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**Consider an $f$-algebra $A$ with identity (i.e., $a, b, c \in A$, $a \wedge b = 0$, $c \geqq 0 \Rightarrow ca \wedge b = ac \wedge b = 0$) over the rationals $\boldsymbol{Q}$. Let $\mathscr{M}$ be the maximal $\ell$-ideals of $A$.**

THEOREM I. **If $1 \leqq a \in A \Rightarrow 1/a \in A$, then each $A/M$, $M \in \mathscr{M}$ is a totally ordered division ring, and $A$ can be embedded into a real $f$-algebra.**

THEOREM II. **$A$ fiber-bundle or sheaf-like structure**

$$\pi: E \equiv \bigcup \{A/M \mid M \in \mathscr{M}\} \rightarrow \mathscr{M}, \pi^{-1}(M) = A/M \subset E$$

**is constructed. Assume $\bigcap \mathscr{M} = \{0\}$; $\mathscr{M}$ has the hull-kernel topology. All continuous cross sections $\sigma: \mathscr{M} \rightarrow E$ ($\pi \circ \sigma =$ identity) form a partial algebra $\Gamma(\mathscr{M}, E)$ containing an isomorphic copy of $A \cong \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$. Let $A^* = \{a \in A \mid |a| < n1$, some integer $n\}$. If**

**(i) $1 \leqq a \in A^* \Rightarrow 1/a \in A^*$**

**then $A \cong \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$, where $\hat{A}$ is order dense in $\Gamma(\mathscr{M}, E)$. If in addition**

**(ii) $A^*$ is complete with respect to the absolute value $|a|$, $a \in A$,**

**then $A \cong \hat{A} = \Gamma(\mathscr{M}, E)$.**

The purely algebraic result I is established first and completely independently of II. M. Henriksen and D. G. Johnson [9] proved that a $\Phi$-algebra $A$ (a real archimedean $f$-algebra without nilpotents) is isomorphic to a subalgebra of the partial algebra $D(\mathscr{M})$ of all continuous functions $f: \mathscr{M} \rightarrow \boldsymbol{R} \cup \{\pm \infty\}$ into the two point compactification of the reals $\boldsymbol{R}$, where $f$ is finite on a dense open subset of $\mathscr{M}$. This representation suffers from the defect that it is not onto. In general, $\mathscr{M}$ is a compact Hausdorff space and $D(\mathscr{M})$ is not closed under addition or multiplication. However, if $\mathscr{M}$ is extremally disconnected, then $D(\mathscr{M})$ is an algebra, but even in this case, $A$ need not be all of $D(\mathscr{M})$. Many different algebras can have the same maximal ideal space $\mathscr{M}$. Thus the correspondence $A \rightarrow D(\mathscr{M})$ is not one to one. By associating with an $f$-ring $A$, the two invariants $\mathscr{M}$ and $E$ rather than just $\mathscr{M}$ alone, the nonuniqueness of the representation $A \rightarrow D(\mathscr{M})$ is overcome. The underlying cause of the nonuniqueness of the latter representation is that the set $\boldsymbol{R} \cup \{\pm \infty\}$ is too small. The space $\boldsymbol{R} \cup \{\pm \infty\}$ must be replaced by the bigger space $\bigcup \{A/M \mid M \in \mathscr{M}\}$, where in general each $A/M$ contains $\boldsymbol{R}$ properly. An immediate advantage of this is that $D(\mathscr{M})$, which in general is not closed under addition, is replaced by $\Gamma(\mathscr{M}, E)$, which is a group. Neither of these two need be closed under multiplication. In general, the stalks $A/M$ are integral domains contained in the hyper-reals. A structure $\pi: E \to \mathscr{M}$ called a "field" will be constructed. Fields are generalizations of sheaves, the main difference being that the stalks $\pi^{-1}(M)$ of $E$ need no longer necessarily be discrete, as would be the case for sheaves. The space $\mathscr{M}$ has the already familiar hull-kernel topology. The topologies on $E$ and $\mathscr{M}$ are not objects arbitrarily constructed for sake of convenience but rather are intrinsically associated with the ring $A$. In fact, the topologies on $E$ and $\mathscr{M}$ are the unique minimal ones subject to the two requirements that $\pi: E \to \mathscr{M}$ as well as all the maps $\hat{a}: \mathscr{M} \to E$ of $\hat{A}$ be continuous. The whole theory of such fields is treated in a very general framework in [6] and will not be repeated here. The construction of these fields, though elementary, is somewhat lengthy and involved. One of the major objectives of this paper is to give a very concise, direct, yet completely rigorous construction of the field by utilizing the available additional structure whenever possible to shorten proofs that otherwise would be long if everything was done in as great a generality as possible. By avoiding reference to towers of previous theorems, but rather by outlining the steps that would be necessary to develop the subject from its beginnings and keeping topological considerations to a minimum, the author hopes to make the representation $A \cong \hat{A} = \Gamma(\mathscr{M}, E)$ not only more attractive, but perhaps also more widely used.

The representation $A \cong \hat{A} \subseteq \Gamma(\mathscr{M}, E)$ applies to a wider class of rings than $\Phi$-algebras. Indeed, $A$ need not be commutative nor does it have to be an algebra over the reals. Perhaps some parts of the theory that at the present have been established only for subalgebras of $D(\mathscr{M})$ ([9], [10], [16]) have analogues in the more general class of $f$-rings, that could be proved by using the representation $A \cong \hat{A} \subseteq \Gamma(\mathscr{M}, E)$. In fact, only very recently P. Nanzetta ([16]) has made a study of the maximal $\Phi$-subalgebras of $D(\mathscr{M})$ and it would be interesting to know how far his results generalize to subalgebras of $\Gamma(\mathscr{M}, E)$.

The representation methods used here are stated for an $\ell$-group $A$ where the subgroups $\mathscr{M}$ are not assumed to be normal in the hope that this method can be applied to other ordered algebraic structures. These and similar methods have only very recently been used sucessfully in [13], [14], [19], [5], and [6].

## 2. $F$-rings closed under bounded inversion.

A partially ordered ring is said to be closed under bounded inversion if $0 < 1 < a$ implies there exists $a^{-1}$. The main objective of this section is to show that a semisimple $f$-ring that is closed under bounded inversion is a subdirect product of totally ordered division rings. Then § 4 describes with as much accuracy as can be expected this subdirect product inside the full direct product.

NOTATION 2.1. In any $\ell$-group $A$, by an *$\ell$-subgroup* $M$ will be meant a convex subgroup $M$ that also is a sublattice; $M$ is *prime* will mean that $M$ is an $\ell$-subgroup such that for $0 < a, c \in A$ with $a \wedge c \in M$ implies that either $a \in M$ or $c \in M$. In this case the right coset space $A/M$ is totally ordered by $M + a < M + c$ provided coset representatives $a$ and $c$ can be chosen with $a < c$. If $A$ is an $\ell$-ring, then $M$ is an $\ell$-ideal, if $M$ is a ring ideal and an additive $\ell$-subgroup of $A$. Thus a maximal $\ell$-ideal $M$ of $A$ may be properly contained in a proper ideal $J$ of $A$, $M \subset J \subset A$, but then $J$ is not an $\ell$-ideal. The word "prime" here will always be used in the above sense of $\ell$-groups and never in the purely ring theoretic sense. Although in an arbitrary $\ell$-ring maximal $\ell$-ideals need not be prime, in $f$-rings they are always prime (see [7]; pp. 146–149, Theorems 9, 6, and 4). A subset $A_1$ of a partially ordered ring $A$ is said to be *closed under bounded inversion* if $0 < 1 \leqq a \in A_1$ implies there exists $a^{-1} \in A$.

The next proposition establishes I for the special case when $A$ is totally ordered. Throughout the logical symbol "$\forall$" will be used for the phrase "for any."

PROPOSITION 2.2. *Consider a totally ordered ring $A$ closed under bounded inversion. Define $I$ and $N$ to be the invertible and non-invertible elements. Let $S$ be the small elements $S = \{s \in A \mid \forall i, 0 < i \in I, |s| < i\}$. Then*

(i) $N = S$;

(ii) *$N$ is a maximal ideal and also an $\ell$-ideal;*

(iii) *$A/N$ is a totally ordered division ring.*

*Proof.* (i) Clearly, $A = N \cup I$, with $N \cap I = \varnothing$. Note that $0 < s \in S \cap I$ implies $s < s$. Thus $S \subseteq N$. To show that $S = N$, it only remains to show that $A\backslash S \subseteq I$. If $0 < a \in A\backslash S$, then there is an $i \in I$ with $0 < i \leqq a$. Hence

$$1 = i^{-1}i \leqq i^{-1}a \Rightarrow \exists x \in A \text{ with } xi^{-1}a = 1\,,$$
$$1 = ii^{-1} \leqq ai^{-1} \Rightarrow \exists y \in A \text{ with } ai^{-1}y = 1\,.$$

Thus $xi^{-1} = (xi^{-1})a(i^{-1}y) = i^{-1}y = a^{-1}$.

(ii) First, $N$ is a convex $\ell$-subgroup of $A$. For if $x, y \in N$, then $2x, 2y \in N$. Thus for any $i \in I$, $|2x| < i$ and $|2y| < i$, and consequently

$$|x - y| \leqq |x| + |y| \leqq 2(|x| \vee |y|) = 2\max(|x|, |y|) < i\,.$$

If $N$ were not an ideal, then for some $0 < n \in N$ and some $0 < x \in A$

either $nx \notin N$ or $xn \notin N$. Assume $xn \notin N$, the case when $nx \notin N$ is similar and will be omitted. Then $xn \in I$. Let us show that $x \in I$. If $x \notin I$, then for any $0 < i \in I, 2x < i$. Since $n < 1$, we have $xn < 2xn \leqq i$. Thus $xn \in N$, a contradiction. Hence $x \in I$. Define $y = (xn)^{-1}$, i.e., $yxn = xny = 1$. Then $nyx = x^{-1}(xny)x = x^{-1}x = 1$. Thus $(yx)n = n(yx) = 1$ gives a contradiction. Thus $N$ is an ideal, $A = N \cup I$, $N \cap A^+ < I \cap A^+$, and $A/N = I \cup \{0\}$ is a totally ordered division ring.

The next theorem gives the main application of the last proposition. It is not assumed that $\bigcap \mathscr{M} = \{0\}$ in the next theorem.

THEOREM I 2.3. *Consider an f-ring A with an identity such that* $1 < a \in A$ *implies* $1/a \in A$. *Then for each maximal* $\ell$*-ideal M, A/M is a totally ordered division ring.*

*Proof.* Suppose $1 + M < a + M \in A/M$. Then $(1 + M) \vee (a + M) = 1 \vee a + M = a + M$, and $1 < 1 \vee a$ implies $1/(1 \vee a) \in A$. Thus $A/M$ satisfies the hypotheses of 2.2 with $N = \{0\}$. Hence $A/M$ is a totally ordered division ring.

In general, there do not seem to be any easily describable necessary and sufficient conditions for an $f$-ring $A$ to be embeddable in an $f$-ring containing the reals (see [11; p. 351]).

