

# Pacific Journal of Mathematics

**ON ILYEFF'S CONJECTURE**

A. MEIR AND AMBIKESHWAR SHARMA

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**An apparently easy problem due to Ilyeff states: If all zeros  $z_1, z_2, \dots, z_n$  of a complex polynomial  $P(z)$  lie in  $|z| \leq 1$  then there is always a zero of  $P'(z)$  in each of the disks  $|z - z_j| \leq 1, j = 1, \dots, n$ . If true, the conjecture is best possible as one can see from the example  $P(z) = z^n - 1$ . In full generality the conjectured result was proved only for polynomials of degree  $\leq 4$ . In this paper the conjecture is proved for quintics and extensions of earlier results are obtained for zeros of higher derivatives of polynomials having multiple roots.**

The above conjecture of Ilyeff was published in Hayman's *Research Problems in Function Theory*. Its validity for polynomials of degree  $\leq 4$  was proved in [1] and [5]. Rubinstein has shown in [5] that the statement holds in general if  $|z_j| = 1$ . A conjecture stronger than that of Ilyeff was announced in [2] and was proved for those zeros  $z_j$  of  $P(z)$  for which  $|z_j| = 1$ .

### 2. Zeros of multiplicity $k$ on the boundary.

**THEOREM 1.** *Let  $P(z) = (z - z_0)^k Q(z)$ ,  $Q(z) = \prod_{j=1}^{n-k} (z - z_j)$  with  $|z_0| = 1$ , and  $|z_j| \leq 1, z_j \neq z_0$  ( $j = 1, \dots, n - k$ ). Then at least one zero of  $P^{(\nu)}(z)$  ( $1 \leq \nu \leq n - 1$ ) lies in the disk*

$$(2.1) \quad \left| z - \frac{k}{\nu + 1} z_0 \right| \leq 1 - \frac{k}{\nu + 1}.$$

*For  $\nu > k$ , strict inequality will hold in (2.1) except when  $\nu = n - 1$  and  $P(z) = (z - z_0)^k (z - z_1)^{n-k}$  with  $|z_0| = |z_1| = 1$ .*

**REMARK.** The conjectured result of Goodman, Rahman and Ratti [2] for zeros on the boundary is included in Theorem 1 as a special case when  $k = 1, \nu = 1$ .

*Proof.* Without loss of generality, we may assume  $z_0 = 1, \nu \geq k$ . Then we easily have

$$(2.2) \quad \frac{p^{(\nu+1)}(1)}{p^{(\nu)}(1)} = \frac{\nu + 1}{\nu - k + 1} \cdot \frac{Q^{(\nu-k+1)}(1)}{Q^{(\nu-k)}(1)}.$$

Denoting the zeros of  $P^{(\nu)}(z)$  by  $\zeta_1, \dots, \zeta_{n-\nu}$  and those of  $Q^{(\nu-k)}(z)$  by  $w_1, \dots, w_{n-\nu}$ , we have from (2.2)

$$\operatorname{Re} \sum_{j=1}^{n-\nu} \frac{1}{1-\zeta_j} = \frac{\nu+1}{\nu-k+1} \operatorname{Re} \sum_{j=1}^{n-\nu} \frac{1}{1-w_j}.$$

Since by Gauss-Lucas theorem we have  $|w_j| \leq 1$ , it follows that

$$\operatorname{Re} (1-w_j)^{-1} \geq \frac{1}{2}$$

for all  $j$ . Thus

$$(2.3) \quad \frac{1}{n-\nu} \sum_{j=1}^{n-\nu} \operatorname{Re} \frac{1}{1-\zeta_j} \geq \frac{1}{2} \cdot \frac{\nu+1}{\nu-k+1},$$

so that

$$(2.4) \quad \max_j \operatorname{Re} (1-\zeta_j)^{-1} \geq \frac{1}{2} \cdot \frac{\nu+1}{\nu-k+1}$$

which is equivalent to (2.1).

