A NOTE ON GROUPS WITH FINITE DUAL SPACES

LAWRENCE WASSON BAGGETT
A NOTE ON GROUPS WITH FINITE DUAL SPACES

LARRY BAGGETT

If a locally compact group has only a finite number of inequivalent irreducible unitary representations, then one is tempted to conjecture that it is a finite group. The conjecture is known to be true in certain special cases. We present here a proof in case the group satisfies the second axiom of countability.

PROPOSITION 1.1. If $G$ is an abelian locally compact group having only a finite number of inequivalent irreducible unitary representations, then $G$ is a finite group.

This follows immediately from the Pontrjagin duality theorem.

PROPOSITION 1.2. If $G$ is a compact group having only a finite number of inequivalent irreducible unitary representations, then $G$ is a finite group.

We may deduce a proof of this from the Peter-Weyl theorem, for example, as follows: $L^2(G)$ is the direct sum $\sum_i I_i$ of finite dimensional subspaces $[I_i]$, where, for each $i$, $I_i$ is a minimal two-sided ideal in $L^2(G)$. Further, there is a one-to-one correspondence between the set $[I_i]$ of these ideals and the set of all equivalence classes of irreducible unitary representations of $G$. If the latter set is finite, as assumed, then $L^2(G)$ is finite dimensional, and $G$ is necessarily a finite group.

The proof we give here for the second countable case depends on Dixmier's theory of square-integrable representations, which, in turn, depends on some rather technical results concerning Hilbert algebras. It would be desirable, of course, to discover an elementary proof to what appears to be such an elementary theorem. I have devised a fairly elementary proof—"elementary" in the sense that, beyond the notion of Haar measure, the only deep result needed is Kadison's theorem on the algebraic irreducibility of a topologically irreducible *-representation of a $C^*$-algebra. This proof, however, still suffers from being quite long, so I do not include it.

Regarding the situation when $G$ is an arbitrary locally compact group, there is no direct integral theory available in general, and we therefore lose an important tool for moving from hypotheses about the dual space to conclusions, for example, about the regular representation. I can not make headway in resolving this conjecture even
in the case when $G$ is an uncountable discrete group. Of course, if the conjecture is true, then the hypothesis of a finite dual must imply that the dual space is actually discrete. Even this subsidiary implication is apparently nontrivial in general.

2. Discrete decomposition of the regular representation. Let $G$ be a unimodular group, and let $R$ denote the left regular representation of $G$. The following theorem can be deduced from §14 of [1].

**Theorem 2.1.** Let $M$ be a closed subspace of $L^2(G)$ which is irreducible under $R$. Then the mapping $(f, \phi) \rightarrow f*\phi$ is defined on $L^2(G) \times M$ into $M$, and there exists a constant $k_M$ such that $||f*\phi||_2 \leq k_M ||f||_2 ||\phi||_2$ for every $f$ in $L^2(G)$ and every $\phi$ in $M$.

**Lemma 2.2.** Let $M$ be a closed irreducible subspace of $L^2(G)$, and let $k_M$ be a constant which satisfies the inequality in Theorem 2.1 above. Suppose $N$ is a closed subspace of $L^2(G)$ for which $R|_N$ is equivalent to $R|_M$. If $f$ is an element of $L^2(G)$ and $\phi$ is an element of $N$, then $||f*\phi||_2 \leq k_M ||f||_2 ||\phi||_2$, so that $k_N$ may be taken to equal $k_M$.

**Proof.** Let $\theta$ be an equivalence between $R|_M$ and $R|_N$. Let $\phi$ be an element of $N$, $f$ be an element of $L^2(G)$, and $[f_n]$ be a sequence of elements of $L^2(G) \cap L^2(G)$ which converges to $f$ in $L^2(G)$. Then:

$$\begin{align*}
||f*\phi||_2 &= (\text{by Theorem 2.1}) \lim ||f_n*\phi||_2 = \lim ||f_n*\theta(\theta^{-1}(\phi))||_2 \\
&= \lim ||\theta(f_n*\theta^{-1}(\phi))||_2 = ||f_n*\theta^{-1}(\phi)||_2 \leq \lim k_M ||f_n||_2 ||\theta^{-1}(\phi)||_2 \\
&= k_M ||f||_2 ||\phi||_2.
\end{align*}$$

**Theorem 2.3.** Let $G$ be a unimodular group. Assume that $R$ is a direct sum of irreducible subrepresentations and that only a finite number of inequivalent irreducible representations occurs in this decomposition. Then $G$ is a finite group.

