

Pacific Journal of Mathematics

QUASI-INJECTIVE MODULES AND STABLE TORSION CLASSES

EFRAIM PACILLAS ARMENDARIZ

QUASI-INJECTIVE MODULES AND STABLE TORSION CLASSES

E. P. ARMENDARIZ

In this note we examine the \mathcal{T} -torsion submodule of quasi-injective R -modules, R a ring with unit, where \mathcal{T} is a torsion class in the sense of S. E. Dickson. We show that for a stable torsion class \mathcal{T} , the \mathcal{T} -torsion submodule of any quasi-injective module is a direct summand, while if \mathcal{T} contains all Goldie-torsion modules, then every epimorphic image of a quasi-injective module has its \mathcal{T} -torsion submodule as a direct summand. In addition, we show that for a stable torsion class \mathcal{T} , all \mathcal{T} -torsion-free modules are injective if and only if $R = T(R) \oplus K$ (ring direct sum), with K Artinian semisimple.

All R -modules will be unitary left R -modules. Originally our results were obtained for torsion classes closed under submodules. However, the referee has kindly pointed out how this assumption can be omitted throughout, supplying a proof in the case of Theorem 2.3. We take this opportunity to express our gratitude.

1. Following S. E. Dickson [2], a class \mathcal{T} ($\neq \emptyset$) of R -modules is a *torsion class* if \mathcal{T} is closed under factors, extensions, and arbitrary direct sums. The torsion class \mathcal{T} is *stable* if \mathcal{T} is closed under essential extensions. Every torsion class \mathcal{T} determines in every R -module A a unique maximal \mathcal{T} -submodule $T(A)$, the *\mathcal{T} -torsion submodule* of A , and $T(A/T(A)) = 0$, i.e., $A/T(A)$ is *\mathcal{T} -torsion-free*. The R -module A *splits* if $T(A)$ is a direct summand of A . For further properties of torsion classes the reader is referred to [6].

The class \mathcal{G} of Goldie-torsion modules is the smallest torsion class containing all factor modules A/B where B is essential in A , and their isomorphic copies. As shown in [1], $\mathcal{G} = \{A \mid Z_2(A) = A\}$ where $Z_1(A) =$ singular submodule of A and $Z_2(A)/Z_1(A) =$ singular submodule of $A/Z_1(A)$ (see also [4]).

An R -module A is *quasi-injective* provided every homomorphism from any submodule of A into A can be extended to an endomorphism of A . For any R -module A , $E(A)$ will denote the injective envelope of A .

2. The proof of the following is straightforward and so will be omitted.

PROPOSITION 2.1. *If the torsion class \mathcal{T} is stable then every*

injective R -module splits. If \mathcal{T} is closed under submodules, the converse holds and either condition is equivalent to $T(E(A)) = E(T(A))$ for all R -modules A .

The next lemma can be found in [5, Proposition 2.3].

LEMMA 2.2. *If A is a quasi-injective R -module and $E(A) = M \oplus N$ then $A = (M \cap A) \oplus (N \cap A)$.*

We now have

THEOREM 2.3. *Let \mathcal{T} be a stable torsion class. Then every quasi-injective R -module A splits, $A = T(A) \oplus N$ where N is quasi-injective and \mathcal{T} -torsion-free.*

Proof. Choose a submodule N of A maximal with respect to $T(A) \cap N = 0$. Then $E(A) = E(T(A)) \oplus E(N)$, hence by Lemma 2.2, $A = A \cap E(T(A)) \oplus A \cap E(N)$. Since \mathcal{T} is stable $A \cap E(T(A)) = T(A)$ and hence $A = T(A) \oplus N$ with $N = A \cap E(N)$ quasi-injective and \mathcal{T} -torsion-free.

Since the class \mathcal{G} of Goldie-torsion modules is stable, it follows that $G(A)$ is a direct summand of A whenever A is quasi-injective; this was obtained by M. Harada in [5, Th. 1.7].

Let \mathcal{T} be a torsion class; a submodule B of an R -module A is \mathcal{T} -closed if $T(A/B) = 0$.

LEMMA 2.4. *Let \mathcal{T} be a torsion class and let B be a \mathcal{T} -closed submodule of the R -module A . If M is any R -module and $f \in \text{Hom}_R(M, A)$ then $N = f^{-1}(B)$ is \mathcal{T} -closed in M .*

Proof. If C/N is a \mathcal{T} -submodule of M/N then $f(C)/B$ is a \mathcal{T} -submodule of A/B . Hence $f(C) \subseteq B$ since B is \mathcal{T} -closed and so N is \mathcal{T} -closed.

