $L'_{M\Phi}$. Without loss of generality we may assume that g is a characteristic function of a nonatomic set ω. For two disjoint sets ω', ω'' and χ', χ'' their characteristic functions, $\| U(\chi' + \alpha\chi'') \| = \| U\chi' + \alpha U\chi'' \| = \| \chi' + \chi'' \|$ where $|\alpha| = 1$ and $\omega = \omega' \cup \omega''$. Thus $U\chi'$ and $U\chi''$ cannot be both nonzero at $\{a\}$. Since ω is nonatomic, we may replace it by subset of arbitrarily small measure; $Ug = 0$. This contradicts the fact that $\chi \neq 0$. Hence $UL'_{M\Phi} \subseteqq L'_{M\Phi}$; simiarly $U^{-1}L'_{M\Phi} \subseteqq L'_{M\Phi}$. $U(L'_{M\Phi}) = L'_{M\Phi}$. It follows that $Ul_{M\Phi} \subseteqq l_{M\Phi}$ with an application of Lemma 12.

Now if $l_{M\Phi}$ is not a Hilbert space and does not contain a two-dimensional Hilbert subspace, then Lemma 2 and Theorem 4 imply that H is a hermitian operator on $L_{M\Phi}$ if and only if it is of the form as stated in case (3) of Theorem 4. Hence case (3) holds for all g in $L^b_{M\Phi}$ by Lemma 13 and Lemma 14. Since $L^b_{M\Phi}$ is dense, the proof is thus complete.

As a special case of the theorem, we record the following result as a corollary.

COROLLARY. *With the conditions as before and assume that the atoms in the measure space have equal mass, either*

(1) *There is a regular set isomorphism* T *and a fixed* a.e. *nonzero function* u *such that* $Uf(.) = u(.)f(T^{-1}.)$ *all* f *in* $L_{M\Phi}$, *or else*

(2) U_1 *is of the form as stated in* (1) (*T and u in this case are defined only on the nonatomic part*) *and* U_2 *is unitary on* $l_{M\Phi}$ *which is a Hilbert space.*

REMARK. U_1 is always characterized in (3) of the Theorem 5 if $L_{M\Phi}$ is not a L_2 space.

5. An example. The following example shows that the Theorem

3 does not hold if the symmetry condition is not present. It also shows that isometries other than the type in Theorem 5 occur if the atoms in the underlying measure space have unequal mass.

Let (Ω, Σ, μ) be a measure space with contains two atomic sets m_1 and m_2 each with measure 16 and at least one other measurable set m_3 of mass 1. With $\Phi(x) = \int_0^x \psi(t)dt$ where

$$\psi(t) = \begin{cases} 2t & 0 \leqq t < 1/2 \\ t^2 + 3/4 & t > 1/2 , \end{cases}$$

the obtained $L_{M\Phi}$ is not a Hilbert space. Specifically the two dimensional subspace on $\{m_2, m_3\}$ is not a Hilbert space, because the convex curve $\{(y, z): 16\Phi(|y|) + \Phi(|z|) = 1\}$ is not an ellipse. Now write $L_{M\Phi} = l_1 + l_2$ where l_2 is the two dimensional space of functions vanishing on Ω-$\{m_1, m_2\}$ and l_1 of those being zero on $\{m_1, m_2\}$. Define $U = U_1 + U_2$ where U_2 on l_2 in matrix form is

$$U_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and U_1 is identity on l_1. Then for any $L_{M\Phi}$ such that $\|f\| = 1$, we have $0 \leqq |f(m_1)|, |f(m_2)| \leqq 1/4$, so that

$$\begin{aligned} \int \Phi(|Uf|) &= 16\{\Phi(|Uf(m_1)|) + \Phi(|Uf(m_2)|)\} + \int_{\Omega-\{m_1, m_2\}} \Phi(|Uf|) \\ &= 16\{|f(m_1)|^2 + |f(m_2)|^2\} + \int_{\Omega-\{m_1, m_2\}} \Phi(|Uf|) = \int \Phi(|f|) . \end{aligned}$$

Therefore $\|Uf\| = \|f\| = 1$. U is isometric.

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INJECTIVE HULLS OF SEMI-SIMPLE MODULES OVER REGULAR RINGS

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The object of this paper is to provide an explicit construction of the injective hull of a semi-simple module over a commutative regular ring.

The existence of injective hulls of an arbitrary module M and their uniqueness upto isomorphism over M was shown by B. Eckmann and A. Schopf in 1953 [6]. But only in few cases these hulls have been described explicitly [1, 2].

In the special case when the ring is regular as well as Noetherian, the problem is already solved since over such a ring every module is known to be semi-simple [9] and hence is its own injective hull [11, 10]. To begin with we show that every monotypic component of the module is injective and then prove a topological lemma about T_1-spaces. The Zariski topology of the maximal ideal space of the basic ring being T_1, we make use of the lemma to obtain the desired construction of an injective hull of the module. We show by an example that a semi-simple module over a regular ring need not always be injective and obtain finally a necessary and sufficient condition for the injectivity of the module.

DEFINITION 1. A ring R is called (von Neumann) *regular* if for every $a \in R$, there exists an element $x \in R$ such that $axa = a$. This condition reduces to $a^2x = a$ if R is commutative. A Boolean ring is an example of a commutative regular ring. It is well known that a commutative ring R with unit is regular if and only if every simple R-module is injective [11].

Throughout this paper we shall consider R to be a commutative regular ring with unit 1. Let Ω denote the set of maximal ideals of R. For each $a \in R$ define Ω_a by $\Omega_a = \{P \in \Omega \mid a \notin P\}$. It follows that $\Omega_a \cap \Omega_b = \Omega_{ab}$. Thus Ω can be made into a topological space with $\{\Omega_a \mid a \in R\}$ as the system of basic open sets. This topology of Ω is known as the Zariski topology. Ω is clearly a T_1-space since if P and Q are any two distinct points in Ω, there exists $a \in P - Q$ which implies that Ω_a is a neighbourhood of Q not containing P.

DEFINITION 2. Let M be a semi-simple R-module. For any simple submodule S of M, there exists exactly one $P \in \Omega$ with $S \cong R/P$. The sum of all those simple submodules of M which are isomorphic to R/P, will be denoted by M_P and will be called the *R/P-monotypic componet* of M. The *support of M*, to be denoted by Supp (M) is the set of all those maximal ideals P in Ω for which M_P is nonzero.

In our discussion M will always denote a semi-simple R-module with supp $(M) = S$. As usual for any function f, the symbol supp (f) will mean the set of all those elements in domain (f) for which $f(x) \neq 0$. We shall write $E = H(M)$ to express the fact that E is an injective hull of M. Where no ambiguity can arise, we let $H(M)$ stand for an arbitrary injective hull of M. If α is any cardinal number and L any module, the sombol $\alpha \odot L$ will stand for the external sum of α copies of L.

THEOREM 1. *For any $P \in S$, the associated monotypic component M_P is an injective module.*

Proof. Let α be the length of M_P and T a set with $|T| = \alpha$. Then $M_P \cong \alpha \odot R/P = E$. Let π be the set of all functions from T into R/P. Now each factor R/P of π being injective [11], π is injective; hence there exists an $H(E) \subseteqq \pi$. Without loss of generality we can take α to be an infinite cardinal. Assume E is not injective. Then $E \subset H(E) \subseteqq \pi$. Take any element $f \in H(E) - E$. Since $H(E)$ is an essential extension of E, one has $Rf \cap E \neq 0$ which implies $0 \neq rf \in E$ for some $r \in R - P$. As R/P is a field and $f(t) \neq 0$ for infinitely many $t \in T$, we have $0 \neq (r + P)f(t) = rf(t)$ for infinitely many $t \in T$. But this contradicts the fact that $rf \in E$. Hence E is injective.

REMARK 1. $\prod_{P \in S} M_p$ is injective since each factor M_P is injective.

DEFINITION 3. Let X be any topological space and A any subset of X. An element $x \in A$ is called an *isolated point* of A if there exists a neighbourhood U of x such that $U \cap A = \{x\}$, i.e., if $\{x\}$ is an open set in the relative topology of A. A subset A of X is said to be *discrete* if every element x in A is an isolated point of A.

LEMMA 1. *Let $f \in \prod_{P \in S} M_P$ and $a \in R$ such that $0 \neq af \in \bigoplus_{P \in S} M_P$, then every element in* supp (af) *is an isolated point of* supp (f).

Proof. Let supp $(af) = \{P_1, P_2, \cdots, P_n\}$ where $P_i \neq P_j$ if $i \neq j$. This implies that there exist elements $a_i \in P_i - P_1$ $(i = 2, 3, \cdots, n)$. Put $b = aa_2a_3 \cdots a_n$. Then $b \notin P_1$ and $b \in P$ for each $P \in$ supp(f) with $P \neq P_1$. Hence $\Omega_b \cap$ supp$(f) = \{P_1\}$ showing that P_1 is an isolated point of supp(f). Similar argument will prove that $P_2, \cdots, P_n$ are also isolated points of supp(f).

REMARK 2. It follows from the lemma that the support of any nonzero element in an essential extension of $\bigoplus_{P \in S} M_P$ contains an isolated point.

