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ON A BOUNDARY PROPERTY OF PRINCIPAL FUNCTIONS

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A behavior of $(P)L_1$-principal functions on some compactifications of a Riemann surface is studied. The main result in this paper is that a $(P)L_1$-principal function is extended almost everywhere continuously to some compactifications and the extention is almost everywhere constant on each part of P. If the genus of the surface is finite and P is the canonical $\tilde{P}$, $(\tilde{P})L_1$-principal function can be extended continuously to the Kerékjártó-Stoilow compactification.

The method of normal operators on open Riemann surfaces was developed by Sario [10] and others, and they established the existence theorems of harmonic functions with given singularities and prescribed modes of behavior near the ideal boundary. Especially, principal functions constructed by principal operators have been taken up by several authors, and many interesting results have been proved. Kusunoki [4] introduced the notion of canonical potentials and canonical differentials, and developed the theory of Abelian integrals on open Riemann surfaces. This canonical potential is readily shown to be $(\tilde{P})L_1$-principal function corresponding to the canonical partition of the boundary, if it is single-valued (paragraph 3). On a compact bordered surface, a $(P)L_1$-principal function is defined by the property that it is constant on each part of partition P of the boundary and has zero flux over each P-dividing cycle. The main purpose of the present paper is to study such property of $(P)L_1$-principal functions on the boundary of arbitrary open Riemann surfaces.

Nakai and Sario [8] showed that, in the case of identity partition, corresponding L_1-principal function can be extended finitely continuously and it is almost everywhere constant on the Royden boundary, and this property and vanishing of flux over the boundary characterize L_1-principal functions corresponding to the identity partition. Kusunoki [5] proved that a canonical potential on a Riemann surface has a constant value quasi-everywhere on each connected component of the Kuramochi boundary. We shall show in the present paper that a $(P)L_1$-principal function can be extended almost everywhere continuously so that the extention is almost everywhere constant on each part of a given regular partition P of the boundary, where compactification of the Riemann surface may be of Martin, Kuramochi, Royden, Wiener, or a Q-compactification, denoting by Q any sublattice of HP which contains constant (Theorem 1). In the case of Martin compactification, each polar set on the boundary is also of harmonic measure zero. In the case of Kuramochi compactification, the condition 'almost everywhere' can be strengthened to 'quasi-everywhere'. Further, if the set of constant values of the extended function on the parts of P satisfies a certain condition, this property, vanihing of the flux over any P-dividing cycle and Dirichlet integrability on a boundary neighborhood characterize the $(P)L_1$-principal functions (Theorem 2).

In the case of Riemann surfaces of finite genus, a $(\tilde{P})L_1$-principal function corresponding to the canonical partition $\tilde{P}$ is *everywhere* constant on each part of $\tilde{P}$ (Theorem 3). However, we can show by an example that this condition is not sufficient to characterize the $(\tilde{P})L_1$-principal functions on Riemann surfaces of finite genus.

1. Let R be an open Riemann surface and let us denote by P a regular partition of the ideal boundary of R (Ahlfors-Sario [1]) and by I the identity partition. Let U be a boundary neighborhood such that $R - \bar{U}$ is a regular region, and Ω a canonical region containing $\overline{R - U}$. Let us also denote by the same P the partition of the boundary $\partial\Omega$ of Ω which is induced by the original P. For given singularities s in $R - \bar{U}$ with vanishing flux on R, there corresponds a *$(P)L_1$-principal function $f_{P\Omega}$* on Ω. This $f_{P\Omega}$ is defined by the following conditions:

(i) $f_{P\Omega}$ has the singularities s.

(ii) $f_{P\Omega}$ is constant on each part of P on $\partial\Omega$.

(iii) the flux of $f_{P\Omega}$ vanishes over any part of P. By (ii) and (iii), we have

$$\text{(iv)} \qquad \| df_{P\Omega} \|_{\Omega\cap\bar{U}} = -\int_{\partial U} f_{P\Omega} df^*_{P\Omega} < \infty \, .$$

According to Rodin-Sario [9], *the suitably normalized family $\{f_{P\Omega}\}_\Omega$ converges uniformly on a compact set to a $(P)L_1$-principal function f_P on R with the singularities s.*