If \mathcal{T} is a torsion class containing the class \mathcal{G} , then a \mathcal{T} -closed submodule B of the R -module A has no essential extension in A ; hence if A is quasi-injective then B is a direct summand of A by [3, Corollary 3, p. 24]. Another way of showing this has been suggested by the referee: Choose K maximal in A with respect to $K \cap B = 0$. Then $E(A) = E(K) \oplus E(B)$ so $A = (A \cap E(K)) \oplus (A \cap E(B)) = K \oplus (A \cap E(B))$. Now $A/B \cong K \oplus (A \cap E(B))/B$ and since $\mathcal{G} \subseteq \mathcal{T}$, $A \cap E(B) = B$.

THEOREM 2.5. *If \mathcal{T} is a torsion class containing \mathcal{G} , and the R -module A is an epimorphic image of a quasi-injective R -module then A splits.*

Proof. Let M be quasi-injective, $f: M \rightarrow A$ an epimorphism. Then $T(A)$ is a \mathcal{F} -closed submodule of A , hence by Lemma 2.4, $N = f^{-1}(T(A))$ is \mathcal{F} -closed in M . By the previous remark, N is a direct summand of M , say $M = N \oplus P$. Then $f(P) \cap T(A) = 0$ and so $A = T(A) \oplus f(P)$.

We note that the previous theorem is a generalization of [7, Th. 1.1] and the method employed is that of [8, Th. 2.10].

3. In [1, Th. 3.1] it was shown that a ring $R = G(R) \oplus K$ (ring direct sum), where K is semisimple with minimum condition, if and only if all \mathcal{F} -torsion-free modules are injective. In this section we prove this result for any stable torsion class \mathcal{F} .

LEMMA 3.1. *Let $R = S \oplus K$, where S is semisimple with minimum condition. Then any R -module A satisfying $KA = 0$ is an injective R -module.*

Proof. If $KA = 0$ then A is an injective S -module. Let I be a left ideal of R ; then $I = S_1 \oplus K_1$ where $S_1 \subseteq S, K_1 \subseteq K$ are left ideals of R . Also $1 = u + v, u \in S, v \in K$. If $f: I \rightarrow A$ is an R -homomorphism, then for any $b \in K_1, 0 = vf(b) = f(b)$. There is an S -homomorphism $g: S \rightarrow A$ coinciding with f on I , and this yields an R -homomorphism $g^*: R \rightarrow A$ coinciding with f on I if we define $g^*(s + k) = g^*(s)$.

THEOREM 3.2. *Let \mathcal{F} be a stable torsion class. Then all \mathcal{F} -torsion-free R -modules are injective if and only if $R = T(R) \oplus K$, where K is a semisimple ring with minimum condition.*

Proof. Assume A is injective whenever $T(A) = 0$. Since the class \mathcal{F} of \mathcal{F} -torsion-free modules is closed under submodules [2], every submodule of any $A \in \mathcal{F}$ is injective, hence is a direct summand, and so every $A \in \mathcal{F}$ is completely reducible. Let M be any R -module and assume $T(M) \neq 0$. If $T(M)$ is essential in M then $T(M) = M$, since \mathcal{F} is stable. Otherwise select B maximal relative to $B \cap T(M) = 0$. Then $T(B) = 0$ and so B is injective. Thus $M = B \oplus U$. Now $M/U \cong B \in \mathcal{F}$ so that $T(M) \subseteq U$ by [2, Proposition 2.4]. The maximal property of B ensures that $T(M)$ is essential in U and so $U = T(M)$. In particular $R = T(R) \oplus K$, where K is a completely reducible R -module since $K \in \mathcal{F}$. The decomposition is two-sided since right multiplications are R -homomorphisms and both classes \mathcal{F} and \mathcal{F} are closed under factors.

Conversely, assume $R = T(R) \oplus K$, where K is a semi-simple ring with minimum condition. We note that for any R -module A , if $T(A) = 0$ then $T(R)A = 0$. For if $A \in \mathcal{F}$ and $0 \neq a \in A$ then $T(R)a$ is an epimorphic image of $T(R)$ and so $T(R)a \subseteq T(A) = 0$. That every \mathcal{F} -torsion-free module is injective now follows from Lemma 3.1.