In (2.3), strict inequality will hold unless all the zeros of  $Q^{(\nu-k)}(z)$  lie on the unit circle. This can happen only if  $z_1 = z_2 = \dots = z_{n-k}$  and  $|z_1| = 1$ . For suppose  $Q^{(\nu-k)}(z)$  has  $p$  distinct zeros  $w_1, \dots, w_p$  with multiplicities  $m_1, \dots, m_p$ . Then by the Gauss-Lucas theorem  $Q(z)$  must have the same zeros with multiplicities  $m_1 + \nu - k, \dots, m_p + \nu - k$  so that the degree of  $Q(z)$  will be  $n - \nu + p(\nu - k) = n - k$ . Hence  $p = 1$ , i.e.,  $Q^{(\nu-k)}(z) = (z - w_1)^{n-\nu}$  and so  $w_1 = z_1$  and  $Q(z) = (z - z_1)^{n-k}$ . Thus  $P(z) = (z - 1)^k(z - z_1)^{n-k}$  and so all zeros of  $P^{(\nu)}(z)$  must lie on the line segment connecting  $z_1$  and 1. Strict inequality will hold in (2.4) unless

$$(2.5) \quad \left| \zeta_j - \frac{k}{\nu+1} \right| = \frac{\nu-k+1}{\nu+1}, \quad j = 1, \dots, n-\nu,$$

so that  $\zeta_1 = \zeta_2 = \dots = \zeta_{n-\nu} = (k/(\nu+1) + (\nu-k+1)/(\nu+1))z_1$ . Since the centroid of the zeros of polynomial is invariant under differentiation, we must also have

$$\zeta_1 = \frac{k + (n-k)z_1}{n} = \frac{k}{\nu+1} + \frac{\nu+k+1}{\nu+1}z_1,$$

so that  $\nu = n - 1$ , which proves the assertion.

Taking  $P_a(z) = (z - 1)(z^2 - 2az + 1)$  with  $-1/2 \leq a \leq 1$ , we see that the zeros of  $P'_a(z)$  fill the entire circumference of the circle  $|z - 1/2| = 1/2$ , so that for  $\nu = k = 1$ , the result (2.1) cannot be improved.

**3. Some lemmas.** If the polynomial  $P(z) = (z - z_0)^k Q(z)$ ,  $Q(z) = \prod_{j=1}^{n-k} (z - z_j)$ ,  $z_0 \neq z_j$ ,  $j = 1, \dots, n - k$ , then as in (2.2), we have for

$\nu \geq k$  (if  $P^{(\nu)}(z_0) \neq 0$ ),

$$(3.1) \quad \frac{P^{(\nu+1)}(z_0)}{P^{(\nu)}(z_0)} = \frac{\nu + 1}{\nu - k + 1} \cdot \frac{Q^{(\nu-k+1)}(z_0)}{Q^{(\nu-k)}(z_0)}.$$

Denoting the zeros of  $P^{(\nu)}(z)$  by  $\zeta_1, \dots, \zeta_{n-\nu}$  and of  $Q^{(\nu-k)}(z)$  by  $w_1, \dots, w_{n-\nu}$  and setting

$$|z_0 - w_j| = r_j, |z_0 - \zeta_j| = \rho_j, j = 1, \dots, n - \nu,$$

$(r_1 \leq r_2 \leq \dots \leq r_{n-\nu}; \rho_1 \leq \rho_2 \leq \dots \leq \rho_{n-\nu})$  we have

$$(3.2) \quad \sum_{j=1}^{n-\nu} \frac{1}{z_0 - \zeta_j} = \frac{\nu + 1}{\nu - k + 1} \sum_{j=1}^{n-\nu} \frac{1}{z_0 - w_j}.$$

$$(3.3) \quad \prod_{j=1}^{n-\nu} r_j = \frac{\binom{n}{k}}{\binom{\nu}{k}} \prod_{j=1}^{n-\nu} \rho_j,$$

where the last relation follows from the fact that

$$\begin{aligned} P^{(\nu)}(z_0) &= \binom{n}{\nu} \nu! \prod_{j=1}^{n-\nu} (z_0 - \zeta_j) \\ &= \binom{n-k}{\nu-k} \nu! \prod_{j=1}^{n-\nu} (z_0 - w_j). \end{aligned}$$

In the sequel we shall need the following lemmas.

