

Pacific Journal of Mathematics

FUNCTIONALLY COMPACT SPACES

RAYMOND FRANK DICKMAN AND ALAN ZAME

FUNCTIONALLY COMPACT SPACES

R. F. DICKMAN, JR., AND A. ZAME

The purpose of this note is to introduce a property which is weaker than compactness but stronger than minimality which closely relates to some filter properties and some mapping properties of compact spaces. We also indicate some procedures for constructing absolutely closed and minimal spaces.

All of the definitions used but not given in this paper may be found in [2].

DEFINITION. An open filter base on a topological space X is a filter base consisting of open subsets of X .

A Hausdorff topological space X is called *functionally compact* if whenever \mathcal{U} is an open filter base on X such that the intersection A of the elements of \mathcal{U} is equal to the intersection of the closures of the elements of \mathcal{U} , then \mathcal{U} is a base for the neighborhoods of A .

THEOREM 1. *There exists a noncompact Hausdorff space which is functionally compact.*

Proof. Let Z^+ = positive integers, $I = [0, 1]$. Let $0 < a_1 < a_2 < \dots < a_n < \dots$ be an increasing sequence in I with $\lim a_n = a_0$. For each $i \in Z^+$ let a_i^j be a strictly increasing sequence (in (a_{i-1}, a_i) for $i \geq 2$; in $(0, a_1)$ for $i = 1$) with $\lim_j a_i^j = a_i$. Let

$$C = \bigcup_{i>0} \{a_i\} \cup \bigcup_{\substack{i>1 \\ j>1}} \{a_i^j\}.$$

Let $\alpha_0 = a_0$ and for $i \geq 1$ let

$$\alpha_i = \{a_1^i, a_2^{i-1}, \dots, a_i^1, a_i\}.$$

Let $C^* = \{\alpha_i; i \geq 0\}$ and put

$$F = (I \setminus C) \cup C^*.$$

Let \mathcal{F} be the topology on F such that $\mathcal{F} \upharpoonright (I \setminus C)$ is the usual topology on $I \setminus C$ and a basic open neighborhood of $\alpha_n (n \geq 0)$ is a set of the form $N \cup \{\alpha_n\}$, where $N = O \cap (I \setminus C)$, O open in I and $\{a_1^n, \dots, a_n^n, a_n\} \subset O$.

It is easy to see that (F, \mathcal{F}) is a Hausdorff space. It is not compact, however, because C^* is an infinite discrete closed set in F .

In fact, if $O_1, O_2 \in \mathcal{S}$ and $O_1 \cap C^* \neq \emptyset, O_2 \cap C^* \neq \emptyset$ then $cl_F O_1 \cap cl_F O_2 \cap C^*$ is infinite, so that points in C^* cannot be separated by closed neighborhoods. We will now show that (F, \mathcal{S}) is functionally compact.

Define the set function $\psi: F \rightarrow I$ via

$$\psi(\{a\}) = \begin{cases} a & \text{if } a \in I \setminus C \\ \alpha_0 & \text{if } a = \alpha_0 \\ \{\alpha_1^i, \dots, \alpha_i^1, \alpha_i\} & \text{if } a = \alpha_i, i \geq 1 \end{cases}$$

and $\psi(A) = \bigcup_{a \in A} \psi(\{a\})$ for $A \subseteq F$. ψ does not preserve openness or closedness. However, if $A \subseteq F$ is closed then there exists a set A^c , closed in I , such that

$$(i) \quad \psi(A) \cap (I \setminus C) = A^c \cap (I \setminus C)$$

and

$$(ii) \quad \psi(A) \supseteq A^c,$$

namely,

$$A^c = cl_I \psi(A) \cap (I \setminus C).$$

Hence, if $U \subseteq F$ is open there exists an open set $U^0 \subseteq I$ such that

$$(iii) \quad U^0 \cap (I \setminus C) = \psi(U) \cap (I \setminus C)$$

$$(iv) \quad U^0 \supseteq \psi(U);$$

namely, the set $U^0 = ((U')^c)'$ (where $'$ denotes complementation in either F or I).

