

# Pacific Journal of Mathematics

**A RATE OF GROWTH CRITERION FOR UNIVERSALITY OF  
REGRESSIVE ISOLS**

JUDITH LEE GERSTING

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Let  $\alpha$  be an infinite retraceable set having the property that if  $a_n$  is the retraceable function ranging over  $\alpha$ , then for each partial recursive function  $p(x)$ , there is a number  $m$  such that  $p(a_n) < a_{n+1}$  whenever  $n \geq m$  and  $p(a_n)$  is defined. Recently, T. G. McLaughlin proved the existence of retraceable sets having this property and also of such sets having recursively enumerable complements. In addition, he showed that sets of this kind will be immune and that each of their regressive subsets will be retraceable. The main result of this paper states that (infinite) regressive isols that contain a retraceable set with this property will be universal. As corollary to this result we obtain the existence of cosimple universal regressive isols.

We will assume that the reader is familiar with the terminology and main results of the papers listed in the references. We let  $E$  denote the collection of all nonnegative integers,  $I$  the collection of all isols, and  $A_R$  the collection of all regressive isols. If  $f: E \rightarrow E$  is a recursive and combinatorial function, then we let  $C_f$  denote its canonical extension to  $I$ . If  $\alpha \subseteq E$ , then we say that  $\alpha$  is *cofinite* if the complement of  $\alpha$  is a finite set, i.e., if there is a number  $m$  such that  $n \geq m \Rightarrow n \in \alpha$ . If  $f$  is a partial function (from a subset of  $E$  into  $E$ ) then we denote the domain and range of  $f$  by  $\delta f$  and  $\rho f$ , respectively. If  $f$  is a partial function and  $x$  and  $y$  any numbers, then we write " $f(x) < y$ " to mean either that  $f(x)$  is undefined or else  $f(x)$  is defined and  $f(x) < y$ ; we interpret " $f(x) \leq y$ " in a similar manner. We recall from [7] that an infinite isol  $A$  is *universal* if for each pair of recursive combinatorial functions  $f$  and  $g$ , one has

$$C_f(A) = C_g(A) \implies \{n \mid f(n) = g(n)\} \text{ is cofinite.}$$

2. *T*-regressive isols. We call a retraceable function  $a_n$  *T*-retraceable if it has the property that for each partial recursive function  $p(x)$ , there is a number  $m$  such that

$$n \geq m \implies p(a_n) < a_{n+1}.$$

We call an infinite retraceable set *T*-retraceable if it is the range of a *T*-retraceable function. A useful result of T. G. McLaughlin, [8], is

(1) cosimple *T*-retraceable sets exist. McLaughlin also observed that

(2)  $T$ -retraceable sets are immune.

We call an infinite regressive isol  $T$ -regressive if it contains a  $T$ -retraceable set. By (1) and (2) it follows that both  $T$ -regressive isols and cosimple  $T$ -regressive isols exist. We let  $A_{TR}$  denote the collection of all  $T$ -regressive isols. Let  $\alpha$  be a  $T$ -retraceable set and  $\delta$  any finite set. Then it can be easily shown that the set  $\alpha \cup \delta$  is also  $T$ -retraceable. It follows from this property that

$$(3) \quad T \in A_{TR} \text{ and } n \in E \implies T + n \in A_{TR} .$$

REMARK. We wish to give next an example of a  $T$ -retraceable set. In a proof not yet published, T. G. McLaughlin used movable markers to obtain the existence of a cosimple  $T$ -retraceable set. Our proof here will be a little easier because we do not require that the  $T$ -retraceable set that we construct be cosimple.

