APPROXIMATION BY INNER FUNCTIONS

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Let $L^\infty(T)$ denote the complex Banach algebra of (equivalence classes of) bounded measurable functions on the unit circle $T$, relative to Lebesgue measure $m$. The norm $\|f\|_\infty$ of an $f$ in $L^\infty(T)$ is the essential supremum of $|f|$ on $T$. The collection of all bounded holomorphic functions in the open unit disc $U$ forms a Banach algebra which can be identified (via radial limits) with the norm-closed subalgebra $H^\infty$ of $L^\infty(T)$.

A function $f$ in $L^\infty(T)$ is unimodular if $|f| = 1$ a.e., on $T$. The inner functions are the unimodular members of $H^\infty$. It is well known that they play an important role in the study of $H^\infty$.

The main result (Theorem 1) is that the set of quotients of inner functions is norm-dense in the set of unimodular functions in $L^\infty(T)$. One consequence of this (Theorem 7) is that the set of radial limits of holomorphic functions of bounded characteristic in $U$ is norm-dense in $L^\infty(T)$. It is also shown (Theorem 3, 4) that the Gelfand transforms of the inner functions separate points on the Silov boundary of $H^\infty$, and this is used to obtain a new proof (and generalization) of a theorem of D. J. Newman (Theorem 4).

Our proof of the main result uses only one nontrivial property of $H^\infty$, beyond the fact that $H^\infty$ is a norm-closed subalgebra of $L^\infty$. It therefore applies, without any extra effort, to a much more general situation which we now describe.

Let now $L^\infty$ denote the Banach algebra of all bounded measurable functions on some measure space $X$, normed by the essential supremum, and let $B$ be a norm-closed subalgebra of $L^\infty$. We say that $B$ has the annulus property if the following is true:

If $X$ is the union of disjoint measurable sets $E_1$ and $E_2$ and if $0 < r_1 < r_2 < \infty$, then there exists $h$ in $B$ such that

1. $1/h$ is in $B$, and
2. $|h| = r_i$ a.e., on $E_i$, for $i = 1, 2$.

That $H^\infty$ (in the classical setting described above) has the annulus property is well known: to see it, put $u = r_i$ on $E_i$ (now $T = E_1 \cup E_2$), and define

$$h(z) = \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta})d\theta\right\} \quad (z \in U).$$

Then $h$ maps $U$ into the annulus $\{w: r_1 < |w| < r_2\}$, and the radial limits of $h$ have modulus $r_i$ a.e., on $E_i$.
Furthermore, the $H^\infty$-algebras associated with weak*-Dirichlet algebras also have the annulus property. This is a special case of Lemma 2.4.3 of [11]. We shall have no opportunity to use any other property of these algebras, and will therefore not even define them here. An excellent account of them is given in [11].

In order to avoid repetition we now state what our standing assumptions will be. Theorems 1 to 5 will deal with the general situation just described. $H^\infty$ will simply denote some subalgebra of some $L^\infty$, the only other hypothesis being that $H^\infty$ has the annulus property. The “inner functions” will again be the unimodular members of $H^\infty$. Theorems 6, 7, 8 are more special and deal with the classical situation of the unit circle.

**THEOREM 1.** The set of all quotients of inner functions is norm-dense in the set of all unimodular functions in $L^\infty$.

**Proof.** Since the measurable unimodular functions taking finitely many values are norm-dense in the set of all unimodular functions in $L^\infty$, and since each function of the latter type is a product of finitely many unimodular functions each taking at most two values, it is sufficient to prove the following.

**PROPOSITION.** If $E_1$ and $E_2$ are disjoint measurable subsets of $X$ whose union is $X$, if $\lambda_1$ and $\lambda_2$ are complex numbers of modulus 1, and if $\epsilon > 0$, then there exist inner functions $\phi_1$ and $\phi_2$ such that

$$|\lambda_i - \phi_1(x)/\phi_2(x)| < \epsilon \quad \text{a.e., on } E_i (i = 1, 2).$$