Now, suppose \mathcal{U} is an open filter in F such that $\bigcap \{U: U \in \mathcal{U}\} = \bigcap \{cl_F U: U \in \mathcal{U}\} = A$ and that $V \supseteq A, V \in \mathcal{S}$. We claim that there exist $U_1, U_2 \in \mathcal{S}$ such that $V \supseteq U_1 \cap U_2$.

First of all, if $O \in \mathcal{S}$ and $O \cap C^*$ is infinite, then $cl_F O \supseteq C^*$. Hence if $A \cap C^*$ is finite then there exists $U \in \mathcal{U}$ such that $U \cap C^* = A \cap C^*$, while if $A \cap C^*$ is infinite then $A \supseteq C^*$. Thus, in either case there exists $U_1 \in \mathcal{U}$ such that $V \cap C^* \supseteq U_1 \cap C^*$. Furthermore,

$$V^0 \supseteq \psi(A) = \bigcap \psi(cl_F U) \supseteq \bigcap (cl_F U)^c.$$

Hence, since V^0 is open in I and $(cl_F U)^c$ is closed in I the compactness of I implies that there exists $U_2 \in \mathcal{U}$ such that $V^0 \supseteq (cl_F U_2)^c$. But then

$$\begin{aligned} V^0 \cap (I \setminus C) &= \psi(V) \cap (I \setminus C) = V \cap (I \setminus C) \supseteq (cl_F U_2)^c \cap (I \setminus C) \\ &= (cl_F U_2) \cap (I \setminus C) \supseteq U_2 \cap (I \setminus C). \end{aligned}$$

Hence

$$V = (V \cap (I \setminus C)) \cup (V \cap C^*) \supseteq (U_2 \cap (I \setminus C)) \cup (U_1 \cap C^*) \supseteq U_1 \cap U_2.$$

This proves the result.

There are some obvious generalizations of this construction. The above example F of a functionally compact space also shows that the property of being functionally compact is not closed-hereditary or open-hereditary or even regular-closed-hereditary.

THEOREM 2. *A Hausdorff space X is minimal Hausdorff if and only if for every point $x \in X$ and every open filter-base \mathcal{U} on X such that $x = \bigcap \{U : U \in \mathcal{U}\}$ and $x = \bigcap \{cl_x U : U \in \mathcal{U}\}$, \mathcal{U} is a base for the neighborhoods of x .*

Proof of the necessity. Suppose that X is minimal Hausdorff and that \mathcal{U} is an open filter-base on X and $x \in X$ such that

$$x = \bigcap \{U : U \in \mathcal{U}\} \quad \text{and} \quad x = \bigcap \{cl_x U : U \in \mathcal{U}\}$$

and let R be any open set containing x . We note that x is the unique cluster point of \mathcal{U} . Thus by Theorem 1.3 of [1], \mathcal{U} converges to x and so there exists $U \in \mathcal{U}$ such that $U \subset R$. Of course this implies that \mathcal{U} is a base for the neighborhoods of x .

Proof of the sufficiency. We need to show that every open filter-base with a unique cluster point converges to that point. To this end let \mathcal{V} be an open filter-base on X with a unique cluster point x and let R be any open subset of X containing x . Let \mathcal{W} be the collection of all open subsets of X containing x and let

$$\mathcal{U} = \{V \cap W : V \in \mathcal{V} \text{ and } W \in \mathcal{W}\}.$$

Then \mathcal{U} is an open filter-base on X ,

$$x = \bigcap \{U : U \in \mathcal{U}\} \quad \text{and} \quad x = \bigcap \{cl_x U : U \in \mathcal{U}\}.$$

Thus by our hypothesis there exists $U \in \mathcal{U}$ such that $U \subset R$. Of course this implies that \mathcal{V} converges to x and this completes the proof.

COROLLARY 2.1. *Every functionally compact Hausdorff space is minimal Hausdorff.*

Proof. This result is an immediate consequence of the definition of functionally compact and Theorem 2.