Let G be a regular region containing all singular points of f_P, then

$$\| df_P \|_{R-\bar{G}} = -\int_{\partial G} f_P df^*_P < \infty$$

(Ahlfors-Sario [1]). Therefore, for an arbitrary given positive ε, we can choose a sufficiently large compact set K with a smooth boundary so that

$$\| df_P \|_{R-K} < \varepsilon \, ,$$

and we can find Ω so large that

$$\| df_P - df_{P\Omega} \|_K < \varepsilon \quad \text{and} \quad \left| \int_{\partial K} f_P df_P^* - \int_{\partial K} f_{P\Omega} df_{P\Omega}^* \right| < \varepsilon .$$

Then we have

$$\begin{aligned} \| d(f_P - f_{P\Omega}) \|_\Omega &= \| d(f_P - f_{P\Omega}) \|_K + \| d(f_P - f_{P\Omega}) \|_{\Omega-K} \\ &\leqq \| d(f_P - f_{P\Omega}) \|_K + \| df_P \|_{R-K} + \| df_{P\Omega} \|_{\Omega-K} \\ &< 4\varepsilon , \end{aligned}$$

because

$$\| df_{P\Omega} \|_{\Omega-K} = \left| \int_{\partial K} f_{P\Omega} df_{P\Omega}^* \right| < \left| \int_{\partial K} f_P df_P^* \right| + \varepsilon = \| df_P \|_{R-K} + \varepsilon .$$

Thus we get

LEMMA 1. *A $(P)L_1$-principal function f_P can be approximated in norm by principal functions $\{f_{P\Omega}\}_\Omega$ on canonical regions Ω.*

The same assertion can be proved quite analogously for an L_0-principal function too.

2. By Lemma 1 we readily see

LEMMA 2. *For any regular partition P of the ideal boundary, we have*

$$f_P = f_I + h_P$$

on R, where $dh_P \in (P)\Gamma_{hm}$.

Here $(P)\Gamma_{hm}$ is the Hilbert space spanned by the differentials of harmonic measures associated with P-dividing cycles. If γ_P is a P-dividing cycle, γ_P divides a sufficiently large Ω into two regions Ω_1 and Ω_2. Let w_Ω be a harmonic measure which is 1 on $\partial\Omega_1 - \gamma_P$ and 0 on $\partial\Omega_2 - \gamma_P$. Then $w_\gamma = \lim_{\Omega\to R} w_\Omega$ is a harmonic measure associated with γ_P.

On a compact bordered surface $\bar{\Omega}$, it is obvious that

$$d(f_{P\Omega} - f_{I\Omega}) = dh_{P\Omega} \in (P)\Gamma_{hm}(\Omega) .$$

We have

$$\| d(h_P - h_{P\Omega}) \|_\Omega \leqq \| d(f_P - f_{P\Omega}) \|_\Omega + \| d(f_I - f_{I\Omega}) \|_\Omega \to 0 \text{ as } \Omega \to R .$$

Therefore $dh_P \in (P)\Gamma_{hm}$.

Let us remark that if $R \in 0_{HD}$, $(P)\Gamma_{hm} \subset \Gamma_{he} = \{0\}$, and therefore $(P)L_1$-principal functions coincide each other for all P.

LEMMA 3. *An $(I)L_1$-principal function f_I has the following representation on a boundary neighborhood U.*

$$f_I = r_0 + cu ,$$

where c is a constant, u is a harmonic measure of the ideal boundary with respect to U, and r_0 is a harmonic function on U satisfying

$$c_1(1-u) \leqq r_0 \leqq c_2(1-u)$$

with suitable constants c_1 and c_2.

Proof. By the definition of $(I)L_1$ on R, f_I is a limit function of g_Ω as Ω tends to R, where g_Ω satisfies

(i) $f_I = g_\Omega$ on ∂U.

(ii) g_Ω is constant on $\partial\Omega$.

(iii) $\int_{\partial\Omega} dg_\Omega^* = 0$.

Let us denote by $r_{0\Omega}$ the harmonic function on $\Omega \cap U$ which is equal to f_1 on ∂U and 0 on $\partial\Omega$. Then $\lim_{\Omega\to R} r_{0\Omega} = r_0$ exists and it is harmonic on U. The function $g_\Omega - r_{0\Omega}$ is 0 on ∂U and constant, say c_Ω, on $\partial\Omega$. Then it is equal to $c_\Omega u_\Omega$, where u_Ω is a harmonic measure of $\partial\Omega$ with respect to $\Omega \cap U$. We have

$$\lim_{\Omega\to R} (g_\Omega - r_{0\Omega}) = f_I - r_0 = \lim_{\Omega\to R} c_\Omega u_\Omega$$

and

$$\lim_{\Omega\to R} u_\Omega = u .$$

Hence $c = \lim_{\Omega\to R} c_\Omega$ exists and is equal to $(\lim c_\Omega u_\Omega)/\lim u_\Omega$. Further we have

$$\left(\min_{\partial U} f_I\right)(1-u) \leqq r_0 \leqq \left(\max_{\partial U} f_I\right)(1-u) \text{ on } U .$$

3. Lemmas 2 and 3 show that in order to know the behavior of $(P)L_1$-principal functions on the boundary, it is enough to study that of harmonic measures.

