FACTORIZABLE SEMIGROUPS

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A multiplicative semigroup S is said to be factorizable
if it can be written as the set product AB of proper subsemi-
groups A and B. If this is possible, AB is called a factoriza-
tion of S, with factors A and B. The factors are not required
to be unique.

The following problems have been considered:

(I) Given a factorizable semigroup $S = AB$, where A and
B are members of the semigroup classes P and Q, respectively
(P and Q not necessarily distinct), to what semigroup class
does S belong?

(II) What are sufficient conditions for the factorizability
of a semigroup?

(III) Can the concept of factorizability be used in charac-
terizing direct products of semigroups?

In § 2 various cases of (I) are considered; for example, if P or Q
is the class of groups, or if P and Q are each the class of cyclic
semigroups, what type of semigroup is S? Section 3 is also devoted
to (I); in this section certain inverse semigroups, as well as com-
pletely simple semigroups, are characterized by means of factorizations.
Converses are also included in the results of § 3.

Section 4 gives sufficient conditions for the factorizability of a
semigroup. These solutions to (II) involve the existence of magnify-
ing elements in the semigroup.

Problem (III) is considered in an attempt to obtain an internal
characterization, involving factorizability, of a direct product of semi-
groups. The possibility of using the concept of factorizability for this
is suggested by the occurrence, in group theory, of set products of
groups in certain characterizations of direct products of groups. In
§ 5 semigroup direct products $A \times B$ in which A and B contain a cer-
tain type of idempotent are characterized in this manner.

The reader is referred to [1] for basic concepts and terminology
of algebraic semigroup theory. In addition, the duals of any nonself-
dual result will be taken for granted without comment.

As for notation, that of [1] is used. By $S \backslash A$ is meant the set of
all elements in S which are not in A. As is customary, $|S|$ will de-
note the cardinal of the set S. If S is a semigroup without identity
then S^1 will denote the semigroup obtained by adjoining an identity
element, say 1, to S. The empty set will be denoted by $\square$. Finally,
$\langle x \rangle$ will denote the cyclic semigroup generated by x.

2. **Problem (I).** A logical starting point in a consideration of factorizable semigroups is to look at (I), and in particular to consider those factorizations in which one of the semigroup factors is a group. [One reason for this approach is that there exist group-theoretic results involving factorizable groups and, furthermore, almost all such results give partial solutions to the group-theoretic analogue of (I).] Such factorizations, as well as those involving cyclic factors, are examined in this section.

LEMMA 2.1. *If a semigroup S is factorizable as $S = AB$, where A and B are groups, then S is a group.*

Proof. Since the identity elements of A and B are left and right identity elements, respectively, of S, and are thus equal, S has an identity. That S is a group follows easily.

It has been shown [1, p. 38] that any right group is the direct product of a group and a right zero semigroup. The next lemma considers nondegenerate right groups.

LEMMA 2.2. *Let S be a semigroup which is neither a group nor a right zero semigroup. If S is a right group, then S is factorizable as $S = AB$, where A is a group and B is a right zero semigroup; thus, in particular, B is right simple. Conversely, if S is factorizable as $S = AB$, where A is a group and B is right simple, then S is a right group.*

Proof. Assume S is a right group and let B denote its nonnull set of idempotents. Since S is right simple, every element of B is a left identity of S and B is a right zero semigroup. Let e be arbitrary but fixed in B and set $A = Se$. Then A is a group and S is factorizable as $S = AB$.

Conversely, if S has the given factorization, then the identity 1 of A is a left identity of S. Moreover, $1 = a_0 b_0$ for some a_0 in A and b_0 in B, and $b_0 b_1 = b_0$ for some b_1 in B, so $1 = a_0 b_0 = 1 b_1 = b_1 \in B$. Hence an arbitrary element ab of S has a right inverse $b'a^{-1}$ in S relative to 1, where $aa^{-1} = 1$ in A and $bb' = 1$ in B, so S is a right group.