It involves no loss of generality to assume that $\lambda_1 \neq \lambda_2$. Let $\alpha_1$ and $\alpha_2$ be closed disjoint subarcs of $T$, of length less than $\epsilon$, containing $\lambda_1$ and $\lambda_2$, respectively. Let $\Omega$ be the complement of $\alpha_1 \cup \alpha_2$ in the Riemann sphere. Then there is an annulus

$$D = \{z: r_1 < |z| < r_2\}$$

and a continuous function $\phi$ on its closure $\overline{D}$ whose restriction to $D$ is a one-to-one conformal map of $D$ onto $\Omega$ ([2], p. 247). If $|z| = r_i$ then $\phi(z)$ is in $\alpha_i (i = 1, 2)$. The reflection principle shows that $\phi$ is holomorphic on $\overline{D}$, except for a simple pole at some point $z_0$ in $D$. By a theorem of Ahlfors [1] there exists a function $\phi_2$, holomorphic on $\overline{D}$, such that $\phi_2$ has a zero at $z_0$ and $|\phi_2(z)| = 1$ on $\partial D$. Define $\phi_1 = \phi \cdot \phi_2$. Then $\phi_1$ is holomorphic on $\overline{D}$, $|\phi_1(z)| = 1$ on $\partial D$, and $\phi = \phi_1/\phi_2$.

By the annulus property which $H^\infty$ satisfies, there exists $h$ in $H^\infty$ such that $|h| = r_i$ a.e., on $E_i$, and $1/h$ is in $H^\infty$. Thus $h$ maps $X$ into $\partial D$, $\|h\|_\infty = r_2$ and $\|1/h\|_\infty = 1/r_1$. Since $\phi_1$ and $\phi_2$ are holomor-
phic on $\tilde{D}$, their Laurent expansions converge uniformly on $\tilde{D}$. Since $H^\infty$ is norm-closed, this implies that the compositions $\phi_1 = \Phi_1 \circ h$ and $\phi_2 = \Phi_2 \circ h$ are in $H^\infty$. Clearly, they are also inner. Finally, $\phi_1/\phi_2 = \Phi \circ h$, and $(\Phi \circ h)(x)$ is in $\alpha_i$ for almost every $x$ in $E_i (i = 1, 2)$.

This proves the proposition, and hence Theorem 1.

**Theorem 2.** Let $Q$ be the set of all functions of the form $\psi \tilde{\phi}$, where $\psi$ is a finite linear combination of inner functions and $\phi$ is inner. Then $Q$ is norm-dense in $L^\infty$.

**Proof.** By Theorem 1, the norm-closure $\bar{Q}$ of $Q$ contains all unimodular functions in $L^\infty$. Let $\chi_E$ be the characteristic function of a measurable set $E \subset X$. Note that $2\chi_E - 1$ is unimodular, and hence is in $\bar{Q}$. Since $\bar{Q}$ is a linear space, it follows that $\chi_E$ is in $\bar{Q}$ for every measurable $E \subset X$, and hence $\bar{Q} = L^\infty$.

Since $Q$ is the algebra generated by the inner functions and their complex conjugates, Theorem 2 may be restated as follows:

**Corollary.** The self-adjoint algebra generated by the inner functions is norm-dense in $L^\infty$.

**Remark.** The subgroup $G$ consisting of those unimodular functions which are quotients of inner functions has already occurred in certain studies ([5], [7, p. 12]). Theorem 1 shows how delicate the question of membership in $G$ is. Note that $G \subset Q$ (see Theorem 2) and that $Q \subset \bar{Q}$, where $\bar{Q}$ denotes the set of those functions in $L^\infty$ which are of the form $\phi \tilde{\psi}$, where $\phi$ and $\psi$ are in $H^\infty$. In the classical situation, every nonconstant $f$ in $\bar{Q}$ satisfies

$$\int_X \log |f| \, dm > -\infty.$$ 

We doubt that this necessary condition is also sufficient (even for unimodular $f$) but we have no counterexample.

In connection with Theorem 2, we recall that it is still an open question whether the closure $J$ of the set of finite linear combinations of inner functions is $H^\infty$ (cf. [3], p. 348). Actually, $J$ is a subalgebra of $H^\infty$ which in the classical case of the circle has the same maximal ideal space and Šilov boundary as $H^\infty$ (see the footnote to Theorem 3 and the proof of Theorem 4).

