SEMI-SQUARE-SUMMABLE FOURIER-STIELTJES TRANSFORMS

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I. GLICKSBERG

For $G$ a locally compact abelian group with dual $\Gamma$, let $\mu$ be a (finite regular Borel) measure on $G$ with Fourier-Stieltjes transform $\hat{\mu}$. Doss has recently shown that when $\Gamma$ is (algebraically) a totally ordered abelian group and $\hat{\mu}$ is square integrable on the negative half $\Gamma_-$ of $\Gamma$ then its singular component $\sigma$ has $\sigma = 0$ on $\Gamma_-$; in particular $\mu E = 0$ for each common null set $E$ of the analytic measures (those with transforms 0 on $\Gamma_-$), such $E$ being Haar-null.

In the similar (but usually distinct) case in which $\Gamma$ is partially ordered by a nonzero homomorphism $\psi: \Gamma \rightarrow \mathbb{R}$ with $\Gamma_- = \psi^{-1}(-\infty, 0]$ the common null sets $E$ are known, and our purpose is to note in this setting how function algebra results apply to show $\mu E = 0$ when $\hat{\mu} \in L^2(\Gamma_-)$, and when $\hat{\mu}$ satisfies sometimes weaker (but more obscure) hypotheses.

Doss' results appear in [2], and the function algebra results we apply are those in [4, §1], [5, §2], with which we shall assume the reader familiar. The common null sets mentioned above are given in [1, 5].

THEOREM 1. Let $\psi, \Gamma_-$ be as above and let $\varphi: \mathbb{R} \rightarrow G$ be the homomorphism dual to $\psi$. If

\[
\int_{\Gamma_-} |\hat{\mu}(\gamma)|^2 \, d\gamma < \infty
\]

then $\mu$ vanishes on all Borel $E \subset G$ for which

\[
\{t \in \mathbb{R}: x + \varphi(t) \in E\}
\]

has linear measure 0, for all $x \in G$, i.e. (by definition [3, §2]) $\mu$ is absolutely continuous in the direction of $\varphi$.

Proof. Let $G^a$ be the Bohr compactification of $G$, with dual $\Gamma_d$, the discrete version of $\Gamma$. Dual to $\psi: \Gamma_d \rightarrow \mathbb{R}$ we have a map of $\mathbb{R}$ into $G^a$, the composition $\mathbb{R} \xrightarrow{\varphi} G \rightarrow G^a$, which we still call $\varphi$. Note that each Borel $E$ in $G$ is Borel$^1$ in $G^a$, and if $E \subset G$ satisfies (2) for $x \in G$ it does for all $x$ in $G^a$ (the set is void for $x \in G^a \setminus G$). As in [5] we are forced to transfer our attention to $G^a$ to apply the function algebra results.

\[^1\] We take the $\sigma$-ring generated by compacta as our Borel sets.
Let $A$ be the closed span of $\Gamma = \psi^{-1}[0, \infty)$ in $C(G^a)$, a subalgebra of $C(G^a)$. As usual we can shift $\mu$ to a measure on $G^a$ carried by its subset $G$ [6] with the same Fourier-Stieltjes transform as before. Let $\hat{f}$ be the element $\hat{\mu} \chi$ of $L^1(\Gamma')$, where $\chi$ is the characteristic function of $\Gamma^-$, and $\hat{f}$ the element of $L^1(G)$ corresponding to $\hat{f}$.

For any trigonometric polynomial $p = \sum c_i \gamma_i$ in $A$ (i.e., with $\psi(\gamma_i) \geq 0$) we have
\[
(p \mu)^{\wedge}(\gamma) = \int \gamma p d \mu = \sum c_i \hat{\mu}(\gamma - \gamma_i) = (\sum c_i \delta_{-\gamma_i}) \hat{\mu}(\gamma),
\]
and since
\[
(\sum c_i \delta_{-\gamma_i}) \hat{\mu}(\gamma) = \sum c_i \hat{\mu}(\gamma - \gamma_i) \chi(\gamma - \gamma_i)
\]
if $\psi(\gamma) \leq 0$, we have
\[
\int_{r^-} |(p \mu)^{\wedge}|^2 d\gamma \leq \int |(\sum c_i \delta_{-\gamma_i}) \hat{f}|^2 d\gamma = \int |pf|^2 dx \leq \|p\|_\infty \|f\|_2,
\]
or
\[
(3) \quad \| (p \mu)^{\wedge} \chi \|_2 \leq \|f\|_2 \|p\|_\infty.
\]

Now (3) continues to hold for any $a \in A$ in place of $p \in A$: for if $p_n \to a$ in $A$ then $(p_n \mu)^{\wedge} \to (a \mu)^{\wedge}$ uniformly, so that for any compact $K \subset \Gamma^-$
\[
\int_K |(a \mu)^{\wedge}|^2 d\gamma = \lim \int_K |(p_n \mu)^{\wedge}|^2 d\gamma \leq \|f\|_2^2 \lim \|p_n\|_\infty
\]
whence $\|(a \mu)^{\wedge} \chi\|_2 \leq \|f\|_2 \|a\|_\infty$. Indeed this clearly follows whenever $\|p_n\|_\infty \leq \|a\|_\infty$ and $(p_n \mu)^{\wedge} \to (a \mu)^{\wedge}$ uniformly.

