A STUDY OF ABSOLUTE EXTENSOR SPACES

CARLOS JORGE DO REGO BORGES
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In the enclosed paper, we will prove, among others, the following results: (a) A sufficient condition that a space \( L \) be an absolute extensor for the class of stratifiable spaces in that \( L \) be hyperconnected (this is a refinement of the concept of equiconnected space). This condition is also necessary if \( L \) is a metrizable space. (b) Every hypogeodesic space is hyperconnected. (c) Every equiconnected space is \( \infty \)-hyperconnected. (d) Every \( \infty \)-hyperconnected space is an absolute extensor for the class of \( CW \)-complexes of Whitehead.

Some of the merits of our results are the following: (1) The well-known extension theorem of Dugundji (Theorem 4.1 of [2]) as well as part of Theorem 4.3 of [1] are immediate consequences of our Theorem 4.1. (2) It is easily verified that the space \( Y \) of Theorem 3.4 of [3] (of course, we need to interchange the roles of the first and second variables of \( F \) in order to agree with Dugundji’s definition of \( \lambda \)-stable), is locally hypogeodesic and, therefore, Theorem 3.4 of [3] is an easy consequence of our Theorems 3.3 and 4.4. (3) Our results generalize Theorems 3 and 4 of Himmelberg [4] and Theorem 3.4 of Dugundji [3] by removing the stringent hypothesis that the range space be metrizable.

Our many attempts to solve the question “Is every equiconnected metrizable space an AE (metrizable)?”, which is raised in [3], have, so far, ended in failure. However, our Theorem 4.3 offers a partial solution which leads us to conjecture an affirmative answer to this question, especially in view of the “replacement-by-polytopes” technique of Dugundji [2] (If only we could do it!).

2. Definitions. Throughout, let \( P_{n-1} \) denote the unit simplex in Euclidean \( n \)-space \( R^n \) (i.e., \( P_{n-1} = \{ t \in R^n \mid \sum_{i=1}^{n} t_i = 1 \text{ and each } t_i \geq 0 \} \), \( I \) the closed unit interval, and \( A^n \) the \( n \)-fold cartesian product of any set \( A \). Furthermore, let \( \delta_i: A^n \rightarrow A^{n-1} \) be the map defined by

\[
\delta_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)
\]

for \( i = 1, 2, \ldots, n \).

It seems appropriate to start our definitions by recollecting the concept of an “equiconnected space”, which was introduced by Fox [4] and is called a \( UC \)-space by Serre [9, p. 490], not only because

\[1\] We use the following abbreviations: AE \( \equiv \) absolute extensor, ANE \( \equiv \) absolute neighborhood extensor, AR \( \equiv \) absolute retract, ANR \( \equiv \) absolute neighborhood retract.
it is closely related to the new concepts we will introduce but also because it will make them more plausible. It should also be observed that the similarity of our Definitions 2.2 and 2.3 with Definitions 1.1 and 5.1 of Michael [8] is, by far, not accidental.

**Definition 2.1.** A space $L$ is equiconnected if there exists a continuous map $F: L \times L \times I \rightarrow L$ such that $F(a, b, 0) = a, F(a, b, 1) = b$ and $F(a, a, t) = a$ for all $(a, b) \in L \times L$ and $t \in I$. (The function $F$ will be called an equiconnecting function.) The space $L$ is said to be locally equiconnected if $F$ is defined only on $U \times I$ with $U$ some neighborhood of the diagonal of $L \times L$.