We now consider the maximal ideal space $M$ of $H^\infty$. The annulus property implies that 1 is in $H^\infty$, so $M$ is compact. The Gelfand transform $\hat{f}$ of an $f$ in $H^\infty$ is a continuous function of $M$, such that $||\hat{f}|| = ||f||_\infty$, where $||\hat{f}||$ denotes the maximum of $|\hat{f}|$ on $M$, and $||f||_\infty$ is the essential supremum of $|f|$ on $X$. We shall use the following
notations:
If $\phi$ is inner, then
\[ K_{\phi} = \{ \gamma \in M : |\hat{\phi}(\gamma)| = 1 \} . \]
If $\Sigma$ is a set of inner functions, then
\[ K_{\Sigma} = \bigcap_{\phi \in \Sigma} K_{\phi} . \]
If $\Sigma$ is the set of all inner functions in $H^\infty$, we write $K$ in place of $K_{\Sigma}$.

The Šilov boundary of $H^\infty$ will be denoted by $\partial$.

**Theorem 3.** The Gelfand transforms of the inner functions separate points on $K$.

*Proof.* Let $\gamma_0$ and $\gamma_1$ be distinct points of $K$. There exists $f$ in $H^\infty$ with $\hat{f}(\gamma_0) = 0$ and $\hat{f}(\gamma_1) = 1$. By Theorem 2, one can find $\phi$ and $\psi$ such that $\phi$ is inner, $\psi$ is a finite linear combination of inner functions, and $\|\phi f - \psi\|_* < 1/3$. Hence
\[ |\hat{\phi}(\gamma)\hat{f}(\gamma) - \hat{\psi}(\gamma)| < \frac{1}{3} \]
for every $\gamma \in M$, in particular for $\gamma_0$ and $\gamma_1$. So $|\hat{\psi}(\gamma_0)| < 1/3$, and $|\hat{\psi}(\gamma_1)| > 2/3$ since $|\hat{\phi}(\gamma_1)| = 1$. This shows that $\hat{\psi}$ separates $\gamma_0$ and $\gamma_1$.

Theorem 3 leads directly to a generalization of a theorem which D. J. Newmann proved in the classical case [9] and which characterizes the Šilov boundary $\partial$ of $H^\infty$ in terms of inner functions:

**Theorem 4.** $\partial = K$.

*Proof.* Let $\phi$ be inner. Choose $f$ in $H^\infty$, not identically 0. Since $|\hat{\phi}| = 1$ on $X$, $\|f\phi\|_\infty = \|f\|_\infty = \|\hat{f}\|$. There exists $\gamma_0$ in $M$ at which $|\hat{f}\phi|$ attains its maximum, $\|f\phi\|_\infty$, so that
\[ \|\hat{f}\| = |\hat{f}(\gamma_0)\hat{\phi}(\gamma_0)| \leq \|\hat{f}\| \cdot |\hat{\phi}(\gamma_0)| = \|\hat{f}\| . \]
This implies that $|\hat{\phi}(\gamma_0)| = 1$ (i.e., $\gamma_0$ is in $K_\phi$) and that $|\hat{f}(\gamma_0)| = \|\hat{f}\|$. Thus every $|\hat{f}|$ attains its maximum (relative to $M$) at some point of $K_\phi$. This says: $\partial \subset K_\phi$. Since $K$ is the intersection of all $K_\phi$, we have $\partial \subset K$.

To prove that $\partial$ fills all $K$, let $E$ be a proper compact subset of $K$, choose $\gamma_1$ in $K$ but not in $E$. It then follows from Theorem 3 that

\footnote{Kenneth Hoffman has communicated to us a proof which together with Theorem 3 shows that in the classical case of the circle the inner functions separate points on all of $M$.}
there exist finitely many inner functions, say \( \phi_1, \ldots, \phi_n \), such that
\[ \phi_i(\gamma) = 1 \quad \text{for} \quad 1 \leq i \leq n, \]
but
\[ \inf_i \text{Re} \hat{\phi}_i(\gamma) < 1 \quad \text{for every} \gamma \in E. \]

Then \( f = 1 + \phi_1 + \cdots + \phi_n \) is in \( H^\omega \), \( \hat{f}(\gamma) = n + 1 = \|\hat{f}\| \), but \( |\hat{f}(\gamma)| < n + 1 \) for every \( \gamma \) in \( E \). Hence \( E \) does not contain \( \delta \). This completes the proof.

The following result about function algebras was stated without proof in [4] by the first author. We point out that it does not depend on the annulus property.