LEMMA 2. *Let E be a proper essential extension of $\bigoplus_{P \in S} M_P$. Then for any $f \in E - \bigoplus_{P \in S} M_P$, $\operatorname{supp}(f)$ contains infinitely many isolated points.*

Proof. Since E is an essential extension of $\bigoplus_{P \in S} M_P$ and $0 \neq f \in E$, we can find an element $a \in R$ such that $0 \neq af \in \bigoplus_{P \in S} M_P$. Let $\operatorname{supp}(af) = \{P_1, P_2, \cdots, P_n\}$. By Lemma 1, each P_1 is an isolated point of $\operatorname{supp}(f)$. Choose an element $Q \in \operatorname{supp}(f) - \operatorname{supp}(af)$. As $P_i \not\subseteq Q$, there exist elements $r_i \in P_i - Q (i = 1, 2, \cdots, n)$. Then

$$r = r_1 r_2 \cdots r_n \in (P_1 \cap P_2 \cap \cdots \cap P_n) - Q .$$

It follows that $0 \neq rf \in E$. Since for some $s \in R$, $0 \neq srf \in \bigoplus_{P \in S} M_P$, we can apply Lemma 1 to show that the elements in $\operatorname{supp}(srf)$ are isolated points of $\operatorname{supp}(f)$ and they are all distinct from $P_1, P_2, \cdots, P_n$. Now $\operatorname{supp}(f)$ being infinite, we can find an element in

$$\operatorname{supp}(f) - (\operatorname{supp}(af) \cup \operatorname{supp}(srf))$$

which will give rise to another set of finitely many elements isolated points of $\operatorname{supp}(f)$ each being different from the ones obtained before. Proceeding thus we get infinitely many isolated points of $\operatorname{supp}(f)$. This proves the lemma.

We now prove the following topological fact about T_1-spaces:

LEMMA 3. *In any T_1-space X, if A and B are nonvoid subsets such that A as well as every nonvoid subset of B has an isolated point, then there exists an isolated point in $A \cup B$.*

Proof. Let the complement of a subset C of X be denoted by C'. Since A is given to have an isolated point p, there exists an open neighbourhood U of p such that $U \cap A = \{p\}$. From

$$U \cap (A \cup (B \cap U')) = U \cap A$$

we conclude that p is also an isolated point of $A \cup (B \cap U')$. If $B \cap U$ is empty, then p is an isolated point of $A \cup B$ and so the lemma holds. We have therefore to consider only the case when $B \cap U$ is nonvoid.

By hypothesis $B \cap U$ contains an isolated point q which can be assumed to be distinct from p without any loss in generality. This assumption, together with the fact that X is T_1 implies that $\{p\}'$ is an open set containing q. Now q being an isolated point of $B \cap U$, we have $V \cap B \cap U = \{q\}$ for some neighbourhood V of q. Thus we obtain

$$U \cap V \cap \{p\}' \cap (A \cup B) = U \cap V \cap \{p\}' \cap B = \{q\} \cap \{p\}' = \{q\} .$$

Since $U \cap V \cap \{p\}'$ is a neighbourhood of q, the above relation implies that q is an isolated point of $A \cup B$.

REMARK 3. From Lemma 3 we immediately have the following

(i) Let B be a discrete subset of a T_1-space X and A any subset of X with an isolated point, then $A \cup B$ has an isolated point.

(ii) If A and B are nonvoid subsets of a T_1-space X with the property that each of their nonvoid subsets has an isolated point then $A \cup B$ has the same property.

LEMMA 4. *Let $A = \bigcup_{i \in I} A_i$ where each A_i is without an isolated point. Then A has no isolated point.*

Proof. Suppose A has an isolated point p. Then $p \in A_i$ for some $i \in I$ and $\{p\} = U \cap A$ for some neighbourhood U of p. Hence $\{p\} = U \cap A_i$ contrary to the hypothesis that A_i is without an isolated point. Thus A has no isolated point.

LEMMA 5. *If A has no isolated point, then $\bar{A}$, the closure of A also has no isolated point.*

Proof. Assume p is an isolated point in $\bar{A}$ with $V \cap \bar{A} = \{p\}$ for some neighbourhood V of p, then $p \in \bar{A} \cap A'$ implies the existence of an element $q \in V \cap A \subseteqq V \cap \bar{A}$ with q distinct from p, a contradiction. Hence A has no isolated point.

REMARK 4. We know that the semi-simple module $M = \sum_{P \in S} M_P$ (direct) hence $M \cong \bigoplus_{P \in S} M_P$. Since the injective module $\prod_{P \in S} M_P$ contains $\bigoplus_{P \in S} M_P$ as a submodule, it also contains an $H(\bigoplus_{P \in S} M_P)$. Thus to find an injective hull of M, it is sufficient to obtain one of $\bigoplus_{P \in S} M_P$ inside $\prod_{P \in S} M_P$. This is done in the following:

THEOREM 2. *Let $H = \{f \in \prod_{P \in S} M_P \mid$ Every nonvoid subset of* supp (f) *has an isolated point$\}$. Then H is an injective hull of $\bigoplus_{P \in S} M_P$.*

Proof. Let f, g be any two elements in H, then since

$\operatorname{supp}(f+g) \subseteqq \operatorname{Supp}(f) \cup \operatorname{supp}(g)$, we have $f+g \in H$ by Remark 3 (ii) following Lemma 3. Now if $a \in R, f \in H$, then $\operatorname{supp}(af) = \Omega_a \cap \operatorname{supp}(f)$ implies that $af \in H$. Hence H is an R-submodule of $\prod_{P \in S} M_P$ and it contains $\bigoplus_{P \in S} M_P$ since every nonvoid subset of a finite set is discrete. Now let $0 \neq f \in H$, then $\operatorname{supp}(f)$ is nonempty and hence contains an isolated point P so that for some

$$a \in R, \operatorname{supp}(af) = \Omega_a \cap \operatorname{supp}(f) = \{p\} .$$

Thus $0 \neq af \in \bigoplus_{P \in S} M_P$. Hence H is an essential extension of $\bigoplus_{P \in S} M_P$.

As to the injectivity of H assume by way of contradiction that H has a proper essential extension E. Then $H \subset E \subseteqq \prod_{P \in S} M_P$. Take $f \in E, f \notin H$. Then there exists a nonvoid subset of $\operatorname{supp}(f)$ without isolated points. Denote by X, the union of all those subsets of $\operatorname{supp}(f)$ which have no isolated points. By Lemma 4, X has no isolated point. Let $Y = \operatorname{supp}(f) \cap X'$ where X' is the complement of X in S. Then Y is nonvoid since by Remark 2, Lemma 1, $\operatorname{supp}(f)$ contains an isolated point which cannot belong to X. Thus $\operatorname{supp}(f) = X \cup Y$ is a decomposition of $\operatorname{supp}(f)$ into disjoint nonempty subsets X and Y. Moreover every nonvoid subset of Y contains an isolated point for otherwise it will have to be contained in X which is not possible. Now for any subset $A \subseteqq \operatorname{supp}(f)$, define f_A to be the function such that

$$f_A(P) = \begin{cases} f(P) & \text{if } P \in A \\ 0 & \text{if } P \in S - A \end{cases}$$

we can then write $f = F_X + f_Y$. Since $\operatorname{supp}(f_Y) = Y$, one has $f_Y \in H$ and hence from $f_X = f - f_Y$, it follows that $f_X \in E$. The fact that f_X is a nonzero element in an essential extension E of $\bigoplus_{P \in S} M_P$, then implies that $X = \operatorname{supp}(f_X)$ has an isolated point. We thus arrive at a contradiction. Hence H is injective. This completes the proof.

COROLLARY 1. *$\prod_{P \in S} M_P$ is an injective hull of $\bigoplus_{P \in S} M_P$ if and only if every nonvoid subset of S has an isolated point. In particular if S is discrete in Ω, then $\prod_{P \in S} M_P \cong H(M)$.*

Proof. If S has the property that each of its nonvoid subsets has an isolated point, then for every $f \in \prod_{P \in S} M_P$, $\operatorname{supp}(f)$ has the same property. Hence by Theorem 2, $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$. On the other hand let $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$. Suppose that some non-empty subset A of S has no isolated point. Then A must be an infinite set. We can find a function $f \in \prod_{P \in S} M_P$ with $\operatorname{supp}(f) = A$. Then $f \notin \bigoplus_{P \in S} M_P$ and hence $f \neq 0$. Since $\prod_{P \in S} M_P$ is an essential extension of $\bigoplus_{P \in S} M_P$, by

Remark 2, supp (f) has an isolated point contrary to the assumption that A has no isolated point. Hence every nonvoid subset of S has an isolated point. The last part of the corollary follows immediately from the fact that every element in a discrete set is an isolated point.

COROLLARY 2. *If S contains only principal ideals, then*

$$\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P) .$$

Proof. Let Ra be any maximal ideal in S. If P in S is different from Ra, then $a \notin P$ since $a \in P$ would mean $Ra \subseteqq P$, hence $Ra = P$, a contradiction. Regularity of R implies that $a = a^2x$ for some $x \in R$. Since $0 = a(1 - ax)$ belongs to every P in S, $1 - ax$ belongs to every element in S different from Ra. Also $1 - ax \notin Ra$ since other wise $1 \in Ra$. It follows that $\Omega_{1-ax} \cap S = \{Ra\}$. Thus every element in S is an isolated point. By Corollary 1, we have $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$.