**Definition 2.2.** A space $L$ will be called respectively hyperconnected, $m$-hyper connected, if there exists functions $h^i: L^i \times P_{i-1} \rightarrow L$, for $i = 1, 2, \ldots$, which satisfy conditions (a), (b), (c) and conditions (a), (b), (d) respectively:

(a) $t \in P_{n-1}$ and $t_i = 0$ implies $h_n(x, t) = h_{n-1}(\delta_i x, \delta_i t)$ for each $x \in L^n$ and $n = 2, 3, \ldots$,

(b) for each $x \in L^n$, the map $t \rightarrow h_n(x, t)$, from $P_{n-1}$ to $L$, is continuous,

(c) for each $x \in L$ and neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ such that

$$\bigcup_{i=1}^{\infty} h_i(V^i \times P_{i-1}) \subset U$$

and $V \subset U$,

(d) for each $x \in L$ and neighborhood $U$ of $x$, there exists a neighborhood $V$ of $x$ such that

$$\bigcup_{i=1}^{m} h_i(V^i \times P_{i-1}) \subset U$$

and $V \subset U$ (we should observe that, in this case, the functions $h_k$, for $k \geq m + 1$, may be assumed to be constant functions). The space $L$ will be called $\infty$-hyperconnected provided that $L$ is $m$-hyperconnected for $m = 1, 2, \ldots$. The space $L$ will be called locally hyperconnected provided that, for each $x \in X$, there exists a neighborhood of $x$ which is hyperconnected. We similarly define locally $m$-hyperconnected and locally $\infty$-hyperconnected.

**Definition 2.3.** A space $L$ is said to be hypogeodesic if there exists a function $F: L \times L \times I \rightarrow L$ satisfying the following conditions:

(a) for $x, y \in L$ and $t \in I$, $F(x, y, 0) = x, F(x, y, 1) = y$ and $F(x, x, t) = x$,

(b) for each $y \in L$, the map $(x, t) \rightarrow F(x, y, t)$, from $L \times I$ to $L$, is continuous,
(c) for each \( x, a, y \in L \) and neighborhood \( U \) of \( y \), there exist neighborhoods \( V \) of \( a \) and \( W \) of \( y \) (\( V \) and \( W \) depend on \( x \)) with \( W \subset U \), such that

\[
F(a, x, t) \in W \implies F(b, x, t) \in U
\]

for any \( b \in V \),

(d) for each \( x \in L \) there exist neighborhood bases \( \{V_{\alpha}\} \) and \( \{U_{\alpha}\} \) of \( x \), with \( V_{\alpha} \subset U_{\alpha} \) for each \( \alpha \), such that

\[
y \in V_{\alpha}, z \in U_{\alpha} \implies F(z, y, t) \in U_{\alpha}
\]

for each \( t \in I \).

(The function \( F \) will be called an hypogeodesic function for \( L \).)

The space \( L \) is said to be locally hypogeodesic if \( F \) is defined only on \( U \times I \) with \( U \) some neighborhood of the diagonal on \( L \times L \).

Clearly, every locally convex linear topological space is hypogeodesic, and every linear topological space is equiconnected. Furthermore, every equiconnecting function is easily seen to satisfy conditions (a), (b), and (c) of Definition 2.3.

3. Hypo, equi and hyper.

**Theorem 3.1.** If \( L \) is hypogeodesic then \( L \) is hyperconnected.

**Proof.** (Similar to the proof of Proposition 5.3 of Michael [8].) Throughout this proof we will make use of the following notation:

1. If \( x \in L^{n+1} \), then \( \hat{x} \in L^n \) is defined by \( \hat{x}_i = x_i \) for \( i = 1, \ldots, n \).
2. If \( t \in P_n \) and \( t_{n+1} \neq 1 \) then \( \hat{t} \in P_{n-1} \) is defined by

\[
\hat{t}_i = \frac{t_i}{1 - t_{n+1}} \quad \text{for} \quad i = 1, \ldots, n.
\]

Clearly \( \hat{\delta}_i \hat{x} = (\hat{\delta}_i x) \) and \( \hat{\delta}_i \hat{t} = (\hat{\delta}_i t) \), whenever \( t \in P_{n+1} \), \( x \in L^n \) and \( 1 < i < n \).

