

# Pacific Journal of Mathematics

## **QUASI-INJECTIVE MODULES AND STABLE TORSION CLASSES**

EFRAIM PACILLAS ARMENDARIZ

## QUASI-INJECTIVE MODULES AND STABLE TORSION CLASSES

E. P. ARMENDARIZ

In this note we examine the  $\mathcal{T}$ -torsion submodule of quasi-injective  $R$ -modules,  $R$  a ring with unit, where  $\mathcal{T}$  is a torsion class in the sense of S. E. Dickson. We show that for a stable torsion class  $\mathcal{T}$ , the  $\mathcal{T}$ -torsion submodule of any quasi-injective module is a direct summand, while if  $\mathcal{T}$  contains all Goldie-torsion modules, then every epimorphic image of a quasi-injective module has its  $\mathcal{T}$ -torsion submodule as a direct summand. In addition, we show that for a stable torsion class  $\mathcal{T}$ , all  $\mathcal{T}$ -torsion-free modules are injective if and only if  $R = T(R) \oplus K$  (ring direct sum), with  $K$  Artinian semisimple.

All  $R$ -modules will be unitary left  $R$ -modules. Originally our results were obtained for torsion classes closed under submodules. However, the referee has kindly pointed out how this assumption can be omitted throughout, supplying a proof in the case of Theorem 2.3. We take this opportunity to express our gratitude.

1. Following S. E. Dickson [2], a class  $\mathcal{T}$  ( $\neq \emptyset$ ) of  $R$ -modules is a *torsion class* if  $\mathcal{T}$  is closed under factors, extensions, and arbitrary direct sums. The torsion class  $\mathcal{T}$  is *stable* if  $\mathcal{T}$  is closed under essential extensions. Every torsion class  $\mathcal{T}$  determines in every  $R$ -module  $A$  a unique maximal  $\mathcal{T}$ -submodule  $T(A)$ , the  *$\mathcal{T}$ -torsion submodule* of  $A$ , and  $T(A/T(A)) = 0$ , i.e.,  $A/T(A)$  is  *$\mathcal{T}$ -torsion-free*. The  $R$ -module  $A$  *splits* if  $T(A)$  is a direct summand of  $A$ . For further properties of torsion classes the reader is referred to [6].

The class  $\mathcal{G}$  of Goldie-torsion modules is the smallest torsion class containing all factor modules  $A/B$  where  $B$  is essential in  $A$ , and their isomorphic copies. As shown in [1],  $\mathcal{G} = \{A \mid Z_2(A) = A\}$  where  $Z_1(A) =$  singular submodule of  $A$  and  $Z_2(A)/Z_1(A) =$  singular submodule of  $A/Z_1(A)$  (see also [4]).

An  $R$ -module  $A$  is *quasi-injective* provided every homomorphism from any submodule of  $A$  into  $A$  can be extended to an endomorphism of  $A$ . For any  $R$ -module  $A$ ,  $E(A)$  will denote the injective envelope of  $A$ .

2. The proof of the following is straightforward and so will be omitted.

PROPOSITION 2.1. *If the torsion class  $\mathcal{T}$  is stable then every*

*injective  $R$ -module splits. If  $\mathcal{T}$  is closed under submodules, the converse holds and either condition is equivalent to  $T(E(A)) = E(T(A))$  for all  $R$ -modules  $A$ .*

The next lemma can be found in [5, Proposition 2.3].

**LEMMA 2.2.** *If  $A$  is a quasi-injective  $R$ -module and  $E(A) = M \oplus N$  then  $A = (M \cap A) \oplus (N \cap A)$ .*

We now have

**THEOREM 2.3.** *Let  $\mathcal{T}$  be a stable torsion class. Then every quasi-injective  $R$ -module  $A$  splits,  $A = T(A) \oplus N$  where  $N$  is quasi-injective and  $\mathcal{T}$ -torsion-free.*

*Proof.* Choose a submodule  $N$  of  $A$  maximal with respect to  $T(A) \cap N = 0$ . Then  $E(A) = E(T(A)) \oplus E(N)$ , hence by Lemma 2.2,  $A = A \cap E(T(A)) \oplus A \cap E(N)$ . Since  $\mathcal{T}$  is stable  $A \cap E(T(A)) = T(A)$  and hence  $A = T(A) \oplus N$  with  $N = A \cap E(N)$  quasi-injective and  $\mathcal{T}$ -torsion-free.

Since the class  $\mathcal{G}$  of Goldie-torsion modules is stable, it follows that  $G(A)$  is a direct summand of  $A$  whenever  $A$  is quasi-injective; this was obtained by M. Harada in [5, Th. 1.7].

Let  $\mathcal{T}$  be a torsion class; a submodule  $B$  of an  $R$ -module  $A$  is  $\mathcal{T}$ -closed if  $T(A/B) = 0$ .