Let  $\{p_i(x)\}$  be an enumeration of all partial recursive functions of one variable such that each partial recursive function appears exactly once in the enumeration. Let the function  $u_n$  be defined by

$$u_n = \sum_{0 \leq i, x \leq n} p_i(x) ,$$

where we set  $p_i(x) = 0$  if  $p_i(x)$  is undefined. By [3, Lemma 2], there is a retraceable function  $t_n^*$  such that

$$t_n^* > u_n , \quad \text{for each } n \in E .$$

Let  $t_n = 2t_n^*$ . Then  $t_n$  is also a retraceable function, and ranges over a set of even numbers. In addition, for each partial recursive function  $p_e(x)$  ( $e$  denoting the index in the enumeration), we see that

$$(4) \quad n \geq e \implies p_e(n) \leq u_n < t_n .$$

Let the function  $a_n$  be defined by

$$\begin{cases} a_0 = t_1 = t(1) , \\ a_n = t^{n+1}(2n + 1) , \quad \text{for } n \geq 1 . \end{cases}$$

Because the retraceable function  $t_n$  assumes only even values, it is readily seen that  $a_n$  is a retraceable function. Also, for each partial recursive function  $p_e(x)$ ,

$$n \geq e \implies t^{n+1}(2n + 1) \geq e ,$$

and therefore by (4) we have, for  $n \geq e$

$$\begin{aligned} p_e(a_n) &= p_e(t^{n+1}(2n + 1)) \\ &< t^{n+2}(2n + 1) \\ &< t^{n+2}(2n + 3) = a_{n+1} . \end{aligned}$$

Hence  $\alpha_n$  is a  $T$ -retraceable function, and its range will be a  $T$ -retraceable set.

3. **The main result.** The main result that we wish to prove is that  $T$ -regressive isols are universal. For this purpose we will need two lemmas, each of which involves a relation  $T \leq^* u_n$  between infinite regressive isols  $T$  and functions  $u_n$ ; the relation was introduced in [3] and we now recall how it is defined. If  $T$  is an infinite regressive isol and  $u_n$  any function from  $E$  into  $E$ , then  $T \leq^* u_n$  if there is a regressive function  $t_n$  that ranges over a set in  $T$  such that  $t_n \leq^* u_n$ ; here  $t_n \leq^* u_n$  means that the mapping  $t_n \rightarrow u_n$  has a partial recursive extension. It can be shown that if  $T \leq^* u_n$ , then  $t_n \leq^* u_n$  for every regressive function  $t_n$  that ranges over a set in  $T$  [3]. Also if  $T$  is any infinite regressive isol and  $u_n$  any recursive function, then  $T \leq^* u_n$ . The first lemma we will state without proof because it can be readily obtained from results in [3].

LEMMA 1. *Let  $T$  be an infinite regressive isol and let  $u_n$  be any function such that  $T \leq^* u_n$ . Let  $u_n \geq 1$ , for each number  $n \in E$ . Then  $\sum_T u_n \in A_R$  and if  $t_n$  is any regressive function that ranges over a set in  $T$ , then*

$$j(t_0, 0), \dots, j(t_0, u_0 - 1), j(t_1, 0), \dots, j(t_1, u_1 - 1), \dots,$$

*represents a regressive enumeration of a set belonging to  $\sum_T u_n$ .*

LEMMA 2. *Let  $T$  be a  $T$ -regressive isol and let  $u_n$  and  $\tilde{u}_n$  be any functions such that both  $T \leq^* u_n$  and  $T \leq^* \tilde{u}_n$ . Let  $u_n \geq 1$  and  $\tilde{u}_n \geq 1$ , for each number  $n \in E$ . Let*

$$(5) \quad \sum_T u_n = \sum_T \tilde{u}_n.$$

*Then the set  $\{n \mid u_n = \tilde{u}_n\}$  is cofinite.*

*Proof.* Let  $t_n$  be a  $T$ -retraceable function that ranges over a set in  $T$ , and let  $\tau = \rho t_n$ . By Lemma 1,

$$j(t_0, 0), \dots, j(t_0, u_0 - 1), j(t_1, 0), \dots, j(t_1, u_1 - 1), \dots,$$

$$j(t_0, 0), \dots, j(t_0, \tilde{u}_0 - 1), j(t_1, 0), \dots, j(t_1, \tilde{u}_1 - 1), \dots,$$

