AFFINE COMPLEMENTS OF DIVISORS

Mario Borelli
AFFINE COMPLEMENTS OF DIVISORS

MARIO BORELLI

Recently Goodman and Hartshorne have considered the question of characterizing those divisors in a complete linear equivalence class whose support has an affine complement. However their characterization is not clearly "linear", and in fact we have to resort to Serre's characterization of affine schemes to prove that, indeed, the condition "the support of an effective divisor has an affine complement" is, in the language of Italian geometry, expressed by linear conditions. In the language of Weil this means that the set of effective divisors, in a complete linear equivalence class, whose supports have affine complements is a linear system. This is our first result. Subsequently we study the intersection of all such affine-complement supports of effective divisors in the multiples of a given linear equivalence class, and prove the following: if the ambient scheme is a surface or a threefold, or if the characteristic of the groundfield is 0, (or assuming that we can resolve singularities!) then a minimal intersection cannot have zero-dimensional components, nor irreducible components of codimension 1, whose associated sheaf of ideals is invertible.

In particular we obtain anew Zariski's result (see [11]) that every complete nonsingular surface is protective, and that the examples of nonsingular, nonprojective threefolds given by Nagata and Hironaka (see [9] and [5]) are optimal, in the sense that no examples can be given of nonsingular, nonprojective threefolds in which the "bad" subsets are either closed points or two-dimensional subschemes.

The notation and terminology we use are, unless otherwise specifically stated, those of [4]. We consider only algebraic schemes, with an arbitrary, algebraically closed ground field \( k \). For the sake of convenience we drop the adjective "algebraic", and speak simply of schemes.

When we refer to, say, Lemma 2.3 without further identification, we mean Lemma 2.3 of the present work, to be found as the third statement of \( \S \ 2 \).

1. Let \( X \) be a scheme, \( \mathcal{L} \) an invertible sheaf over \( X \). A regular section \( s \in \Gamma(X, \mathcal{L}) \) identifies an exact sequence

\[
0 \rightarrow \mathcal{L}^{-1} \xrightarrow{\theta(s)} \mathcal{O}_X \rightarrow \mathcal{K} \rightarrow 0
\]

with \( \text{Supp} (\mathcal{K}) = \text{Supp} (s) = \{ x \in X \mid s(x) \in \mathcal{M}_x \} \), \( \mathcal{M}_x \) denoting the uni-

595
que maximal submodule of the stalk $\mathcal{L}_x$ of $\mathcal{L}$ at $x$.

With these notations we quote a result of Goodman and Hartshorne (see [3], Proposition 3):

\[ X_s = X - \text{Supp}(s) \text{ is affine if, and only if, } X_s \text{ contains no complete curves and the following condition holds:} \]

\[ (1) \quad \dim_k \lim_{n} H^i(X, \mathcal{L}^n \otimes \mathcal{F}) < \infty \]

for every coherent sheaf $\mathcal{F}$ over $X$, where the maps in the inductive system above are those induced by the injection $\mathcal{L}^{-1} \subset \theta(\theta) \mathcal{O}_x$.

In this part we shall prove the following

**Theorem 1.1.** Let $\mathcal{L}$ be an invertible sheaf over a scheme $X$. Let $A(\mathcal{L}) = \{ s \in \Gamma(X, \mathcal{L})_{\text{reg}} | X_s \text{ is affine} \}$. Then $A(\mathcal{L}) \cup \{ 0 \}$ is a vector space over $k$.

**Remark.** While it is quite easy to show that the set of sections $A_s(\mathcal{L}) = \{ s \in \Gamma(X, \mathcal{L}) | X_s \text{ contains no complete curve} \}$ is a vector space over $k$, we were not able to show that those elements of $\Gamma(X, \mathcal{L})$ which obey (1) also form a vector space. Leaving this open question aside, to prove Theorem 1.1 we make use instead of Serre’s well known characterization of affine schemes, (see [10], and following lemma of Goodman and Hartshorne (see [3], Lemma 4).

**Lemma 1.2.** Let $\mathcal{L}$ be an invertible sheaf over a scheme $X$, and let $s \in \Gamma(X, \mathcal{L})$. Then, for every coherent sheaf over $X$, and every $i \geq 0$,

\[ \lim_{n} H^i(X, \mathcal{L}^n \otimes \mathcal{F}) = H^i(X_s, \mathcal{F} | X_i). \]

The following lemma is needed in the proof of Theorem 1.1.

**Lemma 1.3.** Let $\{ V_n \}_{n \geq 0}$ be vector spaces over $k$, and let

\[ \theta_n: V_0 \longrightarrow \text{Hom}_k(V_n, V_{n+1}) \quad n > 0 \]

be linear transformations such that, for all $v, w \in V_0$,

\[ \theta_{n+1}(w) \circ \theta_n(v) = \theta_{n+1}(v) \circ \theta_n(w). \]

For all $p > 0, q > 0$, let $\theta_{p+q,q}(v) = \theta_{p+q,q}(v) \circ \cdots \circ \theta_q(v)$. Then, for all $p > 0, q > 0$, and for all $v, w \in V_0$, and for all $\lambda, \mu \in k$,

\[ \theta_{p+q,\lambda}(\lambda v + \mu w) = \sum_{i=0}^{q+1} \binom{q+1}{i} \lambda^i \mu^{q+1-i} \theta_{p+q,q-q-i+1}(v) \circ \theta_{p+q-i,p}(w). \]
Proof. We proceed by induction on \( q \). For \( q = 1 \) we have
\[
\theta_{p+1}(\lambda v + \mu w) = \theta_p(\lambda v + \mu w)
\]
which gives us our assertion. Now
\[
\theta_{p+q}(\lambda v + \mu w)
\]
\[
= \theta_{p+q}(\lambda v + \mu w) \left[ \sum_{j=0}^{q-1} \left( \binom{q}{j} \right) \theta_{p+q-j-1}(v) \circ \theta_{p+q-j-1}(w) \right]
\]
and the lemma is proved.