THEOREM 2.3. *Let the semigroup S be factorizable as $S = AB$, where A is a group. If B is contained in the semigroup class $P_i (i = 1, 2, \cdots, 8)$, then so is S, where*
   P_1: *Regular semigroups;*
   P_2: *Simple semigroups;*

P_3:   *Completely simple semigroups;*
P_4:   *Left simple semigroups;*
P_5:   *Right simple semigroups;*
P_6:   *Left groups;*
P_7:   *Right groups;*
P_8:   *Groups.*

REMARK. Theorem 2.3 is not meant to be an all-inclusive result, i.e., the semigroup classes P listed are not all those classes for which such a result is true.

REMARK. Note in the proofs of some of the cases of Theorem 2.3 that stronger conclusions than those stated are obtainable.

Proof. In each case the identity 1 of A will be a left identity of S.

(1) Assume B is a regular semigroup and let $x = ab$ be arbitrary in $S = AB$. Then there exist elements a^{-1} of A and b' of B such that $a^{-1}a = 1$ and $b = bb'b$. Setting $x' = b'a^{-1}$ and recalling that 1 is a left identity of S then gives $x = xx'x$, so S is regular.

(2) Assuming B is simple, let $z = a_1b_1$ and $y = a_2b_2$ be arbitrary in $S = AB$. Then there exist elements a_1^{-1} of A and b_3, b_4 of B such that $a_1^{-1}a_1 = 1$ and $b_2 = b_3b_1b_4$. Thus $y = a_2b_2 = (a_2b_3a_1^{-1})(a_1b_1)b_4 \in SzS$, so S is a simple semigroup.

(3) If B is completely simple then by (2) S is simple. In addition, B contains a minimal left ideal [1, p. 78]; denote it by M. Then $T = AM$ is a left ideal of S, since

$$ST = (AB)(AM) \subseteqq ASM = A(AB)M = A^2BM \subseteqq AM = T \ .$$

If T is not a minimal left ideal of S then there exists a left ideal W of S such that $W \subset T$. Then $M \cap W$ is a left ideal of B contained in M so, by the minimality of M, $M \cap W = M$, i.e., $M \subseteqq W$. Therefore $T = AM \subseteqq AW \subseteqq W$, contrary to $W \subset T$.

Thus T is a minimal left ideal of S, so since S contains an idempotent, it follows that S is completely simple [4]. (This easily-proved result is also stated as an exercise in [1, p. 84].)

(4) and (6). Let $1 = a_0b_0$. Since B is left simple there exists some b_1 in B such that $b_1b_0 = b_0$. But $b_0a_0 = a_0b_0 = 1$, so $b_11 = 1$. Thus $(b_1)^2 = b_1$, so b_1 is a right identity of B and hence of S. This implies $1 = b_1$ is an identity of S. Then B is a group, so S is a group by Lemma 2.1.

(5) and (7). Lemma 2.2.

(8) Lemma 2.1.

It is natural now to examine the possibility of generalizing Theorem 2.3 to obtain a theorem characterizing all factorizations in which a group occurs as a factor. In this regard consider the following conjectures and accompanying remarks, which indicate somewhat the difficulty of obtaining such a generalization.

CONJECTURE 1. Let S be a semigroup factorizable as $S = AB$, where A is a group and B is contained in an arbitrary semigroup class P. Then S is contained in P.

As a counterexample to this first conjecture, let S be a right group which is neither a group nor a right zero semigroup and let P be the class of all right zero semigroups. By Lemma 2.2, S is factorizable as $S = AB$, where A is a group, $B \in P$, and $S \notin P$.

A somewhat weaker generalization also thought to be false is

CONJECTURE 2. Let S be a semigroup factorizable as $S = AB$, where A is a group. If B is contained in a semigroup class P which contains the class of all groups as a subclass, then S is contained in P.

A generalization similar to Conjecture 2 but not requiring one factor to be a group is

CONJECTURE 3. Let S be a semigroup factorizable as $S = AB$, where A and B are contained in the semigroup classes P_1 and P_2, respectively. If P_1 is a subclass of P_2, then S is contained in P_2.

This conjecture is also false-for let S be the bicyclic semigroup (with generators p and q, $pq = 1$) and let P_1 and P_2 each be the class of all semigroups having only finitely many idempotents [alternatively, let P_1 and P_2 each be the class of all nonsimple semigroups]. If $A = \langle q \rangle^1$ and $B = \langle p \rangle^1$, then S is factorizable as $S = AB$, where $A \in P_1$, $B \in P_2$, $P_1 \subsetneqq P_2$, and $S \notin P_2$.