COROLLARY TO THEOREM I 2.4. *Consider an f-ring A with* $1 \in A$ *and* $\mathscr{M}$ *as its set of maximal* $\ell$*-ideals. If A is closed under bounded inversion and if* $\bigcap \mathscr{M} = \{0\}$, *then*

(i) *A is a subdirect product of totally ordered division rings;*

(ii) *A can be embedded into a real f-algebra.*

*Proof.* (i) By the previous theorem, $A$ is a subdirect product $A \subseteqq \prod \{A/M \mid M \in \mathscr{M}\}$ of totally ordered division rings $A/M$. (ii) By a difficult result from [17], any totally ordered division ring $A/M$ can be embedded in a division ring containing the reals.

Having established I as quickly as possible, some additional facts needed for the proof of II are next derived.

2.5. Let $A$ be an $\ell$-group and $\mathscr{M}$ any set of normal prime subgroups $M$ ($M \lhd A$, $M$ is convex, $M$ is a sublattice of $A$, and $0 < a$, $b \in A$, $a \wedge b \in M$ implies $a \in M$ or $b \in M$). The hull-kernel topology $\mathfrak{H}$ on $\mathscr{M}$ has a subbasis consisting of the sets $P(a) \equiv \{M \in \mathscr{M} \mid a \notin M\} = P(a^+) \cup P(a^-)$, where $a = a^+ - a^-$, $a^+, a^- > 0$, $a^+ \wedge a^- = 0$ and $P(a^+) \cap P(a^-) = \varnothing$. The group operation is written additively even though $A$ is not assumed to be abelian.

Let $E$ be the disjoint union $E = \bigcup \{A/M \mid M \in \mathscr{M}\}$ of the right coset spaces $A/M$. Each element $a \in A$ gives a map $\hat{a}: \mathscr{M} \to E$, where $\hat{a}(M) = M + a$. For $a, c \in A$ and any subset $K \subseteqq \mathscr{M}$, we say $a \geqq c$ on $K$ if $M + a \geqq M + c$ for all $M \in K$. If $M + a = M + c$ for all $M \in K$, write $\hat{a} \mid K = \hat{c} \mid K$. If $A$ is a ring and $c = 0$ or 1, then simply write $\hat{a} \mid K = 0$ or $\hat{a} \mid K = 1$.

Although in later applications the subsequent material is needed only for the additive $\ell$-group of an $f$-ring, it will be developed here more generally, because it is hoped that these results in conjunction with the methods of § 4 will be applicable to other partially ordered algebraic structures.

LEMMA 2.6. *Let $A$ be any $\ell$-group and $\mathscr{M}$ any set of normal prime subgroups. For any $x_1, x_2 \in A$ define $W_1 = \{M \in \mathscr{M} \mid M + x_1 < M + x_2\}$, $W_2 = \{M \in \mathscr{M} \mid M + x_2 < M + x_1\}$.*

(i) *Then $W_1 = \{M \mid (x_2 - x_1)^+ \notin M\}$, $W_2 = \{M \mid (x_2 - x_1)^- \notin M\} \in \mathfrak{H}$.*

(ii) *For any $M_1 \neq M_2 \in \mathscr{M}$ with $M_i \not\subseteqq M_j$ for $i \neq j$, there exist $0 < x_i \in A$ with $x_i \in M_i$ and $x_2 > x_1 \bmod M_1$, $x_1 > x_2 \bmod M_2$. Furthermore, $M_i \in W_i$.*

(iii) *If $\mathscr{M}$ has the property that for any $M_1 \neq M_2 \in \mathscr{M}$, $M_1 \not\subseteqq M_2$, then $\mathscr{M}$ is Hausdorff.*

*Proof.* (i) Since $M$ is prime, any $a = a^+ - a^- \in A$ satisfies $0 = a^+ \wedge a^- \in M$, and hence either $a^+ \in M$ or $a^- \in M$. Since $M$ is normal in $A$, we have $M + a^+ - a^- = a^+ + M - a^-$. Now it follows that $M < M + a \Leftrightarrow M + a = M + a^+ \Leftrightarrow a^+ \notin M$. Hence

$$M + x_2 < M + x_1 \Leftrightarrow M < M + x_1 - x_2 \Leftrightarrow (x_1 - x_2)^+ \notin M .$$

(ii) Since $M_i$ is generated by its positive elements, it follows that there is an $x_i \in M_i^+ \backslash M_j^+$ with $M_i + x_j = M_i + x_j^+ > M_i$ for $i \neq j$.

(iii) Let $x_i \in M_i$ and $W_i$ be as above. Then $M_i \in W_i \in \mathfrak{H}$. If $M \in W_1 \cap W_2$, then $M < M + x_1 - x_2$ and $M < M + x_2 - x_1$ which is not possible.

LEMMA 2.7. *Consider any $\ell$-group $A$ and any set $\mathscr{M}$ of normal prime subgroups satisfying the following conditions:*

(a) $\forall M_1 \neq M_2 \in \mathscr{M}$, $M_1 \not\subseteqq M_2$.

(b) $A \backslash \bigcup \mathscr{M} \neq \varnothing$.

(c) *If $I \subset A$ is any proper, convex, $\ell$-subgroup generated by the members of $\mathscr{M}$, then there exists an $M \in \mathscr{M}$ with $I \subseteqq M$.*

*Then $(\mathscr{M}, \mathfrak{H})$ is compact Hausdorff.*

*Proof.* Let $\{F_\lambda\}$ be any indexed family of closed subsets $F_\lambda = \bar{F}_\lambda \subset \mathscr{M}$ with $\bigcap F_\lambda = \varnothing$. It has to be shown that then already a finite intersection of these is empty, i.e., that $F_{\lambda(1)} \cap \cdots \cap F_{\lambda(n)} = \varnothing$

for some choice of $\lambda(1), \cdots, \lambda(n)$. Let $k(F_\lambda) = \bigcap\{M \mid M \in F_\lambda\} = \{a \in A \mid \hat{a} \mid F_\lambda = 0\}$. Let $I$ be the convex, $\ell$-subgroup generated by $k(F_\lambda)$. (Note that in case $A$ is an $\ell$-ring and $\mathscr{M}$ consists of convex prime $\ell$-ideals, that then the $k(F_\lambda)$ and $I$ in (c) would be convex $\ell$-ideals.) Suppose $I \neq A$. Then by assumption (c), there exists an $M \in \mathscr{M}$ with $I \subseteqq M$. But then for any $\lambda$, $k(F_\lambda) \subset I \subseteqq M$. Thus $M \in \bar{F}_\lambda = F_\lambda$ for all $\lambda$. But then $M \in \bigcap F_\lambda = \varnothing$ is a contradiction. Thus $I = A$. Take $e \in A \backslash \cup \mathscr{M}$. Then there are $\lambda(1), \cdots, \lambda(n)$ such that

$$e \leqq a_{\lambda(1)} + \cdots + a_{\lambda(n)} \qquad 0 < a_{\lambda(j)} \in k(F_{\lambda(j)}) .$$

It is asserted that $F_{\lambda(1)} \cap \cdots \cap F_{\lambda(n)} = \varnothing$. For if $M \in F_{\lambda(1)} \cap \cdots \cap F_{\lambda(n)}$, then for all $j$, $a_{\lambda(j)} \in k(F_{\lambda(j)}) = \bigcap\{M \mid M \in F_{\lambda(j)}\} \subseteqq M$. Thus also $e \in M$, a contradiction. Hence $F_{\lambda(1)} \cap \cdots \cap F_{\lambda(n)} = \varnothing$.

LEMMA 2.8. *Let $A$ and $\mathscr{M}$ be as in the previous lemma. Suppose $0 < e \in A \backslash \cup \mathscr{M}$. If $K_0, K_1 \subset \mathscr{M}$ are closed subsets with $K_0 \cap K_1 = \varnothing$, then there is a $k \in A$ in $0 < k \leqq e$ such that $\hat{k} \mid K_0 = 0$ but $\hat{k} \mid K_1 = \hat{e} \mid K_1$.*

*Proof.* The proof proceeds in several steps.

(i) We first show that if $K_0 = \{M_0\}$, $K_1 = \{M_1\}$ are singletons that then there is an $a \in A$ with

$$M_0 + a < M_0 - e \,, \qquad M_1 + e < M_1 + a \,.$$

It follows that there are $a_0, m_0 \in M_0$ and $a_1, m_1 \in M_1$ for which

$$A = M_0 + M_1 \Rightarrow \quad 2e = a_0 + m_1$$
$$A = M_1 + M_0 \Rightarrow -2e = a_1 + m_0 \,.$$

Setting $a = a_0 + a_1$ and using that $M_0, M_1 \lhd A$, we get

$$M_0 + a = M_0 + a_1 = M_0 - 2e < M_0 - e$$
$$M_1 + e < M_1 + 2e = M_1 + a_0 = M_1 + a \,.$$

(ii) Having shown that points can be separated in the above strong sense, now take $K_0 = \{M_0\}$ and $K_1 = \bar{K}_1$ arbitrary. Then for any $k \in K_1$, there is an $a(k) \in A$ with

$$M_0 + a(k) < M_0 - e \,, \qquad k + e < k + a(k) \,.$$

Now define $U(k) = \{M \mid M + e < M + a(k)\} \in \mathfrak{H}$ and $W(k) = \{M \mid M + a(k) < M - e\} \in \mathfrak{H}$. Then $\{U(k) \mid k \in K_1\}$ is an open cover of the compact set $K_1$. Thus there are a finite number $a_j \equiv a(k_j)$, $U_j \equiv U(k_j)$, and $W_j = W(k_j)$ with $K_1 \subseteqq U_1 \cup \cdots \cup U_n$. Set $W = \bigcap\{W_j \mid j = 1, \cdots, n\}$. Define $a = a_1 \vee \cdots \vee a_n$. Then

$$\forall M \in U_j,\, M + e < M + a_j \Rightarrow \forall M \in K_1,\, M + e < M + a\,;$$
$$\forall M \in W,\, M + a_j < M - e \Rightarrow \forall M \in W,\, M + a < M - e\,.$$

Note that since $A/M$ is totally ordered, all of these inequalities are sharp. Thus $a < -e$ on $W$ and in particular at $M_0$ while on $K_1$, $e < a$ holds.

(iii) Finally, let $K_0 = \bar{K}_0$ and $K_1 = \bar{K}_1$ be arbitrary closed sets. For each $M_0 \in K_0$ choose a neighborhood $W = W(M_0)$ and $a = a(M_0)$ as above in (ii). Since $K_0 \subseteqq \bigcup \{W(M_0) \mid M_0 \in K_0\}$ there is a finite subcover

$$K_0 \subseteqq W_1 \cup \cdots \cup W_m;\, a_j < -e \text{ on } W_j;\, a_j > e \text{ on } K_1;\, j = 1, \cdots, m\,.$$

The element $a = a_1 \wedge \cdots \wedge a_m$ satisfies the following

$$a_j < -e \text{ on } W_j \Rightarrow a < -e \text{ on } K_0\,,$$
$$a_j > e \text{ on } K_1 \Rightarrow a > e \text{ on } K_1\,.$$

Note that $0 < e$ and set $k = (a \vee 0) \wedge e$. Thus $0 \leqq k \leqq e$ and finally

$$M \in K_0 \Rightarrow M + k = M + a \vee 0 = M\,,$$
$$M \in K_1 \Rightarrow M + k = M + e\,.$$

DEFINITION 2.9. Consider a partially ordered group $A$ with an indexed set $\mathscr{M}$ of subgroups topologized in some way by a topology $\mathfrak{H}$, and let $0 < e \in A \backslash \cup \mathscr{M}$ be a distinguished element. A subset $A_1 \subseteqq A$ is said to contain *positive bounded partitions* of $e$ on $(\mathscr{M}, \mathfrak{H})$, provided that for any finite $\mathfrak{H}$-open cover $\mathscr{M} = U_1 \cup \cdots \cup U_n$, there are $e_1, \cdots, e_n \in A_1$ satisfying:

( i ) $e = e_1 + \cdots + e_n$;

( ii) $e_i \in \bigcap \mathscr{M} \backslash U_i$, $i = 1, \cdots, n$;

(iii) $0 \leqq e_i \leqq e$.