Proof of Theorem 1.1. For any \( s \in \Gamma(X, \mathcal{F}) \) and any coherent sheaf \( \mathcal{S} \) let
\[
\theta_s(s): H^i(X, \mathcal{S} \otimes \mathcal{F}) \rightarrow H^i(X, \mathcal{S} \otimes \mathcal{F})
\]
denote the homomorphism corresponding to the injection
\[
0 \rightarrow \mathcal{S} \otimes \mathcal{F} \rightarrow \mathcal{O}_X.
\]
By Lemma 1.2, and Theorem 1 of [10], it clearly suffices to prove the following statement: If \( s_i \in \Gamma(X, \mathcal{S}) \), \( i = 1, 2 \), are such that
\[
\lim_{n} [H^i(X, \mathcal{S} \otimes \mathcal{F}), \theta_s(s_i)] = 0, i = 1, 2
\]
for all coherent sheaves of ideals \( \mathcal{S} \) over \( X \), then, for all \( \lambda, \mu \in k \),
\[
\lim_{n} [H^i(X, \mathcal{S} \otimes \mathcal{F}), \theta_s(\lambda s_1 + \mu s_2)] = 0.
\]
Now the homomorphisms \( \theta_s(s) \) define a homomorphism
\[
\theta_s: H^0(X, \mathcal{S}) \rightarrow \text{Hom}_k[H^i(X, \mathcal{S} \otimes \mathcal{F}), H^i(X, \mathcal{S} \otimes \mathcal{F})]
\]
with the additional property that the following diagram commutes:
We can therefore apply Lemma 1.3, and obtain, with the same notations as in the lemma:

\[ \theta_{p+1,s}(\lambda s_i + \mu s_i) = \sum_{j=0}^{i+1} \binom{i+1}{j} \lambda^j \mu^{i+1-j} \theta_{p+i+i-j-1,s_i} \circ \theta_{p+i-j,s}(s_i), \]

for all \( p \geq 1, \) and all \( i \geq 1. \)

The theorem now follows from the above equation, the commutativity of diagram (1.3.1), and the fact that

\[ \lim_{n \to \infty} \left[ H^i(X, \mathcal{L}^\otimes n \otimes \mathcal{F}), \theta_n(s_i) \right] = 0 \quad i = 1, 2 \]

implies that, for all \( z \in H^i(X, \mathcal{L}^\otimes n \otimes \mathcal{F}), \) and all \( p > 0, \)

\[ \theta_{p+n,s}(s_i)(z) = 0 \]

for \( n \gg 0 \) (depending possibly on \( z \)).

**Remarks.** Clearly \( A(\mathcal{L}) \subset A_*(\mathcal{L}) \), but the question “does \( A(\mathcal{L}) \neq \emptyset \) imply \( A_*(\mathcal{L}) = A(\mathcal{L})? \)” has a negative answer. In fact, if \( A(\mathcal{L}) \neq \emptyset \), and \( s \in A_*(\mathcal{L}), \) \( X \) need not even be quasi-affine. A counterexample can be found in the birational blow-up of a point on the exceptional divisor of the blow-up of a point of the projective plane. To the author’s knowledge, however, there are no counterexamples to the affirmative answer with \( \text{Supp}(s) \) irreducible. One might therefore conjecture with Goodman that \( A(\mathcal{L}) \neq \emptyset, s \in A_*(\mathcal{L}), \text{Supp}(s) \) irreducible imply \( s \in A(\mathcal{L}). \)

2. We continue with the notations introduced in § 1.

**Definition 2.1.** Let \( X \) be a scheme, \( \mathcal{L} \) an invertible sheaf over \( X, U \) an open subset of \( X. \) We define, for all \( n > 0, \)

(a) \[ X_n(\mathcal{L}) = \bigcup_{s \in A(\mathcal{L}^\otimes n)} X_s \]

(b) \[ X(\mathcal{L}) = \bigcup_{n > 0} X_n(\mathcal{L}). \]

Furthermore, we say that \( U \) is \( \mathcal{L} \)-projective if \( U \subset X(\mathcal{L}). \)

**Remark.** If \( U \) is \( \mathcal{L} \)-projective, then \( \mathcal{L} | U \) is ample, but the converse need not be true. In fact \( H^0(X, \mathcal{L}^\otimes n) \) may have base points
for all \( n > 0 \), while choosing \( U \) sufficiently small will always result in \( \mathcal{L} | U \) being ample. Of course, to say that \( X \) is \( \mathcal{L} \)-projective is to say that \( \mathcal{L} \) is ample.

We proceed to study \( \mathcal{L} \)-projective open subsets. If \( (\mathcal{F}, \mathcal{O}_F) \) is a closed subscheme of the scheme \( X \), we say that the invertible sheaf \( \mathcal{L} \) is ample on \( F \) when the invertible sheaf \( \mathcal{L} \otimes \mathcal{O}_F \) over \( F \) is ample.