One semigroup class not included in Theorem 2.3 was the class of cyclic semigroups. Two results concerning such semigroups are given in the next theorem.

THEOREM 2.4. *Let the semigroup S be factorizable as $S = AB$.*
(1) *If A is a group and B is a cyclic semigroup, then S is a group.*
(2) *If A and B are cyclic semigroups, then S is a group.*

Proof. (1) Since the identity 1 of A is a left identity of S, it suffices to show that each element of S has a left inverse in S rela-

tive to 1. Let $B = \langle b \rangle$. Since every element s of S can be written as $s = ab^i$, some a in A and $i \geq 1$, it suffices to show that b has a left inverse in S. This follows from the fact that $1 = a_1 b^k$ for some a_1 in A and $k \geq 1$.

(2) Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Since $S = AB$ there exist minimal positive integers i and u such that $a = a^i b^r$ and $b = a^u b^s$ for some positive integers r and s. Then let j and v be minimal integers in $J = \{k \geq 1 \mid a = a^i b^k\}$ and $V = \{w \geq 1 \mid b = a^u b^w\}$, respectively, and consider the "minimal" representations $a = a^i b^j$ and $b = a^u b^v$. This part of the theorem is proved by considering the possible relationships between u and i and between v and j.

Case (i). If $i = u$ and $j = v$, then $S = A^2 \subsetneqq A$, contradicting $A \subset S$.

Case (ii). If $i = u$ and $j < v$, then $b = a^u b^v = ab^{v-j}$. Thus $u = 1$, by the minimality of u, and $v \leq v - j$, by the minimality of v, so $j \leq 0$, a contradiction.

Case (iii). If $i = u$ and $v < j$, then $a = a^i b^j = b^{j-v+1} \in B$, so $A \subseteqq B$. Then $S = B^2 \subseteqq B$, which is a contradiction of $B \subset S$.

Case (iv). If $i < u$ and $v = j$, then $b = a^{u-i+1} \in A$, so $S = A$, a contradiction.

Case (v). If $u < i$ and $v = j$, then $a = a^{i-u} b$, so $i \leq i - u$ by the minimality of i. Hence $u \leq 0$, again a contradiction.

Case (vi). If $i < u$ and $j < v$, then $b = a^u b^v = a^{u-i+1} b^{v-j}$, so $u \leq u - i + 1$ by the minimality of u. Thus $i = 1$, so $b = a^u b^{v-j}$. Then the minimality of v implies $j \leq 0$, a contradiction.

Case (vii). If $u < i$ and $v < j$, then $a = a^{i-u} b^{j-v+1}$, contradicting the minimality of i.

Case (viii). Assume $i < u$ and $v < j$. Then $a^{u-i+1} = a^u b^j = b^{j-v+1}$. If $i \geq 2$, then $b = a^{i-1} b^{j+1}$; then the minimality of u implies $u \leq i - 1$, contradicting the assumption that $i < u$. Thus $i = 1$, so $b = b^{j+1}$. Then b has index 1 so B is a cyclic group [1, p. 20]. Hence by the left-right dual of part (1) of this theorem S is a group.

Case (ix). Assume $u < i$ and $j < v$. It will be shown that $a^u b^j$ is a right identity of S and that each element in $S = AB$ has a right inverse in S relative to $a^u b^j$. Since $a = a^i b^j = a^i a^u b^v b^{j-1} = a^u a^i b^j b^{v-1} = a^{u+1} b^{v-1}$, the minimality of i, together with $u < i$, implies that $i - u = 1$. This implies $b = a^u b^{j+1}$, so similarly $v - j = 1$. Collecting results gives

$$(2.1) \qquad i - u = 1, \qquad v - j = 1.$$

From the fact that $b^j a^i = a^m b^n$ for some positive integers m and n, it follows that $a^{i+1} = a^i b^j a^i = a^{i+m} b^n$ and $a^2 = a(a^i b^j) = a^{i+1} b^j = a^{i+m} b^{n+j}$. Thus by (2.1) $a = a^i b^j = a^{i-2} a^2 b^j = a^{i-2} a^{i+m} b^{n+j} b^j = a^m b^n$, so

$$(2.2) \qquad a = a^i b^j = b^j a^i = a(a^u b^j) = (b^j a^u)a.$$

Similarly, using $b = a^u b^v$, (2.1), and the fact that $b^v a^u = a^s b^t$ for some positive integers s and t, it follows that

$$b = a^u b^v = a^u b^2 b^{v-2} = a^u a^{s+u} b^{t+v} b^{v-2} = a^s b^t ,$$

so

(2.3) $b = a^u b^v = b^v a^u = b(b^j a^u) .$

Finally, (2.2) implies that $a^u b^j = (b^j a^u a^u) b^j = b^j (a^u a^u b^j) = b^j a^u$, i.e.,