The example due to Urysohn in Remark 1.5 of [1] is a minimal Hausdorff space which is not functionally compact. In the next section we indicate a method of constructing many such spaces.

COROLLARY 2.2. *A functionally compact Hausdorff space is com-*

compact if and only if it is regular.

Proof. The proof follows immediately from the fact that a minimal Hausdorff space is compact if and only if it is regular.

DEFINITIONS. By a *mapping* we will always mean a continuous function. A mapping from a space X to a space Y is *closed* provided for every closed set C in X , $f(C)$ is a closed set in Y .

THEOREM 3. *A Hausdorff space X is functionally compact if and only if every mapping of X into any Hausdorff space is closed.*

Proof of the necessity. Suppose that X is functionally compact and let f be a mapping of X into a Hausdorff space Y . Let C be a closed set in X and suppose there exists a point y in $cl_Y f(C) \setminus f(C)$. Let \mathcal{V} be the collection of all open subsets of Y containing y and let $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$. By Theorem 2, X is minimal Hausdorff and hence absolutely closed. Since continuous images of absolutely closed spaces are absolutely closed, $f(X) = cl_Y f(X)$ and so $y \in f(X)$. Thus \mathcal{U} is a collection of nonempty open subsets of X and therefore is an open filter-base on X . Furthermore since Y is a Hausdorff space, $f^{-1}(y) = \bigcap \{U : U \in \mathcal{U}\}$ and $f^{-1}(y) = \bigcap \{cl_X U : U \in \mathcal{U}\}$. Thus by our hypothesis \mathcal{U} is a base for the neighborhoods of $f^{-1}(y)$ and so there exists $U \in \mathcal{U}$ such that $U \subseteq X \setminus C$. But then $f(U)$ is an open subset of $f(X)$ that contains y and misses $f(C)$ (since $U = f^{-1}(f(U))$ if $U \in \mathcal{U}$) and this is a contradiction. Thus $f(C)$ is closed and this completes the proof of the necessity.

Proof of the sufficiency. Suppose that every mapping of X into a Hausdorff space is closed and let \mathcal{U} be an open filter-base on X such that the intersection A of the elements of \mathcal{U} equals the intersection of the closures of the elements of \mathcal{U} . Suppose further that there exists an open set R of X containing A such that for every $U \in \mathcal{U}$, $(X \setminus R) \cap U \neq \emptyset$. Let Y be the decomposition of X whose only nondegenerate element is A and let f be the natural transformation of X onto Y defined by $x \in f(x)$. We topologize Y by defining a base \mathcal{B} for a topology as follows:

$B \in \mathcal{B}$ if and only if (i) $f^{-1}(B)$ is an open subset of $X \setminus A$;

or

(ii) $f^{-1}(B) \in \mathcal{U}$.

Then Y with this topology is a Hausdorff space and f is a mapping of X onto Y . By our hypothesis f must be a closed map ; however

$f(X \setminus R)$ is not closed since $f(A)$ is a limit point of $f(X \setminus R)$ and $f(A) \notin f(X \setminus R)$. This is a contradiction and this completes the proof.

REMARK. The space F of Theorem 1 and Theorem 3 show that the property of being functionally compact is not productive. In fact, even if \mathcal{C} is compact $F \times \mathcal{C}$ will not usually be functionally compact since the projection $\pi: F \times \mathcal{C} \rightarrow \mathcal{C}$ is not a closed map unless \mathcal{C} is finite.

COROLLARY 3.1. *Let X be a Hausdorff space, Z a functionally compact Hausdorff space and h a mapping of Z onto X . Then X is functionally compact.*

Proof. Let f be a mapping of X into a Hausdorff space Y and let C be a closed subset of X . Since Z is functionally compact, the mapping $f \circ h$ is a closed map of Z into Y and so $f(C) = (f \circ h)(f^{-1}(C))$ is a closed subset of Y . Thus f is closed and X is functionally compact.