REMARK 5. For any module M over a regular and Noetherian ring R, $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P) = \bigoplus_{P \in S} M_P$ since every ideal of R is a principal ideal [9] and every R-module is injective [10, 11].

COROLLARY 3. *There exist semi-simple modules over a regular ring which are not injective.*

Proof. Let R_0 be the two-element Boolean ring $\{0, e_0\}$, I an infinite index set and R, the set of all functions $f: I \longrightarrow R_0$. Then R is a complete Boolean ring and hence a commutative regular ring. For each $\alpha \in I$, define P_α by $P_\alpha = \{f \in R \mid f(\alpha) = 0\}$. It is easily seen that P_α is a maximal ideal of R[7]. Let $M = \bigoplus_{\alpha \in I} R/P_\alpha$. Then M is a semi-simple module with Supp $(M) = \{P_\alpha \mid \alpha \in I\}$. Take any $P_{\alpha_0} \in$ Supp(M) and define f by

$$f(\alpha) = \begin{cases} e_0 & \text{if} \quad \alpha = \alpha_0 \\ 0 & \text{if} \quad \alpha \neq \alpha_0 \end{cases}$$

then $f \in R - P_{\alpha_0}$ and $f \in P_\beta$ for all $\beta \in I$ with $\beta \neq \alpha_0$. Thus

$$\Omega_f \cap \text{Supp}\,(M) = \{P_{\alpha_0}\}$$

which implies that Supp (M) is discrete. Hence by Corollary 1, $\prod_{\alpha \in I} (R/P_\alpha) = H(\bigoplus_{\alpha \in I} (R/P_\alpha))$. The fact that I is infinite then shows that $\bigoplus_{\alpha \in I} (R/P_\alpha)$ is not injective.

COROLLARY 4. *If $S = A \cup D_1 \cup D_2 \cup \cdots \cup D_n$ where A has an* *isolated point and* $D_i (i = 1, 2, \cdots, n)$ *are discrete sets, then* $\prod_{P \in S} M_P \cong H(M)$.

Proof. It follows immediately from Lemma 3 and Corollary 1.

In Corollary 3 we have a concrete example showing that not every semi-simple R-module is injective. It is therefore worthwhile to ask under what conditions a semi-simple R-module is injective. The following theorem gives a characterisation for the injectivity of a semi-simple module.

THEOREM 3. *M is injective if and only if S has only finite discrete subsets.*

Proof. Let M be injective. Assume that $D \subseteqq S$ is an infinite discrete subset. We can find $f \in \prod_{P \in S} M_P$ with $\operatorname{supp}(f) = D$. Since D is infinite, $f \notin \bigoplus_{P \in S} M_P$. The fact that $\operatorname{supp}(f)$ is discrete implies by Theorem 2, that $f \in H(\bigoplus_{P \in S} M_P) = \bigoplus_{P \in S} M_P$ and so we get a contradiction. Hence S contains only finite discrete subsets.

Conversely suppose that S has only finite discrete subsets. Assume that M is not injective. Then $\bigoplus_{P \in S} M_P$ has a proper essential extension E inside $\prod_{P \in S} M_P$. Hence for any $f \in E - \bigoplus_{P \in S} M_P$, $\operatorname{supp}(f)$ contains an infinite discrete subset by Lemma 2. This contradiction then proves that M is injective.

Added in Proof.

REMARK 6. Under the assumptions of Theorem 3, S is a compact subset of Ω.

Proof. Let $S \subseteqq U_{i \in I} \Omega_{a_i}$ so that $S = U_{i \in I} (S \cap \Omega_{a_i})$ where we assume without loss of generality that each $S \cap \Omega_{a_i}$ is nonvoid. For each i in I, pick one P_i from $S \cap \Omega_{a_i}$ and let A be the set of all such P_i. Then $\Omega_{a_i} \cap A = \{P_i\}$ for each i in I. This implies that A is a discrete subset of S and hence by Theorem 3, A is finite. Consequently S is compact.

As a consequence of the above remark, we obtain as a corollary of Theorem 3, the following result of J. Levine, announced in an abstract in the Notices:

COROLLARY. (Levine) *If an injective module M over a commutative regular ring R is a direct sum of simple submodules, then there are only finitely many nonisomorphic simples in the sum.*

Proof. Let $M^* = \sum_P X_P$ be the sum of nonisomorphic simple submodules in the direct sum decomposition of M. Then for each X_P, there exists exactly one P in S with X_P isomorphic to R/P and hence the R/P-monotypic component of M^* is X_P. Moreover, M^* being a direct summand of M, is injective and, therefore, by Remark 6, its support S^* is compact. Any nonvoid subset of S^* also has this property since it is injective. We propose to show that S^* is discrete. Take any P in S^* and let $\{P\}'$ be the complement of $\{P\}$ in S^*. Then $\{P\}'$ being open and compact, we have $\{P\}' = U_{i\in 1}^{n} S_{c_i}$, where $S_{c_i} = \Omega_{c_i} \cap S^*$. Now, c_i in R implies that there exists x_i in R with $c_i = c_i^2 x_i$ $i = 1, 2, \cdots, n$. Put $d_i = 1 - c_i x_i$. Then from $c_i d_i = 0$, it follows that $d = d_1 d_2 \cdots d_n$ belongs to every Q in S^*, different from P and does not belong to P. Hence $\{P\} = S_d$. Thus every point in S^* is an isolated point as was required. By Theorem 3 we have S^* finite.

REMARK 7. Theorem 1 is a special case of a more general Proposition of C. Faith [Proposition 3, Rings with ascending condition on annihilators, Nagoya Math. J. 27 (1966), 179–181]. Let a module M be called *Σ-injective* if it is injective and every direct sum of copies of M is also injective. Then Proposition 3 of Faith has the following corollaries:

COROLLARY 1. *Let R be any ring, and let M be any injective simple module. Then if M is finite dimensional over the field $K = End\, M_R$, then M is Σ-injective.*

COROLLARY 2. *If R is any commutative ring, and M is an injective simple module, then M is Σ-injective.*

Theorem 1 is a special case of Corollary 2 when R is a regular ring.

REMARK 8. Corollary 3 of Theorem 2 provides an example of a semisimple module over a commutative regular ring which is not injective. C. Faith has sketched an example of a simple module over a noncommutative regular ring which is not injective [Chapter 15, "Lectures on Injective Modules and Quotient Rings" Springer Verlag, New York 1967].

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CONCERNING CONTINUA NOT SEPARATED BY ANY NONAPOSYNDETIC SUBCONTINUUM

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Certain theorems that apply to compact, metric continua that are separated by none of their subcontinua can be generalized and strengthened in those continua that are separated by none of their nonaposyndetic subcontinua. For those of the former type, if the continuum is aposyndetic at a point, it is locally connected at the point. The same conclusion is possible if the continuum is not separated by any nonaposyndetic subcontinuum. Also, if a continuum is separated by no subcontinuum and cut by no point, it is a simple closed curve. A second result of this paper is to prove that if no nonaposyndetic subcontinuum separates and no point cuts the continuum, then it is a cyclically connected continuous curve; in fact this yields a characterization of hereditarily locally connected, cyclically connected continua.

A third theorem characterizes an hereditarily locally connected continuum as an aposyndetic continuum that is separated by no nonaposyndetic subcontinuum. This is a somewhat stronger result than the known equivalence of hereditary local connectedness and hereditary aposyndesis.

A *continuum* is a closed, connected point set and the theorems of this paper are true for those continua that are compact and metric. If x is a point in the continuum M, then the continuum is *aposyndetic at* x if for every point y in $M - x$, there exists an open set U and continuum H such that $x \in U \subset H \subset M - y$. If M is aposyndetic at x for each point x in M, then M is *aposyndetic*, and M is *nonaposyndetic* if there is a point x in M such that M is not aposyndetic at x. By this definition a degenerate continuum is an aposyndetic continuum. The set S in M is said to *separate* M if $M - S$ is not connected and is said to *cut* M if for some pair of points $x, y \in M - S$, every subcontinuum of M intersecting both x and y must also intersect S. If every pair of points in M is contained in some simple closed curve lying in M, then M is *cyclically connected*. The continuum M is *hereditarily locally connected* if M is locally connected and every subcontinuum of M is locally connected, and M is *hereditarily aposyndetic* if it as well as each of its subcontinua is aposyndetic. In what follows, a subcontinuum of M is aposyndetic or nonaposyndetic if, with the relative topology from M, it is aposyndetic or nonaposyndetic respectively.

Bing has proved that if a continuum that is separated by no subcontinuum is aposyndetic at a point, it is locally connected at the point [1, Th. 2]. The next theorem shows that the same conclusion follows if the continuum is separated by no nonaposyndetic subcontinuum.