(2.4) $a^u b^j = b^j a^u .$

By (2.2), (2.3), and (2.4) $a^u b^j$ is an identity of S. Let $a^x b^y$ be an arbitrary element of S, and choose positive integers k and h such that $kj \geqq y + 1$ and $hu \geqq x + 1$. Then $khj \geqq y + 1$ and $khu \geqq x + 1$, so by (2.4)

$$(a^x b^y)(z)(a^x) = a^x , \quad \text{where} \quad z = b^{khj-y} a^{khu-x} \in S .$$

Thus $(a^x b^y)z = a^u b^j$, so every element of S has a right inverse in S relative to the right identity $a^u b^j$ of S. Hence S is a group.

REMARK. Theorem 2.4 (1) cannot be generalized to the case in which B is a finitely-generated commutative semigroup. For consider the multiplicative semigroups $A = \{2^i \mid i$ is an integer$\}$ and $B = \{3^j \mid j$ is a nonnegative integer$\}$, and let $S = AB$. Then S is a semigroup and $S = AB$ is a factorization of S, where A is a group and B is a finitely-generated commutative semigroup. However, S is not a group.

3. **Problem (I): continued.** Certain inverse semigroups, as well as completely simple semigroups, are characterized in this section. The converse of (I) is also considered in connection with these characterizations.

Recall that the natural partial ordering of the set E of idempotents of a semigroup S is defined by

$$e \leqq f \quad \text{if and only if} \quad ef = fe = e .$$

THEOREM 3.1. *Let S be a semigroup whose set E of idempotents is finite ($|E| > 1$) and forms a chain relative to its natural partial ordering. Denote the maximal element of E (relative to this ordering) by e_n and assume $|H_n| > 1$, where H_n is the group of units of $e_n S e_n$. Then the following conditions are equivalent:*
  (1) *S is an inverse semigroup;*
  (2) *S is a union of groups;*
  (3) *S is factorizable as $S = AB$, where A is a group and B is an inverse semigroup;*

(4) S is factorizable as $S = AB$, where A is an inverse semigroup and B is a group;

(5) S is factorizable as $S = AB$, where A and B are inverse semigroups;

(6) S is a semilattice of completely simple semigroups;

(7) S is a semilattice of groups;

(8) S is a linearly ordered set of completely simple semigroups;

(9) S is a linearly ordered set of right groups;

(10) S is a linearly ordered set of groups.

Proof. (1) $\Rightarrow$ (2). Clifford and Preston [2, p. 41] have given one proof of this. An independent proof is given by the author in his doctoral dissertation.

(2) $\Rightarrow$ (1). Clear.

(1) $\Rightarrow$ (3) $\Rightarrow$ (5). If (1) is satisfied, then S is an inverse semigroup and $S = \bigcup_{k=1}^{n} H_k$, where $E = \{e_1, \cdots, e_n\}$ and H_k is the group of units of $e_k S e_k$ for $k = 1, 2, \cdots, n$. Let $A = H_n$ and $B = (S \backslash A) \cup \{e_n\}$, where e_n is maximal in E. Since e_n acts as an identity on $S = A \cup B$ and since $e_n \in A \cap B$, $S = AB$ is a factorization of the type given in (3). Clearly (3) implies (5).

(1) $\Rightarrow$ (4) $\Rightarrow$ (5). Similar to the above

(5) $\Rightarrow$ (1). Let $x = ab$ be an arbitrary element of $S = AB$, and let a' and b' be the unique elements of A and B, respectively, such that $aa'a = a$, $a'aa' = a'$, $bb'b = b$, and $b'bb' = b'$. Then letting $x' = b'a'$, the commutativity of E implies $xx'x = x$ and $x'xx' = x'$. Thus S is regular and hence an inverse semigroup.