Since the following general fact will not only be used in the proof of the next proposition, but also on several later occasions, it is worthwhile to isolate it out of context.

2.10. Suppose $A$ is any $\ell$-group and $\Delta$ any set of prime subgroups (not necessarily normal) such that $\bigcap \{M \mid M \in \Delta\} = \{0\}$. Then for $0 \neq a \in A$,

$$a > 0 \Leftrightarrow \forall M \in \Delta,\, M + a \geqq M\,.$$

For if $a > 0$, then, clearly, $M + a \geqq M$. If $a \not> 0$, but $M + a \geqq M$ for all $M \in \Delta$, then $a^- = -a \vee 0 > 0$. Pick $M \in \Delta$ with $a^- \notin M$. Since $a^+ \wedge a^- = 0$, $a^+ \in M$. Thus $M + a = M + a^+ - a^- = M - a^- < M$, a contradiction.

PROPOSITION 2.11. *Suppose $A$ is an f-ring with identity and $\mathscr{M}$ its set of maximal $\ell$-ideals in the hull-kernel topology. Define $A^* = \{a \in A \mid |a| < n1$ for some integer $n\}$. Assume that*

$$1 < a \in A^* \Rightarrow 1/a \in A .$$

*Then there are positive bounded partitions of* 1 *on* $\mathscr{M}$.

*Proof.* If $\mathscr{M} = U_1 \cup \cdots \cup U_n$ is any $\mathfrak{H}$-open cover, choose open sets $W_i$ whose closures $\overline{W}_i$ satisfy $W_i \subset \overline{W}_i \subset U_i$ such that still $\mathscr{M} = W_1 \cup \cdots \cup W_n$. Apply Lemma 2.8 with $K_1 = \overline{W}_i$ and $K_0 = \mathscr{M} \backslash U_i$ to obtain elements $k_i$ satisfying

$$0 \leqq k_i \leqq 1,\ \hat{k}_i \mid \overline{W}_i = 1, \text{ and } \hat{k}_i \mid (\mathscr{M}\backslash U_i) = 0,\ i = 1, \cdots n .$$

Set $k = k_1 + \cdots + k_n \in A^*$. Since $M + 1 \leqq M + k$ for all $M \in \mathscr{M}$ it follows that $\bigcap \mathscr{M} + 1 \leqq \bigcap \mathscr{M} + k$ in $A/\bigcap \mathscr{M}$. Thus there exists an $m \in \bigcap \mathscr{M}$ with $1 \leqq k + m$. Since $\bigcap \mathscr{M}$ is an $l$-ideal, $|m| \in \bigcap \mathscr{M}$; take $0 < m = |m|$. Then

$$k + m \geqq k + m \wedge 1 = (k + m) \wedge (k + 1) \geqq 1 .$$

Let $0 < m = m \wedge 1$. Set $a = k + m$. Then $1 \leqq a \in A^*$. Since $A^*$ is closed under bounded inversion, $A$ is an algebra over the rationals and $(1/n)m \in \bigcap \mathscr{M}$. Define $e_i = (k_i + (1/n)m)a^{-1}$, clearly conditions (i)–(iii) of Definition 2.9 are satisfied.

## 3. The representation of an $L$-group as cross sections.

In order to obtain a faithful representation of an arbitrary $\ell$-group $A$ as a group of cross sections in a field, first a method of Banaschewski ([1]) is used to introduce a group topology on $A$. Then this topology on $A$ is used in the construction of the field. If the topology on $A$ is discrete, then the resulting field is an ordinary sheaf. Although written additively, in this section the group $A$ is not assumed to be abelian.

3.1. A subset $S$ of the positive elements $A^+\backslash\{0\}$ of an $\ell$-group $A$ is called a set of *topological units* (cf., [1]) if for any $\varepsilon, \varepsilon' \in S$ and $x \in A$ there is a $\delta \in S$ satisfying:

$$\text{I.}\quad \delta \leqq \varepsilon \wedge \varepsilon' ,$$
$$\text{II.}\quad 2\delta \leqq \varepsilon ,$$
$$\text{III.}\quad \delta \leqq x + \varepsilon - x .$$

Although in [1] the condition that

$$\text{IV.}\quad \inf \{s \mid s \in S\} = 0$$

is imposed, here this will not be assumed. The sets $(-\varepsilon, \varepsilon) \equiv \{a \in A \mid -\varepsilon < a < \varepsilon\}$, $\varepsilon \in S$, form a basis for the open neighborhoods of zero for a group topology on $A$ which depends on the particular choice of $S$. All the proofs in [1] are valid without IV. However, the resulting group topology is Hausdorff if and only if IV holds (see [1]). Any basis for the set of all open neighborhoods of zero in $A$ will be denoted by $\mathfrak{W}$.

The subset $A^* \subseteqq A$ consisting of all $a \in A$ such that there exists some finite subset $\varepsilon_1, \cdots, \varepsilon_k$ of $S$ and some integer $n$ (all depending on the element $a$) for which $|a| < n(\varepsilon_1 \vee \cdots \vee \varepsilon_k)$ is a convex normal $\ell$-subgroup of $A$; the elements of this $\ell$-subgroup are called the *bounded elements* of $A$ (with respect to $S$).

3.2. Suppose $\mathscr{M}$ is any indexed set of prime subgroups of any $\ell$-group $A$ with $\bigcap \mathscr{M} = \{0\}$. There always exists such a set for any $\ell$-group $A$; for example, $\mathscr{M}$ could be taken as the set of all regular subgroups of $A$. As a consequence of the fact that each $M \in \mathscr{M}$ is prime, the right coset space $A/M$ of $A$ modulo $M$ is totally ordered. For $a, c, d \in A$, there is an order preserving transformation $R(a)$ of $A/M$, where $(M + d)R(a) = M + (d + a)$ and $R(a + c) = R(a)R(c)$. Thus $a \longrightarrow R(a)$ is an $\ell$-homomorphism of $A$ onto a transitive subgroup of the $\ell$-group of all order preserving transformations. Thus $A$ is $\ell$-isomorphic to a subdirect sum of $\ell$-subgroups of order preserving permutations of totally ordered sets. The groups in $\mathscr{M}$ are not assumed to be normal in $A$; and several isomorphic copies of the same subgroup may appear several different times in $\mathscr{M}$, even if $A$ contains only one such subgroup. If $E$ is the disjoint union $E = \bigcup \{A/M \mid M \in \mathscr{M}\}$, then there is a natural projection $\pi: E \longrightarrow \mathscr{M}$, $\pi(A/M) = \{M\}$. Each $a \in A$ gives a function $\hat{a}: \mathscr{M} \longrightarrow E$ defined by $\hat{a}(M) = M + a \in \pi^{-1}(M) \subset E$. For any subset $A_1 \subseteqq A$, $\hat{A}_1$ will denote $\hat{A}_1 = \{\hat{a} \mid a \in A_1\}$. Define $E \vee E$ as $E \vee E = \{(x, y) \in E \times E \mid \pi(x) = \pi(y)\}$. For $x = M + a \in A/M$ and any subset $U \subset A$, the notation "$x + U$" will mean $x + U = \{xR(u) \mid u \in U\} = \{M + (a + u) \mid u \in U\}$. Each zero neighborhood $U \in \mathfrak{W}$ gives an entourage which will always be denoted as $U'$ with a prime, where $U' \equiv \{(x, y) \in E \vee E \mid y \in x + U\}$. (I.e., $(M + a_1, M + a_2) \in U'$ with $a_1, a_2 \in A$, if and only if there exists a $u \in U$ such that $M + a_2 = M + (a_1 + u)$. If some $M \in \mathscr{M}$ is normal in $A$, then $\{U' \cap [\pi^{-1}(M) \times \pi^{-1}(M)] \mid U \in \mathfrak{W}\}$ is a basis for the left invariant uniform structure on $A/M$. The reader, who is mainly interested in applications to the case when $A$ is the additive group of a ring, will loose nothing by assuming $A$ to be abelian. All such $U'$ as $U$ ranges over $\mathfrak{W}$ are a filter basis for a so-called *field uniform structure* $\mathfrak{U}$ on $E \vee E$. Although $\mathfrak{U}$ is not a uniform structure on $E$ in the usual sense of the word, nevertheless it has many of the properties of a uniform structure. For $a \in A$ and $V \in \mathfrak{U}$, the usual notation associated with uniform spaces will be used for $V^{-1} = \{(x, y) \in E \vee E \mid (x, y) \in V\}$ and for $V(\hat{a}) = \{x \in E \mid (\hat{a}(\pi(x)), x) \in V\}$. Note that for $U \in \mathfrak{W}$, the above becomes $U'(\hat{a}) = \{x \in E \mid x \in \pi(x) + a + U\}$.

3.3. The set of all $U'(\hat{a})$ with $U \in \mathfrak{W}$ and $a \in A$ is a subbasis for the open sets for a topology $N(\hat{A})$ on $E$. If each $M \in \mathscr{M}$ is normal in $A$, and if $U = \{a \in A \mid -\varepsilon < a < \varepsilon\}$ with $\varepsilon \in S$, then $U'(\hat{a}) = \{x \in E \mid | -x + \hat{a}(\pi(x)) | < \pi(x) + \varepsilon\}$; and $U'(\hat{a})$ may be intuitively described as a "tube of width $\varepsilon$ around the section $\hat{a}(\mathscr{M}) \subset E$." Thus, according to this definition, a typical basic neighborhood of a point $x \in E$ is a set of the form $U_1'(\hat{a}) \cap \cdots \cap U_n'(\hat{a}_n)$ with $U_1, \cdots, U_n \in \mathfrak{W}$ and $a_1, \cdots, a_n \in A$, provided that this set contains $x$. Two simplifications should be noted. First, all the $U_j$ may be taken as the same element. Second, all the $a_j$ may be taken to satisfy $x = \pi(x) + a_j$. There are $u_j \in U_j$ with $\pi(x) + a_j + u_j = x$. Choose $W \in \mathfrak{W}$ with $u_j + W \subseteqq U_j$ and set $c_j = a_j + u_j$. Then

$$z \in W'(\hat{c}_j) \Rightarrow z \in \pi(z) + c_j + W \subseteqq \pi(z) + a_j + U_j \subseteqq U_j'(\hat{a}_j) ;$$
$$\bigcap_{j=1}^{n} W'(\hat{c}_j) \subseteqq \bigcap_{j=1}^{n} U'(\hat{a}_j) .$$

It is clear that the topology $N(\hat{A})$ is not Hausdorff if the condition IV that $\inf S = 0$ fails.

