

# Pacific Journal of Mathematics

**A NOTE ON THE SIMILARITY OF MATRIX AND ITS  
CONJUGATE TRANSPOSE**

JOHN WALTER DUKE

## A NOTE ON THE SIMILARITY OF A MATRIX AND ITS CONJUGATE TRANSPOSE

JOHN W. DUKE

**It is well-known that each square matrix  $A$  over a field is similar to its transpose  $A^t$  and there exists a nonsingular symmetric matrix  $P$  for which  $PA^t = AP$ . The purpose of this note is to show that if  $A$  is similar to its conjugate transpose  $A^*$  then, under certain conditions, there exists a nonsingular Hermitian matrix  $Q$  for which  $QA^* = AQ$ .**

Let  $f$  be an automorphism of order 2 on a field  $F$  and let  $K$  be the fixed field of  $f$ . For each  $x$  in  $F$ , we denote  $f(x)$  by  $\bar{x}$ . If  $A = (a_{ij})$  is a matrix over  $F$ , let  $A^* = (b_{ij})$  where  $b_{ij} = \overline{a_{ji}}$ . A matrix  $M$  is called Hermitian (skew-Hermitian) provided  $M^* = M$  ( $M^* = -M$ ).

Taussky and Zassenhaus [1] have shown that for each square matrix over a field, there exists a nonsingular symmetric matrix which transforms the given matrix into its transpose. Our main result is

**THEOREM 1.** *Suppose  $F$  is an infinite field whose characteristic is different from 2. If a matrix  $A$  over  $F$  is similar to  $A^*$ , there exists a nonsingular Hermitian matrix  $Q$  over  $F$  for which  $QA^* = AQ$ .*

We shall utilize the following lemmas in both of which  $z$  denotes an element of  $F$  which is not in  $K$ .

**LEMMA 1.** *Every element of  $F$  can be expressed uniquely in the form  $a + bz$  where both  $a$  and  $b$  lie in  $K$ .*

*Proof.* If  $c$  belongs to  $F$ , it is clear that

$$c = c - (c - \bar{c})(z - \bar{z})^{-1}z + (c - \bar{c})(z - \bar{z})^{-1}z$$

since  $z \neq \bar{z}$ . This is the required form since both

$$a = c - (c - \bar{c})(z - \bar{z})^{-1}z$$

and

$$b = (c - \bar{c})(z - \bar{z})^{-1}$$

lie in  $K$ . The uniqueness of the expression follows from the fact that  $z$  does not belong to  $K$ .

LEMMA 2. *If  $c = r + sz$  and  $d = t + uz$  with  $r, s, t,$  and  $u$  in  $K$  and  $c/\bar{c} = d/\bar{d}$ , then  $ru = st$ .*

Lemma 2 implies that there exists a one-to-one correspondence between  $K$  and the set of all elements  $c/\bar{c}$  where  $c = r + z$  and  $r$  ranges over  $K$ . If  $F$  is infinite, Lemma 1 implies that  $K$  is infinite.

*Proof of Theorem 1.* Suppose  $PA^* = AP$  with  $P$  nonsingular. Since the characteristic of  $F$  is not 2, the matrix  $P$  can be expressed as the sum of an Hermitian matrix  $H$  and a skew-Hermitian matrix  $S$ . Hence  $HA^* = AH$  and it remains to show that  $H$  may be chosen nonsingular.

Since  $(cP)A^* = A(cP)$  for all  $c$  in  $F$ , we want to choose  $c$  so that  $M = cP + \bar{c}P^*$  is nonsingular. The matrix  $M$  is nonsingular if and only if  $-c/\bar{c}$  is distinct from all of the eigenvalues of  $P^{-1}P^*$ . Since there exist infinitely many values of  $-c/\bar{c}$ , an element  $c$  can be properly chosen and the proof is complete.

In regard to finite fields, we have

THEOREM 2. *Suppose  $A$  is a square matrix of order  $n$  over a field  $F$  whose characteristic is different from 2 and  $PA^* = AP$  with  $P$  nonsingular. If there exists an element  $y$  in  $F$  such that  $y^m$  does not belong to  $K$  for  $1 \leq m \leq n + 1$ , there exists a nonsingular Hermitian matrix  $Q$  for which  $QA^* = AQ$ .*

*Proof.* Utilizing the same decomposition of  $P$  as in the proof of Theorem 1, it is sufficient to show there exists an element  $c$  in  $F$  such that  $cP + \bar{c}P^*$  is nonsingular. For  $c$  nonzero,  $cP + \bar{c}P^*$  is nonsingular if and only if  $-\bar{c}/c$  is not an eigenvalue of  $P(P^*)^{-1}$ . Hence let  $k_1, k_2, \dots, k_t$  be the distinct eigenvalues of  $P(P^*)^{-1}$  in  $F$  and let

$$W = \{1, -k_1, -k_2, \dots, -k_t\}.$$

If for each nonzero  $x$  in  $F$  there exists  $k$  in  $W$  such that  $\bar{x} = kx$ , then  $k^r$  belongs to  $W$  for all positive integers  $r$  since  $\bar{x^r} = k^r x^r$ . In particular, for the element  $y$  mentioned in the hypothesis of the theorem,  $\bar{y} = dy$  for some  $d$  in  $W$  and hence the elements  $d^i$ , for  $1 \leq i \leq n + 2$ , all belong to  $W$ . Since  $W$  contains only  $t + 1$  elements and  $0 \leq t \leq n$ , it follows that  $d^i = d^j$  for some integers  $i$  and  $j$ ,  $i < j$ , between 1 and  $n + 2$ , inclusively. Hence  $j - i \leq n + 1$  and  $d^{j-i} = 1$  since  $d \neq 0$ . Therefore

$$f(y^{j-i}) = d^{j-i}y^{j-i} = y^{j-i}$$

implies  $y^{j-i}$  belongs to  $K$ . This contradiction shows the existence of

some  $c$  in  $F$  such that  $\bar{c} \neq kc$  for all  $k$  in  $W$ : hence  $c$  does not belong to  $K$  and  $cP + \bar{c}P^*$  is nonsingular as required.

As a simple application of Theorem 2, suppose  $F = GF(p^{2s})$  with  $p \neq 2$  and let  $f(x) = x^{p^s}$  for all  $x$  in  $F$ . By considering a generator of the multiplicative group of  $F$ , one may verify the result for matrices over  $F$  of order less than  $p^s$ .

#### REFERENCE

1. Olga Taussky and Hans Zassenhaus, *On the similarity transformation between a matrix and its transpose*, Pacific J. Math. **9** (1959), 893-896.

Received January 3, 1969. This note is included in the author's doctoral dissertation prepared under the direction of Professor Burton W. Jones at the University of Colorado.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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December, 1969

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