COROLLARY 3.2. *If a Hausdorff space X is the union of finitely many functionally compact spaces X_1, X_2, \dots, X_n , X is functionally compact.*

Proof. Let f be a mapping of X into a Hausdorff space Y and let C be a closed subset of X . Then each of the restricted mappings $f|X_i, i = 1, 2, \dots, n$, is closed and so $f(C) = \cup \{(f|X_i)(C \cap X_i): i = 1, 2, \dots, n\}$ is closed in Y . Thus f is closed and X is functionally compact.

DEFINITION. A closed subset C of a space X is said to be r -closed if whenever B is closed in $C, x \in B$ there exist disjoint open sets in X containing x and B , respectively.

THEOREM 4. *An r -closed subset C of a functionally compact space X is functionally compact.*

Proof. Let \mathcal{U} be an open filter-base on C such that

$$\bigcap \{U: U \in \mathcal{U}\} = \{cl_x U: U \in \mathcal{U}\} = A.$$

Let \mathcal{V} be the open filter-base on X consisting of all open sets V of X such that $V \cap C \in \mathcal{U}$.

Then since C is an r -closed subset of $X, \bigcap \{V: V \in \mathcal{V}\} = A = \{cl_x V: V \in \mathcal{V}\}$ and since X is functionally compact, \mathcal{V} is a base for

the neighborhoods of A . Of course this implies that \mathcal{U} is a base for the neighborhoods of A relative to C . Hence C is functionally compact.

It is also easy to see that if C is an open and closed subset of a functionally compact space X then C is functionally compact.

2. Some related examples. Urysohn has given an example of an nonminimal absolutely closed space (Example 1.4 of [3]) and an example of a noncompact minimal space (Remark 1.5 of [1]). We will give here two general methods for constructing such spaces. In fact, our minimal spaces will be nonfunctionally compact, as is Urysohn's.

EXAMPLE 1. Let (X, \mathcal{K}) be a compact Hausdorff space such that $Y \subseteq X$ is an infinite closed subset, $\text{int}_x Y = \emptyset$. Let \mathcal{F} be any topology on Y strictly stronger than $\mathcal{K} \upharpoonright Y$, e.g., the discrete topology, and let \mathcal{U} be the following topology on X : $\mathcal{U} \upharpoonright (X \setminus Y) = \mathcal{K} \upharpoonright (X \setminus Y)$ and a basic open set intersecting Y is of the form $(\mathcal{O} \cap (X \setminus Y)) \cup T$, where $T \subseteq \mathcal{O} \cap Y$, $\mathcal{O} \in \mathcal{K}$, $T \in \mathcal{F}$. Then (X, \mathcal{U}) is a nonminimal Hausdorff space (since \mathcal{U} is stronger than \mathcal{K}) and $\text{int}_x Y = \emptyset$. However, we claim that (X, \mathcal{U}) is absolutely closed.

Let $\mathcal{C} = \{C_\alpha : \alpha \in \mathcal{A}\}$ be a \mathcal{U} -open covering of X . Notice that for each $U \in \mathcal{U}$ there exists $K \in \mathcal{K}$ such that $K \supseteq U$ and $K \cap (X \setminus Y) = U \cap (X \setminus Y)$. Thus, if K_α is such a set for C_α then $\{K_\alpha : \alpha \in \mathcal{A}\}$ is a \mathcal{K} -open covering of X . Hence there exist $\alpha_1, \dots, \alpha_n$ such that

$$K_{\alpha_1} \cup \dots \cup K_{\alpha_n} \supseteq X$$

so that

$$C_{\alpha_1} \cup \dots \cup C_{\alpha_n} \supseteq X \setminus Y$$

and hence

$$\text{cl}_x C_{\alpha_1} \cup \dots \cup \text{cl}_x C_{\alpha_n} = X.$$

This is a generalization of one of Urysohn's examples (Example 1.4 of [3]): Let $a_i^j \rightarrow a_i$ be disjoint convergent sequences and let $X = \bigcup_{i,j} \{a_i^j\} \cup \bigcup_i \{a_i\} \cup \{\infty\}$ be the 1-point compactification of the union of these sequences and their limit points. Let $Y = \{a_1, a_2, \dots\} \cup \{\infty\}$ and let \mathcal{F} be the discrete topology on Y . Then (X, \mathcal{U}) is Urysohn's example.