Let us remind that a *Q-compactification* of R, where Q is a class of continuous functions, is a compactification R_Q^* of R, on which all functions of Q can be extended continuously and the extended functions separate the points on $\Delta_Q = R_Q - R$. For each potential q, let δ_p be the set of points $b \in \Delta_Q$ such that

$$\lim_{a\to b} q(a) = 0 ,$$

and let $\delta_Q = \cap \delta_q$ where q runs through the class of all potentials. We call δ_Q *a harmonic boundary* of R_Q^*. *If R_Q^* is a resolutive compactification, δ_Q is a carrier of harmonic measure ω on Δ_Q, and therefore* $\omega(\Delta_Q - \delta_Q) = 0$ (p. 92, Constantinescu-Cornea [2]).

Let us denote by D and W the class of Dirichlet functions and Wiener functions respectively, then by definition R_D^* is a Royden compactification and R_W^* is a Wiener compactification. Of course, we have $D \subset W$. Let Y be a sublattice of HP which contains constant, then $Y \subset W$. *If $Y \subset W$, R_Y^* is resolutive* (p. 99, Constantinescu-Cornea [2]). Further *all points of δ_D, δ_W and δ_Y are regular* (p. 101, [2]).

Let R_Q^* be a resolutive compactification. A P-dividing cycle γ_P divides R into two parts R_1 and R_2. Let $\Delta_1 = \Delta_Q \cap \bar{R}_1$ and $\Delta_2 = \Delta_Q \cap \bar{R}_2$, then harmonic measure associated with γ_P are given by

$$w_{\gamma_P} = \int_{\Delta_1} h d\omega \,, \qquad \tilde{w}_{\gamma_P} = 1 - w_{\gamma_P}$$

where h is a function on Δ_Q which is 1 on Δ_1 and 0 on Δ_2. Hence, if all points of δ_Q are regular, $w_{\gamma_P} = 1$ on $\delta_Q \cap \Delta_1$ and 0 on $\delta_Q \cap \Delta_2$.

LEMMA 4. *An integral of any element of $(P)\Gamma_{hm}$ is constant almost everywhere on each part of partition P, if the compactification is resolutive and all points of harmonic boundary are regular.*

For the harmonic measure u of the ideal boundary with respect to U, it is easily seen that $u = 1$ *on δ_Q under the same conditions.*

In the case of Martin compactification, the set of points b on the boundary such that

$$\overline{\lim_{a \to b}}\, g(a, a_0) > 0$$

is polar, where $g(a, a_0)$ is a Green function on the Riemann surface with a pole at a_0. Comparing with this Green function in a boundary neighborhood, we can show that $\lim_{a \to b} w_{\gamma_P}(a) = 0$ *or* 1 *and* $\lim_{a \to b} u(a) =$ 1 *quasi-everywhere on the boundary.*

If R is hyperbolic, a function f which is equal to $1 - u$ on U and $\equiv 1$ on $R - U$ is a Dirichlet potential on R, which is seen by the fact that the greatest harmonic minorant of f vanishes identically. There is also a Dirichlet potential on R, which is equal to w_{γ_P} on R_2 and $\equiv 0$ in a neighborhood of Δ_1. *On a Kuramochi compactification a Dirichlet function is a Dirichlet potential if and only if whose quasi-continuous extention vanishes quasi-everywhere on the boundary* Δ (p. 193, Constantinescu-Cornea [2]). Hence we obtain the fact that *an integral of any element of* $(P)\Gamma_{hm}$ *has a constant value quasi-everywhere on each part of* P *on* Δ, *and* $u = 1$ *quasi-everywhere on* Δ.