**LEMMA 1.** *Let  $f(z) = \sum_{j=0}^n \binom{n}{j} a_j z^j$ ,  $g(z) = \sum_{j=0}^n \binom{n}{j} b_j z^j$ ,  $h(z) = \sum_{j=0}^n \binom{n}{j} a_j b_j z^j$ , and suppose that the zeros of  $f(z)$  lie in the annulus  $p \leq |z| \leq q$ , and those of  $g(z)$  lie in  $r \leq |z| \leq s$ , then the zeros of  $h(z)$  lie in  $pr \leq |z| \leq qs$ .*

This lemma is a special case of a theorem due to Szego [4; p. 65, Th. 16.1]. In particular if  $R(t)$  is a polynomial of degree  $n - k$ , and  $f(t) = d^\nu/dt^\nu \{t^k R(t)\}$  and  $h(t) = R^{(\nu-k)}(t)$  ( $\nu \geq k$ ), then an easy computation shows that the polynomial  $g(t)$  of the above lemma may be chosen, except for a constant factor, as follows:

$$(3.3a) \quad g(t) = \sum_{j=0}^{n-\nu} \frac{\binom{n-\nu}{j} \binom{n}{k}}{\binom{\nu+j}{k}} t^j.$$

**LEMMA 2.** *Let  $r_1, \dots, r_m$  and  $a, b, c$  ( $a^m \leq c \leq b^m$ ) be positive numbers satisfying*

$$(3.4) \quad a \leq r_j \leq b$$

$$(3.5) \quad \prod_{j=1}^m r_j \geq c.$$

Then

$$(3.6) \quad \sum_{j=1}^m \frac{1}{r_j^2} \leq \frac{m - \mu}{a^2} + \frac{\mu - 1}{b^2} + \left( \frac{a^{m-\mu} b^{\mu-1}}{c} \right)^2$$

where

$$(3.7) \quad \mu = \min \{ \nu \mid b^\nu a^{m-\nu} \geq c, \nu \text{ integer} \}.$$

*Proof.* We first observed that the maximum of  $\sum_{j=1}^m r_j^{-2}$  is not attained unless equality holds in (3.5) for if  $\prod_{j=1}^m r_j > c$ , then at least one of the  $r_j$ 's say  $r_1$  is strictly greater than  $a$  and so replacing it by  $(1 - \epsilon) \cdot r_1$  with a suitable  $\epsilon$ , we can increase the sum  $\sum r_j^{-2}$ .

Also at most one of the  $r_j$ 's can lie in the open interval  $(a, b)$ . For if we had for some  $i$  and  $j$ ,  $a < r_i \leq r_j < b$ , then replacing  $r_i$  by  $r_i/1 + \epsilon$ , and  $r_j$  by  $r_j(1 + \epsilon)$  with suitable  $\epsilon$ , such that (3.4) and (3.5) remain valid, the sum  $\sum r_j^{-2}$  would be increased by

$$\frac{(1 + \epsilon)^2 - 1}{r_i^2} + \frac{(1 + \epsilon)^{-2} - 1}{r_j^2}$$

which is strictly positive.

So to maximize  $\sum r_j^{-2}$ , we must have

$$r_1 = r_2 = \dots = r_{m-\nu} = a \leq r_{m-\nu+1} \leq r_{m-\nu+2} = \dots = r_m = b$$

so that from  $a^{m-\nu} r_{m-\nu+1} b^{\nu-1} = c$  we obtain

$$a^{m-\nu+1} \cdot b^{\nu-1} < c \leq a^{m-\nu} b^\nu$$

which gives (3.7).

**LEMMA 3.** *Let  $0 < \alpha \leq 1$  and suppose  $w$  is a point in the closed unit disk. Then*

$$(3.8) \quad \operatorname{Re} \frac{1}{\alpha - w} \geq \frac{1}{2\alpha} - \frac{1 - \alpha^2}{2\alpha} \cdot \frac{1}{r^2}, \quad r = |\alpha - w|.$$

The proof follows from elementary geometric considerations.

#### 4. Zeros inside the disk. We shall prove the theorems:

**THEOREM 2.** *If  $P(z) = (z - z_0)^k Q(z)$ , ( $k \geq 1, n \geq 2 + k$ ),  $|z_0| \leq 1$ ,  $Q(z) = \prod_{j=1}^{n-k} (z - z_j)$ ,  $z_j \neq z_0, |z_j| \leq 1$  ( $j = 1, \dots, n - k$ ), then at least one zero of  $P^{(n-2)}(z)$  lies in the closed disk*

$$(4.1) \quad |z - z_0| \leq \frac{2(n - k - 1)}{n - 1} \sqrt{\frac{n - 1 + |z_0|}{n}} .$$