THEOREM 1. *Suppose the compact, metric continuum M is separated by no nonaposyndetic subcontinuum. If M is aposyndetic at x, then M is locally connected at x.*

Proof. The continuum M is aposyndetic at each point of $M - x$ with respect to x. To see this let y be any point in $M - x$. Since M is aposyndetic at x there exists a continuum H such that $x \in H^\circ \subset H \subset M - y$. If $M - H$ is connected then $y \in M - H \subset \overline{M - H} \subset M - x$ and M is aposyndetic at y with respect to x. Suppose that $M - H = A + B$, a separation, with y in B. Now $H + B$ is a continuum and if B is connected, then $y \in B \subset \bar{B} \subset M - x$ and again M is aposyndetic at y with respect to x. So let $B = C + D$, a separation, with y in D. Now $D + H$ is a continuum separating M into A and C. Hence $D + H$ is aposyndetic at y with respect to x and therefore so is M.

In the proof that M is locally connected at x no generality will be lost by assuming x to be a nonseparating point of M. For if x separates M, then each component C of $M - x$ is an open set, $C + x$ is a continuum with x a nonseparating point, and the proof would be complete by showing that $C + x$ is locally connected at x.

First let us show that M is connected im kleinen at x. Let K be a closed set in M such that $x \notin K$. Because M is aposyndetic at each point of K with respect to x, every point of K is in the interior of a continuum that does not intersect x, and by compactness K is in the interior of the sum of a finite number of these continua. For each pair of this finite collection of continua, there exists a continuum in $M - x$ intersecting both, due to the fact that x does not separate and hence does not cut M. Therefore, K lies in the interior of a continuum L that does not contain x.

We need to show that there is a continuum H such that $x \in H^\circ \subset H \subset M - K$. Assume such a continuum does not exist. Then $M - L$ is not connected and is the sum of separated sets A and B with x in A. The point x is not in the interior of the component C of A containing it, so there exists a sequence of points $x_1, x_2, \cdots$ converging to x each point of which belongs to a different component C_i of A. Let K_i be an irreducible subcontinuum of $\bar{C}_i + L$ from x_i to L. Now $K_i + L$ separates M and $K_i + L$ is therefore aposyndetic. Because K_i is irreducible from x_i to L, $K_i \cdot L$ is degenerate and K_i is an aposyndetic continuum. Since every point in $K_i - (x_i + L)$ is a cut point of K_i, everyone of these points must be a separating point of K_i and hence K_i is an arc. Let $K' = \lim\sup K_i$ (it is possible to construct the K_i's so that $K' \cdot \sum K_i = \emptyset$). Now let us show that the continuum $K' + \sum K_i + L$ is not aposyndetic at x. Let $y_i = K_i \cdot L$ and let y be a limit point of $y_1 y_2, \cdots$ in $K' \cdot L$. If in $K' + \sum K_i + L$ there were a continuum H with x in its interior that did not contain y, then there would exist an integer n such that

$$(H \cdot K_n) + (H \cdot [(K' + \sum K_i + L) - K_n])$$

is a separation of H. So $K' + \sum K_i + L$ is not aposyndetic at x with respect to y. Hence this continuum does not separate M and therefore $A + L = K' + \sum K_i + L$. So $A + L$ is a continuum not aposyndetic at x with respect to y. But this means that M is a continuum that is not aposyndetic at x with respect to y. For if, on the contrary, H is a continuum in M such that $x \in H^\circ \subset H \subset M - y$, then $H \cdot B \neq \emptyset$ or else $A + L$ is aposyndetic at x with respect to y. Due to the fact that L is an aposyndetic contiuum (it separates M), $H \cdot L$ is contained in the sum S of a finite number of continua of L each of which misses y. Now $H - S = P + Q$, a separation, with $x \in P \subset A$ and $Q \subset B$. But $P + S$ has a finite number of components since S does and therefore x lies in the interior of the component of $P + S$ containing x. This means that $A + L$ is aposyndetic at x with respect to y which is false. Thus M cannot be aposyndetic at x and this contradiction shows that M is connected im kleinen at x.

Finally, let us show that M is locally connected at x. Let C be a subcontinuum such that $M - C = A + B$, a separation of M (if no such subcontinuum exists, then M is locally connected at x as remarked earlier). If $x \in C$, then because M is connected im kleinen at x, there exists a continuum H such that $x \in H^0 \subset H$ and $C + H$ separates M. So $C + H$ is aposyndetic and so is M at each interior point of $C + H$. Therefore M is connected im kleinen at each interior point of $C + H$ which means that M is locally connected at x. If $x \in A$ and $A + C$ is not irreducible about $x + C$, then by the above argument, M is locally connected at x. On the other hand, if $A + C$ is irreducible about $x + C$, then it is well known that M is locally connected at x. This completes the proof of the theorem.

In [4, Th. 5] it is shown that the notions of hereditary aposyndesis and hereditary local connectedness are equivalent. The next theorem uses the result of Theorem 1 to establish a stronger characterization of hereditarily locally connected continua.

THEOREM 2. *A compact, metric continuum M is hereditarily locally connected if and only if M is an aposyndetic continuum that is separated by no nonaposyndetic subcontinuum.*

Proof. Let us prove the sufficiency. The necessity is trivial.

Since a continuum is hereditarily locally connected if and only if it is hereditarily aposyndetic, it is sufficient for us to prove that M contains no nonaposyndetic subcontinuum. Assume that M contains the nonaposyndetic subcontinuum N. The continuum M is aposyndetic and hence, by Theorem 1, is locally connected. Let y be any point in $M - N$, let U be an open set such that $y \in U \subset \bar{U} \subset M - N$, and let C be the component of $M - y$ containing N. Let $V_1, V_2, \cdots, V_m$ be connected open sets of C such that $(\bar{U} - U)\cdot C$ is a subset of $\sum V_i$ and $\bar{V}_i \cdot (y + N) = \varnothing$, $1 \leqq i \leqq m$. Denote by A_i, $1 \leqq i \leqq m$, an arc that intersects V_i, does not contain y, and has only an end point in common with N.

Because N is not an aposyndetic continuum, there exist points p and q in N such that N is not aposyndetic at p with respect to q. Now in the continuum $N' = N + \sum \bar{V}_i + \sum A_i$, the set $\sum \bar{V}_i + \sum A_i$ has only a finite number of points in common with N and therefore N' cannot be aposyndetic at p with respect to q. Furthermore the set $(\bar{U} - U)\cdot C$ separates M into sets E and F with $y \in E$ and $N \subset F$. Since the continuum N' contains $(\bar{U} - U)\cdot C$ but not y, then $F \subset N'$ because N' is nonaposyndetic and cannot separate M. But M is locally connected at p; consequently there is a connected open set V in M containing p and lying in F such that $q \notin \bar{V}$. This means that N' is aposyndetic at p with respect to q and this contradiction establishes the theorem.

Another result due to Bing [1, Th. 10] is that a continuum is a simple closed curve if it is separated by no subcontinuum and cut by no point. Next this is generalized to continua not separated by any nonaposyndetic subcontinuum.

THEOREM 3. *A compact, metric continuum M is both hereditarily locally connected and cyclically connected if and only if M is separated by no nonaposyndetic subcontinuum and cut by no one of its points.*

Proof. Again the proof of necessity is trivial so let us turn to the sufficiency. All we need to prove is that the continuum is aposyndetic because then, by Theorem 2, M will be hereditarily locally connected and since no point cuts M, then no point separates M and continua of this type are cyclically connected [3, p. 138].

Let us suppose that M is not aposyndetic at a point x in M. According to a theorem of Jones' [2, Th. 18] if no point cuts M, then M is aposyndetic on a dense subset of M. Let y, z be two points at which M is aposyndetic. By Theorem 1 there exist continua H and K neither of which contains x such that $y \in H^\circ \subset H$, $z \in K^\circ \subset K$, and $H \cdot K = \varnothing$. If $M - (H + K)$ is connected, then x is in the interior of the continuum $\overline{M - (H + K)}$ that separates y from z in M. So this continuum and therefore M itself is aposyndetic at x. Thus we can assume that $M - (H + K) = A + B$, a separation of M. One of the sets, $H + A + K$ or $H + B + K$, must be a continuum. Let us show that the other is also a continuum. Let $H + A + K$ be a continuum and suppose that $H + B + K = P + Q$, a separation, with $H \subset P$ and $K \subset Q$. Now $H + A + K$ is not irreducible about $H + K$ or else points in A will cut points in P from points in Q. Let T be a proper subcontinuum of $H + A + K$ containing $H + K$. If $P - H \neq \varnothing \neq Q - K$, then the continua $H + A + K$, $P + T$ and $Q + T$ all separate M and hence are aposyndetic continua. This means that M is aposyndetic at each point of $A + B$. But this is impossible since x lies in $A + B$.

Suppose $P - H = \varnothing$ so that $P = H$, $Q = K + B$, and assume that the point of nonaposyndesis x is in B. The continuum Q is not irreducible about $x + K$ or else in M a point of B will cut x from a point of K. Let T be a proper subcontinuum of Q containing $x + K$. By the above argument $Q - T$ is connected. Let the decomposable continuum $\overline{Q - T}$ be written as the sum of continua X and Y. Both X and Y must intersect T or else x is in the interior of a continuum $X + T$ or $Y + T$ that separates M, and hence M is aposyndetic at x. So $X \cdot T \neq \varnothing \neq Y \cdot T$ and therefore each continuum $X + T$ and $Y + T$ separates M. Thus each is aposyndetic, so is the sum Q, and this means that M is aposyndetic at x. Hence x cannot lie in B and must be in A. If A is connected, then $\bar{A}$ separates M and M is aposyndetic at x. On the other hand, if A is not connected, then each point of A is in the interior of some continuum that separates M and hence is in the interior of an aposyndetic continuum. This shows then that M is aposyndetic at each point of $A + B$ and means that the supposition that $H + B + K$ is not connected is false. So $H + B + K$ as well as $H + A + K$ is a continuum.