Thus by Lemmas 2 and 3, we get

THEOREM 1. *Suppose that a compactification* R^* *of* R *is any-one of Royden, Wiener or a Q-compactification where* Q *is a sub-lattice of HP which contains constant. Then a* $(P)L_1$*-pincipal function can be extended almost everywhere continuously on* R^* *so that the extention is constant almost everywhere on each part of partition* P. *In the case of Martin or Kuramochi compactification, the condition, 'almost everywhere' can be replaced more restrictive 'quasi-every-where'.*

Let us notice that *if the partition* P *is canonical* $\tilde{P}$, a $(\tilde{P})L_1$-*principal function is a single-valued canonical potential and vice versa.* Indeed, if f_1 is a $(\tilde{P})L_1$-principal function with canonical $\tilde{P}$, there is a single-valued canonical potential g with the same singu-larities as f_1. On a sufficiently large Ω there are a $(\tilde{P})L_1$-principal func-tion f_Ω and a single-valued potential g_Ω which satisfy

$$\lim_{\Omega\to R} \| df_1 - df_\Omega \|_\Omega = 0 , \qquad \lim_{\Omega\to R} \| dg - dg_\Omega \|_\Omega = 0 .$$

On Ω we have

$$df_\Omega - dg_\Omega \in \Gamma_{hm}(\Omega) \cap \Gamma^*_{hse}(\Omega) = \{0\} ,$$

and therefore we get

$$df_1 - dg \in \Gamma_{hm} \cap \Gamma^*_{hse} = \{0\}$$

on R.

4. Any principal function f, of L_1 or L_0, has a finite Dirichlet integral on a boundary neighborhood and satisfies

$$\lim_{\Omega\to R} \int_{\partial\Omega} f\, df^* = 0 .$$

Further, by Theorem 1, we know that a $(P)L_1$-principal function f_P has a following property (α).

(α) f_P can be extended on R^* almost everywhere continuously so that it is constant almost everywhere on each part of P.

Now we consider under what restriction, the condition (α) and the following (β) and (γ) characterize $(P)L_1$-principal functions.

(β) The Dirichlet integral taken over a boundary neighborhood is finite.

(γ) The flux over each P-dividing cycle vanishes.

If $R \in O_{KD}$, every real function h which is harmonic except polar singularities and has the properties (β) *and* (γ) *with canonical* $\tilde{P}$, *is nothing but a* $(\tilde{P})L_1$*-principal function corresponding to the canonical* $\tilde{P}$. Indeed, if f_1 is a $(\tilde{P})L_1$-principal function with the same singularities as h, then

$$d(f_1 - h) \in \Gamma_{he} \cap \Gamma^*_{hse} = \{0\} .$$

As a sufficient condition for that the three conditions (α), (β) and (γ) characterize $(P)L_1$-principal functions, we can prove

THEOREM 2. *Suppose that a compactification R^* is one of Royden, Wiener, Martin, Kuramochi or a Q-compactification with a sublattice Q of HP containing constant, and a function g which is harmonic except polar singularities, satisfies* (α), (β) *and* (γ). *If the set of constant values which are taken by g on parts of P is discrete except the supremum and infimum, then g is a* $(P)L_1$*-principal function.*

Proof. Let g be a function satisfying the assumption in the theorem, and f_I a $(I)L_1$-principal function with the same singularities as g. Let $h = f_I - g$, then $dh \in \Gamma_{he} \cap (P)\Gamma^*_{hse}$ and h has the property (α). Let c_0 be one constant value of h on a part of P which is not the supremum nor infimum. We put the constant values of h on the boundary in line as follows.

$$\cdots < c^{(M)} < \cdots < c^{(2)} < c^{(1)} < c_0 < c_1 < c_2 < \cdots < c_N < \cdots .$$

Let $h \vee (c^{(M)}) = h_M$, then we have $h_M \geqq h_{M+1}$ and

$$(1) \qquad \| dh_M \| \leqq \| dh \|_{h \geqq c^{(M)}} \leqq \| dh \| .$$

Let $h_M \wedge c_N = h_{MN}$, then $h_{MN} \leqq h_{MN+1}$, and

$$(2) \qquad \| dh_{MN} \| \leqq \| dh_M \|_{h_M \leqq c_N} \leqq \| dh_M \| ,$$

and it is readily seen that $dh_{MN} \in (P)\Gamma_{hm}$. By (2), a proper subsequence of $\{h_{MN}\}_N$ converges to h_M in norm sense and therefore $dh_M \in (P)\Gamma_{hm}$. By (1), a proper subsequence of $\{h_M\}_M$ converges to h in norm sense, and therefore $dh \in (P)\Gamma_{hm}$. Hence we get that $g(=f_I + h)$ is a $(P)L_1$-principal function.