**LEMMA 2.4.** *Let  $\mathcal{T}$  be a torsion class and let  $B$  be a  $\mathcal{T}$ -closed submodule of the  $R$ -module  $A$ . If  $M$  is any  $R$ -module and  $f \in \text{Hom}_R(M, A)$  then  $N = f^{-1}(B)$  is  $\mathcal{T}$ -closed in  $M$ .*

*Proof.* If  $C/N$  is a  $\mathcal{T}$ -submodule of  $M/N$  then  $f(C)/B$  is a  $\mathcal{T}$ -submodule of  $A/B$ . Hence  $f(C) \subseteq B$  since  $B$  is  $\mathcal{T}$ -closed and so  $N$  is  $\mathcal{T}$ -closed.

If  $\mathcal{T}$  is a torsion class containing the class  $\mathcal{G}$ , then a  $\mathcal{T}$ -closed submodule  $B$  of the  $R$ -module  $A$  has no essential extension in  $A$ ; hence if  $A$  is quasi-injective then  $B$  is a direct summand of  $A$  by [3, Corollary 3, p. 24]. Another way of showing this has been suggested by the referee: Choose  $K$  maximal in  $A$  with respect to  $K \cap B = 0$ . Then  $E(A) = E(K) \oplus E(B)$  so  $A = (A \cap E(K)) \oplus (A \cap E(B)) = K \oplus (A \cap E(B))$ . Now  $A/B \cong K \oplus (A \cap E(B))/B$  and since  $\mathcal{G} \subseteq \mathcal{T}$ ,  $A \cap E(B) = B$ .

**THEOREM 2.5.** *If  $\mathcal{T}$  is a torsion class containing  $\mathcal{G}$ , and the  $R$ -module  $A$  is an epimorphic image of a quasi-injective  $R$ -module then  $A$  splits.*

*Proof.* Let  $M$  be quasi-injective,  $f: M \rightarrow A$  an epimorphism. Then  $T(A)$  is a  $\mathcal{S}$ -closed submodule of  $A$ , hence by Lemma 2.4,  $N = f^{-1}(T(A))$  is  $\mathcal{S}$ -closed in  $M$ . By the previous remark,  $N$  is a direct summand of  $M$ , say  $M = N \oplus P$ . Then  $f(P) \cap T(A) = 0$  and so  $A = T(A) \oplus f(P)$ .

We note that the previous theorem is a generalization of [7, Th. 1.1] and the method employed is that of [8, Th. 2.10].

3. In [1, Th. 3.1] it was shown that a ring  $R = G(R) \oplus K$  (ring direct sum), where  $K$  is semisimple with minimum condition, if and only if all  $\mathcal{S}$ -torsion-free modules are injective. In this section we prove this result for any stable torsion class  $\mathcal{S}$ .

**LEMMA 3.1.** *Let  $R = S \oplus K$ , where  $S$  is semisimple with minimum condition. Then any  $R$ -module  $A$  satisfying  $KA = 0$  is an injective  $R$ -module.*

*Proof.* If  $KA = 0$  then  $A$  is an injective  $S$ -module. Let  $I$  be a left ideal of  $R$ ; then  $I = S_1 \oplus K_1$  where  $S_1 \subseteq S, K_1 \subseteq K$  are left ideals of  $R$ . Also  $1 = u + v, u \in S, v \in K$ . If  $f: I \rightarrow A$  is an  $R$ -homomorphism, then for any  $b \in K_1, 0 = vf(b) = f(b)$ . There is an  $S$ -homomorphism  $g: S \rightarrow A$  coinciding with  $f$  on  $I$ , and this yields an  $R$ -homomorphism  $g^*: R \rightarrow A$  coinciding with  $f$  on  $I$  if we define  $g^*(s + k) = g^*(s)$ .

**THEOREM 3.2.** *Let  $\mathcal{S}$  be a stable torsion class. Then all  $\mathcal{S}$ -torsion-free  $R$ -modules are injective if and only if  $R = T(R) \oplus K$ , where  $K$  is a semisimple ring with minimum condition.*

*Proof.* Assume  $A$  is injective whenever  $T(A) = 0$ . Since the class  $\mathcal{F}$  of  $\mathcal{S}$ -torsion-free modules is closed under submodules [2], every submodule of any  $A \in \mathcal{F}$  is injective, hence is a direct summand, and so every  $A \in \mathcal{F}$  is completely reducible. Let  $M$  be any  $R$ -module and assume  $T(M) \neq 0$ . If  $T(M)$  is essential in  $M$  then  $T(M) = M$ , since  $\mathcal{S}$  is stable. Otherwise select  $B$  maximal relative to  $B \cap T(M) = 0$ . Then  $T(B) = 0$  and so  $B$  is injective. Thus  $M = B \oplus U$ . Now  $M/U \cong B \in \mathcal{F}$  so that  $T(M) \subseteq U$  by [2, Proposition 2.4]. The maximal property of  $B$  ensures that  $T(M)$  is essential in  $U$  and so  $U = T(M)$ . In particular  $R = T(R) \oplus K$ , where  $K$  is a completely reducible  $R$ -module since  $K \in \mathcal{F}$ . The decomposition is two-sided since right multiplications are  $R$ -homomorphisms and both classes  $\mathcal{S}$  and  $\mathcal{F}$  are closed under factors.