3.4. Define $T(\pi)$ to be the biggest topology on $\mathscr{M}$ making $\pi$ continuous, i.e., $T(\pi) = \{V \subseteqq \mathscr{M} \mid \pi^{-1}(V) \in N(\hat{A})\}$. There is a smallest topology $T(\hat{A})$ on $\mathscr{M}$, the so-called *weak-star* topology, making all the maps of $\hat{A}$ continuous. A subbasis for $T(\hat{A})$ may be obtained by taking the inverse images under all elements of $\hat{A}$ of a subbasis of $N(\hat{A})$ as follows:

$$\begin{aligned} \hat{h}^{-1}[U'(\hat{g})] &= \{M \in \mathscr{M} \mid (\hat{g}(M), \hat{h}(M)) \in U'\} \\ &= \{M \in \mathscr{M} \mid M + h \in M + g + U\}; \qquad g, h \in A;\, U \in \mathfrak{W} . \end{aligned}$$

Thus if all the groups of $\mathscr{M}$ are normal in $A$, then the above typical subbasic open set takes a simpler form which depends only on the single group element $-g + h$, i.e.,

$$\hat{h}^{-1}[U'(\hat{g})] = \{M \in \mathscr{M} \mid -g + h \in M + U\} .$$

The next two lemmas establish the relationships between the various topologies. Whatever topology is finally to be put on $\mathscr{M}$, it will have to be a topology such that the projection $\pi\colon E \rightarrow \mathscr{M}$ as well as all the maps of $\hat{A}$ are continuous. It should be stressed that the above definitions of $N(\hat{A})$, $T(\pi)$, and $T(\hat{A})$ apply to any topological group. In order to make these definitions, the topology of $A$ need not be derived from a set of topological units.

LEMMA 3.5. *In the notation of* 3.1–3.4, $T(\pi) \subseteqq T(\hat{A})$.

*Proof.* If $V \in T(\pi)$, then $\pi^{-1}(V) \in N(\hat{A})$. But for each $a \in A$, $\hat{a}: (\mathscr{M}, T(\hat{A})) \to (E, N(\hat{A}))$ is continuous. Thus $\hat{a}^{-1}(\pi^{-1}(V)) \in T(\hat{A})$. But $\pi \circ a =$ identity, so $V = \hat{a}^{-1}(\pi^{-1}(V))$, and thus $T(\pi) \subseteqq T(\hat{A})$.

The next lemma is crucial for an understanding of fields. Its proof involves the use of the group structure on the set of maps $\hat{A}$; the reader may wish to illustrate the geometric significance of the proof by a diagram.

LEMMA 3.6. *The map* $\pi: (E, N(\hat{A})) \to (\mathscr{M}, T(\hat{A}))$ *is continuous. Since* $T(\pi)$ *was the biggest topology on* $\mathscr{M}$ *making* $\pi$ *continuous,* $T(\hat{A}) \subseteqq T(\pi)$ *and hence by* 3.5, $T(\hat{A}) = T(\pi)$.

*Proof.* Take a typical subbasic open set for $T(\hat{A})$ of the form $V = \{b \in B \mid (\hat{g}(b), \hat{h}(b)) \in U'\} = \{b \mid b + h \in b + g + U\}$ with $g, h \in A$ and $U \in \mathfrak{W}$. It suffices to show that $\pi^{-1}(V) \in N(\hat{A})$. Take any $z \in \pi^{-1}(V)$. Then a set $W \in N(\hat{A})$ has to be found such that $z \in W \subseteqq \pi^{-1}(V)$. First, set $c = \pi(z)$. By the definition of $V$, there is a $u \in U$ such that $c + h = c + (g + u)$. Secondly, since translation by group elements is a transitive action on the stalks, there exists an element $t \in A$ such that $z = c + (g + t) = (\hat{g} + \hat{t})(c)$. (Note that $\hat{g}(c) + \hat{t}(c)$ is meaningless.) Thirdly, pick $P \in \mathfrak{W}$ such that $t + P - P - t + u \subseteqq U$. Fourthly, set $W = P'(\hat{g} + \hat{t}) \cap P'(\hat{h} - \hat{u} + \hat{t})$. Since both of the sections $\hat{g} + \hat{t}$ and $\hat{h} - \hat{u} + \hat{t}$ pass through $z$, the element $z \in W$. Next it is shown that $W \subseteqq \pi^{-1}(V)$. To prove this, pick any $y \in W$. It has to be shown that $\pi(y) \in V$, or equivalently, that $\pi(y) + g + s = \pi(y) + h$ for some $s \in U$. But it follows from the definition of $P'$ that

$$y \in P'(\hat{g} + \hat{t}) \Rightarrow \exists h_1 \in P, \pi(y) + (g + t + h_1) = y ;$$
$$y \in P'(\hat{h} - \hat{u} + \hat{t}) \Rightarrow \exists h_2 \in P, \pi(y) + (h - u + t + h_2) = y .$$

Thus $\pi(y) + g + s = \pi(y) + h$ with $s = t + h_1 - h_2 - t + u \in U$.

3.7. If $M$ is a convex $\ell$-subgroup of any $\ell$-group $A$, then the right coset space $A/M$ is a distributive lattice under

$$(M + a_1) \vee (M + a_2) = M + a_1 \vee a_2, (M + a_1) \wedge (M + a_2) = M + a_1 \wedge a_2 ,$$

where $a_1, a_2 \in A$. Thus each stalk of $E$ is a distributive lattice; it also inherits a topology as a subset of $E$. The following functions are continuous:

$$\mathscr{M} \to E\colon M \to M \qquad M \in \mathscr{M}\,.$$
$$E \vee E \to E\colon (x, y) \to x \vee y,\ (x, y) \to x \wedge y \qquad (x, y) \in E \vee E\,.$$
$$E \to E\colon M + c \to M + (c + a) \qquad c \in A,\ a \in A \text{ is fixed.}$$
If all groups of $\mathscr{M}$ are normal, $E \vee E \to E$, $(x, y) \to x - y$
$$(x, y) \in E \vee E\,.$$

The proofs are omitted. In general, if a map such as the group operation, join, or meet is continuous in the topological group $A$, then the induced analogous map of $E \vee E \to E$ will also be continuous.

DEFINITION 3.8. A continuous function $\sigma\colon (\mathscr{M}, T(\hat{A})) \to (E, N(\hat{A}))$ such that $\pi \circ \sigma$ is the identity on $\mathscr{M}$ will be called a *section*. Under the obvious pointwise operations these form a distributive lattice that will be denoted by $\Gamma(\mathscr{M}, E)$. If $\operatorname{dom} \sigma$ is the domain of any function $\sigma$ whatever and if $U$ is any set, then the restriction of $\sigma$ to $\operatorname{dom} \sigma \cap U$ will be denoted by $\sigma \mid U$. The notation $\sigma \mid U = 0$ will mean that $\sigma(M) = M \in A/M$ for each $M \in \operatorname{dom} \sigma \cap U$. (If $A$ is a ring with an identity $1 \in A$, then $\sigma \mid U = 1$ means that $\sigma(M) = M + 1 \in A/M$ for $M \in \operatorname{dom} \sigma \cap U$.)

As a consequence of the assumption that $\bigcap \mathscr{M} = \{0\}$, the map $A \to \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$ is an order preserving isomorphism. A section $\tau \in \Gamma(\mathscr{M}, E)$ such that $|\tau| < \hat{a}$ for some $a \in A^*$ (see 3.1) will be called *bounded*. The sublattice of all bounded sections will be denoted by $\Gamma(\mathscr{M}, E)^*$.

The field uniform structure $\mathfrak{U}$ induces in the obvious way an ordinary uniform structure on $\Gamma(\mathscr{M}, E)$ making it into a uniform space, i.e., each $W \in \mathfrak{U}$ gives an entourage $\tilde{W} \subset \Gamma(\mathscr{M}, E) \times \Gamma(\mathscr{M}, E)$ where $\tilde{W} = \{(\sigma, \tau) \mid (\sigma(b)\tau(b)) \in W\}$ for all $b \in \mathscr{M}\}$.

Conclusion (iv) of the next lemma later will be used to show that in certain cases $\hat{A} = \Gamma(\mathscr{M}, E)$.

LEMMA 3.9. *If each group of $\mathscr{M}$ is normal in $\Gamma(\mathscr{M}, E)$, then:*
( i ) *$\Gamma(\mathscr{M}, E)$ is an $\ell$-group, $\hat{A} \subseteqq \Gamma(\mathscr{M}, E)$ is a subgroup.*
( ii ) *$\Gamma(\mathscr{M}, E)$ is a topological group.*
(iii) *$\Gamma(\mathscr{M}, E)^*$ is a convex normal $\ell$-subgroup of $\Gamma(\mathscr{M}, E)$.*
(iv) *$A \to \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$ is a homeomorphism. In particular, if $A$ is complete, then $\hat{A}$ is closed in $\Gamma(\mathscr{M}, E)$.*

*Proof.* Conclusions (i) and (iii) are trivial. Conclusion (ii) is not later used; its proof is omitted (see [6]). If $A$ were an arbitrary topological group, then, in general, the map $A \to \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$ would be a continuous algebraic isomorphism whose inverse might not be continuous. However, in the present situation, if $U = \{a \in A \mid |a| < s\} \in \mathfrak{B}$ with $s \in S$, then $\bigcap \{M + U \mid M \in \mathscr{M}\} = \{c \in A \mid |M + c| < M + s\} = U$. A typical neighborhood of an element $\hat{c} \in \Gamma(\mathscr{M}, E) \cap \hat{A}$ is $\tilde{U}'(\hat{c}) = \{\sigma \in \Gamma(\mathscr{M}, E) \mid (\hat{c}(M), \sigma(M)) \in U', \forall M \in \mathscr{M}\}$. It suffices to show that under the correspondence $A \leftrightarrow \hat{A}$, the neighborhood $c + U$ of $c$ in $A$ corresponds to $\tilde{U}'(\hat{c}) \cap \hat{A}$ in $\Gamma(\mathscr{M}, E)$. Since $M \lhd A$, it follows that

$$\begin{aligned}\tilde{U}'(\hat{c}) \cap \hat{A} &= \{\hat{a} \mid a \in A, M + a \in M + c + U, \forall M \in \mathscr{M}\} \\ &= \{\hat{a} \mid a \in A, M - c + a \in M + U, \forall M \in \mathscr{M}\} \\ &= \{\hat{a} \mid -c + a \in \bigcap \{M + U \mid M \in \mathscr{M}\}\} = c + U .\end{aligned}$$

The results obtained thus far in this section are summarized in the next proposition.

PROPOSITION 3.10. *If $A$ is an $\ell$-group and $\mathscr{M}$ any indexed family of normal prime subgroups of $A$ with $\bigcap \mathscr{M} = \{0\}$, then define $E$ and $\pi$ by*

$$\pi: E = \bigcup \{A/M \mid M \in \mathscr{M}\} \to \mathscr{M}, \pi^{-1}(M) = A/M \text{ for } M \in \mathscr{M} .$$

*For $a \in A$, define $\hat{a}: \mathscr{M} \to E$ by $\hat{a}(M) = M + a$ and set $\hat{A} = \{\hat{a} \mid a \in A\}$. If $S$ is any set of topological units for $A$, then the sets $\{x \in A \mid -\varepsilon < x < \varepsilon\}, \varepsilon \in S$, define a subbasis for the neighborhoods of the identity for a group topology on $A$. The bounded elements $A^*$ (see 2.1) form a normal, convex $\ell$-subgroup of $A$. Let $T(\hat{A}), N(\hat{A})$ be topologies on $\mathscr{M}, E$ defined by the following subbasic sets:*

(a) $\{b \in \mathscr{M} \mid b + h \in b + g + (-\varepsilon, \varepsilon)\}$ $\quad g, h \in A, \varepsilon \in S$;

(b) $\{x \in E \mid x \in \pi(x) + a + (-\varepsilon, \varepsilon)\}$ $\quad a \in A, \varepsilon \in S$.