Let \( h_i : L \times \{1\} \rightarrow L \) be defined by \( h_i(x, 1) = x \), and let \( F \) be an hypogeodesic function for \( L \). By induction, assume we have defined maps \( h_i : L^i \times P_{i-1} \rightarrow L \), for \( i = 1, \ldots, n \), which satisfy parts (a) and (b) of Definition 2.2. Now let

(3) \[
h_{n+1}(x, t) = \begin{cases} x_{n+1} & \text{if } t_{n+1} = 1 \\ F(h_n(\hat{x}, \hat{t}), x_{n+1}, t_{n+1}) & \text{if } t_{n+1} \neq 1. \end{cases}
\]

Let us check that the functions \( h_n \) satisfy conditions (a), (b) and (c) of Definition 2.2.

2.2(a). Clearly \( h_2(x, t) = h_1(\hat{\delta}_i x, \hat{\delta}_i t) \) whenever \( t_i = 0 \). Hence let us
assume that \( h_n(x, t) = h_{n-1}(\delta_i x, \delta_i t) \) whenever \( t_i = 0 \) and let us shown that \( h_{n+1}(x, t) = F(h_n(x, \delta_{n+1} t), x_{n+1}, t_{n+1}) \) whenever \( t_{n+1} = 0 \) and let us shown that \( h_{n+1}(x, t) = \Lambda_{n+1}(\delta_i a^?, \delta_i t) \) whenever \( t_i = 0 \) (1 \( \leq i \leq n + 1 \)): If \( i = n + 1 \) then \( h_{n+1}(x, t) = F(h_n(x, \delta_{n+1} t), x_{n+1}, t_{n+1}) = h_n(\delta_{n+1} x, \delta_{n+1} t) \) by (3) and Definition 2.3(a). If \( i < n + 1 \) and \( t_{n+1} = 1 \), then \( h_{n+1}(x, t) = x_{n+1} = h_n(\delta_i x, \delta_i t) \), by (3). If \( i < n + 1 \) and \( t_{n+1} \neq 1 \), then

\[
\begin{align*}
  h_{n+1}(x, t) &= F(h_n(\delta_i x, \delta_i t), x_{n+1}, t_{n+1}) \\
  &= F(h_{n-1}(\delta_i x, \delta_i t), x_{n+1}, t_{n+1}) \\
  &= F(\delta_i x, \delta_i t), \quad (\alpha_{n+1}, t_{n+1}) = h_n(\delta_i x, \delta_i t) .
\end{align*}
\]

By induction, we get that the functions \( h_n \) satisfy Definition 2.2(a).

### 2.2(b)

**Proof by induction.** Clearly the map \( t \mapsto h_i(x, t) = x \), from \( P_0 \) to \( L \), is continuous. Suppose that the map \( t \mapsto h_n(w, t) \), from \( P_{n-1} \) to \( X \), is continuous, for each \( w \in L^n \). Pick a fixed point \( x \in L^{n+1} \) and let us show that the map \( t \mapsto h_{n+1}(x, t) \), from \( P_n \) to \( L \), is also continuous: Let \( s \in P_n \). Then

**Case 1.** \( s_{n+1} \neq 1 \). For some neighborhood \( U \) of \( s \) in \( P_n \), \( t \in U \) implies \( t_{n+1} \neq 1 \). Therefore, for \( t \in U \), \( h_{n+1}(x, t) = F(h_n(\delta_i x, \delta_i t), x_{n+1}, t_{n+1}) \) and hence the continuity of the map \( t \mapsto h_{n+1}(x, t) \) at \( s \in P_n \), follows from Definition 2.3(b) and the inductive hypothesis. (Indeed, pick sequence \( t(1), t(2), \cdots \) in \( U \subseteq P_n \) such that \( \lim t(i) = s \). Then one easily sees that \( \lim_i t(i) = s \) and thus \( \lim_i h_n(\delta_i x, \delta_i t(i)) = h_n(\delta_i x, \delta_i s) \) by the inductive hypothesis. Consequently, \( \lim_i h_{n+1}(x, t(i)) = \lim_i F(h_n(\delta_i x, \delta_i t(i)), x_{n+1}, t_{n+1}) = h_{n+1}(x, s) \) which shows that the map \( t \mapsto h_{n+1}(x, t) \) is continuous at \( s \).)

