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This paper presents a proof for the Brauer splitting theorem in the context of a commutative ring with no idempotents except 0 and 1 and continues this investigation. The main results in this paper are the Brauer splitting theorem and the classification of all finitely generated projective indecomposable modules over a separable group algebra.

Throughout this paper we assume that the ring R is a commutative ring with no idempotents except 0 and 1, that the group G has order n invertible in R, and that all RG-modules are unitary left RG-modules. We know that the order of G, n, is invertible in R if and only if RG is separable.

1. First, let us recall the following Brauer splitting theorem: Let K be a field and G be a group of order n invertible in K, then $K(\sqrt[m]{1})$ is a splitting field for G, where m is the exponent of G and $\sqrt[m]{1}$ is a primitive m^{th}-root of 1 ([6], Th. 41–1, p. 292 and Corollary 70–24, p. 475). In [8], G. J. Janusz defined a ring R to be a splitting ring for G if the group algebra RG is the direct sum of central separable R-algebras each equivalent to R in the Brauer group of R; that is, $RG \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i)$, where $\{P_i\}$ are finitely generated projective faithful R-modules, the number of different conjugate classes in G is equal to s. He then proved the Brauer splitting theorem for a Noetherian regular domain, R. This section gives a proof for the above theorem when R is any commutative ring with no idempotents except 0 and 1.

LEMMA 1. *Let R_0 be a subring of R. If R_0 is a splitting ring for G, then R is a splitting ring for G.*

Proof. Because R_0 is a splitting ring for G, $R_0G \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_{R_0}(P_i, P_i)$ where $\{P_i\}$ are finitely generated projective faithful R_0-modules. Then we have

$$RG \cong R \otimes_{R_0} R_0G \cong R \otimes_{R_0} \left(\bigoplus \sum_{i=1}^{s} \mathrm{Hom}_{R_0}(P_i, P_i) \right)$$

$$\cong \bigoplus \sum_{i=1}^{s} R \otimes_{R_0} \mathrm{Hom}_{R_0}(P_i, P_i) \cong \bigoplus \sum_{i=1}^{s} \mathrm{Hom}_R(R \otimes_{R_0} P_i, R \otimes_{R_0} P_i) ,$$

where $\{R \otimes_{R_0} P_i\}$ are finitely generated projective faithful R-modules. This follows since $\{P_i\}$ are finitely generated projective faithful R_0-

modules ([1], Proposition 5-5). Thus R is a splitting ring for G.

THEOREM 2. *If R is a commutative ring with no idempotents except 0 and 1 and RG is a separable group algebra, then $R[\sqrt[m]{1}]$ is a splitting ring for G where $\sqrt[m]{1}$ is a primitive m^{th}-root of 1.*

Proof. Let Z be the set of integers, Q be the set of rationals. The proof divides into two cases.

Case 1. The prime ring of R is finite. Let $\operatorname{Char}(R) = p^e$, where p is a prime integer and e is in Z.

$Z/(p^e)$ is a local ring with the maximal ideal $(p)/(p^e)$ which is also nilpotent. For $(Z/(p^e))[\theta]$ where $\theta = \sqrt[m]{1}$, we have

$$\frac{(Z/(p^e))[\theta]}{((p)/(p^e))[\theta]} \cong (Z/(p))(\bar{\theta})$$

where $\bar{\theta}$ is a primitive m^{th}-root of 1 over $Z/(p)$. Now $(Z/(p))(\bar{\theta})$ is a field; so $((p)/(p^e))[\theta]$ is a maximal ideal. On the other hand, since $(p)/(p^e)$ is nilpotent, $((p)/(p^e))[\theta]$ is also nilpotent. But then $((p)/(p^e))[\theta]$ is an unique maximal ideal and a nilpotent ideal of $(Z/(p^e))[\theta]$. Therefore, $(Z/(p^e))[\theta]$ is a complete local ring where the completion is in the sense of m-topology (see [9], p. 254). Then the Brauer group natural map

$$B((Z/(p^e))[\theta]) \longrightarrow B\left(\frac{(Z/(p^e))[\theta]}{((p)/(p^e))[\theta]}\right) \cong B((Z/(p))(\bar{\theta}))$$

is monomorphic ([1], Corollary 6-2). But $(Z/(p))(\bar{\theta})$ is a splitting field for G; so $(Z/(p^e))[\theta]$ is a splitting ring for G. Thus $R[\theta]$ is a splitting ring for G by the lemma.