represent regressive enumerations of sets belonging to  $\sum_T u_n$  and  $\sum_T \tilde{u}_n$  respectively. Let the regressive functions determined by these two enumerations be given by  $g_n$  and  $\tilde{g}_n$  respectively. In light of (5) we see that  $g_n$  and  $\tilde{g}_n$  will be regressive functions that range over sets in the same isol, and therefore by results in [5] it follows that  $g_n \simeq \tilde{g}_n$ , i.e., there is a one-to-one partial recursive function  $p(x)$  such that

$$(6) \quad \rho g \subset \delta p \text{ and } (\forall n)[p(g_n) = \tilde{g}_n] .$$

In addition, because  $T \leq^* u_n$  and  $T \leq^* \tilde{u}_n$  there will also be partial recursive functions  $f_1$  and  $f_2$  such that

$$\begin{aligned} \tau \subset \delta f_1 \text{ and } (\forall n)[f_1(t_n) = u_n] , \\ \tau \subset \delta f_2 \text{ and } (\forall n)[f_2(t_n) = \tilde{u}_n] . \end{aligned}$$

Define the four functions,

$$\begin{aligned} p_1(x) &= kpj(x, 0) , \\ p_2(x) &= kpj(x, f_1(x) \div 1) , \\ q_1(x) &= kp^{-1}j(x, 0) , \\ q_2(x) &= kp^{-1}j(x, f_2(x) \div 1) , \end{aligned}$$

where  $k$  denotes the familiar recursive function having the property that  $kj(x, y) = x$ . Then each of these functions is partial recursive and will map  $\tau$  into  $\tau$  in the following way:

$$\begin{aligned} \text{if } pj(t_n, 0) = j(t_k, y) & \quad \text{then } p_1(t_n) = t_k , \\ \text{if } pj(t_n, u_n - 1) = j(t_k, y) & \quad \text{then } p_2(t_n) = t_k , \\ \text{if } p^{-1}j(t_n, 0) = j(t_k, y) & \quad \text{then } q_1(t_n) = t_k , \text{ and} \\ \text{if } p^{-1}j(t_n, \tilde{u}_n - 1) = j(t_k, y) & \quad \text{then } q_2(t_n) = t_k . \end{aligned}$$

For each number  $n \in E$ , let

$$\begin{aligned} p_1(t_n) &= t_{n'} , & p_2(t_n) &= t_{n''} , \\ q_1(t_n) &= t_{n^*} , & q_2(t_n) &= t_{n^{**}} . \end{aligned}$$

Because  $t_n$  is a  $T$ -retraceable function and  $p_1(x)$  is a partial recursive function, there will exist a number  $m_1$  such that

$$n \geq m_1 \implies p_1(t_n) < t_{n+1} .$$

Combining this with the property that  $t_n$  is a retraceable function and hence strictly increasing, we see that

$$\begin{aligned} n \geq m_1 &\implies p_1(t_n) < t_{n+1} \\ &\implies t_{n'} < t_{n+1} \\ &\implies n' \leq n . \end{aligned}$$

Therefore,

$$(A) \quad n \geq m_1 \implies n' \leq n .$$

In a similar fashion it can be shown that there are numbers  $m_2, k_1$  and  $k_2$  such that

- (B)  $n \geq m_2 \implies n'' \leq n$ ,
- (C)  $n \geq k_1 \implies n^* \leq n$ ,
- (D)  $n \geq k_2 \implies n^{**} \leq n$ .

Let

$$\bar{m} = \max \{m_1, m_2, k_1, k_2\},$$

and let  $m$  be a number chosen such that

$$(7) \quad \begin{cases} m \geq \bar{m} \text{ and} \\ (\forall n)[n \geq m \implies n' \geq \bar{m} \text{ and } n^* \geq \lceil \bar{m} \rceil]. \end{cases}$$

To complete the proof, we now verify that

$$(8) \quad n \geq m \implies u_n = \tilde{u}_n.$$

In view of the definition of the functions  $g_n$  and  $\tilde{g}_n$  and the relation (6), we see that to verify (8), it suffices to prove

$$(*) \quad n \geq m \implies pj(t_n, 0) = j(t_n, 0);$$

and this will be our approach here. To prove the relation (\*), assume that  $n \geq m$  and let

$$(9) \quad pj(t_n, 0) = j(t_r, x).$$

Then we wish to verify

- (a)  $r = n$  and
- (b)  $x = 0$ .