**Lemma 2.2.** Let \( \mathcal{L} \) be an invertible sheaf over the scheme \( X \), let \( U \) be an \( \mathcal{L} \)-projective open subset, let \( x_1, \ldots, x_m \) be a finite subset of \( U \). Then there exists a suitable \( s \in A(\mathcal{L}^\otimes n) \) such that 
\[
x_1, \ldots, x_m \in X_s \subset U.
\]

**Proof.** It clearly suffices to prove the lemma taking \( U = X(\mathcal{L}) \). Using the quasi-compacity of \( X(\mathcal{L}) \) and a well known argument, we see that, for a sufficiently high integer \( n \), and a suitable finite number of elements \( s_0, s_1, \ldots, s_t \in A(\mathcal{L}^\otimes n) \), there exists an injection
\[
X(\mathcal{L}) \hookrightarrow \text{Proj}(k[s_0, s_1, \ldots, s_t]).
\]
By Theorem 1.1, a homogeneous element of degree \( d \) of the ring \( k[s_0, s_1, \ldots, s_t] \) is an element of \( A(\mathcal{L}^\otimes d) \), and the statement of the lemma is trivially true for \( \text{Proj}(k[s_0, s_1, \ldots, s_t]) \). The lemma is proved.

**Proposition 2.3.** Let \( X \) be a scheme proper over \( k \), \( \mathcal{L} \) an invertible sheaf over \( X \), \( s \in A_c(\mathcal{L}^\otimes n) \). If \( \mathcal{L} \) is ample on \( F = X - X_s \), then \( \mathcal{L} \) is ample.

**Proof.** We shall apply the Nakai-Moishezon-Kleiman criterion for ampleness. (See [6] or [7]). We may clearly assume that \( X \) is integral (see [4], Ch. III, 2.6.2), and proceed by induction on \( r = \dim X \). The case \( r = 1 \) is trivial. Now let \( Y \) be an integral closed subscheme of \( X \), and let \( \dim Y = t \). If \( t = r \), i.e., if \( Y = X \), then \( (\mathcal{L} \cdot X) = n(\mathcal{L}^{r-1} \cdot F) > 0 \), since \( \mathcal{L} \) is ample on \( F \). Here \( (\cdot) \) denotes the intersection pairing, as defined in [6]. If \( t < r \), then either \( Y \subset F \) or \( Y \cap X_s = \emptyset \). In the latter case the canonical image of \( s \) in
\[
H^0(Y, \mathcal{L}^\otimes n \otimes \mathcal{O}_Y)
\]
is an element of \( A_c(\mathcal{L}^\otimes n \otimes \mathcal{O}_Y) \), and therefore, by the induction assumption, \( (\mathcal{L}^t \cdot Y) > 0 \). In the former case, since \( \mathcal{L} \) is ample on \( F \) by hypothesis, it is a fortiori ample on \( Y \), and \( (\mathcal{L}^t \cdot Y) > 0 \) follows. The proposition is proved.

**Remark.** The hypothesis \( s \in A_c(\mathcal{L}^\otimes n) \) is essential, in fact \( Y \cap X_s \neq \emptyset \) does not imply \( (\mathcal{L}^t \cdot Y) > 0 \) in general. An easy counterexample
to the proposition can be given, where $\mathcal{L}$ is not ample, but it is ample on $X - X_s$ for some section $s \in H^0(X, \mathcal{L})$. Take, for instance, $X = \text{the blow-up of } \mathbb{P}_2(k)$ at a point, $f: X \to \mathbb{P}_2(k)$ the associated surjection, $\mathcal{L} = f^*[\mathcal{O}_{\mathbb{P}_2(k)}(1)]$.

**Corollary 2.4.** If $U$ is an open, $\mathcal{L}$-projective subset of the scheme $X$, and $\mathcal{L}$ is ample on $F = X - U$, then $\mathcal{L}$ is ample.

**Proof.** We may assume that $X$ is integral, and proceed by induction on $r = \dim X$. The case $r = 1$ is trivial. Let now $s \in H^0(X, \mathcal{L}^\otimes n)$ be such that $X_s \subset U$ is affine. Such $s$ exists by Lemma 2.2. Let

$$0 \to \mathcal{L}^\otimes -n \overset{\theta(s)}{\to} \mathcal{O}_X \to \mathcal{O}_D \to 0$$

be the exact sequence associated to $s$. Let $D_1, D_2, \ldots, D_t$ be the irreducible components of the subscheme $D$. If $D_i \cap U$ is empty, then $\mathcal{L}$ is ample on $D_i$ by hypothesis. If $D_i \cap U \neq \emptyset$, then $\mathcal{L}$ is ample on $D_i$ by the induction assumption. In fact $D_i \cap U$ is clearly $\mathcal{L} \otimes \mathcal{O}_{D_i}$-projective. In either case $\mathcal{L}$ is ample on $D_i$, and therefore, by 2.6.2 of Ch. III of [4], $\mathcal{L}$ is ample on $D$. Since $X$, is affine we can apply Proposition 2.3. The corollary is proved.

**Corollary 2.5.** If $U$ is an open, $\mathcal{L}$-projective subset of the scheme $X$, and $\dim (X - U) = 0$, then $\mathcal{L}$ is ample.

**Proof.** $\mathcal{L}$ is locally free of rank 1, hence trivially ample on $X - U$. Apply the previous corollary.

We proceed to study the behavior of $\mathcal{L}$-projective open subsets under certain types of morphisms.