Case 2. The prime ring of R is $Z(n)$ which is the quotient ring of Z with respect to the multiplicative closed set $\{n, n^2, \cdots\}$. Since $Z(n)[\theta]$ is a Dedekind domain, it is Noetherian and regular. Then the Brauer group natural map $B(Z(n)[\theta]) \to B(Q(\theta))$ is monomorphic ([1], Th. 7-2). But $Q(\theta)$ is the quotient field of $Z(n)[\theta]$ and a splitting field for G by the Brauer splitting theorem. Therefore, $Z(n)[\theta]$ is a splitting ring for G and so $R[\sqrt[m]{1}]$ is a splitting ring for G by the lemma. By combining Cases 1 and 2, the theorem is proved.

REMARK. The above theorem tells us the existence of a splitting ring, $R[\sqrt[m]{1}]$, for G, if RG is a separable group algebra. We also know that $R[\sqrt[m]{1}]$ is a finitely generated projective and separable R-algebra ([8], Corollary 2-4). But there exists a central separable R-

algebra without a finitely generated projective and separable splitting ring. The following example is due to 0. Goldman: Let R be $Z[\sqrt{2}]$, i, j, k be the usual quaternion basis. If $\alpha = (1+i)/\sqrt{2}$ and $\beta = (1+j)/\sqrt{2}$, then $R1 \oplus R\alpha \oplus R\beta \oplus R\alpha\beta$ is central separable over R. But R has no finitely generated projective and separable extension except direct sums of copies of R, and $R1 \oplus R\alpha \oplus R\beta \oplus R\alpha\beta$ cannot be split.

2. In this section, assume RG is a split group algebra,

$$RG \cong \oplus \sum_i \mathrm{Hom}_R (P_i, P_i) , \qquad i = 1, 2, \cdots, s .$$

When $\{P_i\}$ are considered as RG-modules ([3], p. 5), the classification of all finitely generated projective indecomposable RG-modules can be obtained. Observe that the order of the group G, n, is invertible in R if and only if RG is separable. Therefore, any RG-module M is finitely generated and projective over RG if and only if M is finitely generated and projective over R (see the proof of Proposition 1–5 in [8]).

Let RG be a separable R-algebra and M be a finitely generated projective RG-module; for any x in M there exist $X_1, X_2, \cdots X_q$ in M and $F_1, F_2, \cdots, F_q$ in $\mathrm{Hom}_R (M, R)$ so that $x = \sum_{i=1}^q F_i(x)X_i$. We call $\{F_i, X_i, i = 1, 2, \cdots, q\}$ a R-dual basis of M, and $T_M(x) = \sum_{i=1}^q F_i(xX_i)$ the character of M at x in RG ([4], Proposition 3–1). By a group character we mean the restriction of T_M to G. Obviously, a character T_M is completely determined by its restriction to G. In particular, let R be a splitting ring for G; then

$$RG \cong \oplus \sum_{i=1}^s \mathrm{Hom}_R (P_i, P_i) \cong \oplus \sum_{i=1}^s (RG)E_i ,$$

where E_i is the i^{th}-central primitive idempotent of RG. We let

$$T^i = T_{P_i} .$$

PROPOSITION 3. *If M and N are two isomorphic finitely generated projective RG-modules, then they have the same characters.*