(2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). [1, pp. 126–129].

(2) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10). [3, pp. 189–190].

The conditions placed on E in the preceding theorem seem to be essential. In particular, there exists an inverse semigroup (namely, the bicyclic semigroup) whose set of idempotents forms a countably infinite chain relative to its natural partial ordering but which is not a union of groups. On the other hand, as noted in the following theorem, if the set of idempotents of an inverse semigroup S is assumed to be a countably infinite chain with minimal element relative to its natural partial ordering, then some (but not all) of the conclusions of Theorem 3.1 are valid for S.

THEOREM 3.2. *Let S be an inverse semigroup with a countably infinite set E of idempotents, say $E = \{e_1, e_2, e_3, \cdots\}$. Assume E is naturally ordered, with minimal element e_1, as $e_1 < e_2 < e_3 < \cdots$. Then S is a union of groups but is not factorizable as $S = AB$, where A is a group and B is an inverse semigroup.*

Proof. (1) It is first shown that S is a union of groups. For each $n \geq 1$ let H_n denote the group of units of $e_n S e_n$. Suppose there exists an element x in S such that $x \notin H_n$ for all $n \geq 1$. Then $x' \notin H_n$ for all $n \geq 1$, where x' is the (unique) inverse of x in S. Consider the idempotents $e_i = xx'$ and $e_j = x'x$ and assume, without loss of generality, that $e_i > e_j$. [If $e_i = e_j$ then $x \in H_i$, a contradiction.] Here $xe_j = x = e_i x = xe_i$.

Consider the idempotent $x'e_j x = e_m$. If $e_m \geq e_j$, then $e_m = x'e_j x = x'xe_m = e_j e_m = e_j$. But then $x = xe_j = e_j x$, so $e_i = xx' = e_j xx' = e_j e_i = e_j$, a contradiction. Thus $e_j > e_m$. Next consider the idempotent $x'e_m x = e_w$. If $e_w \geq e_m$, then $xe_w x' = x(x'e_m x)x' = e_i e_m e_i = e_m$, so $e_m = e_j e_m = e_j xe_w x' = xe_m e_w x' = xe_m x' = e_j$. This is impossible, since $e_j > e_m$, so

$$e_j > e_m > e_w .$$

Continuing this procedure (i.e., next considering the idempotent $x'e_w x$) gives rise to a proper, decreasing sequence of idempotents less than e_j. But this is impossible, since there are only finitely many idempotents of S less than e_j.

Hence $x \in H_k$ for some positive integer k, and S is the union of the groups H_n, $n = 1, 2, 3, \cdots$.

(2) Assume to the contrary that S is factorizable as $S = AB$, where A is a group and B is an inverse semigroup. Then A, being a subgroup of S, must be contained in one of the maximal subgroups H_n of S, say $A \subseteq H_k$. In this case, $S = H_k B = H_k S$.

Let x in H_k and y in S be arbitrarily chosen. Then $y \in H_i$ and $xy \in H_j$ for some positive integers i and j. Thus there exist elements x^{-1} in H_k and w in H_j such that $xx^{-1} = x^{-1}x = e_k$, $xe_k = x$, and $xyw = e_j$, from which it follows that $e_j = e_k e_j$, i.e., $e_k \geq e_j$. Thus $xy \in \bigcup_{n=1}^{k} H_n$, so $H_k S \subseteq \bigcup_{n=1}^{k} H_n$. This implies $S = \bigcup_{n=1}^{k} H_n$, so E is necessarily finite, a contradiction.

Thus S has no factorization of this type, completing the proof of the theorem.

The remainder of this section is concerned with completely simple semigroups without zero which are not groups. Let S be such a semigroup, and let E denote its set of idempotents. If S is a right group, then by Lemma 2.2 S is a right zero semigroup (in fact, $S = E$) or S is factorizable as $S = AB$, where A is a group (and thus a left group), B is a right zero semigroup (and thus a right group), and $A \cap B \cap E \neq \square$. In addition, if S is a left group, then S is a left zero semigroup or S is factorizable as $S = AB$, where A is a left zero semigroup (hence a left group), B is a group (hence a right group), and $A \cap B \cap E \neq \square$.