REMARKS. (i) When  $n = 3, k = 1$ , the theorem asserts the existence of a zero of  $P'(z)$  in  $|z - z_0| \leq \sqrt{2 + |z_0|/3}$  which implies the Ilyeff's conjecture in this case. A comparison of (4.1) with (2.1) in the special case  $n = 4, k = 1, \nu = 2$  and  $z_0 = 1$  shows that Theorem 1 asserts the existence of a zero of  $P''(z)$  in  $|z - 1/3| \leq 2/3$ , while (4.1) does so in the disk  $|z - 1| \leq 4/3$ . However, Theorem 2 holds even when  $|z_0| < 1$ .

(ii) Under the hypothesis of Theorem 2, it is possible to replace the right side of (4.1) by

$$\frac{n - k - 1}{n - 1} \theta(z_0)$$

where  $\theta(z_0) = |z_0| + \sqrt{2 - |z_0|^2}$ , which for large values of  $n$  yields a disk smaller than the one given by (4.1).

*Proof.* Without loss of generality, we may take  $z_0 = \alpha, 0 \leq \alpha \leq 1$ . Setting in Lemma 1,  $f(t) = P^{(n-2)}(\alpha + t) = (d^{n-2}/dt^{n-2})(t^k Q(\alpha + t))$  and  $h(t) = Q^{(n-2-k)}(\alpha + t)$ , we have by (3.3a)

$$g(t) = t^2 + \frac{2n}{n - k} t + \frac{n(n - 1)}{(n - k - 1)(n - k)} .$$

For the zeros  $\beta_1$  and  $\beta_2$  of  $g(t)$ , we have

$$|\beta_1|^2 = |\beta_2|^2 = \frac{n(n - 1)}{(n - k)(n - k - 1)} .$$

Assuming that  $\rho_1 \leq \rho_2$  and  $r_1 \leq r_2$  (see notation proceeding (3.2)) we have by Lemma 1,

$$(4.2) \quad \rho_1 \sqrt{\frac{n(n - 1)}{(n - k)(n - k - 1)}} \leq r_1 \leq r_2 ,$$

whence

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} \leq \frac{(n - k)(n - k - 1)}{n(n - 1)} \frac{2}{\rho_1^2} .$$

Suppose now the theorem is false. Then

$$(4.3) \quad \rho_2 \geq \rho_1 > \frac{2(n - k - 1)}{n - 1} \sqrt{\frac{n - 1 + \alpha}{n}} ,$$

and thus

$$(4.4) \quad \frac{1}{r_1^2} + \frac{1}{r_2^2} < \frac{1}{2} \frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)}.$$

Also from (4.3) and (4.2) and from  $n-k \geq 2$ , we have  $r_1 > 1$ , which for  $\alpha = 0$  yields the desired contradiction.

If  $\alpha \neq 0$  then from (3.2), (4.4) and Lemma 3 we get

$$(4.5) \quad \begin{aligned} & \frac{1}{2} \operatorname{Re} \sum_{j=1}^2 \frac{1}{\alpha - \zeta_j} \\ & > \frac{n-1}{2(n-k-1)} \left\{ \frac{1}{\alpha} - \frac{1-\alpha^2}{4\alpha} \cdot \frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)} \right\} \\ & \geq \frac{n-1}{2(n-k-1)} \cdot \frac{1}{2\alpha} \left\{ 2 - \frac{(1-\alpha^2)(n-1)}{n-1+\alpha} \right\} \\ & > \frac{n-1}{4\alpha(n-k-1)} \left\{ 1 + \alpha^2 \cdot \frac{n}{n-1+\alpha} \right\} \\ & \geq \frac{n-1}{2(n-k-1)} \sqrt{\frac{n}{n-1+\alpha}}, \end{aligned}$$

observing that  $n-k \leq 2(n-k-1)$ . Since  $1/\rho_j \geq \operatorname{Re} 1/\alpha - \zeta_j$ ,  $j = 1, 2$ , (4.5) yields a contradiction to (4.3) which completes the proof of the theorem.