Proof. Let M and N be two isomorphic finitely generated projective RG-modules and let α be the isomorphism. If $\{F_i, X_i, i = 1, 2, \cdots, q\}$ is a dual basis of M, then we claim that $\{F_i\alpha^{-1}, \alpha X_i, i = 1, 2, \cdots, q\}$ is a dual basis of N. In fact, for any a in N, there exists b in M such that $\alpha(b) = a$; so

$$\begin{aligned} a &= \alpha\Big(\sum_{i=1}^q F_i(b)\, X_i\Big) = \sum_i F_i(b)(\alpha X_i) \\ &= \sum_i F_i\alpha^{-1}\alpha(b)(\alpha X_i) = \sum_i ((F_i\alpha^{-1})\alpha(b))(\alpha X_i) \\ &= \sum_i (F_i\alpha^{-1}(a))(\alpha X_i) . \end{aligned}$$

This means that $\{F_i\alpha^{-1}, \alpha X_i, i = 1, 2, \cdots, q\}$ is a dual basis of N. But the character of any finitely generated projective RG-module is independent of the dual basis chosen; so $T_N(g) = \sum_i F_i\alpha^{-1}(g\alpha X_i) = \sum_i F_i\alpha^{-1}(\alpha g X_i)$, (for α is a RG-isomorphism), and so $= \sum_i F_i(gX_i) = T_M(g)$.

The following proposition will play an important role in our discussion.

Proposition 4. *If N is a finitely generated projective faithful R-module and M a finitely generated projective left* $\mathrm{Hom}_R(N, N)$*-module, then $M \cong N \otimes_R N'$ with N' a finitely generated projective R-module.*

Proof. By the Morita Theorem on *p.* 9 in [3].

Remark. Proposition 4 gives a counter-example to the converse statement of Proposition 3. Because of Proposition 4, let M and N be two finitely generated projective indecomposable RG-modules over the same central component of the split group algebra RG; that is, $\mathrm{Hom}_R(P_i, P_i)$, then $M \cong P_i \otimes_R N'$ and $N \cong P_i \otimes_R N''$, where N' and N'' are finitely generated projective indecomposable R-modules. Suppose N' and N'' are in $P(R)$, the class group of R, then

$$T_M(g) = T_{P_i}(g)T_{N'}(1) = T_{P_i}(g)\cdot 1 = T_N(g) .$$

But $P_i \otimes_R N' \cong P_i \otimes_R N''$ only if $N' \cong N''$.

Lemma 5. *If RG is a split group algebra; that is,*

$$RG \cong \oplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i) \cong \oplus \sum_{i=1}^{s} (RG)E_i ,$$

then

$$E_i = \sum_g \frac{k_i T^i(g^{-1})}{n} g ,$$

where g is in G, $k_i = \mathrm{rank}(P_i)$ and $T^i = T_{P_i}$.

Proof. Since

$$RG \cong \oplus \sum_{i=1}^{s} (RG)E_i \cong \oplus \sum_{i=1}^{s} \mathrm{Hom}_R(P_i, P_i), E_i = \sum_g E_i(g)g ,$$

for all g in G, $E_i(g)$ in R. We then have

$$E_i h^{-1} = \sum_g E_i(g)(gh^{-1})$$

for some h in G. Taking the character afforded by RG, we have

$$T_{RG}(E_i h^{-1}) = \sum_g E_i(g) T_{RG}(gh^{-1}) .$$

But $T_{RG}(gh^{-1}) = 0$ in case $gh^{-1} \neq 1$, and $= n$ in case $gh^{-1} = 1$ or $g = h$. Hence $T_{RG}(E_i h^{-1}) = E_i(h)n, E_i(h) = T_{RG}(E_i h^{-1})/n$ (for n is invertible in R).