**Proof.** We prove that $L^2(G)$ is a convolution algebra, whence, by [5], $G$ is compact, whence, by Proposition 1.2, $G$ is finite.

Thus, decompose $L^2(G)$ as the direct sum $\sum_i M_i$, where, for each $i$, $M_i$ is a closed irreducible subspace. For each $i$, let $k_i$ denote a constant $k_i(M_i)$ as guaranteed by Theorem 2.1. By Lemma 2.2 we may assume that, if $R|_{M_i}$ is equivalent to $R|_{M_j}$, then $k_i = k_j$. By the hypothesis that only a finite number of inequivalent irreducible representations occurs in $R$, we may conclude that the set $[k_i]$ of these constants is finite, whence uniformly bounded by a positive number $k$. 
Now let \( f \) and \( g \) be two elements of \( L^2(G) \). Denote by \( g_i \) the projection of \( g \) onto the subspace \( M_i \). Then:

\[
\| f \ast g \|_2^2 = \| f \ast \sum_i g_i \|_2^2 = (\text{by the orthogonality of the } [M_i] \text{ and by Theorem 2.1}) 
\]

\[
\sum_i \| f \ast g_i \|_2^2 \leq \sum_i (k_i)^2 \| f \|_2^2 \| g_i \|_2^2 \leq k^2 \| f \|_2^2 \sum_i \| g_i \|_2^2 = k^2 \| f \|_2^2 \| g \|_2^2 .
\]

Hence \( f \ast g \) is again an element of \( L^2(G) \).

3. Finiteness properties in the dual space. By the dual space \( \hat{G} \), we shall mean the set of all equivalence classes of irreducible unitary representations of a locally compact group \( G \).

**Theorem 3.1.** Let \( G \) be an infinite, second countable, unimodular group. Then the spectrum of the regular representation \( R \) of \( G \) is infinite, i.e., \( R \) weakly contains an infinite number of elements of \( G \). (For the definitions of "spectrum" and "weak containment" see [2].)

**Proof.** Assume that \( R \) weakly contains only a finite number of inequivalent irreducible representations of \( G \). By the second countability of \( G \), \( R \) is equivalent to a direct integral of irreducible representations, [4], and we may assume, by [2], that the only irreducible unitary representations which occur in this direct integral decomposition are the elements of \( \hat{G} \) which \( R \) weakly contains. Having assumed that \( R \) weakly contains only a finite number of elements of \( \hat{G} \), this direct integral is equivalent to a direct sum of irreducible unitary representations only finitely many of which are inequivalent. We now have the hypotheses of Theorem 2.3. This implies that \( G \) is finite, which is a contradiction.

**Corollary 3.2.** Let \( G \) be a second countable group. Then \( G \) is finite if and only if \( \hat{G} \) is finite.

**Proof.** Of course "\( G \) finite \( \rightarrow \) \( \hat{G} \) finite" is classical. Conversely, if \( \hat{G} \) is finite, then the spectrum of the regular representation is finite, and the proof will be complete, by the theorem above, if we can show that \( G \) is unimodular.

If \( \delta \) denotes the modular function of \( G \), then \( \delta(G) \) is an abelian group (a subgroup of the group of positive reals) whose dual space, being in one-to-one correspondence with a subset of the dual space of \( G \), is finite. Hence, by Proposition 1.1, \( \delta(G) \) is finite, whence \( \delta \) is identically 1.
Corollary 3.3. Let $G$ be a sigma-compact group. Then $G$ is finite if and only if $\hat{G}$ is finite.