Let $\gamma$ be a fixed element of $\Gamma$ with $\psi(\gamma) > 0$, and let $\mu = \nu + \sigma$ be the Lebesgue decomposition of $\mu$ relative to $M^\gamma$ (the probability measures on $G^a$ orthogonal to $\gamma A$, cf. [4, §1]), with $\nu \ll M^\gamma$, $\sigma M^\gamma$-singular. By the argument of the last paragraph of [5, §2], $\nu$ vanishes on Borel sets in $G^a$ satisfying (2), so we can complete our proof by showing $\sigma = 0$. As in [4] $\sigma$ is carried by $\bigcup K_n$, where $K_n$ is a compact $M^\gamma$-null set.

By the abstract Forelli Lemma [4, 1.2] (applied to the algebra $C + \gamma A$) and dominated convergence we have $\{a_n\}$ in the unit ball of $A$ for which $a_n \mu \to \sigma$ in norm, so $(a_n a \mu)^{\wedge} \to (a \sigma)^{\wedge}$ uniformly and again we conclude that $\|(a \sigma)^{\wedge} \chi\|_2 \leq \|f\|_2 \|a\|_\infty$ for $a \in A$.

\textsuperscript{2} It should be noted that when $G$ is compact $f \in L^1(G)$ and the result follows trivially from [1]; for then $\nu(dx) = \mu(dx) - f(x)dx$ defines an analytic measure.
Now by [5, § 2] each measure \( \tau \) on \( G^a \) orthogonal to \( A \) has \( \tau_{K_n} = 0 \) for each \( K_n \) and thus by [3, 4.8] \( K_n \) is an intersection of peak sets of \( A \), and an interpolation set for \( A \); using the regularity of \( \sigma \) one then concludes\(^8\) there is a sequence \( \{a_j\} \) in the unit ball of \( A \) for which \( a_j \sigma \to |\sigma_{K_n}| \) in norm. So again \( \| |\sigma_{K_n}|^\wedge \chi \|_2 \leq \| f \|_2 \cdot 1 \), which of course implies \( |\sigma_{K_n}|^\wedge \in L^2(\Gamma) \) since the absolute value of this function is even. Because \( \mu \) is carried by the subset \( G \) of \( G^a \), the same is true of its restrictions \( \sigma \) and \( \sigma_{K_n} \) and so, as a measure on \( G \) with square summable transform, \( |\sigma_{K_n}| \) is absolutely continuous by the elementary argument given by Doss [2, Th. 1]. Hence \( \sigma \) is absolutely continuous.

To complete our proof we can show \( \sigma = 0 \) by showing \( \sigma \) is carried by a Haar-null set. And since \( \sigma \) is carried by a \( \sigma \)-compact set, it suffices to show \( \sigma_{x_0 + V} \) is carried by a Haar null set for each \( x_0 \in G \) and some compact symmetric neighborhood \( V \) of the identity. But \( \sigma \) and each \( \lambda \in M^\gamma \) are mutually singular, so it suffices to show there is a \( \lambda \) in \( M^\gamma \) equivalent to Haar measure on \( x_0 + V \), and, for example, with \( m \) Haar measure

\[
\lambda E = \int_{-\infty}^{\infty} \int_{\gamma} \frac{1}{m2V} \chi_{x_0 + z_0}(x - \varphi(t))dx d\rho(t)dt
\]

defines such a measure if

\[
\hat{\rho}(s) = \begin{cases} 1 - \frac{|s|}{\psi(\gamma)}, & |s| \leq \psi(\gamma) \\
0 & \text{elsewhere} \end{cases}
\]