Motivated by these observations, a semigroup S without zero is

said to be a nontrivial completely simple semigroup if S is complete-
ly simple and neither a right group nor a left group. Hence to give
factorization characterizations for all completely simple semigroups
without zero, it only remains to consider nontrivial completely simple
semigroups, which is done in Theorem 3.3.

THEOREM 3.3. *Let S be a semigroup without zero and with idem-
potent set E.*

*(1) S is a nontrivial completely simple semigroup if and only
if S is factorizable as $S = AB$, where A is a minimal left ideal of
S and B is a minimal right ideal of S.*

*(2)   If S is factorizable as $S = AB$, where A is a left group, B
is a right group, and $A \cap B \cap E \neq \square$, then S is completely simple.
Conversely, any nontrivial completely simple semigroup has such a
factorization.*

Proof.   (1) If S is a nontrivial completely simple semigroup, then
Se [eS] is a minimal left [right] ideal of S for every idempotent e of
S [1, p. 78]. Suppose S is left simple and assume $xy = wy$, where
$x, y, w \in S$. Then S, being completely simple, is regular [1, p. 79].
Thus $yy' \in E$ for some y' in S, so yy' is a right identity of S. Hence
$x(yy') = w(yy')$ implies $x = w$, so S is right cancellative and thus a
left group. This is a contradiction, so S is not left simple. Similar-
ly, S is not right simple. Thus there exist idempotents e and f in S
such that $B = fS$ and $A = Se$ are proper subsemigroups of S. Since
S is simple, $S = AB$ is then a factorization of S, where A [B] is a
minimal left [right] ideal of S.

Conversely, assume S has a factorization of the type given in the
theorem statement, and let $y = ab \in S$. Since $A = Sa$ and $B = bS$,
$S = AB = SabS = SyS$, so S is simple. S is also completely simple,
since $A \cap B$ contains a primitive idempotent [1, p. 77]. If S is a left
group then S is left simple and $A = S$, a contradiction. Similarly, S
is not a right group. Hence S is a nontrivial completely simple semi-
group.

(2)   Assume S is factorizable as described and let $e \in A \cap B \cap E$.
Using properties of right groups and left groups, it is easily shown
that S is simple and that e is a primitive idempotent, i.e., that S is
completely simple.

Conversely, by part (1) of this theorem $S = AB$, where A [B] is
a minimal left [right] ideal of S.   In addition, $A \cap B \cap E \neq \square$ [1, p.
77], so let $e^2 = e \in A \cap B$. Since A is left simple and B is right simple,
$A = Aa$ for all $a \in A$ and $B = bB$ for all $b \in B$. Thus e is a right
[left] identity of A [B]. Further, $e \in Aa$ for all $a \in A$ and $e \in bB$ for
all $b \in B$, so A is a left group and B is a right group, which gives

the desired factorization.

This section is concluded with the following result, the proof of which is straightforward and is omitted.

THEOREM 3.4. *A semigroup S factorizable as $S = AB$, where A is a right group and B is a left group, is a group.*

4. Problem (II). In seeking to obtain sufficient conditions for the factorizability of a semigroup S, there are two alternatives; on the one hand, one can assume the existence of certain types of elements in S (e.g., identity, idempotent, or regular) or, on the other hand, one can assume that certain relations are satisfied by all the elements of S (e.g., S is commutative). The former approach is used here.

Some definitions (due partially to Lyapin) are first needed. An element x of a semigroup S is said to be a weak right [left] magnifying (WRM) [(WLM)] element of S if there exists a proper subset A of S such that $S = Ax[S = xA]$. Similarly, x is said to be a strong right [left] magnifying (SRM) [(SLM)] element of S if there is a proper subsemigroup A of S such that $S = Ax[S = xA]$. An example of a semigroup having both WRM elements and WLM elements is the bicyclic semigroup.

THEOREM 4.1. *If a semigroup S contains either a* SRM *element or a* SLM *element, then S is factorizable.*

Proof. Assume x is a SRM element of S. Then $S = Ax$ for some proper subsemigroup A of S. If S is noncyclic, then S is factorizable as $S = AB$, where $B = \langle x \rangle$.