Conversely, assume  $R = T(R) \oplus K$ , where  $K$  is a semi-simple ring with minimum condition. We note that for any  $R$ -module  $A$ , if  $T(A) = 0$  then  $T(R)A = 0$ . For if  $A \in \mathcal{F}$  and  $0 \neq a \in A$  then  $T(R)a$  is an epimorphic image of  $T(R)$  and so  $T(R)a \subseteq T(A) = 0$ . That every  $\mathcal{S}$ -torsion-free module is injective now follows from Lemma 3.1.

We conclude with the following example. Let  $R$  be the ring of lower triangular  $2 \times 2$  matrices over a finite field and let  $\mathcal{S}$  be the smallest torsion class containing all projective simple  $R$ -modules. Note that  $\mathcal{S}$  contains nonzero  $R$ -modules since every simple in the socle of  $R$  is in  $\mathcal{S}$ . Moreover  $R$  is a hereditary Artinian ring with  $(\text{rad } R)^2 = 0$  so by [9, Theorem B] every nonprojective simple is injective. Since  $R$  is not semisimple, it has nonzero  $\mathcal{S}$ -torsion-free  $R$ -modules. If  $T(A) = 0$  for an  $R$ -module  $A \neq 0$  then  $\text{socle}(A)$  contains no projective simples. Since  $R$  is Noetherian and  $\text{socle}(A)$  is essential in  $A$ ,  $A$  is injective. It is readily verified that  $\text{socle}(R) = T(R)$ . Thus the condition that  $T$  be stable is needed in Theorem 3.2, even when  $T$  is closed under submodules.

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Received October 7, 1968.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Efraim Pacillas Armendariz, <i>Quasi-injective modules and stable torsion classes</i> .....	277
J. Adrian (John) Bondy, <i>On Ulam's conjecture for separable graphs</i> .....	281
Vasily Cateforis and Francis Louis Sandomierski, <i>On commutative rings over which the singular submodule is a direct summand for every module</i> .....	289
Rafael Van Severen Chacon, <i>Approximation of transformations with continuous spectrum</i> .....	293
Raymond Frank Dickman and Alan Zame, <i>Functionally compact spaces</i> .....	303
Ronald George Douglas and Walter Rudin, <i>Approximation by inner functions</i> .....	313
John Walter Duke, <i>A note on the similarity of matrix and its conjugate transpose</i> .....	321
Micheal Neal Dyer and Allan John Sieradski, <i>Coverings of mapping spaces</i> .....	325
Donald Campbell Dykes, <i>Weakly hypercentral subgroups of finite groups</i> .....	337
Nancy Dykes, <i>Mappings and realcompact spaces</i> .....	347
Edmund H. Feller and Richard Laham Gantos, <i>Completely injective semigroups</i> .....	359
Irving Leonard Glicksberg, <i>Semi-square-summable Fourier-Stieltjes transforms</i> .....	367
Samuel Irving Goldberg and Kentaro Yano, <i>Integrability of almost cosymplectic structures</i> .....	373
Seymour Haber and Charles Freeman Osgood, <i>On the sum <math>\sum \langle n\alpha \rangle^{-1}</math> and numerical integration</i> .....	383
Sav Roman Harasymiv, <i>Dilations of rapidly decreasing functions</i> .....	395
William Leonard Harkness and R. Shantaram, <i>Convergence of a sequence of transformations of distribution functions</i> .....	403
Herbert Frederick Kreimer, Jr., <i>A note on the outer Galois theory of rings</i> .....	417
James Donald Kuelbs, <i>Abstract Wiener spaces and applications to analysis</i> .....	433
Roland Edwin Larson, <i>Minimal <math>T_0</math>-spaces and minimal <math>T_D</math>-spaces</i> .....	451
A. Meir and Ambikeshwar Sharma, <i>On Ilyeff's conjecture</i> .....	459
Isaac Namioka and Robert Ralph Phelps, <i>Tensor products of compact convex sets</i> .....	469
James L. Rovnyak, <i>On the theory of unbounded Toeplitz operators</i> .....	481
Benjamin L. Schwartz, <i>Infinite self-interchange graphs</i> .....	497
George Szeto, <i>On the Brauer splitting theorem</i> .....	505
Takayuki Tamura, <i>Semigroups satisfying identity <math>xy = f(x, y)</math></i> .....	513
Kenneth Tolo, <i>Factorizable semigroups</i> .....	523
Mineko Watanabe, <i>On a boundary property of principal functions</i> .....	537
James Juei-Chin Yeh, <i>Singularity of Gaussian measures in function spaces with factorable covariance functions</i> .....	547