5. Now we restrict R to be of finite genus and the partition $\tilde{P}$ to be canonical, and consider a compactification R^* of type S. We say a compactification R^* is *of type S*, if for any region $G^* \subset R^*$ whose boundary is in R, $G^* - \Delta$ is connected (Constantinescu-Cornea [2]). Let f_1 be a $(\tilde{P})L_1$-principal function. *If a boundary component*

Δ_e of R is weak, $f_1 + if_1^*$ and hence f_1 has a limit at Δ_e, which is shown as Lemma 2 in Mori [7]. Though the f_1 is single-valued on R, the $f_1 + if_1^*$ may not be single-valued. But this function is semi-exact, and we can choose a boundary neighborhood on which a branch of $f_1 + if_1^*$ is single-valued. If Δ_e is not weak, we can show that *a harmonic measure u of the ideal boundary with respect to a boundary neighborhood can be extended continuously so that $u = 1$ on Δ_e. A harmonic measure w_γ associated with a dividing γ is also identically* 1 *or* 0 *on Δ_e.*

A dividing cycle γ divides the boundary Δ into two parts Δ_1 and Δ_2, and a canonical region Ω into Ω_1 and Ω_2 if Ω is sufficiently large. Let w_Ω be a harmonic function on Ω which is 1 on $\partial\Omega_1 - \gamma$ and 0 on $\partial\Omega_2 - \gamma$, then $\lim_{\Omega\to R} w_\Omega = w_\gamma$. Since the genus of R is finite, there is a planar neighborhood U_e of Δ_e, and a conformal mapping φ of U_e by which Δ_e corresponds to a continuum λ^* on a plane. Then by a properly normalized slit mapping ψ of the complement of λ^*, the image of U_e can be mapped conformally on a region on a z-plane so that λ^* corresponds to a segment λ on the real axis which contains the origin as an interior point. Let r be a positive number on λ, and denote by K_r the disk $|z| \leqq r$. Take a parameter $t = \log z$ and consider a harmonic measure $\mu(t)$ of the image of the circle $|z| = r$, $0 \leqq \arg z < 2\pi$ with respect to the half plane $\operatorname{Re} t \leqq \log r$. The function $\mu(\log z)$ is harmonic in $0 < |z| \leqq r$ and single-valued if $0 \leqq \arg z < 2\pi$, and

$$\mu(\log z) \leqq \frac{2}{\pi} \tan^{-1} \left(\log \frac{r}{|z|}\right)^{-1} .$$

Further, for a sufficiently large Ω, we have

$$w_\Omega(\varphi^{-1}(\psi^{-1}(z))) < \mu(\log z)$$

which is seen by the maximum principle. Combining these two inequalities we get the above result.

Thus, by the use of Lemmas 2 and 3 we obtain that a $(\tilde{P})L_1$-principal function has a limit on each boundary component. Therefore we can extend the function to a Kerékjártó-Stoilow compactification. The fact that the extended function f_1 is continuous on R^* is readily seen because each boundary component has a planar neighborhood and the $f_1 + if_1^*$ is conformal on the neighborhood.

THEOREM 3. *If a Riemann surface is of finite genus, a $(\tilde{P})L_1$-principal function associated with the canonical partition $\tilde{P}$ has a continuous extention on a Kerékjártó-Stoilow compactification.*

6. The converse of the theorem is not true, that is, *there is a Riemann surface of finite genus with a function defined on it which can be extended continuously on a Kerékjártó-Stoilow compactification, but is not a* $(\tilde{P})L_1$*-principal function.*

Let us consider a Riemann surface R of finite genus which is not of class O_{KD} but whose all boundary components are weak. The existence of such a Riemann surface was proved in Jurchescu [3]. An L_0-principal function f_0 is a limit of properly normalized family $\{f_{0\Omega}\}_\Omega$, where each $f_{0\Omega}$ is defined by the property that the normal derivative vanishes along $\partial\Omega$ (Rodin-Sario [9]). Obviously f_0 satisfies the conditions (β) and (γ). Moreover, f_0 has a limit at any boundary component of R. This fact can be proved in the quite same way as for a $(\tilde{P})L_1$-principal function f_1. But all the functions $f_0 - f_1$, where f_0 and f_1 have the same singularities, are constant if and only if $R \in O_{KD}$ (Ahlfors-Sario [1]). Therefore there is a function f_0 on R which is not a $(\tilde{P})L_1$-principal function, but can be extended continuously on a Kerékjártó-Stoilow compactification.

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