So assume S is cyclic, say $\langle y \rangle = S$. Then $x = y^k$ for some positive integer k, so $y = y^i y^k = y^{i+k}$ for some $y^i \in A$. Thus S is a finite cyclic group [1, p. 20]. If 1 denotes the identity of S, then it follows that $S = Sx^{-1} = Axx^{-1} = A1 = A$, so $S = A$, a contradiction. Thus S is necessarily a noncyclic semigroup and hence is factorizable.

The proof is similar if S contains a SLM element.

LEMMA 4.2. *Let S be a semigroup and let K be the set of all* WRM *elements of S. If $K \neq \square$ then K is a subsemigroup of S. Moreover, if $J = S \backslash K \neq \square$, then J is a subsemigroup of S and $J \subsetneqq JK$.*

Proof. If $y, z \in K$ there are proper subsets A and B of S such that $S = Ay = Bz$. Then $S = A(yz)$, so $yz \in K$ and K is a semigroup.

Assume u and v are arbitrary in J. If $uv \in K$ then $S = D(uv) =$

$(Du)v$ for some proper subset D of S. But $u \in J$ so $Du \subset S$. This implies $v \in K$, a contradiction, so $uv \in J$. Thus J is a semigroup if J is nonempty.

Furthermore, if j and x are arbitrary in J and K, respectively, then there exists a proper subset C of S such that $S = Cx$ and an element c in C such that $j = cx$. If $c \in K$ then $j = cx \in K$, a contradiction. Hence $c \in J$, so $j \in JK$ and $J \subsetneqq JK$.

THEOREM 4.3. *Let S be a semigroup containing a* WRM *element, and assume S satisfies any one of the following conditions:*
(i) *S has a left identity element;*
(ii) *S contains a* WLM *element;*
(iii) *S is regular.*
Then S is factorizable.

Proof. Throughout this proof let x be a WRM element of S and let K denote the set of all such elements. By Lemma 4.2 K is a semigroup. Also, let $J = S \backslash K$.

(i) Let 1 be a left identity of S. Since $1 \notin K$, J is a semigroup and $J \subsetneqq JK$. In addition, $K = 1K \subsetneqq JK$, so $S = J \cup K$ is factorizable as $S = JK$.

(ii) Assume y is a WLM element of S. Then there exist proper subsets A and B of S such that $S = Ax = yB$. If $y \in K$ then $S = yB = Cy$ for some proper subset C of S, and $y = yb = cy$ for some b in B and c in C. If s is arbitrary in S, then $s = yt = cyt = cs$ for some t in B, so c is a left identity of S. Hence by part (i) S is factorizable.

So suppose $y \notin K$. Then J is a semigroup by Lemma 4.2 and $J \subsetneqq JK$. In addition, $A \subsetneqq JK$. For assume $a \in A$. If $a \in J$ then $a \in JK$. If $a \in K$ then $a = yd$ for some d in B. But d in necessarily in K, so $a \in JK$.

Hence $S = Ax \subsetneqq (JK)K \subsetneqq JK \subsetneqq S$, so S is factorizable as $S = JK$.

(iii) If S is regular then its set E of idempotents is nonnull. Further $E \cap K = \square$, so by Lemma 4.2 J is a semigroup and $J \subsetneqq JK$. Moreover, if $y \in K$ then $y = (yy')y \in EK \subsetneqq JK$ for some y' in S, so $K \subsetneqq JK$. Thus $S = JK$ is a factorization of S.

5. **Problem (III).** One motivation for considering factorizable semigroups is to characterize a semigroup S which is the direct product of semigroups. Theorem 5.1 gives a result in this direction.

THEOREM 5.1. *If a semigroup S is isomorphic to the direct product $A \times B$, where A and B are semigroups with right and left*

identity, respectively, and where $|A| > 1$ *and* $|B| > 1$, *then*

  (i)   *there is a factorization of* S, *say* $S = A^*B^*$, *such that*

  (ii)   *every element* x *of* S *is uniquely representable in the form*
$x = ab$, *where* $a \in A^*$ *and* $b \in B^*$, *and*

  (iii) $(a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2)$ *for all* a_1, a_2 *in* A^* *and* b_1, b_2 *in* B^*.
Conversely, any semigroup satisfying (i), (ii), *and* (iii) *is isomorphic
to a direct product of semigroups.*

REMARK. A direct product of semigroups is not necessarily
factorizable. For let S be the direct product $P \times P$, where $P = \{2^i \mid i \geqq 1\}$. Here P is a multiplicative semigroup and so, also, is $S = \{(2^i, 2^j) \mid i, j \geqq 1\}$. In this case, $(2, 2) \in S$ but $(2, 2) \notin AB$ for any proper
subsemigroups A and B of S. Thus S is not factorizable.