**Proposition 2.6.** Let $f: X \to Y$ be a proper surjective morphism of integral schemes, $U \subset Y$ an open subscheme, and assume that:

(a) $f_*\mathcal{O}_X$ is locally free over $Y$.

(b) $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is a finite morphism.

Let $\mathcal{L}$ be an invertible sheaf over $Y$. Then $U$ is $\mathcal{L}$-projective if, and only if, $f^{-1}(U)$ is $f^*(\mathcal{L})$-projective.

**Proof.** The necessity follows from the fact that $f|_{f^{-1}(U)}$ is an affine morphism, and 1.3.2 of Ch. II of [4].

To prove the sufficiency, let $y \in U$ be a given closed point. Since $f^{-1}(U)$ is $f^*(\mathcal{L})$-projective, and $f^{-1}(y)$ is a finite set of closed points by hypothesis (b) above, for some $n > 0$ there exists, by Lemma 2.2, a section $s \in H^0(X, f^*(\mathcal{L}^\otimes n))$ such that $X_s$ is affine and $f^{-1}(y) \subset$
Let now $N: f^*({\mathcal{O}_Y}) \to {\mathcal{O}_X}$ denote the norm mapping, as defined in [4], Ch. II, 6.5.1 and ff. By 5.4.10 of Ch. O, of [4] we see that $s$ corresponds to a section $t' \in H^0(Y, f_*(\mathcal{O}_Y)) \otimes \mathcal{L}^{\otimes n})$. Let

$$t = (N \otimes I)(t') \in H^0(Y, \mathcal{L}^{\otimes n}).$$

By 6.5.7 of Ch. II of [4], $t$ is such that $y \in Y$, and $f^{-1}(Y) \subset X$. Since $f \mid f^{-1}(U)$ is a finite morphism, it follows from a theorem of Chevalley's (see [4], Ch. II, 6.7.1) that $Y$ is affine. Clearly $Y \subset U$. Therefore the proposition is proved.

The following proposition, which will enable us to obtain our main results, is a generalization of Theorem 1 of [2]. We shall say that a morphism of integral schemes $f: X \to Y$ is dominating (and also say, less precisely, that $X$ dominates $Y$) if the morphism is proper, birational and surjective.

**Proposition 2.7.** Let $Y$ be a normal, integral scheme, $\mathcal{L}$ an invertible sheaf over $Y$, $U$ an open subscheme of $Y$. Then $U$ is $\mathcal{L}$-projective if, and only if, the following condition holds:

There exists an integral scheme $X$ dominating $Y$, an open subscheme $V \subset X$ with $V \approx U$, and invertible sheaves of ideals $I$, $\mathcal{J}$ of $\mathcal{O}_X$ with support off $V$ such that, for $t \gg 0$, $f^*(\mathcal{L}^{\otimes t}) \otimes \mathcal{J}^{\otimes t} \otimes \mathcal{I}$ is ample on $X$, $f$ being the morphism $f: X \to Y$.

Furthermore the scheme $X$ can be obtained from $Y$ by blowing up a suitable sheaf of ideals of $\mathcal{O}_Y$ with support off $U$.

**Proof.** We essentially follow the argument given by Goodman in [2]. The sufficiency is obvious from Proposition 2.6. In fact, since the support of $\mathcal{J}^{\otimes t} \otimes \mathcal{I}$ is off $V$, the fact that $f^*(\mathcal{L}^{\otimes t}) \otimes \mathcal{J}^{\otimes t} \otimes \mathcal{I}$ is ample clearly implies that $V$ is $f^*(\mathcal{L}^{\otimes n})$-projective. Now, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, since $Y$ is normal, and Proposition 2.6 applies.

To prove the necessity, choose $t \gg 0$ so that the scheme

$$Z = \text{Proj} \left[ \bigoplus_{n \geq 0} H^0(Y, \mathcal{L}^{\otimes n}) \right]$$

is birational to $Y$ and contains an open subscheme $W \approx U$. Note that, if $\mathcal{H} = \mathcal{O}_Z(1)$, $\mathcal{H}$ is very ample for $Z$. As in [2], we can obtain a scheme $X$ which dominates both $Z$ and $Y$ by successively blowing up sheaves of ideals of $\mathcal{O}_Z$ with support off $W$ and taking joins (see theorems 3.2, 3.3 of [8]).

Let $Z_1 \to Z$ be obtained by blowing up a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_Z$ with support off $W$. Then the following three statements hold:
(i) \( Z_1 \) contains an open subscheme \( W_1 \approx W \approx U \).
(ii) \( \mathcal{O}_{z_1} \) is an invertible sheaf of ideals of \( O_{z_1} \) with support off \( W_1 \).
(iii) for \( n > 0 \) the invertible sheaf \( g^*(\mathcal{I}^{\otimes n}) \otimes \mathcal{O}_{z_1} \) is ample on \( Z_1 \) (see [4], Ch. II, 4.6.13).

The join \( X \) of a finite number of such blow-ups has therefore the following three properties:
(a) \( X \) dominates \( Z \). Let \( h: X \to Z \) denote the corresponding surjective, birational, proper morphism.
(b) \( X \) contains an open subscheme \( V \approx W \approx U \).
(c) There exists an invertible sheaf of ideals \( \mathcal{I} \) of \( \mathcal{O}_X \) with support off \( V \) such that, for \( n > 0 \), the invertible sheaf \( h^*(\mathcal{I}^{\otimes n}) \otimes \mathcal{I} \) is ample on \( X \).