**Case 2.** \( s_{n+1} = 1 \). Then \( h_{n+1}(x, s) = x_{n+1} \), by (3). We will show that \( h_{n+1}(x, t) \) is close to \( x_{n+1} \) if \( t_{n+1} \) is close to 1, as follows: Let \( A = \{ h_n(\delta_i x, \delta_i t) \mid t \in P_{n-1} \} \) and, for each \( \tau \in I \), let \( f_\tau : A \rightarrow X \) be defined by

\[
  f_\tau(\alpha) = F(\alpha, x_{n+1}, \tau) .
\]

Then

\[
  h_{n+1}(x, t) = f_{t_{n+1}}(a_i) \quad \text{for some} \ a_i \in A \ \text{if} \ t_{n+1} \neq 1 ,
\]

\[
  h_{n+1}(x, t) = x_{n+1} \quad \text{if} \ t_{n+1} = 1 .
\]

Therefore, in order to prove that the map \( t \mapsto h_{n+1}(x, t) \) is continuous at \( s \), it suffices to show that \( f_\tau \rightarrow f_1 \) uniformly as \( \tau \rightarrow 1 \) (note that \( f_1(\alpha) = x_{n+1} \) for each \( \alpha \in A \), by Definition 2.3(a)): From Definition 2.3(b) we get that \( f_\tau \rightarrow f_1 \) pointwise. Moreover \( A \) is compact, because \( h_n \) is continuous and \( \{ x \} \times P_n \) is compact, and the family \( \{ f_\tau \mid \tau \in I \} \) is evenly continuous (see page 235 of [7]), by Definition 2.3(c). (Let \( a \in A \subseteq L \) and \( y \in L \) and let \( U \) be a neighborhood of \( y \). By Definition 2.3(c), pick neighborhoods \( V \) of \( a \) and \( W \) of \( y \) (\( V \) and \( W \) depend on \( x_{n+1} \)) with
$W \subset U$ such that whenever $F(a, x_{n+1}, t) \in W$ then $F(b, x_{n+1}, t) \in U$ for all $b \in V$. Then we obviously get that $f_{\tau}(V) \subset U$ whenever $f_{\tau}(a) \in W$, which shows that the family \{f_{\tau} | \tau \in I\} is evenly continuous.) By Theorem 23 (page 237) of [7], the family \{f_{\tau} | \tau \in I\} is equicontinuous (note that, for each $a \in A$, \{f_{\tau}(a) | \tau \in I\} $\subset F(A \times \{x_{n+1}\} \times I)$ a compact subset of $L$, because of Definition 2.3(b)). Therefore, the topologies of pointwise convergence and uniform convergence coincide on \{f_{\tau} | \tau \in I\}, by Theorem 15 (page 232) of [7], and hence $f_{\tau} \to f_{1}$ uniformly as $\tau \to 1$.

It is now easily seen that the map $t \to h_{n+1}(x, t)$ is continuous at each $s \in P_{n}$.

2.2(c). Let $x \in L$ and open $U \subset L$ such that $x \in U$. Then, by Definition 2.3(d), there exists neighborhoods $V$ and $W$ of $x$ such that $V \subset W \subset U$ and $F(z, y, t) \in W$ whenever $y \in V$ and $z \in W$. By an inductive argument, one easily sees that

$$\bigcup_{n=1}^{\infty} h_{n}(V^{n} \times P_{n-1}) \subset W \subset U,$$

which completes the proof.