Indeed

\[
\rho(t) = \psi(\gamma) \left( \frac{\sin t\psi(\gamma)/2}{t/2} \right)^2 \geq 0
\]

so \( \lambda \geq 0 \) and

\[
\hat{\lambda}(\gamma) = \frac{1}{m2V} \chi_{x_0 + z_0}(\gamma) \cdot \hat{\rho}(\psi(\gamma)),
\]

as is easily verified; so \( \hat{\lambda} \) vanishes off \( \psi^{-1}(-\psi(\gamma), \psi(\gamma)) \) whence \( \lambda \) is orthogonal to \( \gamma A \), the span of \( \{\beta \in \Gamma: \psi(\beta) \geq \psi(\gamma)\} \). And \( \lambda E = 0 \) implies

\[
\int_{\Gamma} \chi_{x_0 + z_0}(x - \varphi(t_0))dx = 0
\]

for some \( t_0 \) with \( \varphi(t_0) \in V \) since \( \rho(t) > 0 \) a.e., \( \varphi(0) \in V \) and \( \varphi \) is con-

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\(^8\) By regularity there is a peak set (an intersection of countably many such) \( F_n \supset K_n \) for which \( \sigma_{F_n} = \sigma_{K_n} \), and if \( f \) peaks on \( F_n \) then \( f_k \to 1 \) a.e. \( |\sigma_{K_n}| \to 0 \) a.e. \( |\sigma_{K_n}^\wedge| \). If \( \sigma_{K_n} = \rho |\sigma_{K_n}^\wedge|, |\rho| \equiv 1 \), then we have \( f_k \) in the unit ball of \( C(K_n) \) for which \( f_k \to \rho \) a.e. \( |\sigma_{K_n}| \), hence \( b_k \) in the \((1 + \varepsilon)\)-ball of \( A \) for which \( b_k = f_k \) on \( K_n \), whence \( b_k f^k \sigma \to |\sigma_{K_n}| \) by dominated convergence.
tinuous, so if \( E \subset x_0 + V \) we have \( E - \varphi(t_0) \subset x_0 + 2V \), and therefore
\[
0 = \int_E \chi_{x_0+2V}(x - \varphi(t_0))dx = \int_E 1dx = mE.
\]
Hence \( m_{x_0+2V} \ll \chi_{x_0+2V} \); the reverse is obvious (and actually unnecessary) and our proof complete.

Variants of theorem 1 can be obtained from the same argument, but seem to require more artificial hypotheses. For example

**Theorem 2.** With \( \psi, \Gamma \) as before, suppose the continuous function \( f = f^* \in L^1(G) \cap L^1(\Gamma) \) never vanishes on \( \mathbb{G} \), and \( \mu \) is a measure for which for some \( k \)
\[
\int \left| \int (p \mu)^*(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta \leq k \left\| p \right\|_{\infty}
\]
for all trigonometric polynomials \( p = \sum a_i \gamma_1 \) with \( \psi(\gamma_i) \geq 0 \). Then \( \mu E = 0 \) for each Borel \( E \subset G \) satisfying (2).

We argue exactly as before that if \( p_n \mu \to a \mu \) and \( \left\| p_n \right\|_{\infty} \leq \left\| a \right\|_{\infty}, a \in A \), one has
\[
\int_K \left| \int (a \mu)^*(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta \leq k \left\| a \right\|_{\infty}
\]
for \( K \) compact, so (4) holds for \( p \) an arbitrary element of \( A \).

With \( \mu = \nu + \sigma \) as before we again obtain (4) for \( p \in A \) and \( \sigma \) in place of \( \mu \), and then for \( 1 = p \in A \) and \( |\sigma_{K_n}| = \tau \) in place of \( \sigma \). But since \( \tau(-\gamma) = \hat{\tau}(\gamma) \) the finite integral
\[
\int \left| \int \hat{\tau}(\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta
\]
coincides with
\[
\int \left| \int \hat{\tau}(-\gamma)\hat{f}(\beta - \gamma)d\gamma \right|^2 d\beta = \int \left| \int \hat{\tau}(\gamma)\hat{f}(\gamma - \beta)d\gamma \right|^2 d\beta
\]
so that, by Minkowski, \( \hat{\tau} \hat{f} \in L^2(\Gamma) \). Trivially one verifies that the transform of the finite measure \( f^* \) on \( G \) is \( \hat{\tau} \hat{f} \): thus \( f^* \) is absolutely

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4 When such an \( f \) exists this contains the preceding result. For when \( \hat{\nu} \in L^2(\Gamma) \) so is \( (p \mu)^* \chi \) and always of norm \( \leq k \left\| p \right\|_{\infty} \) as we saw in the proof of Theorem 1. But then \( \left\| (p \mu)^* \chi \hat{f} \right\|_1 \leq \left\| (p \mu)^* \chi \right\|_1 \| \hat{f} \|_1 \leq k \left\| p \right\|_{\infty} \| \hat{f} \|_1 \) which is (4).
continuous, so \( \tau = |\sigma_{K_n}| \) is since \( f \) never vanishes; again \( \sigma \) is singular with respect to Haar measure, and \( \sigma = 0 \) follows.