Let now \( s \in H^0(Z, \mathcal{O}_{\mathcal{X}^{n+1}}) \), and let

\[
0 \to \mathcal{O}_{\mathcal{X}^{n+1}} \to \mathcal{O}_Z \to \mathcal{O}_D \to 0
\]

be the corresponding sequence of sheaves. We have \( s \in H^0(Z, \mathcal{O}_{\mathcal{X}^{n+1}}) \), and therefore we have an exact sequence of sheaves

\[
0 \to \mathcal{O}_{\mathcal{X}^{n+1}} \to \mathcal{O}_Z \to \mathcal{O}_H \to 0.
\]

Let \( f: X \to Y \) denote the surjective, birational, proper morphism which has been constructed. Since the equations of \( D, H \) at corresponding points \( P, Q \) differ by an element of \( O_{P,Q} \) for some invertible sheaf of ideals \( \mathcal{I} \) with support off \( f^{-1}(U) \)

\[
f^*(\mathcal{I}^{\otimes n}) \otimes \mathcal{I} = h^*(\mathcal{I}).
\]

The above proves the first statement of the proposition. To prove the second we observe that, once more as in [2], we can obtain an integral scheme \( X_i \), which dominates \( X \), by blowing up a suitable sheaf of ideals \( \mathcal{I} \) of \( \mathcal{O}_Y \) with support off \( U \). That \( X_i \) has the desired properties follows from 8.1.7 and 4.6.13 of Ch. II of [4]. The proposition is proved.

We are now in the position of proving our main result, namely:

**Theorem 2.8.** Let \( Y \) be a normal, integral scheme, proper over \( k \). Let \( \mathcal{L} \) be an invertible sheaf over \( Y \), let \( U \) be an \( \mathcal{L} \)-projective open subscheme of \( Y \), and let \( P \) be the generic point of an irreducible component of \( Y - U \).

(I) If \( P \) is a closed point, then \( P \in Y(\mathcal{L}) \).

(II) Let \( \bar{P} = D \), and assume that either \( \dim Y \leq 3 \) or that \( \text{char } k = 0 \).

If \( \text{codim} D = 1 \), and if the sheaf of ideals \( \mathcal{I} \) which defines
the reduced scheme structure on $D$ is invertible, then, for some $r \geq 0$ and $n > 0$, $P \in Y(\mathcal{L}^\otimes n \otimes \mathcal{L}^\otimes r)$.

Proof of (1). Let $Y - U = F \cup \{P\}$, where $P \not\in F$, and let $f: X \to Y$ be the blow-up morphism constructed in Proposition 2.7. Now, $f^{-1}(P)$ is the “antiregular total transform” of the closed point $P$ (see [9] for the definition of antiregular total transforms, and for the existence of the scheme $X'$ below), therefore we can construct a scheme $X'$ with the following properties:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow f & & \downarrow g \\
Y & & \\
\end{array}
\]

and $h$ are surjections.

(a) The diagram commutes, and the morphisms $g$ and $h$ are surjections.

(b) The morphism $g | g^{-1}(Y - F)$ is an isomorphism.

(c) The morphism $h | [X - f^{-1}(P)]$ is an isomorphism.

Let $U_1 = Y - F$, $U_2 = X - f^{-1}(P)$, $U'_1 = g^{-1}(U_1)$, and let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_Y$, constructed in Proposition 2.7, such that $f$ is the blow-up morphism of $Y$ at $\mathcal{I}$.

We then have that, for a suitable invertible sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_x$, for all $n > 0$, and all $t > 0$, the invertible sheaf

\[
f^*(\mathcal{I}^\otimes n) \otimes \mathcal{I}^\otimes t \otimes \mathcal{O}_x
\]

is ample. We proceed in steps.

Case 1. $P \in \text{Supp} (\mathcal{I})$. Then $X - f^{-1}(F) \approx U_1$, and $\text{Supp} (\mathcal{I}) \subset f^{-1}(F)$. (We omit here the case $\dim Y = 1$, the theorem being trivially true in this case). Therefore $\text{Supp} (\mathcal{I} \otimes \mathcal{I}) \subset f^{-1}(F)$, and hence $X - f^{-1}(F)$ is $f^*(\mathcal{I}^\otimes n)$-projective. By Proposition 2.6 $U_1$ is $\mathcal{L}$-projective, and we are done in this case.

Case 2. $P \in \text{Supp} (\mathcal{I})$. We then have

\[
\mathcal{I} \mathcal{O}_x = \mathcal{I}_1 \otimes \mathcal{I}_2
\]

where $\mathcal{I}_1, \mathcal{I}_2$ are invertible sheaves of ideals of $\mathcal{O}_x$, with

\[
\text{Supp} (\mathcal{I}_1) \cup \text{Supp} (\mathcal{I}_1) \subset f^{-1}(F),
\]

and $\text{Supp} (\mathcal{I}_2) \cup \text{Supp} (\mathcal{I}_2) \subset f^{-1}(P)$.