A proof follows from Corollary 3.2 together with the theorem of [3] which states that, if $G$ is a sigma-compact group, then there exists a compact normal subgroup $H$ for which $G/H$ is second countable.

Bibliography


Received March 7, 1969. This research was supported by NSF Grant GP-6869.

University of Colorado
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

BASIL GORDON
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  STANFORD UNIVERSITY
CALIFORNIA INSTITUTE OF TECHNOLOGY  UNIVERSITY OF TOKYO
UNIVERSITY OF CALIFORNIA  UNIVERSITY OF UTAH
MONTANA STATE UNIVERSITY  WASHINGTON STATE UNIVERSITY
UNIVERSITY OF NEVADA  UNIVERSITY OF WASHINGTON
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY  AMERICAN MATHEMATICAL SOCIETY
UNIVERSITY OF OREGON  CHEVRON RESEARCH CORPORATION
OSAKA UNIVERSITY  TRW SYSTEMS
UNIVERSITY OF SOUTHERN CALIFORNIA  NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 108 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
George E. Andrews, *On a calculus of partition functions* ........................................... 555
Silvio Aurora, *A representation theorem for certain connected rings* .......................... 563
Lawrence Wasson Baggett, *A note on groups with finite dual spaces* ....................... 569
Steven Barry Bank, *On majorants for solutions of algebraic differential equations in regions of the complex plane* ................................................................. 573
Klaus R. Bichteler, *Locally compact topologies on a group and the corresponding continuous irreducible representations* .............................................................. 583
Mario Borelli, *Affine complements of divisors* .............................................................. 595
Carlos Jorge Do Rego Borges, *A study of absolute extensor spaces* ......................... 609
Bruce Langworthy Chalmers, *Subspace kernels and minimum problems in Hilbert spaces with kernel function* ................................................................. 619
John Dauns, *Representation of L-groups and F-rings* .................................................... 629
Spencer Ernest Dickson and Kent Ralph Fuller, *Algebras for which every indecomposable right module is invariant in its injective envelope* ...................... 655
Robert Fraser and Sam Bernard Nadler, Jr., *Sequences of contractive maps and fixed points* ................................................................. 659
Judith Lee Gersting, *A rate of growth criterion for universality of regressive isols* ........ 669
Robert Fred Gordon, *Rings in which minimal left ideals are projective* ....................... 679
Fred Gross, *Entire functions of several variables with algebraic derivatives at certain algebraic points* ................................................................. 693
W. J. Kim, *The Schwarzian derivative and multivalence* ............................................... 717
Robert Hamor La Grange, Jr., *On (m – n) products of Boolean algebras* .................... 725
Charles D. Masiello, *The average of a gauge* ................................................................. 733
Stephen H. McCleary, *The closed prime subgroups of certain ordered permutation groups* ................................................................. 745
Richard Roy Miller, *Gleason parts and Choquet boundary points in convolution measure algebras* ................................................................. 755
Harold L. Peterson, Jr., *On dyadic subspaces* ................................................................. 773
Derek J. S. Robinson, *Groups which are minimal with respect to normality being intransitive* ................................................................. 777
Ralph Edwin Showalter, *Partial differential equations of Sobolev-Galpern type* ........ 787
David Slepian, *The content of some extreme simplexes* ............................................... 795
Joseph L. Taylor, *Noncommutative convolution measure algebras* .......................... 809
B. S. Yadav, *Contractions of functions and their Fourier series* .................................. 827
Lindsay Nathan Childs and Frank Rimi DeMeyer, *Correction to: “On automorphisms of separable algebras”* ................................................................. 833
Moses Glasner and Richard Emanuel Katz, *Correction to: “Function-theoretic degeneracy criteria for Riemannian manifolds”* ......................................................... 834
Satish Shirali, *Correction to: “On the Jordan structure of complex Banach algebras”* ................................................................. 834
Benjamin Rigler Halpern, *Addendum to: “Fixed points for iterates”* ....................... 834