If $H + A + K$ and $H + B + K$ are both irreducible about $H + K$, then let us show that the upper semi-continuous decomposition H', whose elements are points of A together with the sets H and K, is an arc. To do this let us use the result that if the compact, metric continuum M is irreducible about two of its points a and b such that no point of M (including a and b) cuts any other point of M from $a + b$, then M is an arc [1, Th. 6]. In our case because M contains no cut points, no point of A cuts any other point of A from $H + K$ in $H + A + K$. In addition neither H nor K cuts the other from a point of A in $H + A + K$. This means that the decomposition H' is an arc and since $H + B + K$ is also irreducible about $H + K$, then it can be similarly decomposed into an arc K'. But then M would be aposyndetic at each point of $A + B$ which is impossible. So we can assume that $H + A + K$ or $H + B + K$ is not an irreducible continuum about $H + K$.

Let N be an irreducible subcontinuum of $H + A + K$ about $H + K$ and let p be a point of $A - N$ at which M is aposyndetic. Let q be any point of B at which M is aposyndetic. Now M is connected im kleinen at both p and q. Therefore there exist continua P and Q such that $p \in P^\circ \subset P$, $q \in Q^\circ \subset Q$, $P \subset A - (N + x)$ and $Q \subset B - x$. By the above argument $M - (P + Q) = C + D$, a separation of M, where $P + C + Q$ and $P + D + Q$ are continua. Since $N \cdot (P + Q) = \emptyset$, N lies in C. So $(P + D + Q) \cdot N = \emptyset$ and therefore $P + D + Q \subset A + B$. This is impossible since $p \in P \cdot A$ and $q \in Q \cdot B$. Thus the assumption that M contains a point at which M is not aposyndetic has led to a contradiction and the proof is complete.

COROLLARY (*Bing*). *If the compact, metric continuum M is separated by no subcontinuum and cut by no point, then M is a simple closed curve.*

This follows easily as an application of Theorem 3.

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DECOMPOSITIONS OF INJECTIVE MODULES

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The main results of this paper concern decompositions of an injective module, either as a direct sum of submodules or as the injective envelope of a direct sum of injective submodules. This second kind of decomposition can be regarded as an ordinary direct sum (coproduct) in a suitable Abelian category—the spectral category of the ring. The results are therefore put in the context of Abelian categories, and the main result is that in an Abelian category satisfying axiom Ab-5 and with infinite direct sums, any two direct sum decompositions of an injective object have isomorphic refinements.

This is particularly strong if decompositions into indecomposable injectives exist, and it enables one to classify the injective modules over a valuation ring. Such strong results as this are not available for more general classes of modules, but in § 3 the methods of Crawley and Jónsson are exploited to obtain results in certain cases; for example, for modules which are direct sums of countably generated modules. The Crawley-Jónsson results are put into the context of category theory and an example is given (involving relatively injective modules) to show how the hypotheses can be weakened by working in a subcategory of the category of R-modules.

A remark should be made on the types of decompositions we consider for injective modules in § 2. For injective modules over Noetherian rings, ordinary direct sums yield excellent results, due primarily to Matlis [7]. In contrast, Faith and Walker [2] have shown that if R is a non-Noetherian ring, there does not exist any set of injective modules such that any injective module can be imbedded in a direct sum of modules isomorphic to members of this set. In the spectral category, however, reasonable decompositions always exist (Theorem 2 below). The spectral category was introduced by Gabriel and Oberst in [4] and exploited in [10]. The author is indebted to Professor J. E. Roos for pointing out the connection between these two papers and the work reported here.

We do not consider Cartesian product decompositions of injective modules, since product decompositions simply do not have the necessary uniqueness properties. For an example let Q and Z denote the additive groups of rationals and integers, respectively, and $(Q/Z)_p$ the p-primary component of Q/Z. Then

$$\prod_p (Q/Z)_p \cong Q \times \prod_p (Q/Z)_p$$

so that we have two product decompositions of an injective Z-module into indecomposables, and these decompositions are in no sense equivalent.

1. Decompositions in Abelian categories. We will work in Abelian categories satisfying the usual axioms (as in MacLane [6, pp. 248–257]) together with the following three conditions:

(i) The set-theoretic axiom [6, p. 253] that for each object A there is a set of subobjects, such that any subobject is equivalent to a member of this set.

(ii) We assume that arbitrary direct sums (coproducts, cocartesian products) exist.

(iii) We assume the axiom Ab-5 in the following form: if A_i is direct family of subobjects of an object C, and B is a subobject of C, then

$$(\bigcup A_i) \cap B = \bigcup (A_i \cap B) .$$

In clarification of condition (iii), we should remark that if $A_i (i \in I)$ is a family of subobjects of an object C then their injection homomorphisms induce a unique homomorphism from their direct sum (coproduct) into C, and the image of this homomorphism is the *union* (or join) of the A_i, and is denoted $\bigcup A_i$. Similarly, if A and B are subobjects, then $A \cap B$ is the kernel of the natural homomorphism $C \to C/A \oplus C/B$.

DEFINITION. An Abelian category satisfying the conditions (i), (ii), (iii) above will be called a reasonable Abelian category.

In general one can work with direct sums in a reasonable Abelian category just as one would with direct sums of modules. The notion of a decomposition of an object into a direct sum of subobjects, $A = \bigoplus_{i \in I} A_i$, has the obvious interpretation, and two decompositions are *isomorphic* if the summands are isomorphic in pairs. If also $A = \bigoplus_{j \in J} B_j$, we say the second decomposition is a *refinement* of the first if there is a surjective map $\phi \colon J \to I$ such that $B_j \subseteq A_{\phi(j)}$, from which it follows that the induced morphism

$$\bigoplus_{\phi(j)=i} B_j \longrightarrow A_i$$

is an isomorphism. The direct sum of two objects A and B will be written $A \oplus B$, and if A and B are subobjects of C with $A \cap B = 0$, then their join is isomorphic to $A \oplus B$ and will be denoted $A \oplus B$.

Two more remarks should be made: First, if $C = A \oplus B$, and $D \subseteq C$, then $C = D \oplus B$ if and only if $\pi_A \circ \phi_D$ is an isomorphism. (Here π_A is the natural projection $C \to A$, and ϕ_D the natural injection $D \to C$.) Secondly, if $C = A \oplus B$ and $A \subseteq D$ (another subobject), then

A is a summand of D and $D = A \oplus (D \cap B)$.

We recall that a subobject $L \subseteqq A$ is *essential* if for any subobject $S \subseteqq A$, if $S \neq 0$, then $S \cap L \neq 0$. If D is an injective subobject of A and D is essential, then $D = A$.

LEMMA 1. *If $A = \bigoplus_{i \in I} A_i$ (in any reasonable Abelian category) and S is a subobject of A, then S is essential in A if and only if $S \cap A_i$ is essential in each A_i.*

Proof. That the condition is necessary is clear. Conversely, suppose that $B \neq 0$ is a subobject of A. By (iii), there is a finite subset $I^* \subseteqq I$ with $B \cap (\bigoplus_{i \in I^*} A_i) \neq 0$. Therefore, to show that our condition implies S is essential, we need only show that S is essential in finite subsums. By iteration, we need only show that if $S \subseteqq C_1 \oplus C_2$ and $S \cap C_i$ is essential in C_i, then S is essential in $C_1 \oplus C_2$. Let $B \neq 0$ be a subobject of $C_1 \oplus C_2$ and let π_1 and π_2 be the projections to C_1 and C_2. Let $\phi: B \rightarrow C_1$ be the restriction of π_1 to B. If $B \subseteqq C_2$ then clearly $B \cap S \neq 0$, so we assume this is not the case, in which case $\operatorname{im}(\phi) \neq 0$. Hence $\operatorname{im}(\phi) \cap S \neq 0$, so we let

$$B' = \phi^{-1}(\operatorname{im}(\phi) \cap S) .$$

Since $B' \neq 0$, we need only show $B' \cap S \neq 0$. Let ψ be the restriction of π_2 to B'. By the same argument as before, we may assume that B' is not contained in C_1, so that $\operatorname{im}(\psi) \neq 0$. We let

$$B'' = \psi^{-1}(\operatorname{im}(\psi) \cap S)$$

and it is clear that $B'' \neq 0, B'' \subseteqq S$, so that $B \cap S \neq 0$ as desired.

LEMMA 2. *(The exchange property) If M is an object in a reasonable Abelian category and D is an injective subobject, and $M = \bigoplus_{i \in I} M_i$, then there subobjects $M_i' \subseteqq M_i$, so that $M = D \oplus (\bigoplus_{i \in I} M_i')$.*

Proof. Let $S \subseteqq M$ be a subobject chosen maximal with respect to the following properties:

(1) $S = \bigoplus_{i \in I} S \cap M_i$

(2) $S \cap D = 0$.