Since $f^*(\mathcal{I}^\otimes n) \otimes \mathcal{I} \otimes \mathcal{I}^\otimes t$ is ample, we see that $U_2$ is $f^*(\mathcal{I}^\otimes n) \otimes \mathcal{I}_2 \otimes \mathcal{I}_2^\otimes t$-projective. Now $U_2 \approx X' - g^{-1}(P)$, and $g^{-1}(P)$ is a closed point. Furthermore the morphism $h | X - f^{-1}(P)$ is an isomorphism of $X - f^{-1}(P)$ onto $X' - g^{-1}(P)$; this, together with the fact that $Y$ is normal, shows that $h_*(\mathcal{O}_x) = \mathcal{O}_x$. Therefore we see that $h_* (\mathcal{I}_1)$ and $h_* (\mathcal{I}_2)$ are invertible sheaves of ideals of $\mathcal{O}_x$, with
supports on \( g^{-1}(F) \). Also, we can now apply Proposition 2.6, and obtain that \( X' - g^{-1}(P) \) is \( g^*(\mathcal{O}_{\mathcal{X}}^{\otimes t}) \otimes h_*(\mathcal{K}) \otimes h_*(\mathcal{F}) \)-projective. Therefore, by Corollary 2.5, the invertible sheaf \( g^*(\mathcal{O}_{\mathcal{X}}^{\otimes t}) \otimes h_*(\mathcal{K}) \otimes h_*(\mathcal{F}) \) is ample on \( X' \). Now \( X' - g^{-1}(F) \) is isomorphic to \( Y - F \), and \( X' - g^{-1}(F) \) is clearly \( g^*(\mathcal{O}_{\mathcal{X}}^{\otimes t}) \)-projective, since \( \text{Supp}(h_*(\mathcal{K})) \) and \( \text{Supp}(h_*(\mathcal{F})) \) are both contained in \( g^{-1}(F) \). Therefore, by Proposition 2.6, applied to the morphism \( g \), we see that \( Y - F \) is \( \mathcal{L} \)-projective, i.e., \( P \in Y(\mathcal{L}) \). Statement (I) of the theorem is proved.

**Proof of (II).** We let \( X, \mathcal{I}, \mathcal{J}, f \) be as in the previous proof. First of all, we observe that we may assume that \( P \in \text{Supp}(\mathcal{I}) \). In fact, if \( P \in \text{Supp}(\mathcal{I}) \), then a simple application of Theorem 14, p. 154, and Corollary, p. 277 of [12] shows that, for some \( r > 0 \), the sheaf \( \mathcal{O}' = \mathcal{I} \mathcal{O}_{\mathcal{Y}}^{\otimes r} \) is a sheaf of ideals of \( \mathcal{O}_{\mathcal{Y}} \) with \( P \in \text{Supp}(\mathcal{I}) \). Now, by 8.1.3 of Ch. II of [4], the blow-ups of \( Y \) at \( \mathcal{O}' \) and \( \mathcal{J} \) respectively are isomorphic. So we can indeed assume \( P \in \text{Supp}(\mathcal{I}) \).

Note that we now have that \( f^{-1}(P) \) is a point of \( X \), which we denote by \( Q \), and we have that \( \mathcal{O}_{P,Y} \approx \mathcal{O}_{Q,X} \). Suppose first that \( Q \in \text{Supp}(\mathcal{I}) \) either. (Clearly \( Q \in \text{Supp}(\mathcal{O}_{\mathcal{Y}}) \)). For \( t \geq 0 \) take a section \( \rho \in H^0[\mathcal{Y}, f^*\mathcal{O}_{\mathcal{X}}^{\otimes t} \otimes \mathcal{O}_{\mathcal{X}} \otimes \mathcal{J}^{\otimes t}] \) which has the following properties:

(a) \( Q \in X, \) and \( X, \) is affine.

(b) \( X, \cap [\text{Supp}(\mathcal{I}) \cap \text{Supp}(\mathcal{J})] = \emptyset \).

The section \( \rho \) exists since the invertible sheaf \( f^*\mathcal{O}_{\mathcal{X}}^{\otimes t} \otimes \mathcal{O}_{\mathcal{X}} \otimes \mathcal{J}^{\otimes t} \) is ample. By 5.4.10 of Ch 0, of [4], the section \( \rho \) corresponds to a section \( \tau \) of the (not necessarily invertible) sheaf

\[ \mathcal{O}_{\mathcal{Y}}^{\otimes t} \otimes f_*(\mathcal{O}_{\mathcal{X}} \otimes \mathcal{J}^{\otimes t}) . \]

Since \( Y \) is normal, the sheaf \( f_*(\mathcal{O}_{\mathcal{X}} \otimes \mathcal{J}^{\otimes t}) \) is a sheaf of ideals of \( \mathcal{O}_{\mathcal{Y}} \), hence we have an injection

\[ 0 \rightarrow \mathcal{O}_{\mathcal{Y}}^{\otimes t} \otimes f_*(\mathcal{O}_{\mathcal{X}} \otimes \mathcal{J}^{\otimes t}) \rightarrow \mathcal{O}_{\mathcal{Y}}^{\otimes t} . \]

Applying (b) we see that the section \( \tau \in H^0(Y, \mathcal{O}_{\mathcal{Y}}^{\otimes t}) \) under the above injection, has the property that \( Y, \approx X, \). Clearly \( P \in Y, \) and statement (II) is proved in this case, simply by taking \( r = 0 \).

Let now \( Q \in \text{Supp}(\mathcal{I}) \). Let \( g: X' \rightarrow X \) be a desingularization of \( X, \) and let \( h = f \circ g \). Since \( Y \) is normal and \( P \in \text{Supp}(\mathcal{I}) \) we see that \( Q \) is simple on \( X, \) and therefore \( g^{-1}(Q) \) is a point \( Q' \) of \( X' \) such that \( \mathcal{O}_{Q',X'} \approx \mathcal{O}_{Q,X} \). We denote by \( \mathcal{H} \) the invertible sheaf of ideals of \( \mathcal{O}_{X} \), which defines the reduced scheme structure on \( Q' \). \( \mathcal{H} \) is indeed invertible, since \( X' \) is nonsingular.