For (a). We first note that  $r = n'$  and therefore by (7) and (A), we have

$$(10) \quad r \leq n.$$

If  $x = 0$ , then  $r^* = n$ . In this event we have by (7) and  $n \geq m$  that  $n' = r \geq k_1$  so by (C) it follows that  $r^* \leq r$ , and hence also that  $n \leq r$ . Combining this with (10), we see that if  $x = 0$  then  $r = n$ , and we are done.

Assume now that  $x > 0$ ; then  $0 < x \leq \tilde{u}_r - 1$ . Consider the diagram,

$$\begin{array}{ccc} j(t_n, 0) & & j(t_s, y) \\ p \downarrow & & \uparrow p^{-1} \\ j(t_r, x) & \text{-----} & j(t_r, \tilde{u}_r - 1). \end{array}$$

We note first that  $n \leq s$  and  $s = r^{**}$ . Also from (7) and  $n \geq m$  we have  $n' = r \geq k_2$ . By (D) it follows then that  $s = r^{**} \leq r$ . Hence

$n \leq r$ ; and combining this relation with (10) implies  $r = n$ . This completes the proof of part (a).

For (b). By part (a), we know that

$$(11) \quad pj(t_n, 0) = j(t_n, x) ,$$

where  $0 \leq x \leq \tilde{u}_n - 1$ . We wish to show here that  $x = 0$ . It can be proven by an argument similar to that in part (a), that one will also have

$$(12) \quad p^{-1}j(t_n, 0) = j(t_n, y) ,$$

for some  $y, 0 \leq y \leq u_n - 1$ ; and we will omit the details. We will therefore have the following diagram,

$$\begin{array}{ccc} j(t_n, y) & & j(t_n, 0) \\ p^{-1} \uparrow & & \downarrow p \\ j(t_n, 0) & \text{-----} & j(t_n, x) \end{array}$$

and this array will only be possible if  $x = y = 0$ . This verifies part (b) and completes the proof.

**COROLLARY 1.** *Let  $T$  be a  $T$ -regressive isol and let  $u_n$  and  $\tilde{u}_n$  be any functions such that both  $T \leq^* u_n$  and  $T \leq^* \tilde{u}_n$ . Then*

$$\sum_T u_n = \sum_T \tilde{u}_n \implies \{n \mid u_n = \tilde{u}_n\} \text{ is cofinite .}$$

*Proof.* Let the symbol  $1$  denote the recursive function identically equal to  $1$ . Then  $T \leq^* 1$ , and by [3, Lemma 3] both  $T \leq^* (u_n + 1)$  and  $T \leq^* (\tilde{u}_n + 1)$ . Consider the following implications:

$$\begin{aligned} \sum_T u_n = \sum_T \tilde{u}_n &\implies \sum_T u_n + \sum_T 1 = \sum_T \tilde{u}_n + \sum_T 1 \\ &\implies \sum_T (u_n + 1) = \sum_T (\tilde{u}_n + 1) \\ &\implies \{n \mid u_n + 1 = \tilde{u}_n + 1\} \text{ is cofinite} \\ &\implies \{n \mid u_n = \tilde{u}_n\} \text{ is cofinite .} \end{aligned}$$

The first implication is clear, the second follows from results in [3], the third from Lemma 2 and the last one is clear. Together they imply the desired result and this completes the proof.