To show such an S exists note that (i) enables us to apply Zorn's lemma and (iii) guarantees that an ascending union of subobjects satisfying (2) still satisfies (2). Let $M_i' = S \cap M_i$. Then we claim $M = D \oplus (\bigoplus_{i \in I} M_i')$.

Let ϕ be the natural map from M to M/S, and let ψ be the restriction of ϕ to D. Condition (2) above implies that ψ is a monomorphism, so $\operatorname{im}(\psi)$ is an injective subobject of M/S. We need only show that $\operatorname{im}(\psi)$ is essential, which is an easy consequence of Lemma 1.

LEMMA 3. *Let D be an injective object in a reasonable Abelian category, and A a subobject of D. Then there is an injective subobject E of D with $A \subseteqq E$, and A essential in E.*

Proof. By axiom (i), we can choose subobjects E and S of D such that E is maximal with respect to the property that A is essential in E, and S is maximal with respect to the property that $S \cap A = 0$. Let $\phi: D \to D/S$ be the natural homomorphism. If ϕ' is the restriction of ϕ to E, then ϕ' is a monomorphism. We therefore have a homomorphism carrying $\operatorname{im}(\phi')$ back to E, and since D is injective, this extends to a homomorphism $\psi: D/S \to D$. Since $E \subseteqq \operatorname{im}(\psi)$ and A is essential in $\operatorname{im}(\psi)$, we have $E = \operatorname{im}(\psi)$. E is therefore a summand of D, with projection $\psi \circ \phi$, and hence E is injective.

THEOREM 1. *If D is an injective object in a reasonable Abelian category, then any two direct sum decompositions of D have isomorphic refinements.*

Proof. We will consider two decompositions of D and we assume the summands are well ordered, so that we can take ordinal numbers as our indices and write

$$D = \bigoplus_{i<N} A_i = \bigoplus_{j<M} B_j \ .$$

We will construct subobjects $C_{ij} (i < N, j < M)$ of D, such that $D = \bigoplus_{ij} C_{ij}$, $A_i \cong \bigoplus_{j<M} C_{ij}$, $B_j \cong \bigoplus_{i<N} C_{ij}$. The construction will be carried out by induction on pairs of indices (n, j). For each pair (n, j) we will want the following statements to hold:

$1(n, j)$. For $i < n$ there are subobjects $A_{ij} \subseteqq A_i$ and for $i < n$, $k < j$, the subobjects C_{ik} have been constructed.

$2(n, j)$. $\bigoplus_{i<n} A_i = (\bigoplus_{i<n} A_{ij}) \oplus (\bigoplus_{i<n} \bigoplus_{k<j} C_{ik})$.

$3(n, j)$. $(\bigoplus_{k<j} B_k) \cap (\bigoplus_{i<n} \bigoplus_{k<j} C_{ik})$ is essential in both $(\bigoplus_{k<j} B_k) \cap (\bigoplus_{i<n} A_i)$ and $\bigoplus_{i<n} \bigoplus_{k<j} C_{ik}$.

Suppose that the C_{ij} have been constructed so that all of the statements $1(n, j)$, $2(n, j)$, $3(n, j)$ hold for $n \leqq N$, $j \leqq M$. Then $2(N, M)$ and $3(N, M)$ together imply that

$$D = \bigoplus_{i<N} \bigoplus_{j<M} C_{ij}$$

and hence by $2(n, M)$ and $2(n + 1, M)$

$$\bigoplus_{i<n} A_i = \bigoplus_{i<n} \bigoplus_{j<M} C_{ij}$$
$$\bigoplus_{i<n+1} A_i = \bigoplus_{i<n+1} \bigoplus_{j<M} C_{ij}$$

and comparing these two statements, we conclude that

$$A_n \cong \bigoplus_{j<M} C_{nj}$$

since two complements to the same summand are isomorphic.

We now need to note that if D is injective and A, B are summands of D with $A \cap B$ essential in both A and B, and if $D = A \oplus C$, then $D = B \oplus C$ also, and in particular, $A \cong B$. To apply this, we note that for any index j,

$$(\bigoplus_{k<j} B_k) \cap (\bigoplus_{k<j} \bigoplus_{i<N} C_{ik})$$

is essential in both, so that the first summand may be replaced by the second. Doing this, and then applying the same remark for the index $j+1$, we obtain the following expressions

$$\begin{aligned} D &= (\bigoplus_{k<j} \bigoplus_{i<N} C_{ik}) \oplus B_j \oplus (\bigoplus_{j<k} B_k) \\ &= (\bigoplus_{k<j} \bigoplus_{i<N} C_{ik}) \oplus (\bigoplus_{i<N} C_{ij}) \oplus (\bigoplus_{j<k} B_k) . \end{aligned}$$

Comparing middle terms, we obtain as before

$$B_j \cong \bigoplus_{i<N} C_{ij}$$

as desired.

We now complete the proof by carrying out the construction of the subobjects C_{ij} in the required way. We first use induction on the index j. The construction is completed for the index j if subobjects A_{ij} have been constructed for all $i < N$, and subobjects C_{ik} for $i < N$, $k < j$, so that the statements $1(i, k), 2(i, k), 3(i, k)$ holds for all $i \leqq N$, $k \leqq j$. Suppose, now, that the construction has been completed for all indices $k, k < j$, and that j is a limit ordinal. Conditions $3(N, k)$ and $3(N, k+1)$ show that C_{ik} is isomorphic to a summand of B_k, so $\bigoplus_{k<j} C_{ik}$ is isomorphic to a summand of D, and hence is injective. $2(i, k)$ and $2(i+1, k)$ imply that the projection of $\bigoplus_{k<j} C_{ik}$ into A_i is a monomorphism, so $\bigoplus_{k<j} C_{ik}$ is isomorphic to a summand of A_i. Summing over i, we find that $\bigoplus_{i<n} \bigoplus_{k<j} C_{ik}$ is isomorphic to a summand of D (for all $n < N$) and hence is also injective. We can therefore apply the exchange property (Lemma 2) for this subobject and obtain subobjects $A_{ij} (i < N)$ such that

$$D = (\bigoplus_{i<N} \bigoplus_{k<j} C_{ik}) \oplus (\bigoplus_{i<N} A_{ij}) .$$

This is not quite good enough, but if we do this inductively for each index n in turn, we can also guarantee that

$$\bigoplus_{i<n} A_i = (\bigoplus_{i<n} \bigoplus_{k<j} C_{ik}) \oplus (\bigoplus_{i<n} A_{ij})$$

for all $n < N$, so that for all $n < N, 2(n, j)$ will hold. Condition

$3(n, j)$ is immediate (since if K_i is an ascending family of subobjects of an object K and L_i is an ascending family of subobjects with $L_i \subseteqq K_i$, L_i essential in K_i, then axiom (iii) implies that $\bigcup L_i$ is essential in $\bigcup K_i$). This completes the induction at a limit ordinal.

To complete the proof, we must show that if the constuction has been completed for an index j, it can be completed for $j+1$. We do this by induction on n, establishing $1(n, j+1), 2(n, j+1), 3(n, j+1)$. Suppose that n is a limit ordinal, so that the $A_{i,j+1}$ are constructed for all $i < n$ and C_{ik} for $i < n, k < j+1$. Applying $2(i, j+1)$ for all $i < n$ and taking an ascending union we see that $2(n, j+1)$ is immediate, and $3(n, j+1)$ is also immediate because (as we saw at the end of the previous paragraph) ascending unions preserve "essentialness".

We must show, finally, that if we can carry out our construction so that $1(n, j+1), 2(n, j+1), 3(n, j+1)$ (and also $1(n+1, j), 2(n+1, j), 3(n+1, j)$) hold, then we can construct $A_{n,j+1}, C_{n,j}$ so that $1(n+1, j+1), 2(n+1, j+1), 3(n+1, j+1)$ also hold. We have

$$\bigoplus_{i<n} A_i = (\bigoplus_{i<n} A_{i,j+1}) \oplus (\bigoplus_{i<n} \bigoplus_{k<j+1} C_{ik})$$

and the equation remains true if we add A_n on the left and

$$A_{nj} \oplus (\bigoplus_{k<j} C_{nk})$$

on the right. We choose C_{nj} in $\bigoplus_{i<n+1} A_i$ to be maximal with respect to the properties that

(1) $C_{nj} \cap [(\bigoplus_{i<n} A_i) \oplus (\bigoplus_{k<j} C_{nk})] = 0$

(2) $(\bigoplus_{i<n+1} \bigoplus_{k<j+1} C_{ik}) \cap (\bigoplus_{k<j+1} B_k)$

is essential in $\bigoplus_{i<n+1} \bigoplus_{k<j+1} C_{ik}$. It is clear from Lemma 3 that C_{nj} is injective. $3(n+1, j+1)$ is clearly satisfied, and applying the exchange property (for the summand $(\bigoplus_{i<n} A_i) \oplus (\bigoplus_{k<j+1} C_{nk})$ in $\bigoplus_{i<n+1} A_i$) we can find a complement $A_{n,j+1}$ so that $2(n+1, j+1)$ also holds, thus completing the induction.