We observe that, first of all, for all \( m > 0 \)
To see the above it suffices to assume that $Y$ is affine, and in this case it becomes an easy verification, using the facts that $Y$ is normal, that $\mathcal{O}$ is principal and selfradical, and that $\mathcal{O}_{P,Y}$ is a discrete valuation ring.

Let $S_Y$ and $S_X$ denote the singular loci of $Y$ and $X$ respectively. Then $Q' \in h^{-1}(S_P) \cup g^{-1}(S_X)$, since both $P$ and $Q$ are simple on $Y$ and $X$ respectively. Since the invertible sheaf $f^*(\mathcal{L}^{\otimes n}) \otimes \mathcal{O}_X \otimes \mathcal{I}^{\otimes t}$ is ample, the open subscheme $X' = g^{-1}(S_X)$ is $h^*(\mathcal{L}^{\otimes n}) \otimes g^*(\mathcal{J}_X) \otimes g^*(\mathcal{I}^{\otimes t})$-projective, and therefore we can find an open subscheme $V'$ of $X'$, containing $Q'$, and having the following two properties:

(i) $V' \cap [\text{Supp}(g^*(\mathcal{J}_X) \cup h^{-1}(S_P) \cup g^{-1}(S_X))] = \emptyset$
(ii) $V'$ is $h^*(\mathcal{L}^{\otimes n}) \otimes g^*(\mathcal{J}_X) \otimes g^*(\mathcal{I}^{\otimes t})$-projective.

From (i) we see that $h| V'$ is an isomorphism of $V'$ onto an open subscheme $V$ of $Y$, with the property that $P \in V$.

Since $g^*(\mathcal{I})$ is an invertible sheaf of ideals over the nonsingular scheme $X'$, and since $Q' \in \text{Supp}[g^*(\mathcal{J})]$, we see that we have $g^*(\mathcal{I}) = \mathcal{H}^{\otimes r} \otimes \mathcal{N}$, where $r$ is some positive integer, and $\mathcal{H}$ is an invertible sheaf of ideals of $\mathcal{O}_{X'}$, with $Q' \notin \text{Supp}(\mathcal{H})$.

By property (ii) of $V'$ we have that the open subscheme

$$W = V - [\text{Supp}(h_*(\mathcal{H})) \cup \text{Supp}(h_*(g^*(\mathcal{J}_X)))]$$

is $\mathcal{L}^{\otimes n} \otimes h_*(\mathcal{H}^{\otimes r})$-projective. Since $P \in W$ we are done, by (2.8.1). The theorem is proved.

Let $X$ be a scheme. We shall say that the open subscheme $U$ of $X$ is divisorially quasi-projective in $X$ if, for some invertible sheaf $\mathcal{L}$ over $X$, $U$ is $\mathcal{L}$-projective. With this terminology we state

**Corollary 2.9.** Let $Y$ be a normal, integral scheme, proper over $k$. Let $U$ be a maximal divisorially quasi-projective open subscheme of $Y$. Then $Y - U$ has no irreducible components of codimension 1 whose associated sheaf of ideals is invertible. (Under the assumption made in (II) of Theorem 2.8).

**Proof.** By assumption $U$ is $\mathcal{L}$-projective, for some invertible sheaf $\mathcal{L}$ over $Y$. Assume that $P$ is the generic point of an irreducible component of $Y - U$, such that the associated sheaf of ideals $\mathcal{G}$ of $P$ in $\mathcal{O}_Y$ is invertible. Let $r$ be fixed as in the proof of (II) of Theorem 2.8. Then, for $n \gg 0$, $U = Y(\mathcal{L}) \subseteq Y(\mathcal{L}^{\otimes n} \otimes \mathcal{G}^{\otimes r})$. In fact, the first equality follows from the fact that $U$ is maximal divisorially quasi-projective, and the inequality from the fact that $P \notin U$, while, by Theorem 2.8, $P$ does belong to the open subscheme $Y(\mathcal{L}^{\otimes n} \otimes \mathcal{G}^{\otimes r})$. 

(2.8.1) $h_*(\mathcal{H}^{\otimes n}) = \mathcal{G}^{\otimes m}$. 

To see the above it suffices to assume that $Y$ is affine, and in this case it becomes an easy verification, using the facts that $Y$ is normal, that $\mathcal{O}$ is principal and selfradical, and that $\mathcal{O}_{P,Y}$ is a discrete valuation ring.
Corollary 2.10. Let $Y$ be a normal, integral scheme, proper over $k$. Let $U$ be a maximal, quasi-projective open subscheme of $Y$ which contains all the singularities of $Y$. Then $\text{codim}(Y - U) > 1$.

Proof. Under the basic assumption made in (II) of the statement of Theorem 2.8, this corollary is an immediate consequence of the previous one. In fact, it suffices to observe that, if $\mathcal{H}$ denotes an ample, invertible sheaf over $U$, then, since $Y$ is nonsingular off $U$, there exists at least one invertible sheaf $\mathcal{L}$ over $Y$ which extends $\mathcal{H}$. Therefore $U$ is $\mathcal{L}$-projective, and hence maximal divisorially quasi-projective. Since $Y$ is nonsingular off $U$, the corollary follows from Corollary 2.9.