**THEOREM 1.** *Let  $T$  be a  $T$ -regressive isol. Then  $T$  is universal.*

*Proof.* Let  $f$  and  $g$  be any recursive combinatorial functions. We wish to show that

$$(13) \quad C_f(T) = C_g(T) \implies \{x \mid f(x) = g(x)\} \text{ is cofinite .}$$

Let the functions  $e_n$  and  $\tilde{e}_n$  be defined by

$$(14) \quad \begin{cases} e_0 = f(0) , \\ e_n = f(n) - f(n - 1) , \end{cases} \quad \text{for } n \geq 1 ,$$

$$(15) \quad \begin{cases} \tilde{e}_0 = g(0) , \\ \tilde{e}_n = g(n) - g(n - 1) , \end{cases} \quad \text{for } n \geq 1 .$$

Clearly  $e_n$  and  $\tilde{e}_n$  are recursive functions, since combinatorial functions are also increasing. In addition to this,  $e_n$  and  $\tilde{e}_n$  will be the *e-difference* functions associated with the functions  $f$  and  $g$  respectively, [see 1]. Hence by [1, Corollary 2] we see that

$$\begin{aligned} C_f(T) &= \sum_{T+1} e_n , \quad \text{and} \\ C_g(T) &= \sum_{T+1} \tilde{e}_n . \end{aligned}$$

To verify (13) assume that  $C_f(T) = C_g(T)$ . Then

$$(16) \quad \sum_{T+1} e_n = \sum_{T+1} \tilde{e}_n .$$

Because  $T$  is a  $T$ -regressive isol, it follows from (3) that  $T + 1$  will also be a  $T$ -regressive isol. Also both  $T + 1 \leq^* e_n$  and  $T + 1 \leq^* \tilde{e}_n$ , since  $e_n$  and  $\tilde{e}_n$  are each recursive functions. In light of Corollary 1, it follows from (16) that there is a number  $m \in E$ , such that

$$(17) \quad n \geq m \implies e_n = \tilde{e}_n .$$

If  $m = 0$ , then it is easy to see from (14) and (15) that  $f(n) = g(n)$  for each number  $n \in E$ ; and the desired result follows. Let us assume now that  $m \geq 1$ . Let

$$\begin{aligned} U &= \sum_{(T+1-m)} e_{m+n} \\ &= \sum_{(T+1-m)} \tilde{e}_{m+n} . \end{aligned}$$

Then, from (16) we have

$$e_0 + \dots + e_{m-1} + U = \tilde{e}_0 + \dots + \tilde{e}_{m-1} + U ;$$

and hence also

$$(18) \quad e_0 + \dots + e_{m-1} = \tilde{e}_0 + \dots + \tilde{e}_{m-1} .$$

In view of (14) and (15), it follows from (18) that  $f(m - 1) = g(m - 1)$ . Finally, combining this fact with (14), (15) and (17) we see that

$$n \geq m - 1 \implies f(n) = g(n) .$$



Therefore the set  $\{n \mid f(n) = g(n)\}$  is cofinite, and this completes the proof of the theorem.

**THEOREM 2.** *There exist cosimple universal regressive isols.*

*Proof.* Use Theorem 1 and the fact that cosimple  $T$ -regressive isols exist.

**4. Concluding remarks.** (A) The existence of universal regressive isols was first proved by E. Ellentuck in some notes not yet published. Also, in some unpublished notes, J. Barback showed that multiple-free regressive isols exist and that these are also universal.

(B) We have also proved the following result, stated here without proof, of which Theorem 1 is a corollary:

Let  $T$  be a  $T$ -regressive isol, and let  $f$  and  $g$  be any recursive combinatorial functions. Then

$$C_f(T) \leq C_g(T) \implies \{x \mid f(x) \leq g(x)\} \text{ is cofinite.}$$

(C) We wish to state without proofs some additional properties of the collection  $A_{TR}$  of all  $T$ -regressive isols. We will assume that the reader is familiar with the three relations  $\leq$ ,  $\leq$  and  $*$  defined between infinite regressive isols; the first two are defined in [6], and the third in [2].