**Theorem 3.2.** If $L$ is equiconnected then $L$ is $\infty$-hyperconnected.

**Proof.** Exactly the same as the proof of Theorem 3.1, except that we must verify that the functions $h_{n}$, $n = 1, 2, \ldots$, satisfy condition (d) of Definition 2.2: Let us first observe that, for each neighborhood $U$ of $x$ one easily finds a neighborhood $V$ of $x$ with $V \subset U$,

$$F(V \times V \times I) \subset U,$$

because $F$ is continuous, $F(x, x, t) = x$ for each $t \in I$ and $I$ is compact. Clearly $L$ is 1-hyperconnected. Therefore let us assume that $L$ is $n$-hyperconnected and show that $L$ is $(n + 1)$-hyperconnected. Pick $x \in L$ and neighborhood $U$ of $x$. Then pick neighborhoods $V$ and $W$ of $x$ such that

$$F(V \times V \times I) \subset U, h_{n}(W^{n} \times P_{n-1}) \subset V, W \subset V \subset U.$$

Using the definition of $h_{n+1}$, it is trivial to check that

$$h_{n+1}(W^{n+1} \times P_{n}) \subset U.$$

This concludes our inductive argument and completes the proof.

Because of the proofs of Theorems 3.1 and 3.2, the following result is clearly valid and easily verified:
**Theorem 3.3.** If \( L \) is locally hypogeodesic (locally equi-connected) then \( L \) is locally hyper connected (locally \( \infty \)-hyper connected).

4. Extension theorems.

**Theorem 4.1.** Every hyperconnected space \( L \) is an \( \AE \) (stratifiable).

**Proof.** The proof of this result is very similar to the proof of Theorem 4.3 in [1]. We will thus indicate the general procedure without details. We briefly comment on some recurring notation throughout ensuing proof: Letting \( U \rightarrow \{U_n\}_{n=1}^{\infty} \) be a stratification of \( X \) (see footnote 1), we let \( n(U, x) = \min \{n \mid x \in U_n\} \) and \( U_x = U_{n(U, x)} - (X - \{x\})^{\sim}_{n(U, x)} \), for each open \( U \subset X \) and \( x \in U \). It is easily seen (see Lemma 4.1 of [1]) that each \( U_x \) is an open neighborhood of \( x \), \( U_x \cap V \neq \emptyset \) implies that \( x \in V \) or \( y \in U \) (indeed, \( n(U, x) \leq n(V, y) \) implies that \( y \in U \)).

Now let \( X \) be a stratifiable space, \( A \) a closed subset of \( X \), \( f: A \rightarrow L \) a continuous function. Let \( W = X - A \), \( W' = \{x \in W \mid x \in U_y \) for some \( y \in A \) and open \( U \) containing \( y \} \) and \( m(x) = \max \{n(U, y) \mid y \in A \) and \( x \in U_y \} \), for each \( x \in W' \). It is easily seen that \( m(x) < n(W, x) < \infty \).

Using the paracompactness of \( W \), let \( \mathcal{V} \) be an open locally finite (with respect to \( W \)) refinement of \( \{W_x \mid x \in W \} \). For each \( V \in \mathcal{V} \) pick \( x_v \in V \) with \( V \subset W_{x_v} \). If \( x_v \in W' \) pick \( a_v \in A \) and open \( S_v \) containing \( a_v \) such that \( x_v \in (S_v)_{a_v} \) and \( n(S_v, a_v) = m(x_v) \); if \( x_v \in W \), let \( a_v \) be the fixed point \( a_0 \in A \).

Let \( \{p_v \mid V \in \mathcal{V}\} \) be a partition of unity subordinated to \( \mathcal{V} \) and define \( g: X \rightarrow L \) by

\[
g(x) = f(x) \quad \text{for} \quad x \in A
\]

\[
g(x) = h_n((f(a_{v_1}), \ldots, f(a_{v_n})), (p_{v_1}(x), \ldots, p_{v_n}(x)) \quad \text{for} \quad x \in W,
\]

where \( V_1, \ldots, V_n \) are the only elements \( V \in \mathcal{V} \) such that \( p_v(x) \neq 0 \), for \( x \in X - A \). Clearly \( g \) is a well defined function from \( X \) to \( L \). It is not quite obvious that \( g \) is continuous anywhere, as was the case in Theorem 4.3 of [1].