*Let $\Gamma(\mathscr{M}, E)$ be the lattice of all continuous sections $\mathscr{M} \to E$ and $\Gamma(\mathscr{M}, E)^*$ the sublattice of bounded sections (see 3.9). Then:*

(i) (a) *$T(\hat{A})$ is the biggest topology on $\mathscr{M}$ making $\pi: (E, N(\hat{A})) \to \mathscr{M}$ continuous; and*

(b) *$T(\hat{A})$ is the smallest topology on $\mathscr{M}$ making all the maps $\hat{A}$ continuous.*

(ii) *$T(\hat{A})$ is the unique smallest topology on $\mathscr{M}$ which satisfies the following two conditions:*

(a) *$\pi: (E, N(\hat{A})) \to \mathscr{M}$ is continuous;*

(b) *All $\hat{a}: \mathscr{M} \to (E, N(\hat{A}))$, $a \in A$, are continuous.*

(iii) *$A^*$ and $A$ are isomorphically embedded as $l$-subgroups in $A^* \cong \hat{A}^* \subseteqq \Gamma(\mathscr{M}, E)$ and $A \cong \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$.*

The intrinsic nature of the topologies $N(\hat{A})$ and $T(\hat{A})$ should be stressed again. The reader can only conclude from the previous proposition that if it is agreed upon beforehand that $E$ is to be endowed with the natural topology $N(\hat{A})$, that then $T(\hat{A})$ is both uniquely determined and has several intrinsic characterizations. However, this is only a half of the story. In fact, the topology of the base space $\mathscr{M}$ uniquely determines the topology on the stalks $E$. Thus in particular, $N(\hat{A})$ is uniquely determined by $T(\hat{A})$. Also in a certain very precise sense $N(\hat{A})$ is minimal just as $T(\hat{A})$ is. Since these facts will not be needed for later purposes, and since the present framework of $\ell$-groups is a setting much too specialized for exhibiting facts that are true in more general circumstances, the reader is again referred to [6] for a complete account.

The next example illustrates that in commutative nonarchimedean $f$-rings there are more desirable choices of topological units than the choice $\{r1 \mid 0 < r \in \boldsymbol{Q}\}$ that is used here throughout.

EXAMPLE 3.11. Choose three distinct totally ordered fields $R \subset F \subset K$, $\boldsymbol{R}$—the reals, $F$ a field having a strictly decreasing positive sequence of elements $\{c\{n\} \mid n = 1, 2, \cdots\}$ converging to zero, while $K$ is an $\eta_1$-field. Then $K$ has a similar subset $\{k(\lambda) \mid \lambda \in \Lambda\}$ converging to zero indexed by a totally ordered set $\Lambda$ which can no longer be countable. Take a locally compact, totally disconnected space $X$, a fixed base point $p \in X$, and some neighborhood filter $\boldsymbol{N}(p)$ of $p$. Viewing $K$ as a discrete space, the ring $A$ of all continuous functions $f: X \rightarrow K$ with $f(p) \in F$ is a nonarchimedean real $f$-algebra. Consider triples $N, \lambda, n$ consisting of an $N \in \boldsymbol{N}(p)$, an index $\lambda \in \Lambda$ and a positive integer $n$. For each such a triple, choose a function $f \in A$ such that $f(p) = c(n)$; such that when restricted to $X \backslash N, f$ satisfies $f \mid (X \backslash N) = k(\lambda)$; and lastly such that $k(\lambda) \leqq f(x) \leqq c(n)$ for all $x \in X$. In case the neighborhood basis of $p$ can be totally ordered by inclusion, these functions can be so chosen that whenever $f$ corresponds to $N, \lambda, n$ and $f'$ to $N', \lambda', n'$ with $N' \subseteqq N$, $\lambda \leqq \lambda'$, and $n \leqq n'$, then $f'(x) \leqq f(x)$ for all $x \in X$. If the cardinality of $\Lambda$ exceeds that of $\boldsymbol{N}(p)$, it is not possible to choose a co-final linearly ordered subset of $S$. Next suppose that the neighborhood filter $\boldsymbol{N}(p)$ of $p$ cannot be linearly ordered, e.g., $X$ can be an uncountable product of discrete two point sets and $p$ the point all of whose coordinates are zero. In this case, irrespective of the cardinality of $\Lambda$, $S$ does not have a cofinal linearly ordered subset.

For certain classes of groups, if $\pi: E \rightarrow \mathscr{M}$ is constructed as above, then the representation $A \cong \hat{A} \subseteqq \Gamma(\mathscr{M}, E)$ is always an isomorphism, i.e., $\hat{A} = \Gamma(\mathscr{M}, E)$. In these cases it is not enough to start with a group $A$ in the class and then show that it gives rise to a field $\pi: E \to \mathscr{M}$ of a certain kind. In order to characterize membership in this class by a necessary and sufficient condition, conversely, it is necessary to start with a sheaf-like structure $\pi: E \to \mathscr{M}$ of a certain specific kind and then prove that $\Gamma(\mathscr{M}, E)$ is a group belonging to the class.

DEFINITION 3.12. Consider a continuous surjective map $\pi: E \to \mathscr{M}$, with $\mathscr{M}$ compact Hausdorff, and each $\pi^{-1}(M)$ a lattice ordered vector space over the rationals $\boldsymbol{Q}$ with a weak order unit $1(M)$. (Later, in addition, each $\pi^{-1}(M)$ will be a ring with $1(M)$ as the identity.) Assume that subtraction, join, and meet are continuous as maps

$$(1) \qquad \begin{aligned} E \vee E \to E: (x, y) &\to x - y \\ (x, y) &\to x \vee y \\ (x, y) &\to x \wedge y \qquad (x, y) \in E \vee E . \end{aligned}$$

Suppose that through each point $x \in E$ there passes through a local section, i.e., a continuous map $\tau_x: \operatorname{dom} \tau_x \to E$, where the domain of $\tau_x$, $\operatorname{dom} \tau_x$, is an open neighborhood of $\pi(x)$, with $\pi \circ \tau_x$ the identity. Assume that the topological spaces $E$, $\mathscr{M}$ and the family of local sections $\{\tau_x \mid x \in E\}$ have the further property that as $V$ ranges over the neighborhoods of $\pi(x)$, the following is a neighborhood basis of each $x \in E$:

$$(2) \qquad \{z \in E \mid \pi(z) \in V \cap \operatorname{dom} \tau_x, \ |\tau_x(\pi(z)) - z| < r1(\pi(z))\} \qquad 0 < r \in \boldsymbol{Q} .$$

Then $\pi$ will be called a *field of vector lattices* over $\boldsymbol{Q}$ with the *weak order unit uniform structure* (over a compact Hausdorff space).

REMARKS. (1) Although as the above definition now stands, it depends upon $\{\tau_x \mid x \in E\}$, nevertheless, it can be shown without too much difficulty that it is independent of the choice of local sections. More generally, if a single point $x \in E$ has a neighborhood basis of the form (2), then $\tau_x$ may be replaced in (2) by any other continuous local section passing through $x$ and the result will still be a neighborhood basis of $x$ (see [6; p. 4, Lemma 1.9]).

(2) The topology on $E$ depends on the stalkwise choice $\{1(M) \mid M \in \mathscr{M}\}$ of weak order units.

**4. Rings of sections over the maximal ideal space.** Previous results are now combined to yield a representation of an $f$-ring $A$ as a ring $\hat{A}$ contained in the group $\Gamma(\mathscr{M}, E)$ of cross sections in a field.

4.1. Consider any $\ell$-group $A$ and any set $\mathscr{M}$ of prime subgroups. Suppose $e$ is an element in $0 < e \in A \backslash \cup \mathscr{M}$. If $0 < a \in A$, $M \in \mathscr{M}$, define $\bar{a}(M) = \inf\{n/j \mid n, j \text{ integers}; M + ja < M + ne\}$. For $a = a^+ - a^-$, set $\bar{a}(M) = \bar{a}^+(M) - \bar{a}^-(M)$. Then $\bar{A} = \{\bar{a} \mid a \in A\}$ is a group of continuous functions $\bar{a}: (\mathscr{M}, \mathfrak{H}) \to \boldsymbol{R} \cup \{\pm\infty\}$ into the two point compactification of the reals. If $A$ is archimedean and $\bigcap \mathscr{M} = \{0\}$, then each $\bar{a}$ is finite valued on a hull-kernel dense open subset of $\mathscr{M}$. For otherwise, if $0 < a, c \in A$, $P(c) \equiv \{M \mid c \notin M\}$, and $\bar{a} \mid P(c) = \infty$, then $M + n(e \wedge c) \leqq M + a$ for all $M \in \mathscr{M}$ and all integers $n > 0$. Then $o \leqq n(e \wedge c) \leqq a$ and $e \wedge c = 0$. But for $M \in P(c), e \wedge c \notin M$ is a contradiction.

Although proved in greater generality, the next lemma will only be needed for the additive group of an $f$-ring.

LEMMA 4.2. *Suppose $A$ is an $\ell$-group and $M \lhd A$ is a prime subgroup. If $0 < r \in A, g \in A$ are elements such that $| M + g | < M + r$, then there exist $m_1, m_2 \in M$ for which $-r < m_1 + g + m_2 < r$.*

*Proof.* Since either $g^+$ or $g^- \in M$, assume $g^- \in M$. Then

$$(g^+ - r) \vee 0 + r = g^+ \vee r \geqq g^+ ,$$

and hence

$$r > -((g^+ - r) \vee 0) + g^+ = (r - g^+) \wedge 0 + g^+ = r \wedge g^+ \geqq 0 .$$

Let $m_1 = (r - g^+) \wedge 0, m_2 = g^-$. Then the above becomes $r > m_1 + g + m_2 \geqq 0$. Now $M < M + r - g^+$ and $M = M + m_1 = (M + r - g^+) \wedge M$ show that $m_1 \in M$.

Some basic facts about $f$-rings that will be used later are summarized below.

4.3. Since squares are positive in any totally ordered ring, they are also positive in any $f$-ring. If an $f$-ring has an identity, then it is necessarily positive and a weak order unit. If $A$ is any $\ell$-ring for which the identity element is a strong order unit, then every additive convex $\ell$-subgroup of $A$ is also necessarily an $\ell$-ideal. An archimedean $f$-ring with or without identity is commutative. The intersection of all the maximal $\ell$-ideals of an $f$-ring $A$ may be nonzero, yet there may be no nilpotents in $A$. For example, this happens in the ring of polynomials over the reals with the inverse lexiographic order.