2. Applications.

COROLLARY 1.1. *Any two direct sum decompositions of an injective R-module have isomorphic refinements.*

To obtain more useful results, we consider another sort of decomposition.

THEOREM 2. *Let R be an associative ring with 1. Then any injective R-module is the injective envelope of a direct sum of injective submodules isomorphic to $E(R/I)$ (for varying I) where R/I is a cyclic left R-module and $E(R/I)$ denotes the injective envelope of R/I.*

Proof. For injective envelopes see [6, p. 102]. We choose a subset S of D such that

(1) the elements of S are nonzero,

(2) the elements of S are independent (that is, the submodule generated by S is the direct sum of the cyclic modules $[x]$, generated by the elements x in S),

(3) S is maximal with respect to properties (1) and (2). We now let B be an injective envelope in D of the submodule generated by S. B is the injective envelope of a direct sum of cyclic modules, and also, by breaking the process into two stages, the injective envelope of a direct sum of injective submodules of the form $E(R/I)$. It is easy to see that $B = D$.

THEOREM 3. *The following conditions on an associative ring R with 1 are equivalent:*

(i) *If I is a left ideal of R, then either I is irreducible or there are left ideals A, B, different from I, such that A is irreducibe and $I = A \cap B$.*

(ii) *Any injective left R-module has a nonzero indecomposable summand.*

(iii) *Any injective left R-module is the injective envelope of a direct sum of indecomposable injective R-modules.*

Proof. Much of this is due to Matlis [7]. Suppose that E is the injective envelope of a cyclic submodule $[x]$ and $E = E_1 \oplus E_2$ where both E_1 and E_2 are nonzero. If in this decomposition $x = x_1 + x_2$, then it is easy to see that E_i is the injective envelope of $[x_i]$ and $o(x) = o(x_1) \cap o(x_2)$, where $o(x)$, and $o(x_i)$ are the order ideals of x and x_i respectively. Further, we see that $o(x) \neq o(x_i)$ for either i. We conclude that $E(R/I)$ is indecomposable if and only if I is an irreducible ideal, and $E(R/I)$ has an indecomposable summand if and only if I satisfies the conclusion of condition (i) above. Since any injective R-module has a summand of the form $E(R/I)$ for some left ideal I, the equivalence of (i) and (ii) is now clear. Clearly (iii) implies (ii) and the proof of the converse is essentially the same as the proof of Theorem 2.

DEFINITION. Let R be an associative ring with unit. $\mathscr{I}(R)$ is the category whose objects are injective R-modules with morphisms defined by

$$\mathrm{Mor}_{\mathscr{I}}(A, B) = \mathrm{Hom}(A, B)/\mathrm{Hom}_0(A, B)$$

where $\mathrm{Hom}_0(A, B)$ is the subgroup of R-homomorphisms whose kernel is essential in A.

THEOREM 4. *$\mathscr{I}(R)$ is a reasonable Abelian category in which all short exact sequences split. Elements of $\mathscr{I}(R)$ are isomorphic in $\mathscr{I}(R)$ if and only if they are isomorphic as modules, and if $A_i (i \in I)$ is a family of injective modules, then their direct sum in the category $\mathscr{I}(R)$ can be identified with the injective envelope of their module direct sum.*

Proof. Most of the proof consists of trivial verifications which will be omitted. We take it as obvious that $\mathscr{I}(R)$ is an additive category satisfying the set theoretic axiom (i). The direct sum of two objects in $\mathscr{I}(R)$ in just their direct sum as R-modules. If $f: A \rightarrow B$, $g: B \rightarrow A$ establish an isomorphism in $\mathscr{I}(R)$ between A and B then $g \circ f$ restricts to the identity function on some essential submodule of A so $g \circ f$ is an automorphism of A, and similarly $f \circ g$ is an automorphism of B, so A and B are isomorphic as modules.

Let us identify the kernels and cokernels. Let $f \in \mathrm{Hom}(A, B)$ and let $[f]$ denote the correspoding element of $\mathrm{Mor}_{\mathscr{I}}(A, B)$. If K is the kernel of f, and E is an injective envelope of K in A, then E is a kernel for $[f]$. We can write $A = E \oplus F$ (in $\mathscr{I}(R)$ or as R-modules). Let π be the projection of A onto F and f' the element of $\mathrm{Hom}(F, B)$ induced by f. Then $[f] = [f'] \circ [\pi]$ since f and $f' \circ \pi$ agree on the essential submodule $K + F$. This is the factorization of $[f]$ into the product of an epimorphism and a monomorphism required in an Abelian category. Finally, $f'(F)$ is a summand of B so $B/f'(F)$ is injective, and if ϕ is the natural homomorphism from B to $B/f'(F)$ then $[\phi]$ is a cokernel for $[f]$. This shows that $\mathscr{I}(R)$ is an Abelian category, so all that remains is to check conditions (ii) and (iii).

The statement on direct sums is a consequence of Lemma 1. For (iii), note that if we have a directed family of $\mathscr{I}(R)$-subobjects of an injective module C, we can choose representative submodules $A_i (i \in I)$ for these subobjects so that the family A_i is a directed family in the usual sense. $\bigcup A_i$ (in the category $\mathscr{I}(R)$) can be identified with any injective envelope of the ordinary union of the A_i. Similarly, $A \cap B$ (in the category $\mathscr{I}(R)$) can be identified with any injective envelope of the ordinary intersection. Both of these are well defined in $\mathscr{I}(R)$. To prove that

$$\left(\bigcup A_i\right) \cap B = \bigcup (A_i \cap B)$$

in the category $\mathscr{I}(R)$, note that the term on the left is any injective envelope of $B \cap K$ K where is an injective envelope of the ordinary union of the A_i. If this ordinary union is denoted by A, then since A is essential in K, the term on the left represents an injective envelope of $A \cap B$. Similarly, the term on the right represents an injective envelope of $A \cap B$, from which the result follows.

We should remark that this theorem carries over to any reasonable Abelian category in which injective envelopes exist.

COROLLARY 4.1. *Any two representations of an injective module as the injective envelope of a direct sum of injective submodules have isomorphic refinements.*

COROLLARY 4.2. *Let M be an injective module which is the injective envelope of a direct sum of indecomposable injective submodules $E_i (i \in I)$. Then any two such decompositions are isomorphic, and furthermore, if N is an injective submodule of M, there is a subset $J \subseteqq I$ such that $M = N \oplus E(\bigoplus_{i \in J} E_i)$.*

The second half of this corollary follows from Lemma 2, in the category $\mathscr{I}(R)$. Theorem 3 gives conditions to which this corollary applies. Another such condition, in terms of transfinite Krull dimension, is given by Gabriel [3, pp. 382, 386]. A similar result in [10].

We recall that a commutative ring R is a valuation ring if it is an integral domain and for any two nonzero elements r and s of R, either r divides s or s divides r. It follows that if I and J are ideals, either $I \subseteqq J$ or $J \subseteqq I$. Hence any ideal is irreducible, and it follows that the injective envelope of a cyclic module, $E(R/I)$, is always indecomposable. One can show further [8] that $E(R/I) \cong E(R/J)$ if and only if there are nonzero elements r, s of R such that $rI = sJ$, (or equivalently, I and J are isomorphic as modules). Applying Theorem 2 and Corollary 4.2, we obtain the following.

COROLLARY 4.3. *An injective module over a valuation ring is the injective envelope of a direct sum of indecomposable injective modules, and any two such decompositions are isomorphic. An injective module is indecomposable if and only if it is of the form $E(R/I)$, and $E(R/I) \cong E(R/J)$ if and only if $I \cong J$.*

Other consequences, not directly involving injective modules, also follow from these results. For the following, let R be a commutative integral domain. We recall that a torsion-free module is reduced if it has no nonzero injective summand, or equivalently if no nonzero element is divisible by all elements of R. If A is a submodule of a module B, A is RD-pure in B if for all $r \in R$, $rA = A \cap rB$. (RD here tands for"relatively divisible".) A module M is RD-injective if it is a summand of any module which contains it as an RD-pure submodule. By [12, Corollary 2] the functor $\mathrm{Hom}(Q/R, \cdot)$ gives a category isomorphism between the category of torsion, injective R-modules and the category of reduced, torsion-free RD-injective R-modules. (Here, Q is the quotient field of R. This result is a corollary of the category isomorphism theorem of Matlis [9, Th. 3.4].) We also have a notion of RD-injective envelope for this theory, and we can actually write down an explicit formula. If A is a reduced torsion-free module, its RD-injective envelope is $\mathrm{Hom}(Q/R, E(A \otimes (Q/R)))$, where $E(M)$ is the ordinary injective envelope of M. All of the previous results for injective modules now carry over because of the category isomorphism theorem mentioned above, but we content ourselves with a version of Corollaries 4.2 and 4.3.

COROLLARY 4.4. *If M is a reduced torsion-free RD-injective module over an integral domain, and if M is the RD-injective envelope of a direct sum of indecomposable RD-injective modules, then any two such decompositions of M are isomorphic. If the domain is a valuation ring, any reduced torsion-free RD-injective module is the RD-injective envelope of a direct sum of ideals, and any two such representations are isomorphic.*

The only additional remark needed to complete the proof of this is that if R is a valuation ring, the RD-injective envelope of an ideal $I, I \neq R$, is $\mathrm{Hom}(Q/R, E(R/I))$ since R/I is essential in Q/I, and there is a natural isomorphism $Q/I \cong (Q/R) \otimes I$.