We will first show that \( g \) is continuous at each point \( b \in A \). Let \( 0 \) be any open subset of \( L \) containing \( f(b) \) and let \( H \) be an open subset of \( 0 \) such that \( f(b) \in H \) and \( \bigcup_{n=1}^{\infty} h_n(H^n \times P_{n-1}) \subset 0 \). Since \( f \) is continuous there exists an open neighborhood \( N \) of \( b \) such that \( f(A \cap N) \subset H \subset 0 \). It is easily seen that \( g((N_b)_b) \subset 0 \), which shows that \( g \) is continuous at each \( b \in A \).

Finally we show that \( g \) is continuous at each \( x \in X - A \). Let \( 0 \) be a neighborhood of

\[
g(x) = h_n((f(a_{v_1}), \ldots, f(a_{v_m})), (p_{v_1}(x), \ldots, p_{v_m}(x)))
\]
and let $N$ be a neighborhood of $x$ which intersects only finitely many $V \in \mathcal{V}$; say $V_1, \ldots, V_m, \ldots, V_{m+k}$. By Definition 2.2(b), there exists a neighborhood $W$ of $(p_{r_1}(x), \ldots, p_{r_m}(x), 0, \ldots, 0) \in P_{m+k-1}$ which is mapped into 0 by the continuous map

$$t \mapsto h_{m+k}(f(a_{r_1}), \ldots, f(a_{r_m+k}), t).$$

Define a map $p: N \to P_{m+k-1} \subset I^{m+k}$ by $p(y) = (p_{r_1}(y), \ldots, p_{r_m+k}(y))$ for each $y \in N$. Since $p$ is clearly continuous, then there exists a neighborhood $U$ of $x$ such that $p(U) \subset W$. It is now easily seen that $g(N \cap U) \subset 0$, with the help of Definition 2.2(a), which shows that $g$ is continuous at each $x \in X - A$, thus completing the proof.

It is easily seen that the preceding proof remains valid if we assume that $L$ is only $(n+1)$-hyperconnected and $X - A$ is $n$-dimensional (in the covering sense), for then we can choose the open cover $\mathcal{V}$ to be of order $n$ and thus define the function $g$ in terms of $f$ and $h_{n+1}$ only. We have, therefore, proved the following result.

**Theorem 4.2.** Let $X$ be a stratifiable space, $A$ a closed subset of $X$, $L$ an $(n+1)$-hyperconnected space and $f: A \to L$ a continuous function. If $\dim(X - A) \leq n$ then $f$ has a continuous extension $g: X \to L$.

The following result, when combined with Theorem 3.2, provides a partial answer to the following question which is raised in [3]: Is every equiconnected matrizable space an AE (metrizable)?

**Theorem 4.3.** Every $\omega$-hyperconnected space $L$ is an AE (CW-complex of Whitehead).

**Proof.** Let $K$ be a CW-complex, $A$ a closed subset of $K$ and $f: A \to L$ a continuous function. For each $n$, let $K_n$ be the $n$-skeleton of $K$. It is well-known that $\dim K_n$ is finite for each $n$. Therefore, by Theorem 4.2 (note that $K$ is stratifiable because of Theorem 7.2 of [1], for example) and induction, we can find continuous functions

$$g_n: A \cup K_n \to L$$

such that

$$g_n|A = f$$

and

$$g_{n+1}|(A \cup K_n) = g_n$$

for each $n$. Now define $g: K \to L$ by

$$g(x) = g_n(x) \quad \text{if } x \in A \cup K_n.$$

It is easily seen that $g$ is a well-defined continuous extension of $f$ to
all of $X$, which completes the proof.

It is easily seen that all the preceding results of this section have local analogues. For the sake of completeness, let us state and prove one.

**Theorem 4.4.** *Every locally hyperconnected space $L$ is an ANE (stratifiable).*

**Proof.** By Theorem 4.1, $L$ is a local AE (stratifiable) (i.e., for each $a \in L$ there exists a neighborhood $N$ of $a$ such that $N$ is an AE (stratifiable)). Consequently, by Theorem 19.2 of Hanner [5], $L$ is an ANE (stratifiable).