EXAMPLE 2. Let (X, \mathcal{K}) be a compact Hausdorff space, $Y \subseteq X$ and φ a function from Y into the 2-element subsets of Y such that

- (i) Y is closed;

- (ii) $\text{int}_{\mathcal{X}} Y = \emptyset$
- (iii) there exist $x_1, x_2 \in Y, x_1 \neq x_2$ such that $x_i \in \text{cl}_{\mathcal{X}} (Y \setminus \{x_1, x_2\})$ ($i = 1, 2$);
 - (α) $\varphi(z_1) \cap \varphi(z_2) = \emptyset$ if $z_1 \neq z_2$;
 - (β) $\bigcup_{y \in Y} \varphi(y) = Y$;
 - (γ) there exist $y_1 \neq y_2$ such that $x_i \in \varphi(y_i), i = 1, 2$;
 - (δ) for each $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{H}$ such that $x_i \in \mathcal{O}_i, i = 1, 2$ there exists $y \in Y$ such that $\varphi(y) \cap \mathcal{O}_i \neq \emptyset, i = 1, 2$.

[For instance, we could take $X = [0, 1], Y = \{a_i^j: i = 1, \dots, 4, j \geq 1\} \cup \{a_1, a_2, a_3, a_4\}$, where the a_i are distinct and the (a_i^j) are disjoint sequences of distinct points with $\lim_j a_i^j = a_i$. We can let φ be a function which "identifies" a_1^j and a_2^j, a_3^j and a_4^j, a_1 and a_3, a_2 and a_4 .]

Let $Y^* = \{\varphi(y): y \in Y\}$ and denote $\varphi(y)$ by y^* . Let $X^* = (X \setminus Y) \cup Y^*$. Let \mathcal{F} be the topology on X^* defined as follows: $U \in \mathcal{F} \Leftrightarrow$ there exists $\mathcal{O} \in \mathcal{H}$ such that

- (1) $U \cap (X \setminus Y) = \mathcal{O} \cap (X \setminus Y)$
- (2) for each $y^* \in U \cap Y^*$ there exists $V \in \mathcal{H}, \varphi(y) \subseteq V, V \subseteq \mathcal{O}$, such that if $\varphi(z) \subseteq V$ then $z^* \in U$.

It is to check that \mathcal{F} is actually a topology and that \mathcal{F} is Hausdorff. However, if $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{F}, y_1^* \in \mathcal{O}_1, y_2^* \in \mathcal{O}_2$ then (δ) implies that $\text{cl}_{\mathcal{F}} \mathcal{O}_1 \cap \text{cl}_{\mathcal{F}} \mathcal{O}_2 \neq \emptyset$. In particular, \mathcal{F} is not compact.

Notice that if $C \subseteq X^*$ is closed then there exists a closed set $C^c \subseteq X$ such that

$$C^c \cap (X \setminus Y) = c \cap (X^* \setminus Y^*)$$

and

$$C^c \subseteq (C \cap (X^* \setminus Y^*)) \cup (\bigcup_{y \in C} \varphi(y)),$$

namely,

$$C^c = \text{cl}_{\mathcal{X}}(C \cap (X^* \setminus Y^*)).$$

PROPOSITION 1. (X^*, \mathcal{F}) is minimal.

Proof. Let \mathcal{U} be an open filter such that

$$\bigcap \{U: U \in \mathcal{U}\} = \bigcap \{\text{cl}_{\mathcal{F}} U: U \in \mathcal{U}\} = \{x\}$$

and let $V \in \mathcal{F}, x \in V$. We claim that there exists $U \in \mathcal{U}$ such that $V \supseteq U$.