**THEOREM 3.** Suppose  $P(z) = (z-z_0)^k Q(z)$ , ( $k \geq 1$ ,  $2k \leq n-2$ ),  $|z_0| \leq 1$ ,  $Q(z) = \prod_{j=1}^{n-k} (z-z_j)$ ,  $z_j \neq z_0$ ,  $|z_0| \leq 1$  ( $j = 1, \dots, n-k$ ). Then at least one zero of  $P^{(n-3)}(z)$  lies in the disk

$$(4.6) \quad |z - z_0| \leq \frac{(n-k-2)}{n-2} \theta(z_0)$$

where  $\theta(z_0) = |z_0| + \sqrt{2 - |z_0|^2}$ .

**REMARK.** (i) In the special case  $n = 4$ ,  $k = 1$ , the above theorem gives an improvement on Theorem 2 of [5], since it guarantees the existence of a zero of  $P'(z)$  in  $|z - z_0| \leq 1/2(|z_0| + \sqrt{2 - |z_0|^2}) < 1$  if  $|z_0| \neq 1$ .

(ii) In case  $2k > n-2$ ,  $n \geq k+3$  we can prove that under the conditions of Theorem 3, the disk  $|z - z_0| \leq (n-k-1/n-1)\theta(z_0)$  will contain at least one zero of  $P^{(n-3)}(z)$ . In particular the disk

$$|z - z_0| \leq \frac{1}{2} \theta(z_0) \leq 1$$

will include at least one zero of  $P^{(n-3)}(z)$  when

$$k > \frac{n-2}{2}.$$

*Proof.* As in Theorem 2, we set  $z_0 = \alpha, 0 \leq \alpha \leq 1$  and identify the polynomials  $f(t), g(t)$  and  $h(t)$  of Lemma 1, as follows:

$$f(t) \equiv P^{(n-3)}(\alpha + t), h(t) = Q^{(n-3-k)}(\alpha + t),$$

and except for a constant factor

$$g(t) = \sum_{j=0}^3 \binom{n}{j} \binom{3}{j} t^{3-j} / \binom{n-k}{j}.$$

Since  $g'(t) > 0$  for real  $t$ , it follows that  $g(t)$  has exactly one real zero. A straightforward substitution yields

$$g\left(-\frac{n}{n-k-1}\right) \leq 0 \leq g\left(\frac{n-2}{n-k-2}\right)$$

on using the assumption  $2k \leq n - 2$ . So denoting the zeros of  $g(t)$  by  $t_1, t_2, t_3$  then for the real zero, say  $t_3$ , we have

$$\frac{n-2}{n-k-2} \leq |t_3| \leq \frac{n}{n-k-1}.$$

Since  $\bar{t}_2 = t_1$ , and  $|t_1 t_2 t_3| = |t_1|^2 |t_3| = \binom{n}{3} / \binom{n-k}{3}$ , we obtain

$$\begin{aligned} \frac{(n-1)(n-2)}{(n-k)(n-k-2)} &\leq |t_1|^2 \\ &\leq \frac{n(n-1)}{(n-k)(n-k-1)} \leq \left(\frac{n-2}{n-k-2}\right)^2 \leq |t_3|^2. \end{aligned}$$

Now by Lemma 1 (using the notation of §3)

$$(4.7) \quad \rho_1^2 \frac{(n-1)(n-2)}{(n-k)(n-k-2)} \leq r_1^2 \leq r_2^2 \leq r_3^2.$$

Suppose the theorem were not true, i.e.,

$$(4.8) \quad \rho_1 > \frac{n-k-2}{n-\alpha} (\alpha + \sqrt{2-\alpha^2}).$$

Then for all  $\alpha, \rho_1 > (n-k-2/n-2)\sqrt{2}$  which would imply that

$$(4.9) \quad r_1^2 > 2 \cdot \frac{(n-1)(n-k-2)}{(n-2)(n-k)} \geq 1.$$

For  $\alpha = 0$ , this already gives a contradiction. If  $0 < \alpha \leq 1$ , then from (3.2) with  $\nu = n - 3$ , from (4.7), and (4.8) and Lemma 3 we have

$$\frac{1}{3} \operatorname{Re} \sum_{j=1}^3 \frac{1}{\alpha - \zeta_j} = \frac{1}{3} \frac{n-2}{(n-k-2)} \operatorname{Re} \sum_{j=1}^3 \frac{1}{\alpha - w_j}$$



$$> \frac{n-2}{n-k-2} \cdot \left[ \frac{1}{2\alpha} - \frac{1-\alpha^2}{2\alpha} \cdot \frac{(n-k)(n-2)}{(n-1)(n-k-2)} \frac{1}{\theta^2(\alpha)} \right].$$