The set $N(A)$ of all nilpotent elements of an $f$-ring is an $\ell$-ideal and $N(A) \subseteqq \bigcap \mathscr{M}$, where $\mathscr{M}$ is the set of all maximal $\ell$-ideals ([12; p. 175–176, Th. 4.8]). The ring $A$ contains a zero divisor if and only if it contains a nilpotent element ([7; p. 147, Th. 6] or alternatively

[12; p. 175, Th. 4.8]). Thus if $\bigcap \mathscr{M} = \{0\}$, then $A$ contains no divisors of zero. In this case each $A/M$, $M \in \mathscr{M}$, is a totally ordered integral domain. Thus any $o$-simple $f$-ring with an identity is a totally ordered integral domain. By [7; p. 132, Th. 9], $A/M$ contains no nontrivial one sided ideals.

4.4. From now on it is assumed that $A$ is an $f$-ring with identity that is also an algebra over the rationals $\boldsymbol{Q}$. It is shown in [11; p. 347] that every $\ell$-ring can be embedded in a lattice ordered algebra over the rationals. It is assumed that the maximal $\ell$-ideals $\mathscr{M}$ satisfy $\bigcap \mathscr{M} = \{0\}$. The set $S$ of topological units of 3.1 will now be taken as $S = \{r1 \mid 0 < r \in \boldsymbol{Q}\}$. For $0 < r \in \boldsymbol{Q}$, let $B(r)$ be defined as $B(r) = \{a \in A \mid |a| < r1\}$. Then the $B(r)$ are a basis for the neighborhoods of zero for a group topology on $A$. This topology may be non-Hausdorff. Ring multiplication in $A$ in general is not even continuous if one of the two variables is held fixed. A subbasis for $T(\hat{A})$ is given by all the sets of the form $\{M \in \mathscr{M} \mid g \in M + B(r)\}$ as $g$ and $r$ range over $A$ and $\boldsymbol{Q}^{+}\backslash\{0\}$.

LEMMA 4.5. *Assume that $A$ is a rational $f$-algebra with an identity. Then $\mathfrak{H} = T(\hat{A})$.*

*Proof.* Clearly, $\{M \mid g \in M + B(r)\} \subseteqq \{M \mid |M + g| < M + r1\}$. Since $M - r1 < M + g < M + r1$, by the previous lemma there is an $m \in M$ such that $-r1 < m + g < r1$. It follows that

$$\{M \mid g \in M + B(r)\} = \{M \mid |M + g| < M + r1\} .$$

Hence $T(\hat{A}) \subseteqq \mathfrak{H}$. Since $\mathfrak{H}$ is a compact Hausdorff topology, in order to show that $T(\hat{A}) = \mathfrak{H}$, it suffices to show that $T(\hat{A})$ is Hausdorff. For arbitrary $M_1 \neq M_2 \in \mathscr{M}$, write $1 = x_1 + x_2$, $x_i \in M_i$. Then $\{M \mid |M + x_i| < M + (1/2)1\}$ are disjoint $T(\hat{A})$-neighborhoods of $M_1$ and $M_2$.

REMARKS 4.6. In the above situation, alternatively, $T(\hat{A})$ can be characterized as the weak $\bar{A} = \{\bar{a} \mid a \in A\}$ topology on $\mathscr{M}$ by the following two observations.

(1) $\{M \mid |\bar{g}(M)| < r\} \subseteqq \{M \mid |M + g| < M + r1\} \subseteqq \{M \mid |\bar{g}(M)| \leqq r\}$

(2) The following sets are a subbasis for $T(\hat{A})$:

$$\{M \mid |\bar{g}(M)| < r\} \qquad 0 < g \in A, 0 < r \in \boldsymbol{Q} .$$

Partitions of identity are needed for the representation of certain $f$-rings as precisely all sections in a field. However, to show the existence of partitions of identity requires that $A^*$ be closed under bounded inversion. The next definition describes a property of the algebra—i.e., uniform closure—that will guarantee that $A^*$ is closed under bounded inversion.

DEFINITION 4.7. With the sets $B(r) = \{a \in A \mid |a| < r1\}$, $0 < r \in \boldsymbol{Q}$, as neighborhoods of zero, the $\boldsymbol{Q}$-algebra $A$ becomes a topological group under addition. In particular, $A$ is a uniform space. In keeping with the terminology already standard for $\Phi$-algebras, a subset $A_1$ of $A$ is *uniformly closed* if it is complete in this uniform structure. If $\Gamma(\mathscr{M}, E)$ are the sections in the field $\pi$ as constructed in 3.10 from $A$ and $\mathscr{M}$, then $\hat{A}$ is *uniformly dense* in $\Gamma(\mathscr{M}, E)$ if $\hat{A}$ is dense in the uniform space $\Gamma(\mathscr{M}, E)$, or equivalently, if for any $\sigma \in \Gamma(\mathscr{M}, E)$ and $0 < r \in \boldsymbol{Q}$, there is an $a \in A$ with $|\sigma - \hat{a}| < r\hat{1}$.

Various conditions equivalent to uniform closure for $\Phi$-algebras are given in [9] and [16]. Note that an algebra may be closed under bounded inversion without being uniformly closed, e.g., piecewise rational functions on $\boldsymbol{R}^+$.

The next proposition establishes the major objective of this paper.

PROPOSITION 4.8. *With $A$ and $\mathscr{M}$ as in 4.4, let $\pi: E \to \mathscr{M}$ be the field associated with $A$ by Proposition 3.10, and $\Gamma(\mathscr{M}, E)$ the $\ell$-group of all continuous sections in this field. Assume that $A^*$ is closed under bounded inversion. Then $\hat{A}$ is uniformly dense in $\Gamma(\mathscr{M}, E)$.*

*Proof.* Let $\sigma \in \Gamma(\mathscr{M}, E)$ and $0 < r \in \boldsymbol{Q}$ be arbitrary. For each $M \in \mathscr{M}$, there is an $a \in A$ with $\sigma(M) = \hat{a}(M)$. The set

$$U = \{b \in \mathscr{M} \mid |\sigma(b) - \hat{a}(b)| < r\hat{1}(b)\}$$

is open because it is $U = \sigma^{-1}[B(r)'(\hat{a})]$ in the notation of 3.7. By compactness, $\mathscr{M}$ can be covered by a finite number $n$ of such sets, i.e., $\mathscr{M} = U_1 \cup \cdots \cup U_n$. Let the corresponding elements $a$ be $a_1, \cdots, a_n$; i.e., $|\hat{a}_j(b) - \sigma(b)| < r1(b)$ for all $b \in U_j$. At this point two crucial pieces of information have to be utilized. First, since by Lemma 4.5 $\mathfrak{H} = T(\hat{A})$, the $U_j$ are also hull-kernel open. Second, by Proposition 2.11. associated with this covering there is at least one positive partition of identity:

$$0 \leqq e_j \leqq 1, \hat{e}_j \mid (\mathscr{M} \backslash U_j) = 0, 1 = e_1 + \cdots + e_n \qquad j = 1, \cdots, n .$$

The element $a \in A$ whose existence has to be established is $a = a_1 e_1 + \cdots + a_n e_n$. At each $b \in \mathscr{M}$ it satisfies

$$|\sigma(b) - a(b)| \leqq \sum_{j=1}^{n} |\sigma(b) - \hat{a}_j(b)| \hat{e}_j(b) < r\hat{1}(b) .$$

Thus $|\sigma - \hat{a}| < r\hat{1}$, and $\hat{A}$ is uniformly dense in $\Gamma(\mathscr{M}, E)$.

DEFINITION 4.9. The ring of all continuous real valued functions on any space $\mathscr{M}$ is denoted by $C(\mathscr{M})$. Suppose that $\mathscr{M}$ is any set of ideals of $A$ with $\bigcap \mathscr{M} = \{0\}$, and that $\mathfrak{H}$ is any topology on $\mathscr{M}$. If for any $a \in A$, any $g \in C(\mathscr{M})$, there is an element $ga \in A$ such that $M + ga = M + g(M)a$ for each $M \in \mathscr{M}$, then $A$ is a *$C(\mathscr{M})$-module.*

The conclusion of the previous proposition is particularly significant if $A$ is uniformly closed. Such an assumption requires that $A$ is a $C(\mathscr{M})$-module and hence in particular a real algebra.

COROLLARY 4.10. *If in addition $A$ is uniformly closed in 4.8, then*

$$\hat{A} = \Gamma(\mathscr{M}, E) \quad \textit{and} \quad \hat{A}^* = C(\mathscr{M})\hat{1} = \Gamma(\mathscr{M}, E)^* .$$

*Proof.* Since $\hat{A}$ is complete and dense in $\Gamma(\mathscr{M}, E)$, it follows from Lemma 3.9 (iv) that $\hat{A} = \Gamma(\mathscr{M}, E)$. The map $a \to \bar{a}: A^* \to \bar{A}^* \subseteqq C(\mathscr{M})$ is an $\ell$-isomorphism. Now $\bar{A}^*$ is a complete real algebra closed under the lattice operations. By the Stone-Weierstrass theorem, $\bar{A}^* = C(\mathscr{M})$. It is asserted that $M + a = M + \bar{a}(M)1$. If $J = \{a \in A \mid \bar{a}(M) = 0\}$, then $M \subseteqq J$. Conversely, since $M$ is a maximal $\ell$-ideal, since $J$ is an $\ell$-ideal, and since $1 \notin J$, it follows that $J = M$. Thus $\hat{A}^* = C(\mathscr{M})\hat{1} = \Gamma(\mathscr{M}, E)^*$.

COROLLARY 4.11. *Under the hypotheses of* 4.8, *the following are equivalent*:

( i ) *$A^*$ is archimedean;*

(ii) *$A$ is Hausdorff;*

(iii) *$E$ is Hausdorff.*

*Proof.* ( i )$\Leftrightarrow$( ii ): The condition that $a \in \overline{\{0\}}$ is equivalent to $|a| < r1$ for all $0 < r \in \boldsymbol{Q}$.

( i )$\Leftrightarrow$(iii): For $x, y \in E$, if $\pi(x) \neq \pi(y)$, then $x$ and $y$ lie in disjoint neighborhoods. If, however, $\pi(x) = \pi(y) = M \in \mathscr{M}$, it can easily be seen that $x$ and $y$ lie in disjoint neighborhoods if and only if $|x - y| > M + r1$ for some $0 < r \in \boldsymbol{Q}$. But the latter is equivalent to $(A^* + M)/M$ or equivalently, $A^*$ being archimedean.

REMARK 4.12. Suppose $A$ is a rational $f$-algebra with $\bigcap \mathscr{M} = \{0\}$; $L$ will denote the $\ell$-ideal of $A^*$ that is the kernel of the $\ell$-homomorphism $a \to \bar{a}: A^* \to \bar{A}^*$.

( i ) Then $L = \{a \in A^* \mid n|a| < 1$ for all integers $n\}$; $ac - ca \in L$ for any $a, c \in A^*$;

(ii) If $A$ is archimedean, then $L = \{0\}$;
(iii) $A/L \cong \bar{A}^*$ is a point separating dense subring of $C(\mathscr{M})$.

The main objective of this paper has been accomplished. Finally, the foregoing results are now summarized in a theorem and its converse.