Case 1. Suppose $x = p^* \in Y^*$. Then there exists $\mathcal{W} \in \mathcal{H}$ with $\varphi(p) \subseteq \mathcal{W}$ such that $\mathcal{W} \cap (X \setminus Y) \subseteq V \cap (X \setminus Y)$ and such that if $y^* \in Y^*, \varphi(y) \subseteq \mathcal{W}$ then $y^* \in V$. Then

$$\mathcal{W} \supseteq \bigcap (\text{cl}_{\mathcal{F}} U)^c = \varphi(p)$$

so there exists $U \in \mathcal{U}$ such that $\mathcal{W} \supseteq (cl_{\mathcal{F}}U)^c$. In particular,

$$V \supseteq \mathcal{W} \cap (X \setminus Y) \supseteq U \cap (X^* \setminus Y^*).$$

However, if $z^* \in (U \cap Y^*) \setminus V$ then $\varphi(z) \notin \mathcal{W}$ but $\varphi(z) \subseteq (cl_{\mathcal{F}}U)^c$ since $U \in \mathcal{F}$. Thus if $z^* \in (U \cap Y^*) \setminus V$ then there exists $y \in (cl_{\mathcal{F}}U)^c \setminus \mathcal{W}$, a contradiction. Thus $V \supseteq (U \cap (X^* \setminus Y^*)) \cup (U \cap Y^*) = U$.

Case 2. Suppose $x \in X^* \setminus Y^*$. We may suppose that $V \subseteq X^* \setminus Y^*$ and in fact that $cl_x V \subseteq X^* \setminus Y^*$ since Y is \mathcal{H} -closed in X . Then, again, there exists $U \in \mathcal{U}$ such that $V \cap (X^* \setminus Y^*) \supseteq U \cap (X^* \setminus Y^*)$. Since $U \in \mathcal{F}$ and $cl_x V \subseteq X^* \setminus Y^*$ we must have that $U \subseteq X^* \setminus Y^*$ so that again $V \supseteq U$.

PROPOSITION 2. (X^*, \mathcal{F}) is not functionally compact.

Proof. Let $A = \{y_1^*, y_2^*\}$. Suppose $\varphi(y_1) = \{x_1, x_1'\}$, $\varphi(y_2) = \{x_2, x_2'\}$. If $\mathcal{O}_1, \mathcal{O}_1', \mathcal{O}_2, \mathcal{O}_2' \in \mathcal{H}$ are sets containing x_1, x_1', x_2, x_2' , respectively then there are $U \in \mathcal{F}$ containing all $y^* \in Y^*$ with

$$\varphi(y) \subseteq \mathcal{O}_1 \cup \mathcal{O}_1' \cup \mathcal{O}_2 \cup \mathcal{O}_2',$$

and if \mathcal{U} is the filter base of all such U then $\bigcap U = \bigcap cl_{\mathcal{F}}U = A$. However, there exist $\mathcal{O} \in \mathcal{F}, A \subseteq \mathcal{O}$ such that \mathcal{O} does not contain any points $y^* \in Y^*$ unless $\varphi(y)$ is contained in a set of the form $\mathcal{O}_1 \cup \mathcal{O}_1'$ or one of the form $\mathcal{O}_2 \cup \mathcal{O}_2'$. Thus the filter generated by \mathcal{U} does not contain all neighborhoods of A .

In [4] G. Viglino studies a property similar to functional compactness. X is C -compact if for each closed set $A \subseteq X$ and each open cover $\mathcal{U} = \{U_\alpha\}$ of A there exist $\alpha_1, \dots, \alpha_n: cl_X(U_{\alpha_1} \cup \dots \cup U_{\alpha_n}) \supseteq A$. It is easy to see that if X is C -compact then X is functionally compact. We do not know whether the two properties are equivalent.