**Theorem 5.1.** *A metrizable space $M$ is an AE (stratifiable) if and only if $M$ is hyperconnected.*

**Proof.** Since the “if” part is an immediate consequence of Theorem 4.1, we direct our attention to the “only if” part (the basic technique is extracted from Michael [8]): Embed $M$ in a Banach space $B$ and let $H$ be the closed convex hull of $M$ in $B$. Then there exists a retraction $r: H \to M$. For each $n$, define $h_n: M^n \times P_n \to M$ by

$$h_n(z, t) = r(\sum_{i=1}^n t_i z_i), \quad \text{for } (z, t) \in M^n \times P_n .$$

Clearly, each $h_n$ satisfies conditions (a) and (b) of Definition 2.2. Since $r$ is continuous, for each $x \in M$ and open subset $U$ of $H$ with $x \in U$, there exists a convex open set $V \subset H$ with $x \in V \subset U$ such that

$$r(V \cap M) \subset U \cap M ;$$

hence $\bigcup_{i=1}^n h_i(V \cap M) \times P_{i-1}) \subset U \cap M$, and hence the functions $h_n$ satisfy condition (c) of Definition 2.2. Consequently $M$ is hyperconnected.

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Pacific Journal of Mathematics
Vol. 31, No. 3  BadMonth, 1969

George E. Andrews, On a calculus of partition functions .................................. 555
Silvio Aurora, A representation theorem for certain connected rings ...................... 563
Lawrence Wasson Baggett, A note on groups with finite dual spaces .......... 569
Steven Barry Bank, On majorants for solutions of algebraic differential equations in regions of the complex plane .................................................. 573
Klaus R. Bichteler, Locally compact topologies on a group and the corresponding continuous irreducible representations ................................................. 583
Mario Borelli, Affine complements of divisors .................................................. 595
Carlos Jorge Do Rego Borges, A study of absolute extensor spaces ........ 609
Bruce Langworthy Chalmers, Subspace kernels and minimum problems in Hilbert spaces with kernel function .................................................. 619
John Dauns, Representation of L-groups and F-rings .................................. 629
Spencer Ernest Dickson and Kent Ralph Fuller, Algebras for which every indecomposable right module is invariant in its injective envelope ........ 655
Robert Fraser and Sam Bernard Nadler, Jr., Sequences of contractive maps and fixed points .................................................. 659
Judith Lee Gersting, A rate of growth criterion for universality of regressive isols .................................................. 669
Robert Fred Gordon, Rings in which minimal left ideals are projective .......... 679
Fred Gross, Entire functions of several variables with algebraic derivatives at certain algebraic points .................................................. 693
W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, Wreath products of ordered permutation groups .................................................. 703
W. J. Kim, The Schwarzian derivative and multivalence .................................. 717
Robert Hamor La Grange, Jr., On (m – n) products of Boolean algebras ........ 725
Charles D. Masiello, The average of a gauge .................................................. 733
Stephen H. McCleary, The closed prime subgroups of certain ordered permutation groups .................................................. 745
Richard Roy Miller, Gleason parts and Choquet boundary points in convolution measure algebras .................................................. 755
Harold L. Peterson, Jr., On dyadic subspaces .................................................. 773
Derek J. S. Robinson, Groups which are minimal with respect to normality being intransitive .................................................. 777
Ralph Edwin Showalter, Partial differential equations of Sobolev-Galpern type ........ 787
David Slepian, The content of some extreme simplexes .................................. 795
Joseph L. Taylor, Noncommutative convolution measure algebras .......... 809
B. S. Yadav, Constructions of functions and their Fourier series ........ 827
Lindsay Nathan Childs and Frank Rimi DeMeyer, Correction to: “On automorphisms of separable algebras” .................................................. 833
Moses Glasner and Richard Emanuel Katz, Correction to: “Function-theoretic degeneracy criteria for Riemannian manifolds” .................................................. 834
Satish Shirali, Correction to: “On the Jordan structure of complex Banach *algebras” .................................................. 834
Benjamin Rigler Halpern, Addendum to: “Fixed points for iterates” ........ 834