THEOREM II 4.13. *Consider an f-algebra $A$ with identity 1 over the rationals $\boldsymbol{Q}$, with $\mathscr{M}$ as its space of maximal $\ell$-ideals; and $A^* \equiv \{a \in A \mid |a| < r1$ for some $r \in \boldsymbol{Q}\}$ its subalgebra of bounded elements. Construct the associated field* (3.10) *and the representations of $A$, $A^*$ as rings of sections contained in the $\ell$-groups of all sections as follows*:

$$\pi\colon E \equiv \cup\{A/M \mid M \in \mathscr{M}\} \to \mathscr{M},\ \pi^{-1}(M) = A/M \qquad M \in \mathscr{M}$$
$$\hat{a}\colon \mathscr{M} \to E,\ \hat{a}(M) = a + M \in A/M \qquad a \in A$$
$$\Gamma(\mathscr{M}, E) = \{\sigma \mid \sigma\colon \mathscr{M} \to E \text{ continuous},\ \pi \circ \sigma = \text{identity}\}$$
$$\Gamma(\mathscr{M}, E)^* = \{\sigma \in \Gamma(\mathscr{M}, E) \mid |\sigma| < r\hat{1},\ \text{some } 0 < r \in \boldsymbol{Q}\}$$
$$A \cong \hat{A} = \{\hat{a} \mid a \in A\} \subseteqq \Gamma(\mathscr{M}, E),\ A^* \cong \hat{A}^* = \{\hat{a} \mid a \in A^*\} \subseteqq \Gamma(\mathscr{M}, E)^* .$$

*Assume that*
(A) $\bigcap \mathscr{M} = \{0\}$;
(B) $1 \leqq a \in A^* \Rightarrow 1/a \in A$.

*Then*:
(i) $\pi\colon E \to \mathscr{M}$ *is a field*
(a) *over the compact Hausdorff space $\mathscr{M}$ in its hull-kernel topology*;
(b) *the topology of $E$ has the subbasis*

$$\{x \in E \mid |\hat{a}(\pi(x)) - x| < \pi(x) + r\hat{1}\},\ a \in A,\ 0 < r \in \boldsymbol{Q} .$$

(ii) *Each $\pi^{-1}(M)$, $M \in \mathscr{M}$, is a totally ordered integral domain.*
(iii) *$\hat{A}$ is uniformly dense in $\Gamma(\mathscr{M}, E)$, i.e., for any $\sigma \in \Gamma(\mathscr{M}, E)$ $0 < r \in \boldsymbol{Q}$, there is an $a \in A$ with $|\sigma - \hat{a}| < r\hat{1}$.*

*If in addition to* (A) *the condition* (C) *that $A$ is uniformly closed holds, then*:
(iv) (B) *holds; $E$ is Hausdorff; $A^* \cong \hat{A}^* = C(\mathscr{M})\hat{1} = \Gamma(\mathscr{M}, E)^*$; $A$ is a $C(\mathscr{M})$-module and in particular, an $\ell$-algebra over the reals.*
(v) $\hat{A} = \Gamma(\mathscr{M}, E)$.

COROLLARY TO THEOREM II 4.14. *With the same notation as in the previous theorem, assume condition (A) and the more restrictive condition* (B′):
(A) $\bigcap \mathscr{M} = \{0\}$,

(B′) $1 \leqq a \in A \Rightarrow 1/a \in A$.
*Then the conclusions* (i), (ii), *and* (iii) *of the previous theorem hold, except that* (ii) *can now be improved as in* (ii′):

(ii′) *Each* $\pi^{-1}(M), M \in \mathscr{M}$, *is a totally ordered division ring.*

(*Furthermore, if* (A), (B′), *and* (*C*) *hold, then all the conclusions* (i)–(v) *of the previous theorem as well as* (ii′) *hold.*)

Perhaps it is not immediately clear just what should be meant by the converse of the previous theorem. In the theorem, the algebraic properties of the ring $A$ determined the properties of the field derived from $A$, namely 4.13 (i) (a), (b), and (ii) or 4.14 (ii′). In the converse, the starting point must be any field $\pi: E \to \mathscr{M}$ whatever subject only to the restriction that it is to have these three properties. Now it has to be proved that $\Gamma(\mathscr{M}, E)$, or an appropriate ring $\Lambda \subseteqq \Gamma(\mathscr{M}, E)$ satisfies the algebraic conditions of the general class of rings under consideration.

The next corollary will be used to prove the converse of the main representation theorem and in particular will give a method for constructing all $f$-rings of the kind considered here. The reader should be warned that in the next corollary $\Gamma(\mathscr{M}, E)$ need not be a ring, as will be shown in Example 5.1. Since the proof of the next corollary is elementary and not very interesting, some of the details are only indicated.

CONVERSE OF THEOREM II 4.15. *Suppose* $\pi: E \to \mathscr{M}$ *is any field of vector lattices over* $\boldsymbol{Q}$ *with the weak order uniform structure* (3.12) *satisfying* 4.13 (i) (a), (b) *and* (ii). *Assume that* $\Lambda \subseteqq \Gamma(\mathscr{M}, E)$ *is a ring having the following properties*:

(a) $E = \bigcup\{\lambda(\mathscr{M}) \mid \lambda \in \Lambda\}$.

(b) $\Lambda$ *is an* $\ell$*-algebra over* $\boldsymbol{Q}$ *with* $1 \in \Lambda$.

(c) $\Lambda$ *separates the points of* $\mathscr{M}$.

(d) $\Lambda^* \equiv \{\lambda \in \Lambda \mid |\lambda| < r\,1$, *some* $0 < r \in \boldsymbol{Q}\}$ *is closed under bounded inversion.*

*For* $M \in \mathscr{M}$, *if* $O(M)$ *is the zero of* $\pi^{-1}(M)$, *define* $\Lambda^{-1}(O(M)) = \{\lambda \in \Lambda \mid \lambda(M) = O(M)\}$ *and* $\mathscr{M}(\Lambda) = \{\Lambda^{-1}(O(M)) \mid M \in \mathscr{M}\}$. *Then*:

(i) $\Lambda$ *is an f-algebra satisfying the same hypotheses as A in the Theorem* II 4.13.

(ii) *The maximal* $\ell$*-ideal space of* $\Lambda$ *in the hull-kernel topology is* $\mathscr{M}(\Lambda) \cong \mathscr{M}$.

(iii) $\Lambda$ *is uniformly dense in* $\Gamma(\mathscr{M}, E)$.

(iv) *The field constructed from* $\Lambda$ *and* $\mathscr{M}(\Lambda)$ (*see* 3.10) *is isomorphic to* $\pi$.

*Proof.* (i) Clearly, $\bigcap\{\Lambda^{-1}(O(M)) \mid M \in \mathscr{M}\} = \{0\}$. Thus 4.13(A) holds. Consequently, $\Lambda$ being a subdirect sum of totally ordered $f$-algebras, is itself an $f$-algebra. Lastly, 4.13(B) follows from (d).

(ii) Conclusion (ii) will follow from two general facts that are useful also in other circumstances. Firstly, for any $\sigma \in \Gamma(\mathscr{M}, E)$, the set $\{b \in \mathscr{M} \mid \sigma(b) - 1(b) > r(b)1(b)$, for some $0 < r(b) \in \boldsymbol{Q}\}$ is open. Secondly, for any subset $I \subset \Lambda$, define $K(I) = \{b \in \mathscr{M} \mid \sigma(b) = O(b)$ for all $\sigma \in I\}$. Suppose $I$ is any convex $\ell$-ideal with $K(I) = \emptyset$. Since each $\pi^{-1}(b)$ is $o$-simple, use of the first fact, followed by a simple covering argument, and finally a lattice operation produces an element in $I$ that exceeds 1; thus $I = \Lambda$. Hence $\mathscr{M}(\Lambda) = \{\Lambda^{-1}(O(M)) \mid M \in \mathscr{M}\}$. Now $\mathscr{M}(\Lambda)$ has a compact Hausdorff topology $\mathfrak{H}$ as the maximal $\ell$-ideal space of the $f$-ring $\Lambda$. Since $\mathfrak{H}$ is contained in the original topology, they both are the same.

(iii)—(iv) Conclusion (iii) follows from 2.11. The proof of (iv) is omitted since it is a straightforward consequence of fundamental definitions.

Specialization of Theorem II to a $\Phi$-algebra $A$ and use of 4.1 and 4.11, gives the usual representation of $A$ as a subalgebra of $D(\mathscr{M})$ as well as some additional information. One advantage of this representation is that the elements of $A$ need no longer take the values $\pm\infty$ on $\mathscr{M}$.

COROLLARY 2 TO THEOREM 4.17. *In the notation of Theorem* II, *if $A$ is a $\Phi$-algebra with $\mathscr{M}$ as its space of maximal $\ell$-ideals, and if*

(B) *$A^*$ is closed under bounded inversion,*

*then*:

(i) *Each $A/M$, $M \in \mathscr{M}$, is an integral domain contained in the hyper-real numbers.*

(ii) *$\hat{A} \cap C(\mathscr{M})\hat{1}$ is dense in $C(\mathscr{M})$.*

(iii) *$E$ is Hausdorff, the residue classes of the constant functions determine a subset $\mathscr{M} \times \boldsymbol{R} \subset E$ that* (*by* ii)) *has the product topology.*

(iv) *Each $\hat{a} \in \hat{A}$ takes values in $\mathscr{M} \times \boldsymbol{R}$ on a dense open subset of $\mathscr{M}$.*

(v) *$\hat{A}$ is uniformly dense in $A \cong \hat{A} \subsetneqq \Gamma(\mathscr{M}, E)$.*

*If* (B) *is replaced by the more stringent condition that*

(B′) *$A$ is closed under bounded inversion,*

*then* (i) *becomes*

(i′) *$A/M$ for $M \in \mathscr{M}$ are either the real or hyper-real numbers.*

*If*

(C) *$A$ is uniformly closed,*

*then $A$ is closed under bounded inversion and*

(v) $\hat{A} = \Gamma(\mathscr{M}, E)$.

5. **Examples.** A few examples and counterexamples will be given. A field $\pi: E \to \mathscr{M}$ is *trivial* or a *product field* if for some ring $F$ which is a topological group, $E = \mathscr{M} \times F$ and $\pi((M, f)) = M$ for $(M, f) \in \mathscr{M} \times F$ with the product topology on $E$. In conclusion, methods are described for constructing $f$-rings $A$ whose associated fields $\pi$ are nontrivial.

The next example gives an $f$-ring $A$ satisfying the hypotheses of the last Theorem II 4.13 so that $\hat{A}$ is uniformly dense in $\Gamma(\mathscr{M}, E)$ but where neither the completion of $A$ nor $\Gamma(\mathscr{M}, E)$ are rings.

EXAMPLE 5.1. Consider the algebra $A$ of all continuous functions $f: \boldsymbol{R}^+ \to \boldsymbol{R}$ ordered point-wise which are eventually rational, i.e., there is a $y$ depending on $f$ such that for all $x > y$, $f(x) = p(x)/q(x)$ where $p, q$ are polynomials and where $q$ has no roots in $(y, \infty)$. Then the space of maximal $\ell$-ideals of $A$ is the one point compactification $\mathscr{M} = \boldsymbol{R}^+ \cup \{\omega\}$, where $\hat{f}(\omega) = p/q$ and $\omega$ is the ideal of all $f$ with $p/q = 0$. It follows from [9; p. 84, 3.6] that it is impossible to embed $A$ as a subalgebra of a uniformly closed $\Phi$-algebra having the same maximal $\ell$-ideal space as $A$.