**Lemma.** Let \( \Sigma \) be a multiplicative semigroup of inner functions. Let \( \mathcal{A}_x \) be the norm-closed subalgebra of \( L^\omega \) which is generated by \( H^\omega \) and the complex conjugates of the members of \( \Sigma \). Then the maximal ideal space \( M_\Sigma \) of \( \mathcal{A}_x \) can be identified with the set \( K_\Sigma \subset M \).

**Proof.** Let \( \Gamma \) be a multiplicative linear functional on \( \mathcal{A}_x \). Restricting \( \Gamma \) to \( H^\omega \), we see that to each such \( \Gamma \) corresponds a unique \( \gamma \) in \( M \), denoted by \( \tau(\Gamma) \), such that \( \Gamma(f) = \hat{f}(\gamma) \) for all \( f \) in \( H^\omega \).

Suppose \( \gamma = \tau(\Gamma) \) and \( \phi \) is in \( \Sigma \). Since \( \phi \hat{\phi} = 1 \), we have
\[ \Gamma(\hat{\phi}) = \Gamma(\phi^{-i}) = 1/\Gamma(\phi) = 1/\hat{\phi}(\gamma). \]
This shows that \( \Gamma \) is determined by \( \gamma \), so \( \tau: M_x \to M \) is one-to-one. It is easy to see that \( \tau \) is continuous. Since both spaces are compact and Hausdorff, \( \tau \) is a homeomorphism. Furthermore, \( \tau(M_x) \subset K_x \), for if \( \gamma = \tau(\Gamma) \), then \( |\hat{\phi}(\gamma)| \leq \|\phi\|_\infty = 1 \), and also
\[ |1/\hat{\phi}(\gamma)| = |\Gamma(\hat{\phi})| \leq \|\hat{\phi}\|_\infty = 1, \]
so that \( |\hat{\phi}(\gamma)| = 1 \) for every \( \phi \) in \( \Sigma \) and every \( \gamma \) in \( \tau(M_x) \).

We want to prove that \( \tau(M_x) = K_x \). To do this, we fix \( \gamma \) in \( K_x \), and show that \( \gamma \) is in \( \tau(M_x) \).

For \( \psi \) in \( H^\omega \) and \( \phi \) in \( \Sigma \), define
\[ \Gamma_0(\psi \hat{\phi}) = \hat{\psi}(\gamma)/\hat{\phi}(\gamma). \]
If \( \psi_1 \hat{\phi}_1 = \psi_2 \hat{\phi}_2 \), then \( \psi_1 \phi_2 = \psi_2 \phi_1 \), which implies \( \hat{\psi}_1(\gamma) \hat{\phi}_2(\gamma) = \hat{\psi}_2(\gamma) \hat{\phi}_1(\gamma) \), and since \( \gamma \) is in \( K_x \), it follows that \( \Gamma_0(\psi_1 \hat{\phi}_1) = \Gamma_0(\psi_2 \hat{\phi}_2) \). In other-words, \( \Gamma_0 \) is well defined on a dense subalgebra of \( \mathcal{A}_x \). It is easy to check that \( \Gamma_0 \) is linear and multiplicative on this subalgebra. Finally (using the fact that \( \gamma \) is in \( K_x \) once more),
\[ |\Gamma_0(\psi \hat{\phi})| = |\hat{\psi}(\gamma)/\hat{\phi}(\gamma)| = |\hat{\psi}(\gamma)| \leq \|\psi\|_\infty = \|\psi \hat{\phi}\|_\infty, \]
so that \( \Gamma_0 \) is bounded and can therefore be extended to a multiplicative linear functional \( \Gamma \) on \( \mathbb{A}_z \). It is clear that \( \tau(\Gamma) = \gamma \), and the proof is complete.

As a consequence, we obtain a theorem of I. J. Schark ([10], [8, p. 174]) which Srinivasan and Wang [11, p. 232] have extended to the context of Weak*-Dirichlet algebras:

**Theorem 5.** The Šilov boundary \( \partial \) of \( H^\infty \) can be identified with the maximal ideal space \( M_\infty \) of \( L^\infty \).