**Theorem 3.** Suppose there are \( \gamma_n \in \Gamma \) for which
\[ \varepsilon_n = \| \tilde{\gamma}_n \mu \|_A \to 0, \]
where the norm is that of \( \tilde{\gamma}_n \mu \) as a functional on \( A = \text{span} \Gamma \). Then \( \mu E = 0 \) for every Borel \( E \) in \( G \) satisfying (2).

We are supposing that \( |(a \mu)^\wedge(\gamma_n)| \leq \varepsilon_n \| a \|_\infty \) for each \( a \in A \), where \( \varepsilon_n \to 0 \). As before we have \( a_j \in A \), \( \| a_j \| \leq 1 \), with \( a_j \mu \to \sigma \), where \( \sigma \) is the \( M^r \)-singular component of \( \mu \), so

\[ (7) \quad |(a \sigma)^\wedge(\gamma_n)| \leq \varepsilon_n \| a \|_\infty \]

follows since \( (a_j \cdot a \mu)^\wedge \to (a \sigma)^\wedge \) uniformly. Now we have \( \sigma \) carried by \( \cup K_j, K_j \) a compact \( M^r \)-null set, and as before an intersection of peak sets of \( A \) and an interpolation set for \( A \). So exactly as before (cf. footnote 3) we have \( \{a_k\} \) in the unit ball of \( A \) for which \( a_k \sigma \to \gamma_n |\sigma_{K_j}| \), whence by (7)

\[ |(\gamma_n |\sigma_{K_j}|)^\wedge(\gamma_n)| = |\sigma_{K_j} |(1) = |\sigma_{K_j}| \leq \varepsilon_n \]

for all \( n \), so \( \sigma_{K_j} = 0 \), \( \sigma = 0 \), completing our proof as before.

As a final remark, we note that for any measure \( \mu \) vanishing on all \( E \) satisfying (2), i.e., for \( \mu \) absolutely continuous in the direction of \( \varphi \), if \( |\varphi(\gamma_n)| \to \infty \), we (at least) have \( \tilde{\gamma}_n \mu \to 0 \) weakly.\(^5\) Indeed since \( \Gamma \mu = \{\gamma \mu: \gamma \in \Gamma\} \) is conditionally weakly compact we need only see any weak cluster point of \( \{\tilde{\gamma}_n \mu\} \) must be 0, so it suffices to show

\[ (\tilde{\gamma}_n \mu)^\wedge(\gamma) = \hat{\mu}(\gamma + \tilde{\gamma}_n) \to 0. \]

But this follows directly from the following easy “Riemann-Lebesgue lemma”: If \( \mu \) is absolutely continuous in the direction of \( \varphi \) then for any \( \varepsilon > 0 \) there is an \( N \) for which \( |\hat{\mu}(\gamma)| < \varepsilon \) if \( |\varphi(\gamma)| > N \).

By [3, 2.4] \( \mu \) translates continuously in the direction of \( \varphi \), i.e., \( \| \mu - \mu_t \| < \varepsilon \) if \( |t| < \delta \), where \( \mu_t E = \mu(\varphi(t) + E) \). Thus for an appropriate continuous \( f \) on \( R \) vanishing off \((-\delta, \delta)\) we have

\[ \| \mu^* f - \mu \| < \varepsilon, \]

where

\[ \mu^* f = \int \mu_t f(t) dt \]

\(^5\) Thus for any measure \( \mu \) on \( G \) one has an analogue of a well known lemma of Helson: if \( |\varphi(\gamma_n)| \to \infty \), any weak cluster point \( \nu \) of \( \{\tilde{\gamma}_n \mu\} \) is carried by a subset \( E \) of \( G \) satisfying (2), i.e., null in the direction of \( \varphi \) in the terminology of [3]. (For \( \nu \) is necessarily a weak cluster point of \( \{\gamma_n \sigma\} \), where \( \sigma \) is the \( M^r \)-singular component of \( \mu \), as always.)
can be interpreted as, say, a Riemann integral. But

\[
(\mu * f)^\wedge(\gamma) = \int \int (x, \gamma) \mu(dx)f(t)dt \\
= \int (x - \varphi(t), \gamma) \mu(dx)f(t)dt \\
= \hat{\mu}(\gamma) \int (\varphi(t), \gamma)f(t)dt \\
= \hat{\mu}(\gamma) \int (t, \psi(\gamma))f(t)dt = \hat{\mu}(\gamma)\hat{f}(-\psi(\gamma))
\]

which shows \((\mu * f)^\wedge\) has the desired property by the Riemann-Lebesgue lemma applied to \(f\). As a uniform limit of such functions \(\hat{\mu}\) of course has the same property.

REFERENCES


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