Since  $(n-k)(n-2) \leq 2(n-1)(n-k-2)$  and  $|\operatorname{Re}(\alpha - \zeta_j)^{-1}| \leq 1/\rho_j$ , we have

$$\begin{aligned} \frac{1}{3} \sum_{j=1}^3 \frac{1}{\rho_j} &> \frac{n-2}{2(n-k-2)} \left\{ \frac{1}{\alpha} \left( 1 - \frac{2}{\theta^2} \right) + \frac{2\alpha}{\theta^2} \right\} \\ &\geq \frac{n-2}{(n-k-2)} \cdot \frac{1}{\theta}. \end{aligned}$$

Therefore

$$\rho_1 < \frac{n-k-2}{n-2} \theta$$

which contradicts (4.8). This completes the proof of Theorem 3.

5. Quintic polynomials. We shall prove the

**THEOREM 4.** *If  $P(z) = (z - z_0)Q(z)$ ,  $Q(z) = \prod_{j=1}^4 (z - z_j)$ ,  $|z_j| \leq 1$  ( $j = 0, 1, \dots, 4$ ), then at least one zero of  $P'(z)$  lies in the disk*

$$(5.1) \quad |z - z_0| \leq \frac{1}{2} \sqrt{2 - |z_0|^2}.$$

**REMARK.** This in particular proves Ilyeff's conjecture for quintics since the right side of (5.1) is  $< 1$  if  $|z_0| < 1$ .

*Proof.* Without loss of generality we may assume  $z_0 \neq z_j$  ( $j = 1, \dots, 4$ ) and  $0 \leq z_0 \leq 1$ .

From (3.3) with  $n = 5, \nu = 1$ , we have

$$(5.2) \quad r_1 r_2 r_3 r_4 = 5 \rho_1 \rho_2 \rho_3 \rho_4.$$

Now identifying in Lemma 1,  $f(t)$  with  $P'(z_0 + t)$ ,  $h(t)$  with  $Q(z_0 + t)$ ,  $g(t)$  becomes, except for a constant factor, the polynomial

$$t^{-1}[(1+t)^5 - 1]$$

whose zeros  $t_1, t_2, t_3, t_4$  satisfy

$$|t_1|^2 = |t_2|^2 = 4 \sin^2 \frac{\pi}{5}, \quad |t_3|^2 = |t_4|^2 = 4 \sin^2 \frac{2\pi}{5}$$

and  $t_1 t_2 t_3 t_4 = 5$ . It follows then from Lemma 1 that

$$(5.3) \quad \rho_1 \cdot |t_1| \leq r_j \leq \rho_4 \cdot |t_4|, \quad (j = 1, \dots, 4).$$

From Lemma 2, (5.2) and (5.3) we conclude that  $\sum_{j=1}^4 r_j^{-2}$  cannot be larger than the corresponding expression for

$$r'_1 = r'_2 = |t_1| \rho_1, r'_3 = \frac{\rho_2 \rho_3}{\rho_1} |t_4|, r'_4 = |t_4| \rho_4.$$

Thus on using  $\rho_1 \leq \rho_2 \leq \rho_3 \leq \rho_4$  and  $|t_1|^{-2} + |t_4|^{-2} = 1$ , we have

$$(5.4) \quad \sum_{j=1}^4 r_j^{-2} \leq \sum_{j=1}^4 r'^{-2} \leq 2\rho_1^{-2}.$$

If  $z_0 \neq 0$ , then on using Lemma 3 and (3.2) with  $k = 1, \nu = 1, n = 5$ , we have from (5.4)

$$\frac{4}{\rho_1} \geq \operatorname{Re} \sum_{j=1}^4 \frac{1}{z_0 - \zeta_j} \geq 2 \left\{ \frac{2}{z_0} - \frac{1 - z_0^2}{2z_0} \cdot \frac{2}{\rho_1^2} \right\},$$

from which the result follows by elementary calculation. If  $z_0 = 0$ , then  $r_j \leq 1$  ( $j = 1, 2, 3, 4$ ) and so by (5.2)  $\rho_1 \leq 5^{-(1/4)} < 2^{-(1/2)}$ . This completes the proof.

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