However, as the Editor has pointed out to the author, the corollary is valid without the basic assumption made in (II) of the statement of Theorem 2.8. In fact, since, as before, $U$ is $\mathcal{L}$-projective for some suitable invertible sheaf $\mathcal{L}$ over $Y$, by Proposition 2.7 there exists an integral scheme $X$ which dominates $Y$, and such that the fundamental locus of the morphism $f: X \to Y$ in $Y$ is, by the maximality of $U$, precisely $Y - U$. However, since $Y$ is normal, such fundamental locus is of codimension $> 1$. The corollary is proved.

In particular, every nonsingular, nonprojective threefold (with no assumptions on the field $k$, other than it be algebraically closed) must have quasi-projective open subschemes whose complements are of pure dimension 1. (See the examples of nonsingular, nonprojective threefolds given by Hironaka and Nagata in [5] and [9] respectively).

To say that the singularities of a normal surface $Y$ are contained in an open affine subscheme is equivalent, by Proposition 1 of [2], to saying that the singularities of $Y$ are contained in an open $\mathcal{L}$-projective subscheme, for a suitable invertible sheaf $\mathcal{L}$ over $Y$. This observation, combined with Corollaries 2.9 and 2.5, give us another proof of the well known result of Zariski (see [11]) that every normal surface, whose singularities are contained in an open affine subscheme is quasi-projective.

In [1] the author has studied divisorial schemes, i.e., schemes which admit a finite open cover of the form $\{X(\mathcal{L}_i)\}_{i=1,\ldots,n}$. Corollary 2.10 implies that, if $Y$ is a normal, divisorial scheme, then the invertible sheaves $\mathcal{L}_i$ can be chosen so that no zero-dimensional subscheme of $Y$, nor integral subschemes of codimension 1, whose associated sheaves of ideals are invertible, appear as components of the closed subsets $Y - Y(\mathcal{L}_i)$. 
BIBLIOGRAPHY


Received January 30, 1969.

UNIVERSITY OF NOTRE DAME
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

BASIL GORDON
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
STANFORD UNIVERSITY
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF TOKYO
UNIVERSITY OF CALIFORNIA
UNIVERSITY OF UTAH
MONTANA STATE UNIVERSITY
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF NEVADA
UNIVERSITY OF WASHINGTON
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
AMERICAN MATHEMATICAL SOCIETY
UNIVERSITY OF OREGON
CHEVRON RESEARCH CORPORATION
OSAKA UNIVERSITY
TRW SYSTEMS
UNIVERSITY OF SOUTHERN CALIFORNIA
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
George E. Andrews, *On a calculus of partition functions* .................................................. 555
Silvio Aurora, *A representation theorem for certain connected rings* ............................ 563
Lawrence Wasson Baggett, *A note on groups with finite dual spaces* ......................... 569
Steven Barry Bank, *On majorants for solutions of algebraic differential equations in regions of the complex plane* ................................................................. 573
Klaus R. Bichteler, *Locally compact topologies on a group and the corresponding continuous irreducible representations* .......................................................... 583
Mario Borelli, *Affine complements of divisors* ................................................................. 595
Carlos Jorge Do Rego Borges, *A study of absolute extensor spaces* ............................ 609
Bruce Langworthy Chalmers, *Subspace kernels and minimum problems in Hilbert spaces with kernel function* ................................................................. 619
John Dauns, *Representation of L-groups and F-rings* .................................................... 629
Spencer Ernest Dickson and Kent Ralph Fuller, *Algebras for which every indecomposable right module is invariant in its injective envelope* ....................... 655
Robert Fraser and Sam Bernard Nadler, Jr., *Sequences of contractive maps and fixed points* ................................................................. 659
Judith Lee Gersting, *A rate of growth criterion for universality of regressive isols* ............ 669
Robert Fred Gordon, *Rings in which minimal left ideals are projective* ....................... 679
Fred Gross, *Entire functions of several variables with algebraic derivatives at certain algebraic points* ................................................................. 693
W. J. Kim, *The Schwarzian derivative and multivalence* ................................................ 717
Robert Hamor La Grange, Jr., *On (m − n) products of Boolean algebras* ..................... 725
Charles D. Masiello, *The average of a gauge* ................................................................. 733
Stephen H. McCleary, *The closed prime subgroups of certain ordered permutation groups* ................................................................. 745
Richard Roy Miller, *Gleason parts and Choquet boundary points in convolution measure algebras* ................................................................. 755
Harold L. Peterson, Jr., *On dyadic subspaces* ................................................................. 773
Derek J. S. Robinson, *Groups which are minimal with respect to normality being intransitive* ................................................................. 777
Ralph Edwin Showalter, *Partial differential equations of Sobolev-Galpern type* ............ 787
David Slepian, *The content of some extreme simplexes* ................................................ 795
Joseph L. Taylor, *Noncommutative convolution measure algebras* ............................... 809
B. S. Yadav, *Contractions of functions and their Fourier series* ................................... 827
Lindsay Nathan Childs and Frank Rimi DeMeyer, *Correction to: “On automorphisms of separable algebras”* ................................................................. 833
Moses Glasner and Richard Emanuel Katz, *Correction to: “Function-theoretic degeneracy criteria for Riemannian manifolds”* ................................................... 834
Benjamin Rigler Halpern, *Addendum to: “Fixed points for iterates”* ............................. 834