3. The Crawley-Jónsson theorems. We wish to review here some important results on direct sum decompositions due to Crawley and Jónsson [1] and to place them in the context of Abelian categories. We should remark that Crawley and Jónsson work with general algebraic systems, and their results are valid in many categories that are not even additive, so that our results do not contain theirs. Our proofs are valid in somewhat more general categories than reasonable Abelian categories, however—in particular in any full subcategory which is closed under summands and direct sums (for example, in the category of torsion-free Abelian groups), and the hypotheses are often weakened by restricting to a subcategory.

DEFINITION. An object D in an Abelian category has the exchange property if for any object A, if we have

$$A = D' \oplus B = \bigoplus_{i \in I} A_i$$

with $D' \cong D$, then there are subobjects $A'_i \subseteqq A_i$ such that

$$A = D' \oplus (\bigoplus_{i \in I} A'_i) .$$

Similarly, D has the finite exchange property if this conditions holds whenever the set I is finite.

THEOREM 5. [1, Th. 4.2]. *If M is an object in a reasonable Abelian category and $M = \bigoplus_{i \in I} A_i = \bigoplus_{j \in J} B_j$ where the sets I and J are countable and the subobjects A_i and B_j have the exchange property, then these two decompositions have isomorphic refinements.*

The proof is a diagonal argument and we refer to [1, pp. 817–818] for details. The countability hypothesis seems to be essential. It can be removed, however, by placing a countability hypothesis on the summands. Crawley and Jónsson therefore assume their summands are countably generated, and the following definition provides a substitute for this in a general setting.

DEFINITION. An object D in an additive category is small if for any direct sum $A = \bigoplus_{i \in I} A_i$, with projections π_i, and any morphism $f: D \to A$, we have $\pi_i \circ f = 0$ for all but a finite number of indices i. D is σ-small if it is a countable ascending union of small subjects. D is countably small if for any direct sum $A = \bigoplus_{i \in I} A_i$, and $f: D \to A$, we have $\pi_i \circ f = 0$ for all but a countable number of indices i.

LEMMA 4. *Let N be a summand of an object M in a reasonable Abelian category such that M is the direct sum of countably small subobjects. Then N is also a direct sum of countably small subobjects.*

This is essentially equivalent to [5, Th. 1]. σ-small can be substituted for countably small, and suitable versions for larger cardinals also are valid.

LEMMA 5. *Let M be an object in a reasonable Abelian category and $M = \bigoplus_{i \in I} A_i = \bigoplus_{j \in J} B_j$, where each of the summands is countably small. Then we can decompose I and J into disjoint, countable subsets I_λ, $J_\lambda (\lambda \in \Lambda)$, such that*

$$\bigoplus_{i \in I_\lambda} A_i \cong \bigoplus_{j \in J_\lambda} B_j \, (\lambda \in \Lambda) .$$

Proof. We outline the proof, which is a straightforward elementary argument. One proceeds by transfinite induction, and the resulting set Λ is a set of ordinal numbers. One first proceeds by induction on ordinals λ, the induction hypothesis being that for $n < \lambda$, the following holds: for each $k < n$, the sets I_k, J_k are defined, so that

$$\bigoplus_{k<n} (\bigoplus_{i \in I_k} A_i) = \bigoplus_{k<n} (\bigoplus_{j \in J_k} B_i)$$

and the sets I_k, J_k are disjoint, countable subsets of I and J respectively. We conclude that there is an ordinal λ such that I is the union of the sets I_k for $k < \lambda$, and J is the union of sets J_k for $k < \lambda$. For any $k < \lambda$, we apply the induction formula for $n = k$ and $n = k + 1$ and obtain

$$\bigoplus_{i \in I_k} A_i \cong \bigoplus_{j \in J_k} B_j$$

as desired, since both are complementary summads to $\bigoplus_{m<k} (\bigoplus_{i \in I_k} A_i)$ in $\bigoplus_{m<k+1} (\bigoplus_{i \in I_k} A_i)$.

THEOREM 6. *If M is an object in a reasonable Abelian category and M is the direct sum of countably small subobjects, then any two direct sum decompositions of M into summands having the exchange property have isomorphic refinements.*

Proof. By Lemma 4, any decomposition refines into one in which the summands are countably small. Since a summand of an object with the exchange property again has the exchange property, one may assume that all summands involved are countably small. By Lemma 5, we may then assume that the index set is countable, and in this case the result follows from Theorem 5.

THEOREM 7. [1, Th. 7.1]. *Let M be an object in a reasonable Abelian category which is the direct sum of σ-small subobjects having the exchange property. Then any two direct sum decompositions of M have isomorphic refinements.*

Proof. By Theorem 6, it suffices to show that if N is a summand of M, then N is also a direct sum of σ-small subobjects having the exchange property. By Lemma 5, it suffices to prove this in the case where $M = \bigoplus_{i=1}^{\infty} A_i$ and each A_i is σ-small, in which case N is also σ-small. We can therefore find subobjects $S_i (i = 0, 1, \cdots)$ of N with S_i small, $S_0 = 0$, $S_{i+1} \supseteqq S_i$, and such that N is the union of the S_i. We proceed by induction on k, choosing for each k a subobject N_k, beginning with $N_0 = 0$. We assume by induction that the N_i are independent, that $\bigoplus_{i=1}^{k} N_i$ is a summand of N, that $S_k \subseteqq \bigoplus_{i=1}^{k} N_i$, and that each N_k has the exchange property. Clearly if we can carry out this construction, the theorem is proved, since $N = \bigoplus_{1=i}^{\infty} N_i$.

By the exchange property for $\bigoplus_{i=1}^{k-1} N_i$, there are submodules

$A'_i \subseteq A_i$ such that

$$M = (\bigoplus_{i=1}^{k-1} N_i) \oplus (\bigoplus_{i=1}^{\infty} A'_i) .$$

Choose $n(k)$ such that

$$S_k \subseteq (\bigoplus_{i=1}^{k-1} N_i) \oplus (\bigoplus_{i=1}^{n(k)} A'_i) .$$

Applying the exchange property to the object on the right, assuming that $M = N \oplus B$, we obtain complementary subobjects $N_k^* \subseteq N$, $B_k \subseteq B$. Let N'_k be the intersection of N with

$$(\bigoplus_{i=1}^{k-1} N_i) \oplus (\bigoplus_{i=1}^{n(k)} A'_i) \oplus B_k .$$

Clearly, $S_k \subseteq N'_k$. Let N_k be a complement to $\bigoplus_{i=1}^{k-1} N_i$ in N'_k. Since N_k is isomorphic to a summand of $\bigoplus_{i=1}^{n(k)} A'_i$, N_k has the exchange property, and since $N = N'_k \oplus N_k^*$, the induction is completed.

COROLLARY 7.1. *Let M be an R-module which is a direct sum of countably generated injective modules. Then any summand of M is a direct sum of injective modules and any two direct sum decompositions of M have isomorphic refinements.*

In the case where M is a direct sum of countably generated indecomposable injective modules, this is contained in results of Faith and Walker [2].

To give another example, we return to our earlier remark that the above proofs are valid in any full subcategory of a reasonable Abelian category which is closed under summands and direct sums. We apply this to the category of torsion-free reduced modules over an integral domain.

COROLLARY 7.2. *If a reduced torsion-free module M over an integral domain is a direct sum of RD-injective modules, then any two direct sum decompositions of M have isomorphic refinements.*

Proof. It is clear that a torsion-free RD-injective module is algebraically compact in the sense of [11]. Algebraically compact modules have very strong completeness properties which make it easy to check that a reduced torsion-free RD-injective module is small in the category of reduced torsion-free modules. The result will follow if we can prove the exchage property for such modules. By [1, Th. 8.2] (or by an elementary argument) we may assume that all of the summands involved are torsion-free and reduced, and using the smallness of the RD-injective modules, we may assume that the total number of summands involved is finite. Since the operation of taking RD-injective envelopes preserves finite direct sums, we may assume that all of the modules involved are RD-injective. Using the category isomorphism theorem mentioned in connection with Corollary 4.4, the result now follows since injective modules have the exchange property.

4. Some unsolved problems. It would be nice to weaken or remove the countability requirements in § 3. In particular, it would be nice to weaken the hypotheses of Theorem 7 and make them agree with those of Theorem 6. By analogy with Theorem 1, one might hope to remove all such hypotheses by assuming that the object being decomposed also has the exchange property.

One would like to prove theorems similar to Corollary 4.4 for other classes of modules defined by relative injectivity properties similar to that defining RD-injective modules. One theorem in this direction which does not follow from our methods is the classification theorem for complete Abelian groups. A reduced Abelian group is RD-injective (or algebraically compact) if and only if it is complete and Hausdorff in its Z-adic topology. Any such group is the completion of a direct sum of indecomposable complete groups, and any two such decompositions are isomorphic. The indecomposable complete groups are just the cyclic groups of prime power order and the additive groups of p-adic integers. A suitable generalization of the results of § 3 might include similar theorems for modules over other rings.

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