In order to find the completion $\tilde{A}$ of $A$ as an additive topological group in the absolute value, consider a Cauchy sequence $\{a_n\} \subset A$. There is a unique polynomial $p$ such that for all sufficiently large $n$, $a_n - p$ tends to zero at $\infty$. Thus $\{a_n - p\}$ converges to a continuous bounded function $f$ that tends to zero at $\infty$. If $C_0(\boldsymbol{R}^+)$ denotes the ring of all such functions $f$, then $\{a_n\}$ is identified with $(f, p)$ and $\tilde{A} = C_0(\boldsymbol{R}^+) \times \boldsymbol{R}[x]$. If also $\{b_n\} \leftrightarrow (g, q) \in \tilde{A}$, then it does not follow that $\{a_n b_n\} \leftrightarrow (fg, pq)$ because $\{a_n b_n - pq\}$ need not be Cauchy. Thus the natural definition of multiplication fails.

The field obtained from $A$ may be written as

$$\pi: E \equiv \boldsymbol{R}^+ \times \boldsymbol{R} \cup \boldsymbol{R}(x) \to \boldsymbol{R}^+ \cup \{\omega\} = \mathscr{M}.$$
$$\pi^{-1}(M) = \boldsymbol{R} \text{ for } M \neq \omega.$$
$$\pi^{-1}(\omega) = \boldsymbol{R}(x).$$

The topology on $\pi^{-1}(\mathscr{M}\backslash\{\omega\})$ is simply the product topology of $\boldsymbol{R}^+ \times \boldsymbol{R}$. A typical point of $\pi^{-1}(\omega) = \boldsymbol{R}(x)$ is of the form $e = s/p + c + q$, where $c \in \boldsymbol{R}$, where $s, p, q \in \boldsymbol{R}[x]$ are polynomials whose degrees satisfy $\partial p > \partial s$, $\partial q \geqq 1$ with $q(0) = 0$, and where one or both of the terms $s/p$, $q$ may be missing. A typical neighborhood of $e$ is obtained by taking any $a \in A$ that eventually equals $c + q$, a rational number $0 < r \in \boldsymbol{Q}$, an interval $(N, \infty)$ for some $N$, and forming

$$B(r)'(\hat{a}) \cap \pi^{-1}[(N, \infty) \cup \{\omega\}] .$$

The above basic neighborhood of $e$ contains the residue classes modulo $\omega$ of all elements of $A$ which are eventually of the form $\bar{s}/\bar{p} + \bar{c} + q$, where $\bar{c} \in \boldsymbol{R}$; where either $\bar{s}/\bar{p}$ is missing or $\bar{s}, \bar{p} \in \boldsymbol{R}[x]$ with $\partial\bar{p} > \partial\bar{s}$, and with $|c - \bar{c}| < r$. Let $h \in \boldsymbol{R}[x]$ be arbitrary and let $\sigma, \tau \in \Gamma(\mathscr{M}, E)$ be the maps

$$\begin{aligned} \sigma(y) &= 1/(y+1) + h(y), \tau(y) = y^2 && y \in \boldsymbol{R}^+ ; \\ \sigma(\omega) &= h , \qquad\qquad \tau(\omega) = x^2 \in \boldsymbol{R}(x) . \end{aligned}$$

The function $\tau\sigma: \mathscr{M} \to E$ is not continuous at $\omega$. In order to see this, take any $a \in A$ with $\hat{a}(\omega) = x^2h \in \boldsymbol{R}(x)$. There is an $N$ such that $\hat{a}(y) = y^2h$ for all $y > N$. It follows from

$$\begin{aligned} (\tau\sigma)(y) &= y^2/(y+1) + y^2h(y) && y \in \boldsymbol{R}^+ , \\ (\tau\sigma)(\omega) &= \tau(\omega)\sigma(\omega) = x^2h = \hat{a}(\omega) , \end{aligned}$$

that for all $y > N$, we have $(\tau\sigma)(y) - \hat{a}(y) = y^2/(y+1)$. Thus for any choice of $r$, we have for all sufficiently large $y$ that $(\tau\sigma)(y) \notin B(r)'(\hat{a})$; or, equivalently, that $|(\tau\sigma)(y) - \hat{a}(y)| > r1$. Thus $\tau\sigma \notin \Gamma(\mathscr{M}, E)$. Hence the latter is an additive group but not a ring. It is well known that the completion of a topological ring is always a topological ring. Although in the above example, additively $A$ is a topological group, multiplication in $A$ is not continuous even in each variable separately.

EXAMPLE 5.2. Consider the lexiographically ordered field $F = R[[t]]$ of real power series in $t$ having only a finite number of terms with positive exponents. Define $\mathscr{M} = \{0\} \cup \{1/n \mid n = 1, 2, \cdots\}$ and $C(\mathscr{M}, F)$ as the ring of all continuous maps of $\mathscr{M}$ into the discrete space $F$. The subring $A \subset (\mathscr{M}, F)$ consisting of all maps $f: \mathscr{M} \to F$ with $f(0) \in R[[t^2]]$ is an $f$-ring satisfying all the hypotheses (A), (B), (C) of Theorem II i.e., $A$ is complete and closed under bounded inversion. The maximal $\ell$-ideal space of $A$ is $\mathscr{M}$ and $A/M \cong F$ for each $M \in \mathscr{M}$. Thus $E = \mathscr{M} \times F$ with $\pi((M, z)) = M$, but $E$ does not have the product topology. Also note that $E$ is not Hausdorff.

EXAMPLE 5.3. Next a general method is described for constructing $f$-rings $A$ whose associated fields $\pi: E \to \mathscr{M}$ have stalks which are not all isomorphic to the same ring. Then the effect that usually can be produced by this method is illustrated by a simple example.

(a) The usual choice $\{r1 \mid 0 < r \in \boldsymbol{Q}\}$ of topological units (in the procedure of 3.1) makes any totally ordered division ring $D$ into an additive topological group, which is Hausdorff if and only if $D$ is archimedean. If $X$ is any compact Hausdorff space, then the $\ell$-group $M(X, D)$ of all continuous maps $X \to D$ will rarely be a topological ring. Suppose $G$ is a given set of homeomorphisms $\alpha: X \to X$ and $\theta$ is some map of $G$ into the semigroup $o -$ End $D$ of all order preserving endomorphisms of $D$. For any $\gamma \in o -$ End $D$ and $f \in M(X, D)$ a function $\gamma f \in M(X, D)$ is defined by $(\gamma f)(x) = \gamma(f(x))$ for $x \in X$. Suppose $B \subseteqq M(X, D)$ is any $f$-subalgebra such that $\theta(\alpha)f \in B$ for all $\alpha \in G$ whenever $f \in B$. (If $X$ is totally disconnected, then a natural choice for $B$ is the algebra of all continuous maps of $X$ into $D$ with the discrete topology on $D$.) Now define $A = \{f \in B \mid f(\alpha(x)) = (\theta(\alpha)f)(x)$ for all $x \in X, \alpha \in G\}$. If $\mathscr{M}$ is the orbit space of $X$ and $p: X \to \mathscr{M}$ is the natural projection defined at $x \in X$ by $p(x) = \{\alpha(x) \mid \alpha \in G\}$, then assume $p$ is open. Then it is known that $\mathscr{M}$ is a compact Hausdorff space. In this case $\mathscr{M}$ is the maximal $\ell$-ideal space of $A$. For $M \in \mathscr{M}$, $A/M$ is a subring of $D$. The set $D(\alpha)$ of all elements left fixed by $\theta(\alpha)$ for $\alpha \in G$ is a division subring of $D$. If $x \in X$ is a fixed point of $\alpha$, i.e., $\alpha(x) = x$, then $f(x) \in D(\alpha)$, in particular if $\alpha(x) = x$ for all $\alpha \in G$, then $A/p(x) \subseteqq \bigcap\{D(\alpha) \mid \alpha \in G\}$.

(b) Now the above situation is specialized. Take $X = \{0\} \cup \{1/n \mid n = \pm 1, \pm 2, \cdots\}$ in the usual topology; set $D = R[[t]]$, the power series field (see 5.2); $G = \{\alpha\}$, where $\alpha$ and $\theta$ are defined by $\alpha(0) = 0$, $\alpha(1/n) = -1/n$, $\theta(\alpha)(t) = t^2$. Note that $\alpha^2 = 1$ while $\theta(\alpha)^2 \neq 1$. Take $B$ as all continuous maps $f: X \to D$ with $D$ discrete. As above, $A = \{f \in B \mid f(\alpha(x)) = (\theta(\alpha)f)(x)$ for all $x \in X\}$. The condition $(\theta(\alpha)f)(0) = f(0)$ forces $f(0) \in \boldsymbol{R}$, the fixed field of $\theta(\alpha)$. If $M(1/n)$, $M(0)$ are defined as the $\ell$-ideals of functions vanishing at $1/n$, 0, then $\mathscr{M} = \{M(0)\} \cup \{M(1/n) \mid n = 1, 2, \cdots\}$ and $A/M(1/n) \cong D$ while $A/M(0) \cong \boldsymbol{R}$.

## REFERENCES

1. B. Banaschewski, *Über die Vervollständigung geordneter Gruppen*, Math. Nach. **16** (1957), 51-71.

2. G. Birkhoff and R. S. Pierce, *Lattice-ordered rings*, Anais. Acad. Bras. **28** (1956), 41-69.

3. P. Conrad and J. Dauns, *An embedding theorem for lattice ordered fields* (to appear)

4. J. Dauns, *Representation of f-rings*, Bull. Amer. Math. Soc. **74** (1968), 249-252.

5. J. Dauns and K. H. Hofmann, *The representation of biregular rings by sheaves*, Math. Zeit. **91** (1966), 103-123.

6. J. Dauns and K. H. Hofmann, *Representation of rings by sections*, A. M. S. Memoir **83** (1968).

7. L. Fuchs, *Partially ordered algebraic systems*, Addison-Wesley, 1963.

8. M. Henriksen and J. R. Isbell, *Lattice-ordered rings and function rings*, Pacific J. Math. **12** (1962), 553-565.

9. M. Henriksen and D. G. Johnson, *Archimedean lattice-ordered algebras*, Fund. Math. **50** (1961), 73-97.

10. M. Henriksen, J. R. Isbell, and D. G. Johnson, *Residue class fields*, Fund. Math. **50** (1961), 107-117.

11. J.R. Isbell, *Embedding two ordered rings in one ordered ring, Part I*, J. Algebra **4** (1966), 341-364.
12. D. G. Johnson, *A structure theory for a class of lattice-ordered rings*, Acta. Math. **104** (1960), 163-215.
13. K. Keimel, *Représentation d'anneaux réticulés dans des faisceaux*, C. R. Acad. Sci. Pari (1968), 124-127.
14. ———, *Anneaux réticulés quasi-réguliers et hyperarchimédiens*, C. R. Acad. Sci. Pari (1968), 524-525.
15. J. E. Kist, *Representations of archimedean function rings*, Illinois J. Math. **7** (1963), 269-278.
16. P. Nanzetta, *Maximal lattice-ordered algebras of continuous functions* Fund. Math. **63** (1968), 53-75.
17. B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202-252.
18. R. S. Pierce, *Radicals in function rings*, Duke Math. J. **23** (1956), 253-261.
19. ———, *Modules over commutative regular rings*, A. M. S. Memoir **70** (1967).

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