We conclude with the following example. Let R be the ring of lower triangular 2×2 matrices over a finite field and let \mathcal{S} be the smallest torsion class containing all projective simple R -modules. Note that \mathcal{S} contains nonzero R -modules since every simple in the socle of R is in \mathcal{S} . Moreover R is a hereditary Artinian ring with $(\text{rad } R)^2 = 0$ so by [9, Theorem B] every nonprojective simple is injective. Since R is not semisimple, it has nonzero \mathcal{S} -torsion-free R -modules. If $T(A) = 0$ for an R -module $A \neq 0$ then $\text{socle}(A)$ contains no projective simples. Since R is Noetherian and $\text{socle}(A)$ is essential in A , A is injective. It is readily verified that $\text{socle}(R) = T(R)$. Thus the condition that T be stable is needed in Theorem 3.2, even when T is closed under submodules.

REFERENCES

1. J. S. Alin and S. E. Dickson *Goldie's torsion theory and its derived functor*, Pacific J. Math. **24** (1968), 195-203.
2. S. E. Dickson, *A torsion theory for Abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 223-235.
3. C. Faith, *Lectures on injective modules and quotient rings*, **49**, Springer-Verlag, Berlin, 1967.
4. A. W. Goldie, *Torsion free modules and rings*, J. Algebra **1** (1964), 268-287.
5. M. Harada, *Note on quasi-injective modules*, Osaka J. Math. **2** (1965), 351-356.
6. J. P. Jans, *Some aspects of torsion*, Pacific J. Math. **15** (1965), 1249-1259.
7. E. Matlis, *Divisible modules*, Proc. Amer. Math. Soc. **11** (1960), 385-391.
8. F. L. Sandomierski, *Semisimple maximal quotient rings*, Trans. Amer. Math. Soc. **128** (1967), 112-120.
9. A. Zaks, *Simple modules and hereditary rings*, Pacific J. Math. **26** (1968), 627-630.

Received October 7, 1968.

UNIVERSITY OF TEXAS
AUSTIN, TEXAS

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

BASIL GORDON
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 31, No. 2

December, 1969

Efraim Pacillas Armendariz, <i>Quasi-injective modules and stable torsion classes</i>	277
J. Adrian (John) Bondy, <i>On Ulam's conjecture for separable graphs</i>	281
Vasily Cateforis and Francis Louis Sandomierski, <i>On commutative rings over which the singular submodule is a direct summand for every module</i>	289
Rafael Van Severen Chacon, <i>Approximation of transformations with continuous spectrum</i>	293
Raymond Frank Dickman and Alan Zame, <i>Functionally compact spaces</i>	303
Ronald George Douglas and Walter Rudin, <i>Approximation by inner functions</i>	313
John Walter Duke, <i>A note on the similarity of matrix and its conjugate transpose</i>	321
Micheal Neal Dyer and Allan John Sieradski, <i>Coverings of mapping spaces</i>	325
Donald Campbell Dykes, <i>Weakly hypercentral subgroups of finite groups</i>	337
Nancy Dykes, <i>Mappings and realcompact spaces</i>	347
Edmund H. Feller and Richard Laham Gantos, <i>Completely injective semigroups</i>	359
Irving Leonard Glicksberg, <i>Semi-square-summable Fourier-Stieltjes transforms</i>	367
Samuel Irving Goldberg and Kentaro Yano, <i>Integrability of almost cosymplectic structures</i>	373
Seymour Haber and Charles Freeman Osgood, <i>On the sum $\sum \langle n\alpha \rangle^{-1}$ and numerical integration</i>	383
Sav Roman Harasymiv, <i>Dilations of rapidly decreasing functions</i>	395
William Leonard Harkness and R. Shantaram, <i>Convergence of a sequence of transformations of distribution functions</i>	403
Herbert Frederick Kreimer, Jr., <i>A note on the outer Galois theory of rings</i>	417
James Donald Kuelbs, <i>Abstract Wiener spaces and applications to analysis</i>	433
Roland Edwin Larson, <i>Minimal T_0-spaces and minimal T_D-spaces</i>	451
A. Meir and Ambikeshwar Sharma, <i>On Ilyeff's conjecture</i>	459
Isaac Namioka and Robert Ralph Phelps, <i>Tensor products of compact convex sets</i>	469
James L. Rovnyak, <i>On the theory of unbounded Toeplitz operators</i>	481
Benjamin L. Schwartz, <i>Infinite self-interchange graphs</i>	497
George Szeto, <i>On the Brauer splitting theorem</i>	505
Takayuki Tamura, <i>Semigroups satisfying identity $xy = f(x, y)$</i>	513
Kenneth Tolo, <i>Factorizable semigroups</i>	523
Mineko Watanabe, <i>On a boundary property of principal functions</i>	537
James Juei-Chin Yeh, <i>Singularity of Gaussian measures in function spaces with factorable covariance functions</i>	547