Next, we find $T_{RG}(E_i h^{-1})$. Because P_i is a finitely generated projective R-module, $\mathrm{Hom}_R(P_i, P_i) \cong P_i \otimes_R \mathrm{Hom}_R(P_i, R)$ ([3], Morita Theorem I). Noting that $\mathrm{rank}(P_i) = \mathrm{rank}(\mathrm{Hom}_R(P_i, R))$, we have

$$T_{(RG)E_i}(g) = T^i(g)k_i \qquad \text{for all } i = 1, 2, \cdots, s .$$

Therefore,

$$T_{RG}(E_i h^{-1}) = \sum_{j=1}^{s} T_{(RG)E_j}(E_i h^{-1}) = \sum_j k_j T^j(E_i h^{-1}) .$$

But $T^j(E_i h^{-1}) = 0$ in case $i \neq j$, so

$$T_{RG}(E_i h^{-1}) = k_i T^i(E_i h^{-1}) = k_i T^i(h^{-1}) .$$

Hence,

$$E_i(h) = \frac{T_{RG}(E_i h^{-1})}{n} = \frac{k_i T^i(h^{-1})}{n} .$$

By substituting $E_i(h)$ in E_i, we have

$$E_i = \sum_g E_i(g) g = \sum_g \frac{k_i T^i(g^{-1})}{n} g .$$

This completes the proof.

LEMMA 6. *For* $i = 1, 2, \cdots, s$, $\mathrm{rank}(P_i)$ *is neither* 0 *nor a zero divisor in* R.

Proof. First, $\mathrm{rank}(P_i)$ is not 0, otherwise E_i is 0 by Lemma 5. This is impossible.

Next, let $\mathrm{rank}(P_i)$ be k_i, and suppose that k_i is a zero divisor in R. We then have a nonzero element, k', in R such that $k'k = 0$. But by Lemma 5,

$$E_i = k_i \sum_g \frac{T^i(g^{-1})g}{n} ;$$

so,

$$k'E_i = k'k_i \sum_g \frac{T^i(g^{-1})g}{n} = (k'k_i) \sum_g \frac{T^i(g^{-1})g}{n} = 0 .$$

Noting that $(RG)E_i \cong \mathrm{Hom}_R(P_i, P_i)$, we have

$$k' \mathrm{Hom}_R(P_i, P_i) \cong k'(RG)E_i = k'E_i(RG) = 0 .$$

On the other hand, P_i is a faithful R-module; so $\mathrm{Hom}_R(P_i, P_i)$ is a faithful R-module. Therefore, $k' \mathrm{Hom}_R(P_i, P_i) = 0$ implies $k' = 0$. This is a contradiction. Thus we have proved that k_i is not a zero divisor in R.

THEOREM 7, *Suppose R is a splitting ring for G and all finitely generated projective indecomposable R-modules are of rank* 1. *Then for any two finitely generated projective indecomposable RG-modules M and N, we have $E_iM \neq 0$ and $E_iN \neq 0$ if and only if $T_M(g) = T_N(g)$ for all g in G, where E_i is the i^{th}-central primitive idempotent of RG.*

Proof. If $E_iM \neq 0$ and $E_iN \neq 0$, then $M \cong E_iM \oplus (1 - E_i)M$ and $N \cong E_iN \oplus (1 - E_i)N$. Since M and N are indecomposable, $(1-E_i)M = 0$ and $(1 - E_i)N = 0$. We have $N = E_iN$ and $M = E_iM$ as left $\mathrm{Hom}_R(P_i, P_i)$-modules. Therefore, by Proposition 4, $M \cong P_i \otimes_R N'$ and $N \cong P_i \otimes_R N''$ where N' and N'' are finitely generated projective R-modules. Since M and N are indecomposable RG-modules, N' and N'' are in $P(R)$. Therefore,

$$\begin{aligned} T_M(g) &= T_{P_i \otimes_R N'}(g) = T_{P_i}(g) \cdot 1 \\ &= T_{P_i}(g) T_{N''}(1) = T_N(g) . \end{aligned}$$