**Proof.** Let \( \Sigma \) be the set of all inner functions. Then
\[
\partial = K = K_\gamma = M_\gamma = M_\infty.
\]
The first of these equalities is Theorem 4, the second is the definition of \( K \), the third is the preceding lemma, and the fourth follows from Theorem 2, since the latter asserts that \( \mathbb{A}_z = L^\infty \).

We now return to the classical situation, i.e., to the unit circle. Recall that an inner function in the open unit disc \( U \) is said to be singular if it has no zero in \( U \).

**Theorem 6.** Suppose \( f \) is in \( L^\infty(T) \), \(| f | = 1, 0 < \varepsilon < 1 \).
(a) There exist Blaschke products \( B_1 \) and \( B_2 \) such that
\[
|| f - B_1/B_2 ||_\infty < \varepsilon.
\]
(b) There exist inner functions \( \phi_1 \) and \( \phi_2 \), with \( \phi_2 \) singular such that
\[
|| f - \phi_1/\phi_2 ||_\infty < \varepsilon.
\]

Of course, the expression \( B_1/B_2 \) in (a) refers to the radial limit function of the quotient of the two Blaschke products, and the norm is the essential supremum over \( T \).

**Proof.** (a) is an immediate consequence of Theorem 1, because of Frostman’s Theorem ([6, pp. 112–113], [8, p. 175]) which asserts that the Blaschke products are norm-dense in the set of all inner functions.

By Theorem 1, it suffices to prove (b) for the case \( f = 1/\psi \), where \( \psi \) is inner. Define
\[
u(w) = \exp \left\{ c \frac{w + 1}{w - 1} \right\}
\]
where \( c > 0 \) is so chosen that \( 3\nu(0) < \varepsilon \), and put
\[ u_i(w) = \frac{u(w) - u(0)}{w[1 - u(0)u(w)]} . \]

Then \( u_1 \) is inner, and one checks easily that
\[ |u(w) - wu_1(w)| < \varepsilon \quad (w \in U) . \]

Put \( w = \psi(z) \) in this inequality, define \( \phi_1 = u_1 \circ \psi \) and \( \phi_2 = u \circ \psi \). Then \( \phi_1 \) and \( \phi_2 \) are inner, \( \phi_2 \) has no zero in \( U \), and
\[ |\phi_2(z) - \psi(z)\phi_1(z)| < \varepsilon \quad (z \in U) . \]

To complete the proof, take radial limits in the last inequality and divide by \( \psi\phi_2 \).

Because of Theorem 6(b), Theorem 2 now takes the following form:

**Theorem 7.** If \( f \) is in \( L^\infty(T) \) and \( \varepsilon > 0 \), then there is a singular inner function \( \phi \) and a finite linear combination \( \psi \) of inner functions, such that
\[ ||f - \psi/\phi||_\infty < \varepsilon . \]

Note that \( \psi/\phi \) is a holomorphic function in \( U \), of bounded characteristic (being a quotient of two \( H^\infty \)-functions). Thus the radial limits of holomorphic functions of bounded characteristic are norm-dense in \( L^\infty(T) \).

We conclude with the observation that the set \( K \) which was described prior to Theorem 3 can be defined (in the classical case) by means of the singular inner functions alone:

**Theorem 8.** If \( \gamma \) in \( M \) is such that \( |\hat\psi(\gamma)| < 1 \) for some inner function \( \psi \), then there is a singular inner function \( \phi \) with \( |\hat\phi(\gamma)| < 1 \).

**Proof.** By Theorem 6(b), with \( \varepsilon = 1 - |\hat\psi(\gamma)| \), there are inner functions \( \phi_1 \) and \( \phi_2 \), with \( \phi_2 \) singular, such that
\[ |\hat\phi_2(\gamma) - \hat\psi(\gamma)\hat\phi_1(\gamma)| \leq ||\phi_2 - \psi\phi_1||_\infty < 1 - |\hat\psi(\gamma)| , \]
which implies that
\[ |\hat\phi_2(\gamma)| < |\hat\psi(\gamma)\hat\phi_1(\gamma)| + 1 - |\hat\psi(\gamma)| \leq 1 . \]

Theorem 8 adds an eighth equivalent condition to the seven that are listed on p. 179 of [8].
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Received March 10, 1969. The First author is a Fellow of the Alfred P. Sloan Foundation. Research supported in part by an NSF Grant GP-8310. The second author was partially supported by NSF Grant GP-6764.

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