Urysohn's two examples to which we have referred can both be embedded as (nondense) subsets of functionally compact spaces. The one referred to in Example 1 can be embedded in F . If we let Q be the quotient space obtained from two disjoint copies of F by identifying the points α_i in the first with α_i in the second, $i = 1, 2, \dots$, we get a functionally compact space in which Urysohn's minimal, non-compact space can be embedded, again as a nondense subset. This is the best that can be expected, since an absolutely closed space C embedded in a Hausdorff space H would have to be a closed subset of H , and hence not dense in H if H is functionally compact unless $C = H$. Also, since open and closed subsets of functionally compact

spaces are functionally compact a nonfunctionally compact absolutely closed space can not be embedded as an open subset of a functionally compact set. We do not know whether for each Hausdorff H there exists a functionally compact X with $H \subseteq X$.

BIBLIOGRAPHY

1. Manuel P. Berri, *Minimal topological spaces*, Trans. Amer. Math. Soc. **108** (1963), 97-105.
2. Bourbaki, *General topology*, Addison-Wesley, 1966.
3. Chen-Tung Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. **130** (1968), 86-104.
4. Giovanni Viglino, *C-compact spaces*, (to appear in Duke Math J.)

Received February 21, 1969.

THE UNIVERSITY OF MIAMI
CORAL GABLES, FLORIDA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

BASIL GORDON
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 31, No. 2

December, 1969

Efraim Pacillas Armendariz, <i>Quasi-injective modules and stable torsion classes</i>	277
J. Adrian (John) Bondy, <i>On Ulam's conjecture for separable graphs</i>	281
Vasily Cateforis and Francis Louis Sandomierski, <i>On commutative rings over which the singular submodule is a direct summand for every module</i>	289
Rafael Van Severen Chacon, <i>Approximation of transformations with continuous spectrum</i>	293
Raymond Frank Dickman and Alan Zame, <i>Functionally compact spaces</i>	303
Ronald George Douglas and Walter Rudin, <i>Approximation by inner functions</i>	313
John Walter Duke, <i>A note on the similarity of matrix and its conjugate transpose</i>	321
Micheal Neal Dyer and Allan John Sieradski, <i>Coverings of mapping spaces</i>	325
Donald Campbell Dykes, <i>Weakly hypercentral subgroups of finite groups</i>	337
Nancy Dykes, <i>Mappings and realcompact spaces</i>	347
Edmund H. Feller and Richard Laham Gantos, <i>Completely injective semigroups</i>	359
Irving Leonard Glicksberg, <i>Semi-square-summable Fourier-Stieltjes transforms</i>	367
Samuel Irving Goldberg and Kentaro Yano, <i>Integrability of almost cosymplectic structures</i>	373
Seymour Haber and Charles Freeman Osgood, <i>On the sum $\sum \langle n\alpha \rangle^{-1}$ and numerical integration</i>	383
Sav Roman Harasymiv, <i>Dilations of rapidly decreasing functions</i>	395
William Leonard Harkness and R. Shantaram, <i>Convergence of a sequence of transformations of distribution functions</i>	403
Herbert Frederick Kreimer, Jr., <i>A note on the outer Galois theory of rings</i>	417
James Donald Kuelbs, <i>Abstract Wiener spaces and applications to analysis</i>	433
Roland Edwin Larson, <i>Minimal T_0-spaces and minimal T_D-spaces</i>	451
A. Meir and Ambikeshwar Sharma, <i>On Ilyeff's conjecture</i>	459
Isaac Namioka and Robert Ralph Phelps, <i>Tensor products of compact convex sets</i>	469
James L. Rovnyak, <i>On the theory of unbounded Toeplitz operators</i>	481
Benjamin L. Schwartz, <i>Infinite self-interchange graphs</i>	497
George Szeto, <i>On the Brauer splitting theorem</i>	505
Takayuki Tamura, <i>Semigroups satisfying identity $xy = f(x, y)$</i>	513
Kenneth Tolo, <i>Factorizable semigroups</i>	523
Mineko Watanabe, <i>On a boundary property of principal functions</i>	537
James Juei-Chin Yeh, <i>Singularity of Gaussian measures in function spaces with factorable covariance functions</i>	547