Conversely, if $T_M(g) = T_N(g)$ for all g in G, then $T_M(a) = T_N(a)$ for all a in RG. Suppose $E_iM \neq 0$ and $E_iN = 0$ for some i; then there exists a $j \neq i$ such that $E_jN \neq 0$. Thus M is a $(RG)E_i$-module and N is a $(RG)E_j$-module, and so we have

$$T_M(E_i) = T_{P_i}(E_i) = T_{P_i}(1) = \mathrm{rank}\,(P_i) .$$

By Lemma 6, $\mathrm{rank}(P_i) \neq 0$ in R, so $T_M(E_i) \neq 0$. Obviously, $T_N(E_i) = 0$. Thus $T_M \neq T_N$ on RG. Consequently, $T_M(g) \neq T_N(g)$ for some g in G. This is a contradiction to $T_M(g) = T_N(g)$ for all g in G, and hence the proof is completed.

COROLLARY 8. *If R is a splitting ring for G, and all finitely generated projective indecomposable R-modules are of rank* 1; *then there are exactly s-classes of finitely generated projective indecomposable RG-modules over different central components each uniquely determined up to an element in $P(R)$.*

Proof. Let M be a finitely generated projective indecomposable

RG-module. From the theorem, we have $M = E_iM \cong P_i \otimes_R N'$ where N' is in $P(R)$. On the other hand, P_i, $i = 1, 2, \cdots, s$, is a finitely generated projective indecomposable RG-module over the i^{th}-central component. Therefore, there are exactly s-classes of finitely generated projective indecomposable RG-modules each uniquely determined up to an element in $P(R)$.

From the above result, we have computed the first Grothendieck group of RG, $K^0(RG)$, in the sense of [2], p. 31.

COROLLARY 9. *If R is a splitting ring for G, then*

$$K^0(RG) = (Z \oplus P(R)) \oplus (Z \oplus P(R)) \oplus \cdots \oplus (Z \oplus P(R)) .$$

A natural question to ask is whether the classification of all finitely generated projective indecomposable RG-modules can be obtained for a nonsplit group algebra. The answer is not known. But for some special rings, we have a definite answer.

For a separable group algebra RG, we have the decomposition, $RG \cong \oplus \sum_{i=1}^{t} A_i$, where A_i has no proper central idempotents and t is an integer.

THEOREM 10. *If R is local or semi-local,* then there are *exactly t-isomorphic classes of finitely generated projective indecomposable RG-modules.*

Proof. From the decomposition of RG, A_i is a central separable C_i-algebra for each A_i, where C_i is the center of A_i ([1], Th. 2-3). Since R is local or semi-local, C_i is semi-local by the lemma on p. 25 in [5]. Therefore any two finitely generated projective indecomposable RG-modules over the i^{th}-component A_i are in an isomorphic class of finitely generated projective indecomposable RG-modules ([7], Th. 1).

COROLLARY 11. *If R is local or semi-local, then*

$$K^0(RG) = Z \oplus Z \oplus \cdots \oplus Z ,$$

t-copies of Z.

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BIBLIOGRAPHY

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring,* Trans. Amer. Math Soc. **97** (1960), 367-409.

2. H. Bass, *K-theory and stable algebra,* Publ. Math. I.H.E.S. Paris, **22** (1964), 5-60.

3. ———, *The Morita theorems,* Mimeographed Notes, University of Oregon, 1964.

4. H. Cartan and S. Eilenberg, *Homological algebra,* Princeton, 1956.

5. L. N. Childs and F. R. DeMeyer, *On automorphisms of separable algebras,* Pacific J. Math. **23** (1967), 25-34.

6. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associated Algebras,* New York, Interscience, 1962.

7. F. R. DeMeyer, *Projective modules over central separable algebras,* Canad. J. Math. **21** (1969), 39-43.

8. G. J. Janusz, *Separable algebra over commutative rings,* Trans. Amer. Math. Soc. **122** (1966), 461-479.

9. O. Zariski and P. Samuel, *Commutative algebra,* Vol. 2, Princeton, D. Van Nostrand Co., 1960.

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