Proof. Assume S is isomorphic to $A \times B$, where the semigroup
A contains a right identity e and the semigroup B a left identity f.
Let μ denote an isomorphism of $A \times B$ onto S, and define

$$A_1 = \{(a, f) \mid a \in A\}, \; B_1 = \{(e, b) \mid b \in B\} \; .$$

Since $A_1 \subseteqq A \times B$ and $B_1 \subseteqq A \times B$, it follows that $A^* = A_1\mu \subseteqq S$ and
$B^* = B_1\mu \subseteqq S$. Also, $A \times B = A_1B_1$. Thus $S = (A \times B)\mu = (A_1B_1)\mu = (A_1\mu)(B_1\mu) = A^*B^*$. Furthermore, A^* and B^* are proper subsemigroups
of S. For assume $A^* = S$ and let $a \in A$, $b \in B$. Then $(a, b)\mu = s \in S = A^* = A_1\mu$, so $(a, b)\mu = s = (a', f)\mu$ for some a' in A. Therefore $(a, b) = (a', f)$, so $b = f$. Hence $B = \{f\}$, so $|B| = 1$, a contradiction. Likewise, $B^* = S$ is impossible. Thus A^*B^* is a factorization of S,
proving (i).

Now suppose $a_1b_1 = a_2b_2$, where

(5.1) $a_i = (a_i', f)\mu \in A^*, \; b_i = (e, b_i')\mu \in B^*, \; i = 1, 2 \; .$

Then

$$\begin{aligned}
(a_1', b_1')\mu &= [(a_1', f)(e, b_1')]\mu = (a_1', f)\mu \cdot (e, b_1')\mu = a_1b_1 \\
&= a_2b_2 = (a_2', f)\mu \cdot (e, b_2')\mu = [(a_2', f)(e, b_2')]\mu \\
&= (a_2', b_2')\mu \; ,
\end{aligned}$$

so $a_1' = a_2'$ and $b_1' = b_2'$. Thus $a_1 = a_2$ and $b_1 = b_2$, proving (ii). Moreover, assuming (5.1), one obtains

$$\begin{aligned}
[(a_1b_1)(a_2b_2)]\mu^{-1} &= (a_1\mu^{-1})(b_1\mu^{-1})(a_2\mu^{-1})(b_2\mu^{-1}) \\
&= (a_1', f)(e, b_1')(a_2', f)(e, b_2') \\
&= (a_1'a_2', b_1'b_2') = (a_1', f)(a_2', f)(e, b_1')(e, b_2') \\
&= (a_1\mu^{-1})(a_2\mu^{-1})(b_1\mu^{-1})(b_2\mu^{-1}) \\
&= [(a_1a_2)(b_1b_2)]\mu^{-1} \; ,
\end{aligned}$$

so $(a_1 b_1)(a_2 b_2) = (a_1 a_2)(b_1 b_2)$, proving (iii).

Conversely, assume S satisfies (i), (ii), and (iii). Consider the mapping μ of the direct product $A^* \times B^*$ onto $S = A^* B^*$ given by

$$\mu: (a, b) \to ab , \quad \text{all} \quad a \in A^*, b \in B^* .$$

That μ is single-valued follows from the definition of equality in direct products. In addition, if (a_1, b_1) and (a_2, b_2) are arbitrary elements of $A^* \times B^*$ then by (iii) and the definition of multiplication in $A^* \times B^*$ it follows that

$$[(a_1, b_1)(a_2, b_2)]\mu = (a_1 a_2, b_1 b_2)\mu = (a_1 a_2)(b_1 b_2)$$
$$= (a_1 b_1)(a_2 b_2) = [(a_1, b_1)\mu][(a_2, b_2)\mu] ,$$

so μ is a homomorphism. That μ is an isomorphism follows from (i) and (ii).

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