**THEOREM A.** *Let  $A \in A_R - E$  and  $T \in A_{TR}$ . Then*

- (a)  $T$  is multiple-free,
- (b)  $A \leq T \implies A \in A_{TR}$ ,
- (c)  $A \leq T \implies A \in A_{TR}$ .

**THEOREM B.** *Let  $A, B, T \in A_{TR}$ . Then*

- (a)  $A * B \implies \min(A, B) \in A_{TR}$ ,
- (b)  $A + B \in A_R \implies \min(A, B) \in A_{TR}$ ,
- (c)  $A, B \leq T \implies \min(A, B) \in A_{TR}$ .

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# Pacific Journal of Mathematics

Vol. 31, No. 3

BadMonth, 1969

George E. Andrews, <i>On a calculus of partition functions</i> . . . . .	555
Silvio Aurora, <i>A representation theorem for certain connected rings</i> . . . . .	563
Lawrence Wasson Baggett, <i>A note on groups with finite dual spaces</i> . . . . .	569
Steven Barry Bank, <i>On majorants for solutions of algebraic differential equations in regions of the complex plane</i> . . . . .	573
Klaus R. Bichteler, <i>Locally compact topologies on a group and the corresponding continuous irreducible representations</i> . . . . .	583
Mario Borelli, <i>Affine complements of divisors</i> . . . . .	595
Carlos Jorge Do Rego Borges, <i>A study of absolute extensor spaces</i> . . . . .	609
Bruce Langworthy Chalmers, <i>Subspace kernels and minimum problems in Hilbert spaces with kernel function</i> . . . . .	619
John Dauns, <i>Representation of L-groups and F-rings</i> . . . . .	629
Spencer Ernest Dickson and Kent Ralph Fuller, <i>Algebras for which every indecomposable right module is invariant in its injective envelope</i> . . . . .	655
Robert Fraser and Sam Bernard Nadler, Jr., <i>Sequences of contractive maps and fixed points</i> . . . . .	659
Judith Lee Gersting, <i>A rate of growth criterion for universality of regressive isols</i> . . . . .	669
Robert Fred Gordon, <i>Rings in which minimal left ideals are projective</i> . . . . .	679
Fred Gross, <i>Entire functions of several variables with algebraic derivatives at certain algebraic points</i> . . . . .	693
W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Wreath products of ordered permutation groups</i> . . . . .	703
W. J. Kim, <i>The Schwarzian derivative and multivalence</i> . . . . .	717
Robert Hamor La Grange, Jr., <i>On <math>(m - n)</math> products of Boolean algebras</i> . . . . .	725
Charles D. Masiello, <i>The average of a gauge</i> . . . . .	733
Stephen H. McCleary, <i>The closed prime subgroups of certain ordered permutation groups</i> . . . . .	745
Richard Roy Miller, <i>Gleason parts and Choquet boundary points in convolution measure algebras</i> . . . . .	755
Harold L. Peterson, Jr., <i>On dyadic subspaces</i> . . . . .	773
Derek J. S. Robinson, <i>Groups which are minimal with respect to normality being intransitive</i> . . . . .	777
Ralph Edwin Showalter, <i>Partial differential equations of Sobolev-Galpern type</i> . . . . .	787
David Slepian, <i>The content of some extreme simplexes</i> . . . . .	795
Joseph L. Taylor, <i>Noncommutative convolution measure algebras</i> . . . . .	809
B. S. Yadav, <i>Contractions of functions and their Fourier series</i> . . . . .	827
Lindsay Nathan Childs and Frank Rimi DeMeyer, <i>Correction to: "On automorphisms of separable algebras"</i> . . . . .	833
Moses Glasner and Richard Emanuel Katz, <i>Correction to: "Function-theoretic degeneracy criteria for Riemannian manifolds"</i> . . . . .	834
Satish Shirali, <i>Correction to: "On the Jordan structure of complex Banach *algebras"</i> . . . . .	834
Benjamin Rigler Halpern, <i>Addendum to: "Fixed points for iterates"</i> . . . . .	834